An Introduction to Point - Set - Topology (Part II)
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Lecture No. 50
The Long Line

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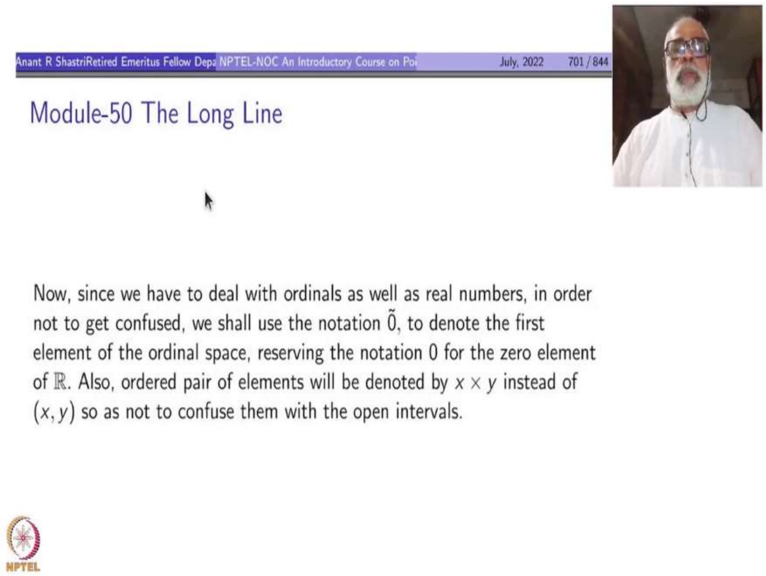

Hello. Welcome to module 50, the NPTEL-NOC course an introductory course on Points Set Topology Part 2. So, continuing with the study of ordinals. We will now construct an example which is called long line. This long line is an example usually in the study of manifolds that we are going to do in the last chapter, here, in this course. The word `line' is used for any topological space which is homeomorphic to, let us say, the whole of the real line or to an open interval.

However, the long line is going to be something which is only locally homeomorphic to open intervals. That means that every point has a neighborhood which is homeomorphic to an open interval. But the entire space is not homeomorphic to an open interval. First of all, that is not the whole point here. We will see that this space is non compact, it is not even second countable that the whole idea. Of course, it is Hausdorff space. So, II-countability axiom is violated here. That is why this is an important example. Let us see now the details.

Once again, now, we have to deal with both the real line as well as the ordinals simultaneously and the notation for 0 will become a conflicting point. So, reserving the notation 0 for the ordinary 0 of the real numbers, the 0 of the ordinals namely, the smallest ordinal will now be denoted by $\tilde{0}$. So, that is the first remark I have to make. Sure.

Also, again we have ordered pairs of elements just like in the previous example of Tychonoff plank. So, we will continue to use this notation $x \times y$ for element of the product set $X \times Y$. so that there is no confusion with intervals (x, y). So, this notation $x \times y$ will continue. In addition, we will have this $\tilde{0}$ will denote the least element in the ordinals. That is the difference.

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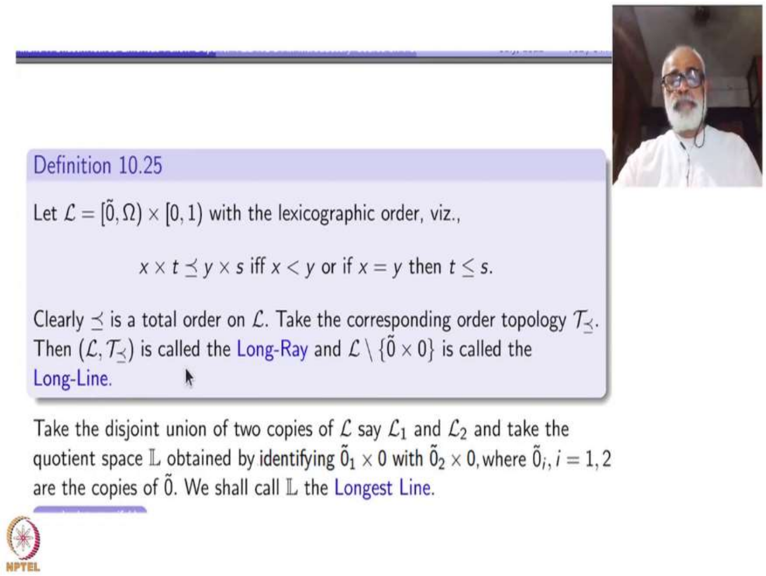

So, I start with again, just like in the Tychonoff plank example, start with the product set $\mathcal{L} = [\tilde{0}, \Omega) \times [0, 1)$. So, both are half open intervals, but there is a difference. Do not get confused. We are not taking both of them as subspaces of ordinals. Here the second factor is the interval in the usual real numbers. The half open interval [0 ,1). Both factors are linearly ordered. Whenever you have a product of two posets, we can take the lexicographic ordering on it.

Namely, I will denote elements of this by symbols such as $x \times t$. Another element y cross, (let us say, x and y are ordinals, t and s are real numbers between 0 and 1), is said to follow the first one if and only if x is less than y in the ordinals or if x is equal to y, then t must be less than equal to s.

That is the lexicographic ordering. Since words are arranged with this rule in any dictionary, that is why this rule itself is called lexicographic ordering or a dictionary ordering. So clearly, this is a total order on $\mathcal{L}$. Because if you have two elements here, first you can compare the first coordinates x and y. If x is less than y, well and good, we have $x \times t$ is precedes $y \times s$. Or it may be other way round. However if $x = y$ then compare t and s etc. No problem because both of them are total order. (Indeed lexicographic order on the product of two totally ordered sets is totally ordered.)

The first one has $\tilde{0} \times 0$ as the least element so we call it a ray.

Then you remove $\tilde{0} \times 0$, the initial point here, throw away that point, that is called a long line. It is just like half closed interval $[0, \infty)$ is a ray and $(0, \infty)$ being called a line. This is just a terminology right now. Not a big deal here. You could have interchanged the definition if you like. But that is not good. Because this is a ray. You can think of this also as ray of course, but this is called long line. Wait for more justiifcation.)

Now, I will define one more thing here, without much effort. I take two disjoint copies of $\mathcal{L}$, let us call them $\mathcal{L}_1$ and $\mathcal{L}_2$. Take the quotient space, of the disjoint union obtained by identifying the two least elements, $(\tilde{0} \times 0)_1$ and $(\tilde{0} \times 0)_2$. Because I have taken two copies of $\mathcal{L}$, I have these two copies of the least elements as well, so identify them.

We shall call the quotient space the longest line and denote it by $\mathbb{L}$. It is just like two copies of the ray $[0, \infty)$ and getting the entire real line by identifying the two 0's. I will justify this name also later along with the other names we have used. You will see that there is no `line' bigger than this.

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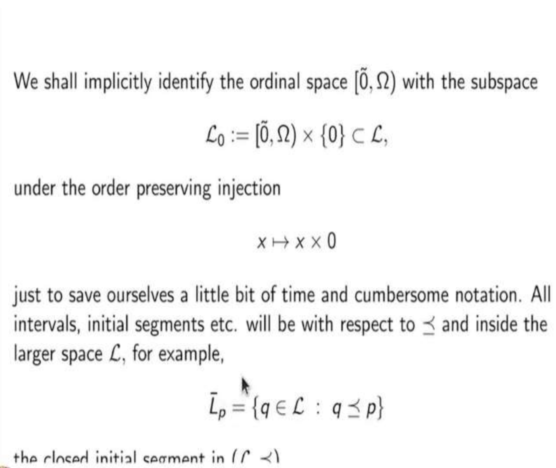

We shall implicitly identify the ordinal space $[\tilde{0}, \Omega)$ with the this coordinate subspace $\mathcal{L}_0$ of $\mathcal{L}$, namely, $[\tilde{0}, \Omega) \times \{0\}$, the subspace of all points second coordinate being 0 of the real numbers. So, I will not use the cumbersome notation every time something cross 0 to denote elements of this subspace. I do not have to write.

So, I am identifying this $[\tilde{0}, \Omega)$ with this subspace under the identification x goes to $x \times 0$. Just to save ourselves a little bit of time and cumbersome notation, that is all.

Terms such as intervals, initial segments etc will be with respect to this new order lexiographic order, on the product space $\mathcal{L}$.

For example, now, $\overline{L_p}$ will denote all the elements q inside $\mathcal{L}$ such that $q \leq p$. This will be the closed left ray. Similarly closed right ray is defined.

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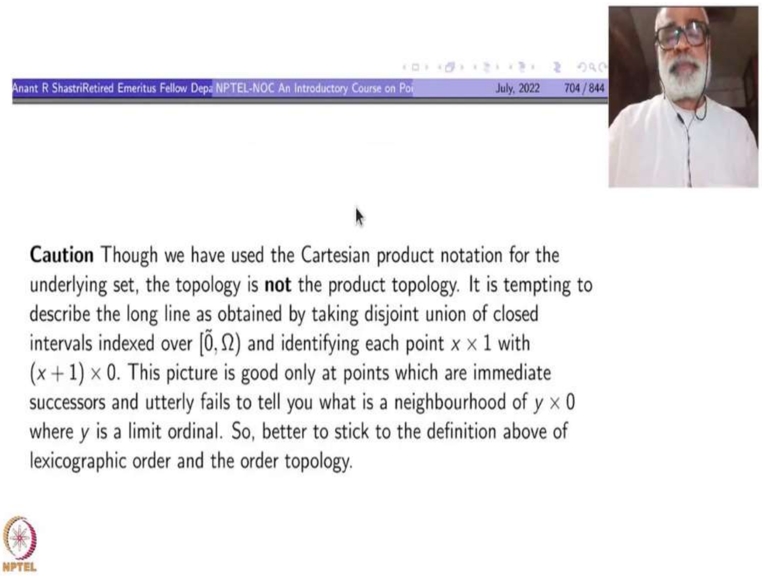

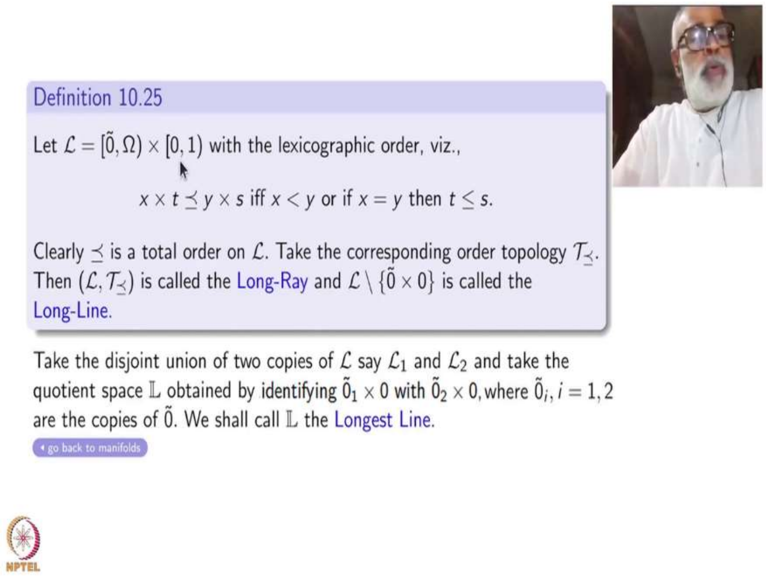

So, here is a caution. Though we have used the Cartesian product notation for the underlying set $\mathcal{L}$, the topology on it is not the product topology. The topology is defined by using the lexicographic order that is my $\mathcal{T}_{\leq}$. It is tempting to describe the long line as obtained by taking disjoint union of closed intervals indexed over $[\tilde{0}, \Omega)$ that is the ordinals, and identifying the endpoint $x \times 1$ with $(x + 1) \times 0$, the starting point of the next interval.

This is alright, because in the lexicographic order the larger set $[\tilde{0}, \Omega] \times [0, 1]$, what is the next element to $x \times 1$? It is $(x + 1) \times 0$. Note that $x \times 1$ itself is not an element of $\mathcal{L}$. First of all, note that $x \times 1$ is not an element of $\mathcal{L}$. That is why we have to work inside $[\tilde{0}, \Omega] \times [0, 1]$.

So, you identify them so that the extra points x cross 1 will all disappear. Duplication will not be there. So, you can think of $x \times 1$ as this point $(x + 1) \times 0$. So, that is another way of defining the long line. Many people especially when they give popular lectures on this topic, follow this practice. That is what I want to tell you.

However, strictly speaking, it is a wrong explanation, wrong presentation of the long line.

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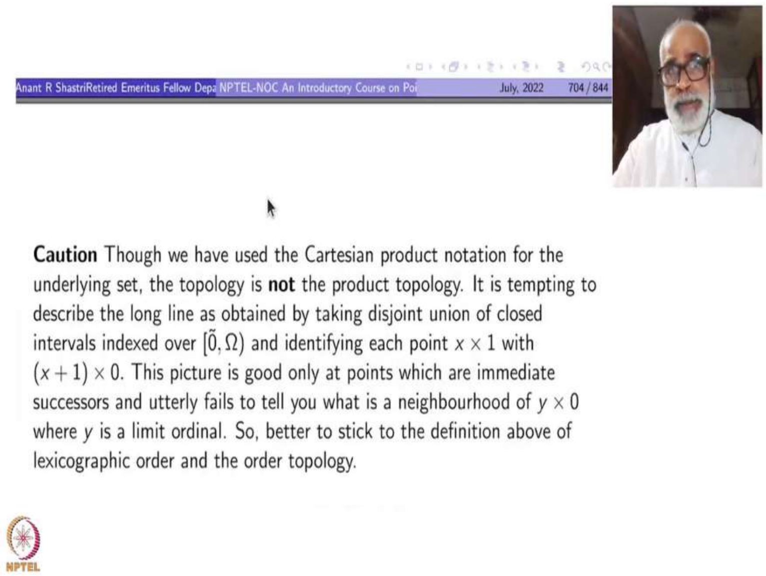

So, it is not a good way to explain it that way because that picture is good only in the initial stage namely $\tilde{0}, \tilde{0}+1$, so on up to ω. As soon as you arrive at a limit ordinal, say little omega, the picture will be a complete failure. So, there you have to strictly follow this rule of lexicographic order. Therefore better define it the way we have done. Take the lexicographic order on $\mathcal{L}$ and take the induced order topology.

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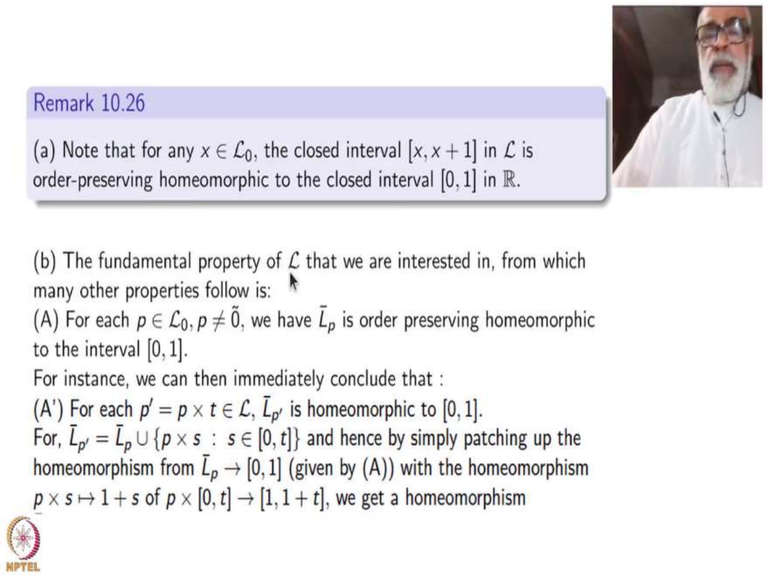

Caution Though we have used the Cartesian product notation for the underlying set, the topology is **not** the product topology. It is tempting to describe the long line as obtained by taking disjoint union of closed intervals indexed over $[\tilde{0}, \Omega)$ and identifying each point $x \times 1$ with $(x+1) \times 0$. This picture is good only at points which are immediate successors and utterly fails to tell you what is a neighbourhood of $y \times 0$ where y is a limit ordinal. So, better to stick to the definition above of lexicographic order and the order topology.

(a) So, note that for any x belonging to $\mathcal{L}_0$, the closed interval $[x, x+1]$ in $\mathcal{L}$, is order preserving homeomorphic to the closed interval $[0,1]$. For each x in $[\tilde{0}, \Omega)$, the open interval $(x \times 0, x \times 1)$ is a nothing the subspace $\{x\} \times (0,1)$ and hence homeomorphic to $(0,1)$. The next point will be precisely $(x+1) \times 0$ in $\mathcal{L}_0$. So, when you add the two end points x and $x+1$, it is homeomorphic to the closed interval $[0,1]$. So, this propery is the one which prompts the above wrong explanation. But if you use that to decribe globally the whole of $\mathcal{L}$, you will get the wrong picture. So, that picture will be good only in part not full picture.

(b) Second point is that there is this fundamental property of $\mathcal{L}$ that we are interested in, from which many other properties follow:

(A) For each p inside $\mathcal{L}_0$ (that is $p = x \times 0$ for some x in $[0, \Omega)$) not equal to $\tilde{0}$, the closed left ray L_p in $\mathcal{L}$, is order preserving homeomorphic to interval $[0,1]$.

Once you have this (A), you can have this (A)' which is an immediate consequence of (A) this can be immediate derive from (A).

(A)' For each p' in $\mathcal{L}$, not equal to $\tilde{0} \times 0$, the closed ray $\overline{L_{p'}}$ is homeomorphic to again the closed interval $[0,1]$. By the way, it is enough to say that something is homeomorphic to a closed interval (other that a singleton), because all of them are homeomorphic to each other. This we have seen several times.

How to see (A')? This $\overline{L_{p'}}$ is nothing but $\overline{L_p}$ union this vertical line segment $[x \times 0, x \times t]$, where $p' = x \times t$, where x is in $[0, \Omega)$ and t in $[0,1)$. We have to go up to t. So, from (A) this is already homeomorphic to closed interval $[0,1]$, and $[x \times 0, x \times t]$ is nothing but $\{x\} \times [0,t]$ and hence homeomorphic $[0,t]$, the usual interval in the real line. The common point is $x \times 0$.

We can put them together you will get a homeomorphism onto $[0, 1+t]$ instead of $[0,1]$. And then you can rescale it, to get homeomorphism from L'_p to $[0,1]$.

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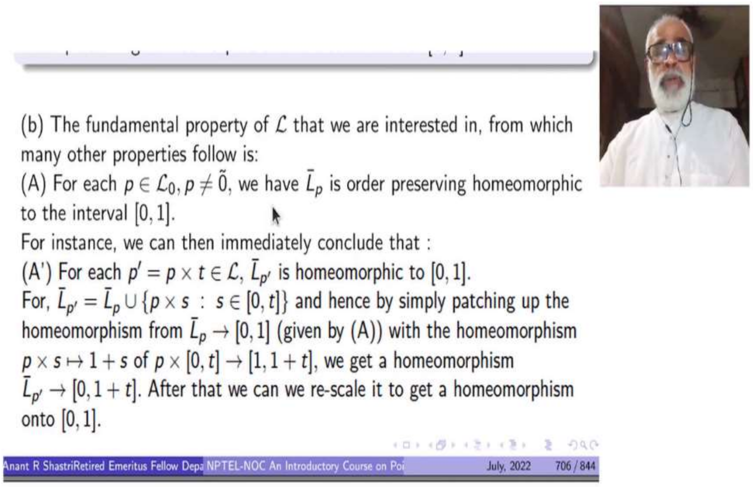

So, as a consequence of (A)’, it follows that this $\mathcal{L}$ is path connected and hence connected.

Not only that $\mathcal{L}$ is locally homeomorphic to an open interval in $\mathbb{R}$, except at the point $\tilde{0} \times 0$ at which we have neighbourhood system consisting of open sets which are homeomorphic to half-closed intervals. The latter claim is obvious. To see the former, all that you have to do is to take some point q which follows p and then p is inside $\overline{L_q}$ which contains a subset homeomorphic to an open interval and which contains p.

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(c) As a consequence, it follows that $\mathcal{L}$ is path connected and hence connected, locally homeomorphic to an open interval in $\mathbb{R}$ except at the point $\tilde{0} \times 0$ at which point we have neighbourhoods of the form $[0, t)$. So, let us prove (A). Suppose property (A) is not true for some $x \in \mathcal{L}_0$. Let y be the least element of all such elements. If $y = z + 1$ for some $z \in \mathcal{L}_0$ then we have $\bar{L}_z$ is homeomorphic to $[0, 1]$. We rescale it to a homeomorphism with $[0, 1/2]$ and patch it up with another homeomorphism of $[z, z + 1]$ with $[1/2, 1]$. Otherwise, y is a limit ordinal and there exists a strictly monotonically increasing sequence $\{x_n\}$ in $\mathcal{L}_0$ which converges to y.

So, let us prove property A. We have not yet proved it. I have indicated a proof already but let us prove this rigorously.

Suppose the property (A) is not true for some x in $(0, \Omega)$. Let y in $(0, \Omega)$ be the least element for which (A) is not true. If you take the set of all points such that this (A) property is not true that is a non-empty set is the assumption. So, this non-empty set will have a least element. So, let y be the least element of all such elements.

If y equal to some $z + 1$ for some $z \in [\tilde{0}, \Omega)$ that means y is a successor for some z. Then we will have $\overline{L_z}$ is homeomorphic to $[0, 1]$ (or a singleton when $z = \tilde{0}$).

because then z is less than y and hence (A) holds or $z = \tilde{0}$. In any case, the interval $[z \times 0, z + 1 \times 0]$ homoemorphic to $[0, 1]$ and hence we can patch these two homeomorphism to get a homeomophism of $\overline{L_y}$ with $[0, 2]$. That will be contradiction.

Next consider the case, when y is not a successor, viz., y is a limit ordinal. so, that is the harder case, what happens? What does it mean? We have seen that there exist a strictly monotonically increasing sequence in $[\tilde{0}, \Omega)$ which converges to y. How to use this to prove (A) in this case.

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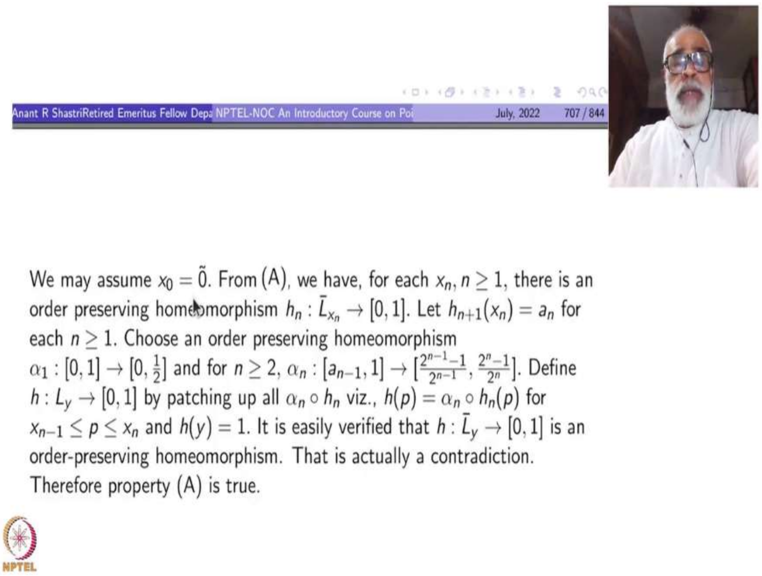

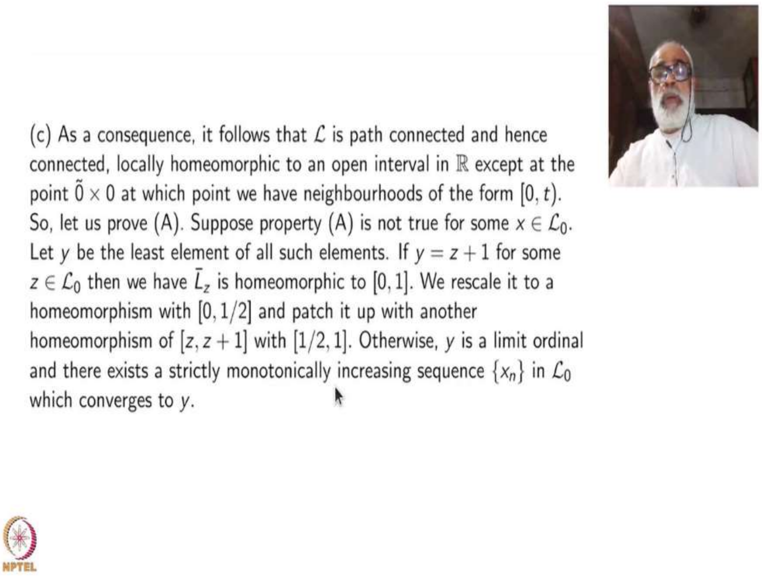

So, we may assume by adding one more point to the sequence in necessary that the sequence starts at $x_0 = \tilde{0}$. From (A), which is applicable to all the points $\{x_n\}, n \geq 1$, there is an order preserving homeomorphism h_n from $\overline{L_{x_n}}$ to $[0,1]$, because x_n is less than y. That is why this is possible.

Now, put $h_{n+1}(x_n) = a_n$, for each $n \geq 1$. So, I am defined this a_n which is an element in the open interval $(0,1)$. Choose an order preserving homeomorphisms, α_1 from $[0,1]$ to $[0,1/2]$ and for $n \geq 2$, α_n from $[a_{n-1}, 1]$ to $[p_n, q_n]$ where $p_n = (2^{n-1}-1)/2^{n-1}$ and $q_n = (2^n - 1)/2^n$.

You can choose these α_i to be affine linear. You can always rescale any two closed intervals. Choose all of them to be order preserving homeomorphisms.

Now define h from L_y to $[0, 1]$ by patching up all these α_1 and α_i's. Namely, I will define $h(p)$ to be $\alpha_n(h_n(p))$ for p between $x_{n-1} \times 0$ and $x_n \times 0$. All points of L_y lie between $x_{n-1} \times 0$ and $x_n \times 0$, starting from $x_0 = 0$.

On the endpoints of these intervals you get two different definitions, but they coincide. So, the function h is well defined. You can easily see now, that h is an order preserving continuous bijection onto $[0, 1)$. Image of any of α_n is never equal to 1, the point 1 is not covered.

So, you to take $h(y) = 1$. Then it is easily verified that h from $\overline{L_y}$ to $[0, 1]$ is an order preserving homeomorphism. The whole point is that the sequence $(2^n - 1)/2^n$ will tend to 1 as n tends to infinity. So that is why, if you define $h(y) = 1$, continuity of h at the point y comes. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Therefore, (A) is true. I recall I took two different cases namely, this y is a limit ordinal and not a limit ordinal y. So, there are two cases. We have proved the statement (A). So, from (A) the rest of the topological aspects will follow.

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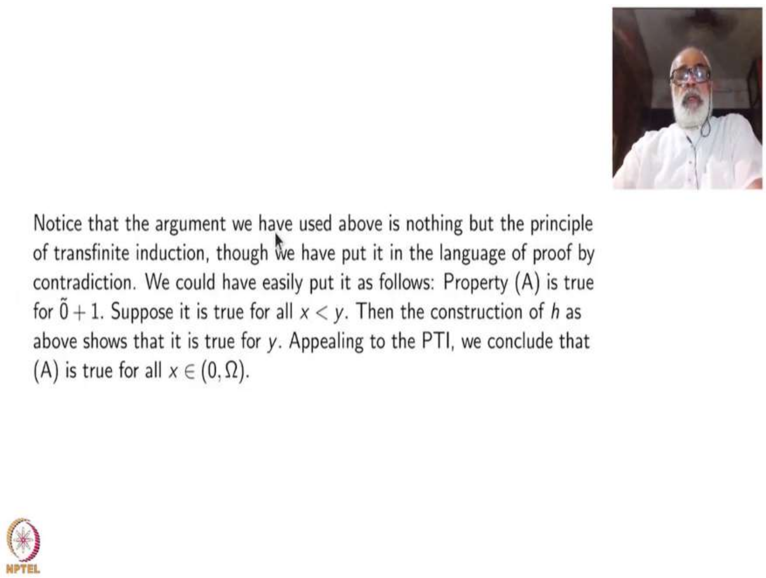

Notice that the argument we have used above is nothing but the principle of transfinite induction though we have put it in the language of proof by contradiction. We could have easily put it as follows. Property A is true for $\tilde{0} + 1$ that is easy to verify. Suppose it is true for all x less than y then the construction of h as above shows that is true for y. That step is of course you have do.

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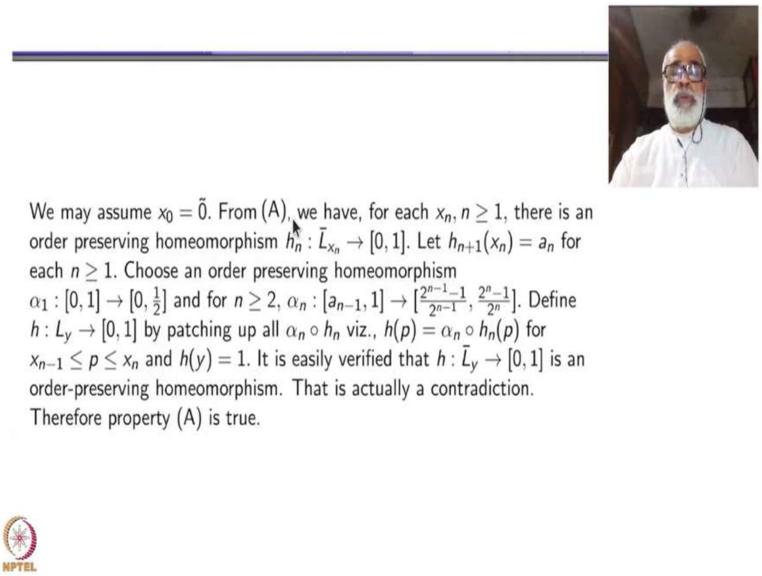

Assuming that it is true for all x less than y, then we have proved for y. So, that is true for all y, by PTI.

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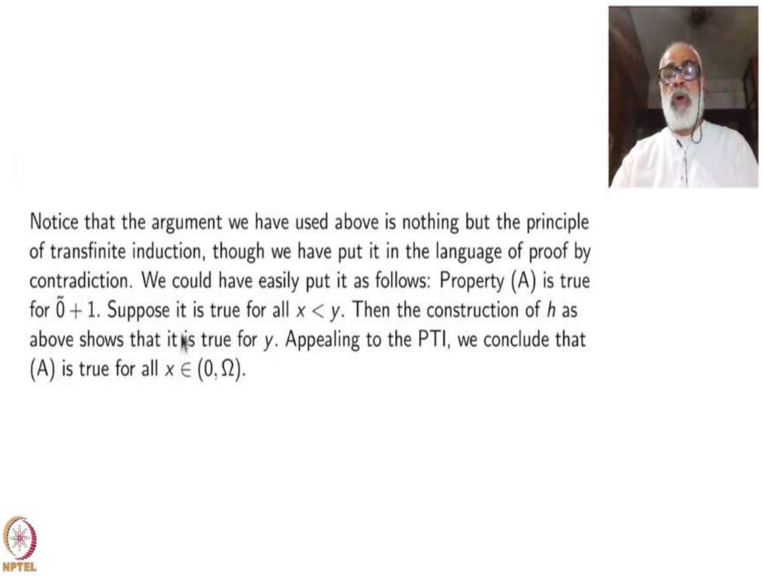

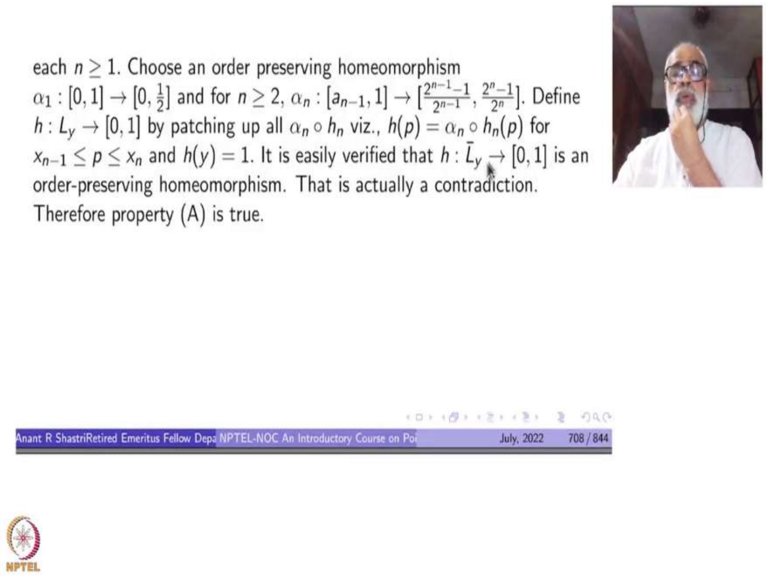

So, here we have used a different language. So, we have got, of course is not true and then take a least upper one. In other words, what have used, we have used the well ordering principle anyway. So, we do not have to go to the principle of transfinite induction here that is all.

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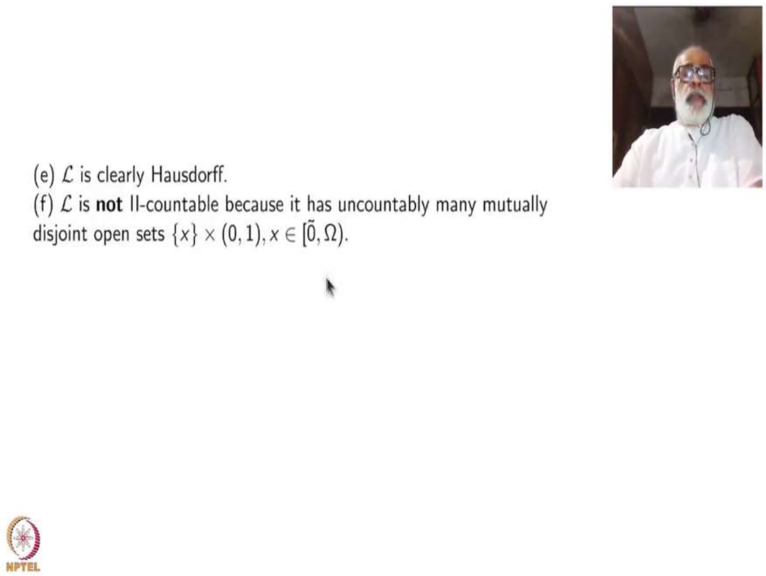

So, this $\mathcal{L}$ is clearly Hausdorff, there is nothing to prove here, because every order topology is Hausdorff. $\mathcal{L}$ is not II-countable. (See the point to which we have come now.) Because it has uncountably many mutually disjoint open sets, $\{x\} \times (0,1)$, the vertical open segments, they are all open intervals in $\mathcal{L}$. They are all disjoint with each other and how many of them are there? As x ranges over $[\tilde{0}, \Omega)$ there will be uncountably many. In a topological space if you have uncountably many disjoint non-empty open sets, inside each of them select a point, that will give you an uncountable discrete set. That is not possible. So, II-countability is violated.

If we extend $\mathcal{L}$ by one more line segment, note that on Ω, we do not have a line, you put one more line there, i.e., start with the set $[\tilde{0}, \Omega] \times [0, 1)$. Then it is not even path connected. Infact, there is no need to put a whole line. You just include $\Omega \times 0$, take $\hat{\mathcal{L}}$ to be the union of $\mathcal{L}$ and $\{\Omega \times 0\}$. Then take the same lexicographic ordering and the corresponding topology that is all. Then this larger space than $\hat{\mathcal{L}}$ not path connected.

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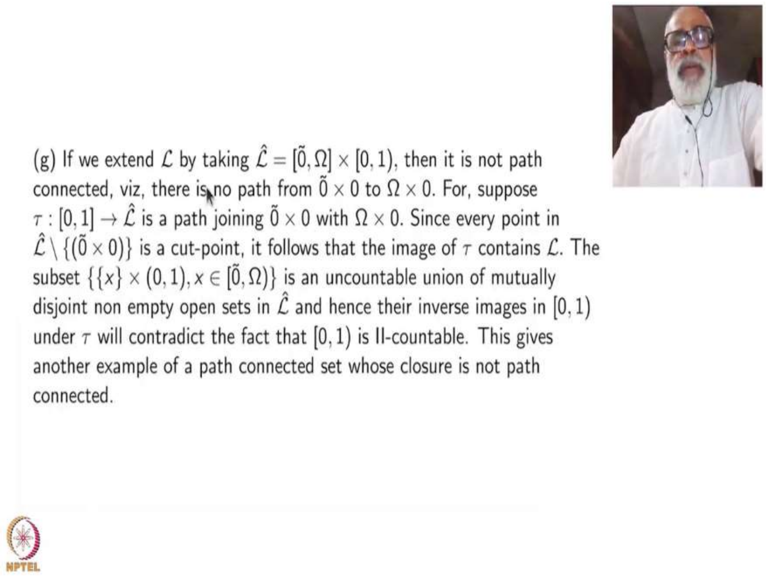

If we extend $\mathcal{L}$ by one more line you see on Ω, we do not have a line you put one more line here. Then it is not path connected. Namely, there is no need to put one more line. You just allow $\tilde{\Omega} \times 0$ this $\Omega \times 0$. Take $\hat{\mathcal{L}}$ to be this $\mathcal{L}$ union this $\Omega \times 0$, one more point. Then the same lexicographic ordering and all that. Then this will be a larger space than $\mathcal{L}$ and that is not path connected.

So, that is what I am going to prove now. Namely, there is no path from $\tilde{0} \times 0$ to $\Omega \times 0$. We know it inside $[\tilde{0}, \Omega]$. Because there are too many discrete points here. But now we are working inside $\hat{\mathcal{L}}$. and claiming there is no path. That is not so obvious.

So, suppose you have a path, path means what, a continuous function from a closed interval say $[0, 1]$ to $\hat{\mathcal{L}}$ joining $\tilde{0} \times 0$ to $\Omega \times 0$.

Note that every point in $\hat{\mathcal{L}}$ other than least element and the greatest element are cut points. If you have a path from one point a to another point b and there is a cut point which separated a and b, then the path must go through that cut point. Just like the Intermediate Value Theorem. So, we are using Intermediate Value Theorem here for that.

That means what? All of these uncountably many discrete points inside $(\tilde{0}, \Omega)$ they should all lie on this path. That is not possible. Because the image of a path is a compact set and it cannot contain an uncountable discrete set. So, I make it clearer here by an alternate argument. If τ from $[0,1]$ to $\hat{\mathcal{L}}$ is the path with $\tau(0) = \tilde{0} \times 0$ and $\tau(1) = \Omega \times 0$, then the cut point argument shows that τ is surjective.

There are uncountably many mutually disjoint open subsets viz. $x \times (0,1)$ in $\hat{\mathcal{L}}$, and so their inverse images under τ in $(0,1)$ will give uncountably many disjoint open subsets of $(0,1)$ contradict the fact that $(0,1)$ open is second countable.

Thus $\mathcal{L}$ is path connected, its clouser in $\hat{\mathcal{L}}$ is the whole of $\hat{\mathcal{L}}$ which is not path connected. We have seen such an example before also, viz., the topologist's sine curve. That is not a great thing, but this example also is of that type.

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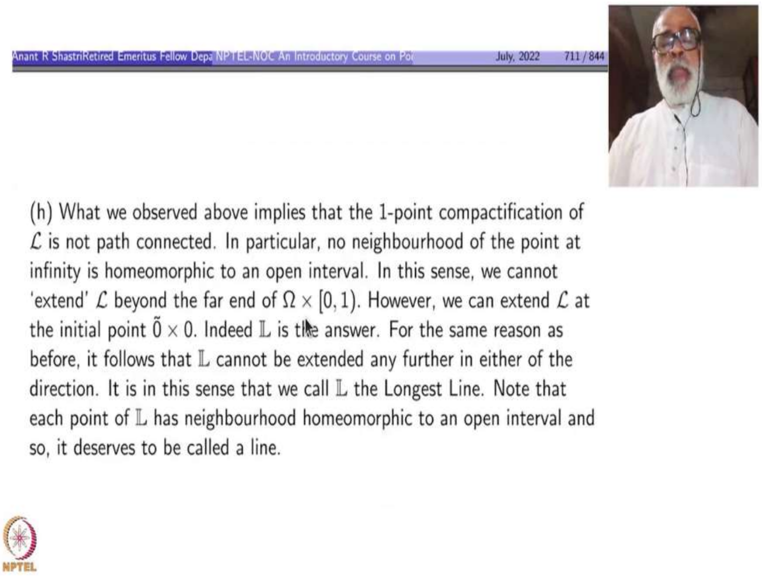

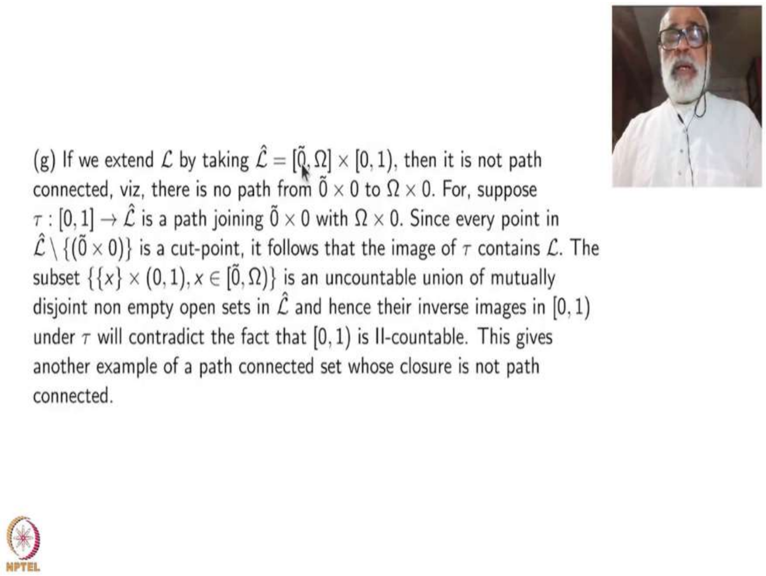

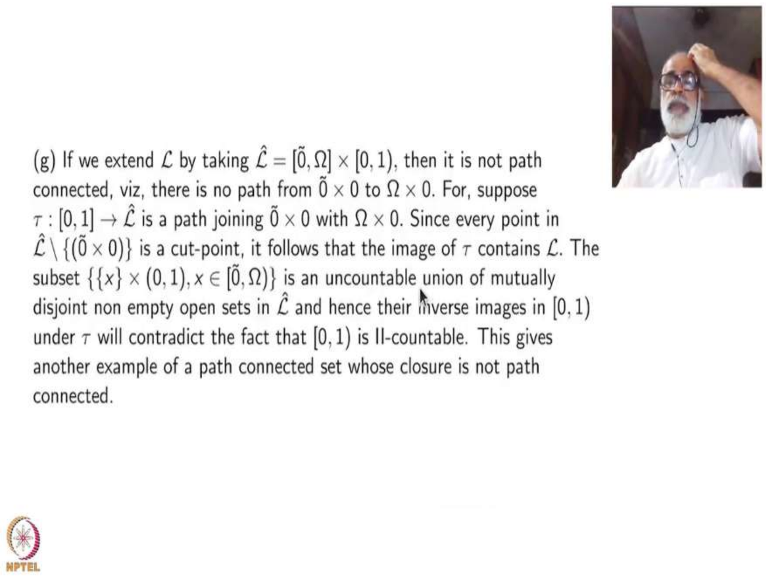

Indeed, what we observed above implies that the one point compactification of $\mathcal{L}$ is not path connected. (That is the stronger conclusion than saying that closure of a path connected set need not path connected.)

In particular, no neighborhood of the point at infinity is homeomorphic to an open interval or a half closed interval. I am talking about one-point compactification. In this sense, we cannot extend $\mathcal{L}$ beyond the far end viz., if we add one more point here there will be a problem about local Euclideanness.

However, we can extend $\mathcal{L}$ at the initial point $\tilde{0} \times 0$. That is what we have done in the definition of this $\mathbb{L}$. That is the answer. Note that every point of $\mathcal{L}$ has a neighbourhood homeomorphic to an open interval.

For the same reason as before, $\mathbb{L}$ cannot be extended any further on the left hand side or on the right hand side. So, this is completely saturated in that sense. Therefore, it is called the longest line.

So, note that each point of $\mathcal{L}$ has a neighbourhood homeomorphic to an open interval. And that is why it deserves to be called a line. It is the longest line because you cannot have another longer than that one. Of course, this will be not second countable either because even the subspace is $\mathcal{L}$ is not second countable.

So, with these remarks and great results here, this chapter comes to an end. Some of the lessons which you have learned here, some of the results, we will use them in the next chapter also. This example $\mathcal{L}$ itself will be quoted as an illustration of the hypothesis that we put namely, second countability in the definition of manifolds. Thank you.