An Introduction to Point-Set-Topology (Part II)

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Lecture 49

Ordinal Topology Continued

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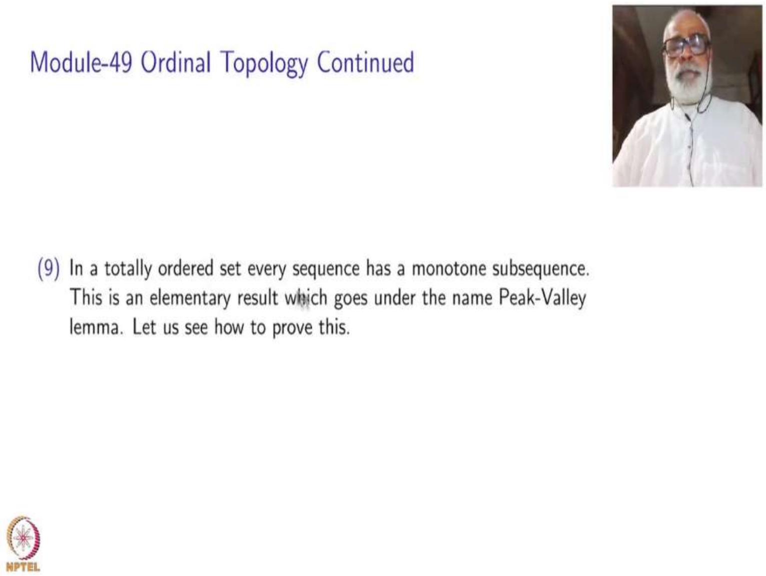

Hello welcome to Module 49 of NPTEL-NOC an introductory course on Point Set Topology Part 2. So, we continue our study of Ordinary Topology. In a totally ordered set every sequence has a monotone subsequence.

This is an elementary result, which goes under the name Peak Valley lemma maybe you have not seen it. On the other hand, this is actually important for us now, so, I would like to recall it completely.

So, what is the meaning? You can imitate what is happening inside real numbers. For real numbers maybe you know this result. But do not use the full properties of real numbers you have to use only that is totally order, no addition, no subtraction, no multiplication and so, on. That is the whole idea. That you can still prove this result just by using totally orderedness is the gist. Every sequence has a monotonically increasing sequence or a monotonically decreasing sequence, a monotone subsequence.

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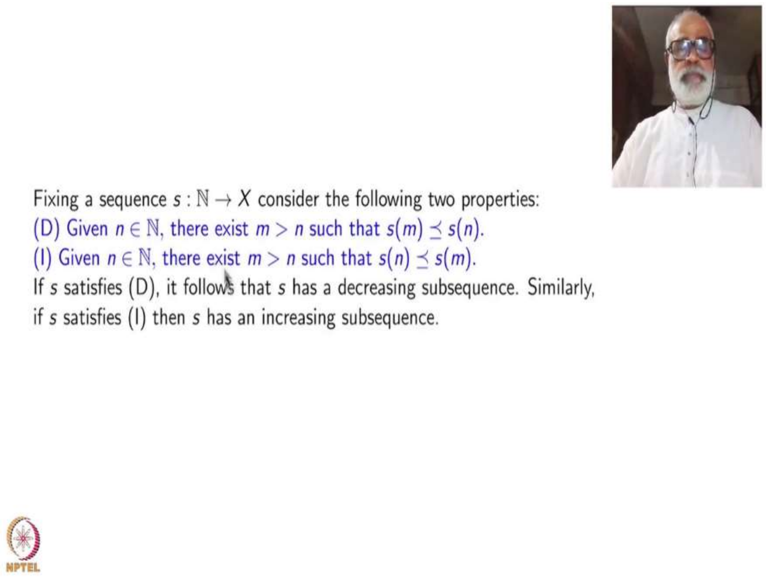

So, fixing a sequence $s = \{s_n\}$ consider the following two properties.

(D) given any $n \in \mathbb{N}$, you will have m bigger than n such that $s_m \leq s_n$. (T his inequality is not to be confused with the order of real numbers that is all, so I have carefully written inequality $s_m \leq s_n$.

(I) the second is that given $n \in \mathbb{N}$ that exist $m > n$ such that $s_n \leq s_m$, the other way round.

If s satisfies (D) it follows that s has a decreasing subsequence. Similarly if s satisfies (I), it will have an increasing subsequence. That is why I have named them (D) and (I), decreasing and increasing respectively. So, now I assume that neither (D) nor (I) is true. Under this let us see what happens to the sequence.

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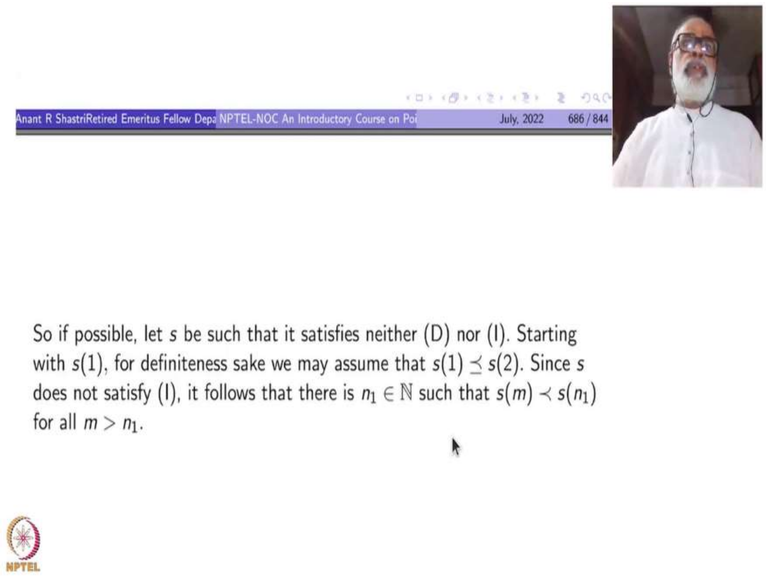

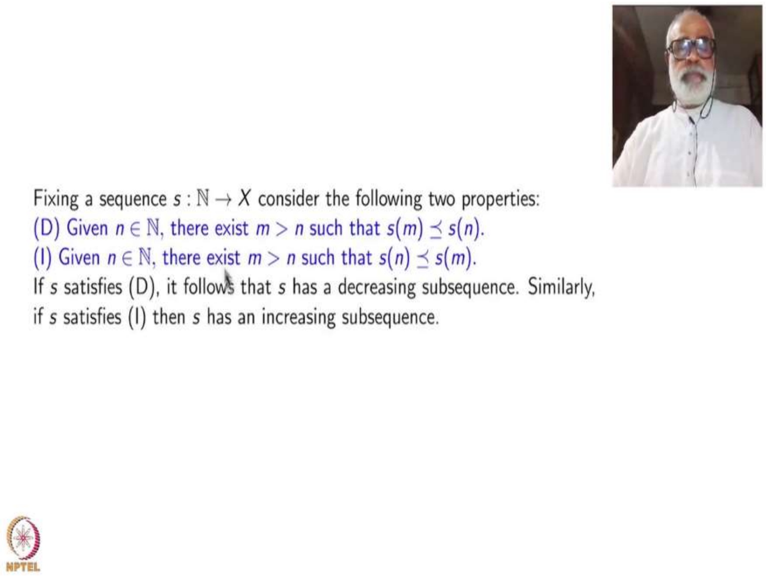

So, if possible let s be such that neither (D) nor (I) is true. Starting with s_1 for definiteness sake, we may assume that $s_1 \leq s_2$. Because the set is totally ordered, $s_1 \leq s_2$ or $s_2 \leq s_1$, one of them is true. By symmetry, we assume $s_1 \leq s_2$. Since s does not satisfy (I), that means what? We cannot go on getting $s_2 \leq s_3$, $s_3 \leq s_4$ and so on. Infact, we cannot go on geeting $j > i$ such that get $s_i \leq s_j$, indefinitely. It follows that there will be some $n_1 \in \mathbb{N}$ such that for all $m > n_1$, s_m will be smaller than s_{n_1}. So, pick up that n_1 what we may assume is that I have started a sequence at n_1 itself. Just forget about the earlier part of the sequence. Now, what happens to s_{n_1+1}? it has to be less than s_{n_1}.

We are back in the first step, where I started with the assumption $s_1 \leq s_2$, except that the inequality is reversed. So, repeating the above step, I get a number $n_2 > n_1$, such that for all $m > n_2$, $s_m \leq s_{n_2}$.

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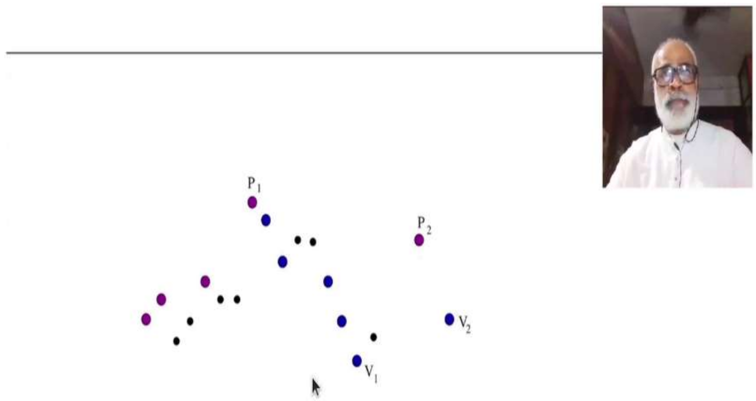

Figure 14: Peak-n-Valley argument to produce monotone subsequences

This is what the picture here. I started with a sequence $s = s_1$ here, keeps going up maybe s_2 is the next and then maybe s_5 is the next, (not necessarily s_3 and s_4) but finally I come to n_1, after that, if you look at the reset of the sequence, everything is smaller than s_{n_1}, there is nothing bigger than this what is the meaning of this one? This is s_1 is peak. So, that is why I denote it by P_1.

So, now the next one is definitely smaller than P_1, because everything is smaller now. So, you keep going down down down if you are successful then you obtain a subsequence which is is decreasing. You keep coming down, in between there may be something which is not decreasing, ignore them and keep going further. Finally, you will be get stuck-up with some s_{n_2} such that all further elements are larger than s_{n_2}. That is denoted by V_1, because it is a Valley. What is the meaning of this? Everything after that is bigger than V_1.

B of course V_1 is smallest that P_1. See because P_1 has the property that it is bigger than everything beyond.

So repeat this process, whatever you have done in these two steps. Next stage you keep going up till you come to another peak P_2 and then keep going down till you hit another valley V_2 and so on.

Keep repeating you get a subsequence P_1, V_1, P_2, V_2 and so on an interlaced subsequence. What is the property of P_1 it is bigger than everything after that, P_2 is bigger than everything after that and so on therefore the sequence $\{P_i\}$ is a monotonically decreasing sequence.

(Similarly, $\{V_1, V_2, V_3, \dots\}$ will be monotonically increasing sequence.) So, either of them contradicts the assumption therefore, there must be a subsequence which is monotone. So, that is the proof.

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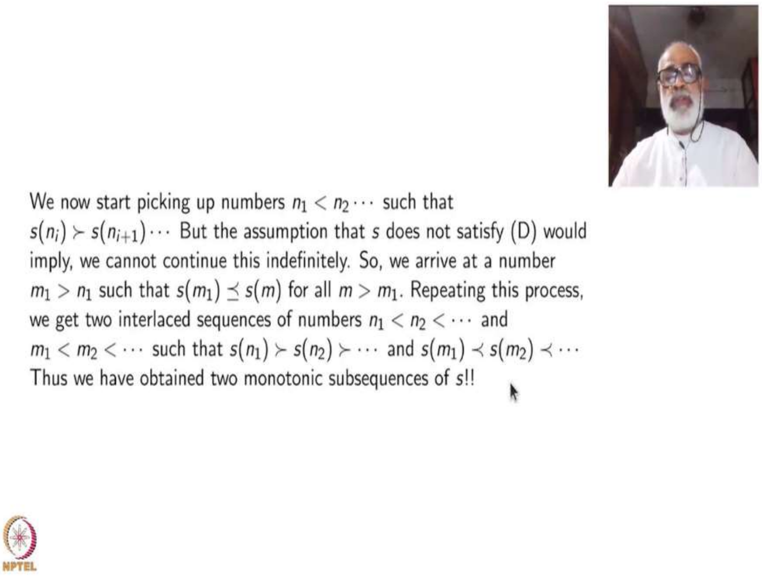

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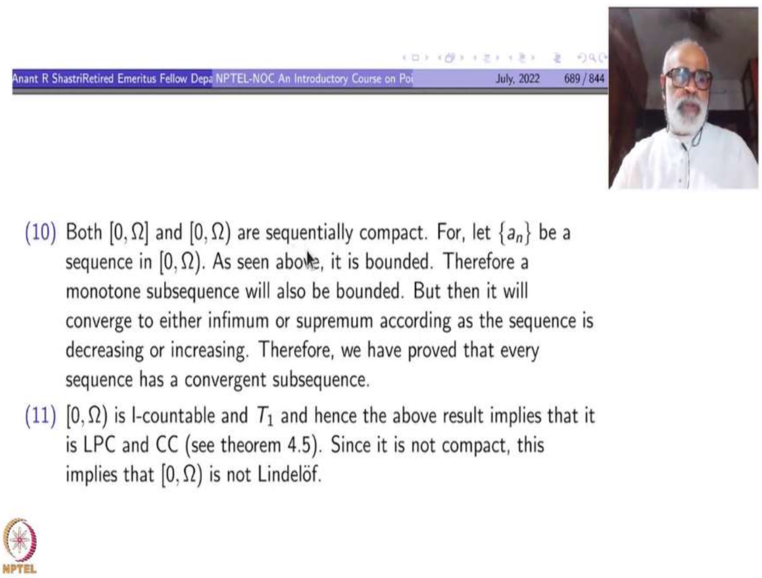

Next, I come back to the ordinal space $[0,\Omega]$. Both $[0,\Omega]$ and $[0,\Omega)$, they are sequentially compact. Remember what sequentially compact means: every sequence has a subsequence which is convergent. So, we are discussing this sequential compactness. So, this point (9) here is done in that background.

So, how do you prove that something is sequentially compact? Start with any sequence $\{a_n\}$ in $[0,\Omega)$. First of all it is bounded. Every sequence is bounded that we have seen before, right? Bounded in $[0,\Omega)$ itself. Also we have seen that $\{a_n\}$ has a monotone subsequence, and that is also bounded. Depending on whther this subsequence is decreasing or increasing, it will converge to the infimum or the supremum. Therefore, we have proved that every sequence has a convergent subsequence. Great. (The proof for $[0,\Omega]$ can be got by first taking a subsequence which takes value inside $[0,\Omega)$.)

So, point (9) is used only to get a monotone subsequence. These monotone subsequences, I do not know whether it is increasing or decreasing. In both the cases, it will converge because it is bounded. This is also standard result in analysis, namely, in $\mathbb{R}$, if you have a bounded monotone sequence it is convergent, follows from the property of the existence of least upper bound and latest lower bound. So, we have we have more or less proved that theorem also here in this approach because all that you to assume, all that you have to do is here is just start with total order and assume that it has LUB property. So, and of course, this assumption that least upper bound or greatest lower bound exists is true for this $[0,\Omega]$.

The next thing is a corollary: $[0,\Omega)$ is I-countable and T_1 (it is actually T_2 we have seen that) and hence, the above result (SC) implies that, it is limit point compact (LPC) and countably compact (CC) also. I will try to recall these things.

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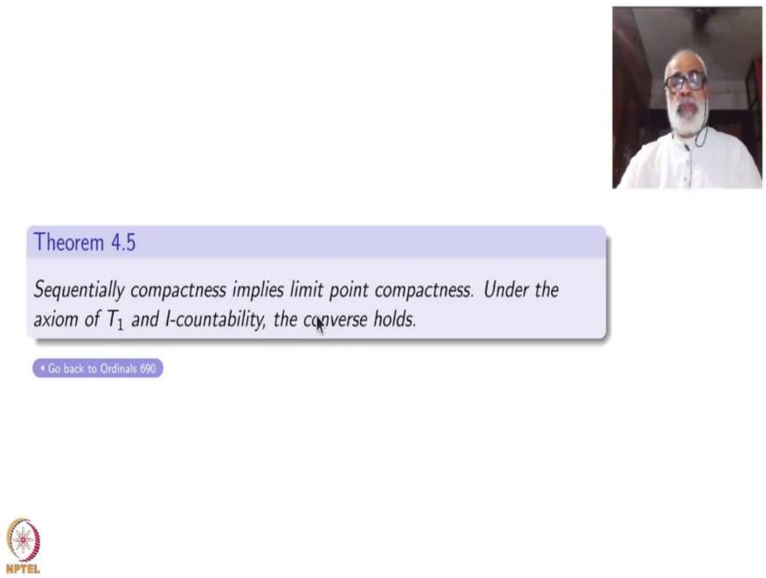

So, let me go through these. This one sequentially compactness implies limit point compactness under the axiom T_1 and first countability. The converse is also true always. Sequential compactness implies these two only under T_1 and first countability. but limit point compactness implies sequential compactness always. These results what we have seen. Actually you can have a look at this picture. Remember.

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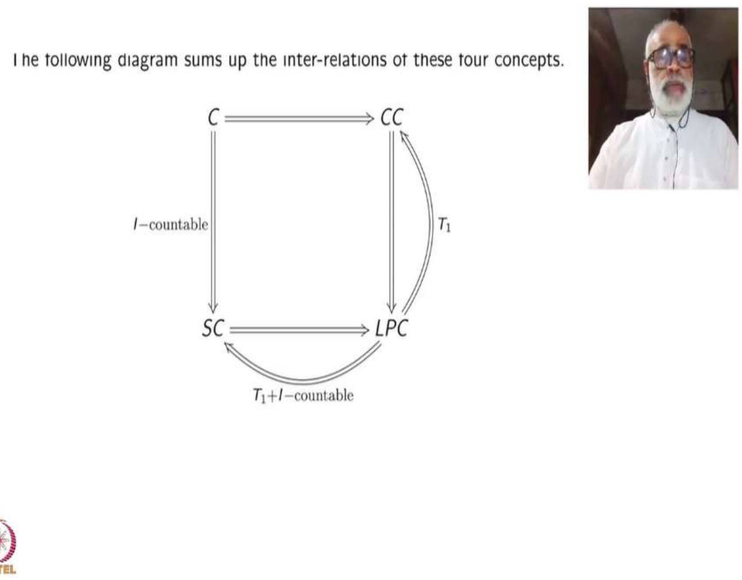

So, this picture tells you that this one, so this is limit point compactness here is countable compactness here plus T_1, limit point compact plus T_1 will imply countable compactness T_1 plus first countable limit point compact implies sequential compactness, from here to here you can come back. So, you can recall, I have just recalled that one for your ready referenced. So, let us go back here now.

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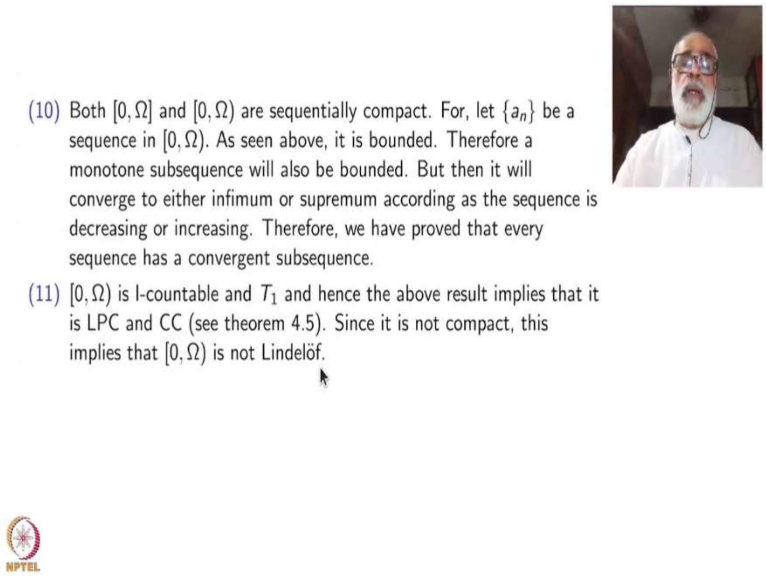

So, $[0, \Omega)$ is I-countable, $[0, \Omega]$ is not I-countable that is what we have seen. And of course, T_1-ness is always there (because it is T_2). Hence, the above result that it is sequential compactness implies that it is limit point compact and countably compact also.

Since it is not compact, this implies that $[0, \Omega)$ is not Lindelöf. Alright? Because once it is Lindelöf and countably compact it will be compact. Remember, Lindelöf means what every open cover has a countable subcover and countably compact means what? Every countable has a finite sub cover. So, combining these two we get compactness.

So one important thing we have derived is that $[0, \Omega)$ is not Lindelöf.

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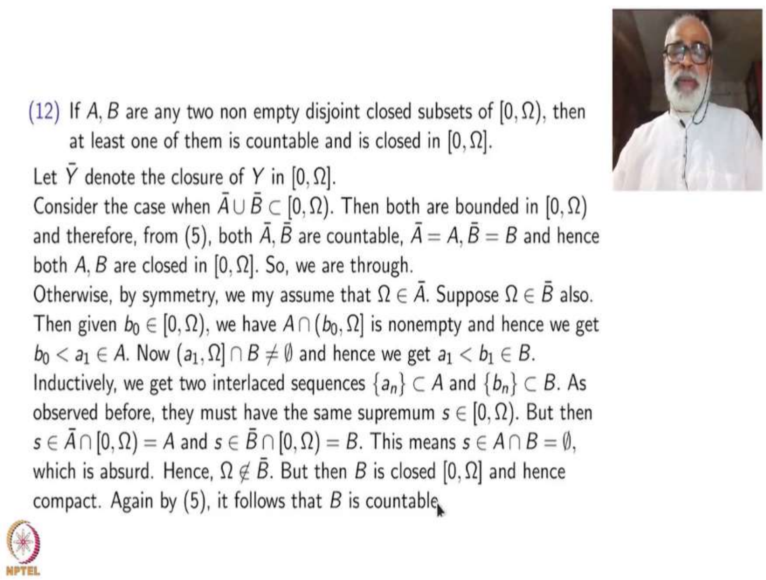

Next, if A and B are any two non-empty disjoint closed subsets in $[0, \Omega)$, then at least one of them is countable and is closed in $[0, \Omega]$. So, you are slowly going towards normality here. So, take two disjoint closed subsets of $[0, \Omega)$, of course, both nonempty. Start with that. At least one of them is countable is a very strong conclusion. That is a closed set in $[0, \Omega)$ to begin with and we are concluding that it is closed in $[0, \Omega]$. How?

Because this Ω may be a limit point after all. So, this says that Ω is not a limit point. This part we know already because any countable subset of $[0, \Omega)$ is bounded inside $[0, \Omega)$. So, this part we know already. So, let us see how the first conclusion works. The second part we have already seen.

Let $\overline{Y}$ denote the closure of Y in $[0, \Omega]$ for any subset Y of $[0, \Omega)$, (a temporary notation, just for in this part. Otherwise, to tell where we are taking the closure, I will have too many notations here, I do not want to have that.) So, for any subset Y of $[0, \Omega)$, $\overline{Y}$ denotes the closure in $[0, \Omega]$.

Now, look at the case wherein $\overline{A} \cup \overline{B}$ is contained in $[0, \Omega)$, (it need not be true always, this is one of the cases I am talking about).

Then both are bounded in $[0, \Omega)$ itself, because they are closed subsets of $[0, \Omega]$ but contained in $[0, \Omega)$. Therefore, from our earlier result, which we have seen last time, both of them are countable. Moreover, $\overline{A}$ is already A and $\overline{B}$ is B, both A and B are closed subsets of $[0, \Omega)$ to begin with. So, in this case, we have concluded that both A and B are countable and closed in $[0, \Omega]$.

Otherwise it means what? Now consider the other case, viz., suppose $\overline{A}$ or $\overline{B}$ contains this element capital Ω. By symmetry, we may assume that Ω is inside $\overline{A}$, by interchanging A and B if necessary, that is all.

Suppose, further that this Ω is in $\overline{B}$ also. That can also happen. Then given b_0 inside $[0, \Omega)$, it follows that $A \cap (b_0, \Omega)$ is non empty because Ω is inside $\overline{A}$. So we get element $a_1 \in A$, such that $b_0 \leq a_1$.

Now, apply the same thing to (a_1, Ω), this is a neighbourhood of Ω, so its intersection B is non-empty. So, you get an element b_1 inside B such that $a_1 \leq b_1$. Repeating this process what you get? You will get two sequences $\{a_n\}$ and $\{b_n\}$ which are interlaced and strictly increasing, a_1 less than b_1 less than a_2 less than b_2 less than and so on.

So, where are they going? So, we observed that two interlaced sequences they must have the same limit s. This is what we have seen earlier and that limit must be inside $[0,\Omega)$. Because no sequence in $[0,\Omega)$ converges to Ω. So it follows that s is inside $[0,\Omega)$.

But then s itself will be in $\overline{A}\cap[0,\Omega)$, which A and s is also in $\overline{B}\cap[0,\Omega)$ which you know is B. This means s is inside $A\cap B$ and that is a contradiction because $A\cap B$ is empty to begin with, That they are disjoint closed subsets of $[0,\Omega)$ to begin with.

Therefore, if we assume Ω is inside $\overline{A}$ then Ω is not inside $\overline{B}$. But that means B itself is closed inside $[0,\Omega]$, hence B is compact. But now this compact subset B is inside $[0,\Omega)$. Therefore, B is countable. This we have seen already. So, we wanted to prove one of them is countable and we have got that. Automatically it will be closed in $[0,\Omega]$ also.

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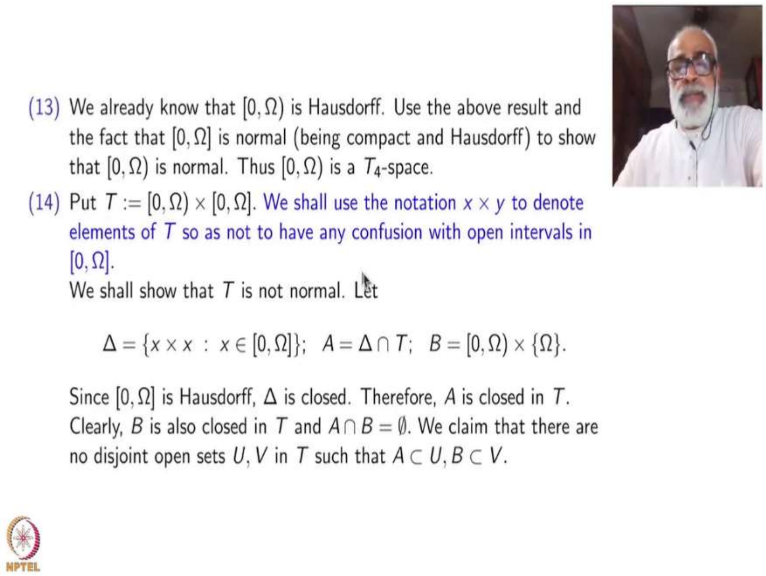

Now, we will use this one in a meaningful way, namely, to prove that $[0,\Omega)$ is normal. We already know that $[0,\Omega]$ is compact and Hausdorff and hence normal. Therefore, starting with the two disjoint subsets in $[0,\Omega)$, you take their closures in $[0,\Omega]$. Just now what we have shown is that these closures are themselve disjoint. Because Ω may belong to the closure of at most one of them. What does that mean? You can separate them inside $[0,\Omega]$ by two disjoint opens subsets, intersect these disjoint open subsets with $[0,\Omega)$, so, that will give you disjoint open subsets of $[0, \Omega)$ each containig one of the original closed subsets. Therefore, $[0, \Omega)$, being already T_2 so it is a T_4 space.

The next thing is the aim of all the work we have been doing these days:

If you take the product space $[0, \Omega) \times [0, \Omega]$, denote this by T, this T is going to be a wonderful example now. (Note that I am not taking $[0, \Omega] \times [0, \Omega]$. That will be compact cross compact, will compact.) So, this T is going to give you a number of counterexamples.

So, now, we have some notational problem here. See, we have been using the ordered pairs to denote elements of a product space earlier and also to denote intervals in a totally ordered set. But now we have to deal with product of two totally ordered spaces. So, I will not use ordered pairs to denote elements of products. I will restrict it to open intervals only and ordered pairs will be denoted by $x \times y$ now. So, this is an element in $X \times Y$, where x in X and y in Y. The standard ordered pair notation would have been (x, y). So, that notation will not be used while we are discussing this example at leat. So, with that convention, we shall show that T is not normal this is normal, though both are T_2 and normal and the second one is actually compact also.

So, that is a strong counter example. An example of product of two normal spaces which is not nor$x$ was promised in part I, but we did not provide any examples there. So, here is an example, a beautiful example.

So, let us have the standard notation: Δ denotes the diagonal subset $x \times x$ where x varies over $[0, \Omega]$. But I want to remain inside T and so I take A equal to $\Delta \cap T$ which is a closed subset of T. For another closed subset B disjoint from T, I choose $[0, \Omega) \times \{\Omega\}$.

So, last point $\Omega \times \Omega$ is not there. So, A and B are closed subsets of T obviously, they are disjoint.

So, here in the picture, this B is the top line segment and A is the diagonal goes all the way close to $\Omega \times \Omega$ but that point is not taken. So, these two are disjoint closed subsets. Since $[0, \Omega]$ is Hausdorff, Δ is closed in the product and therefore, A which is $\Delta \cap T$ is closed in T.

Similarly, since singleton $\{\Omega\}$ is closed in $[0, \Omega]$, it follows that B is a closed subset of the product. Also $A \cap B$ is empty is clear.

So, now we claim that there are no disjoint open subsets U and V inside T, so that A is inside U and B is inside V. That will complete the proof that T is not normal.

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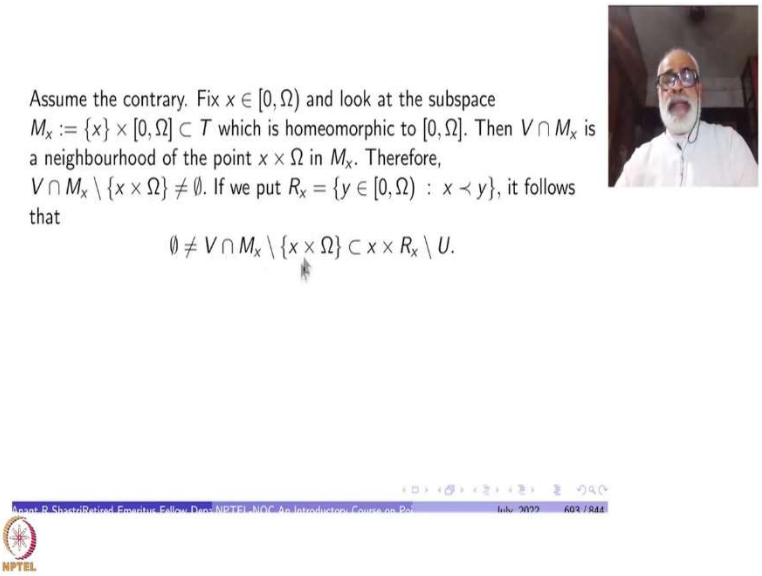

So, we have to do a little bit of work here.

Assume on the contrary. That means what we start with the assumption that two open subsets as above and arrive at a contradiction, that the whole idea.

Assume the contrary. For each $x \in [0,\Omega)$, look at the subspace M_x, the vertical subspace, equal to $\{x\} \times [0,\Omega]$. That is a subset of T.

And it is just homeomorphic to $[0,\Omega]$. If $V \cap M_x$ that will be a neighbourhood of the point $x \times \Omega$ inside M_x, because, $x \times \Omega$ is a point of B and B is contained in V. Therefore, $(V \cap M_x) \setminus \{x \times \Omega\}$, this punctured neighbourhood is non empty because $\{\Omega\}$ is not open in $[0,\Omega]$.

If you put R_x equal the set of all y in $[0,\Omega]$ such that this x is less than y, (this is our familiar old right open ray), it follows that this non empty set $(V \cap M_x) \setminus \{x \times \Omega\}$ is actually contained inside $(x \times R_x) \setminus U$, because U and V are disjoint.

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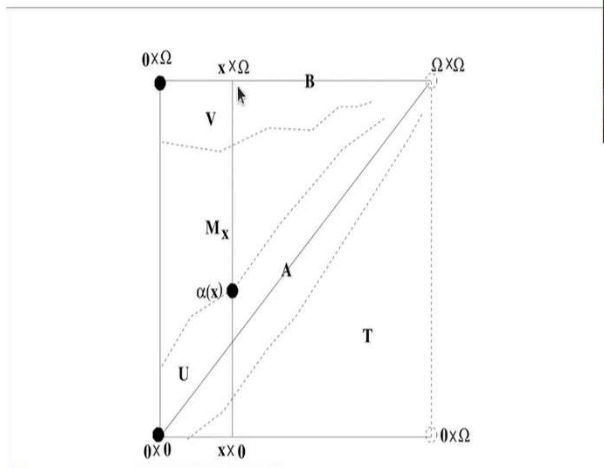

Assume the contrary. Fix $x \in [0,\Omega)$ and look at the subspace $M_x := \{x\} \times [0,\Omega] \subset T$ which is homeomorphic to $[0,\Omega]$. Then $V \cap M_x$ is a neighbourhood of the point $x \times \Omega$ in M_x. Therefore, $V \cap M_x \setminus \{x \times \Omega\} \neq \emptyset$. If we put $R_x = \{y \in [0,\Omega) \,:\, x \prec y\}$, it follows that

$$\emptyset \neq V \cap M_x \setminus \{x \times \Omega\} \subset x \times R_x \setminus U.$$

So, let me show all these things in a picture just to see that you are not completely lost. This is $T = [0,\Omega) \times [0,\Omega]$. A is the diagonal, note that $\{\Omega\} \times \{\Omega\}$ is not in A nor in B. B is the top line $[0,\Omega) \times \{\Omega\}$. U and V are open subsets U containing A and V containing B and we are pretending that $U \cap V$ is non empty.

we feel some thing is going wrong when we keep coming closer and closer to $\{\Omega\} \times \{\Omega\}$. I do not know, so that is where the mysteries lies. Finally we will get a contradiction there.

So, fixing a point x here, I look at $\{x\} \times [0,\Omega]$, the vertical line. That is my M_x. This $V \cap M_x$ this portion is a neighbourhood of this $x \times \Omega$. If you throw at this point, this is part is still non-empty. This set is disjoint from that U whether I throw this point away or not, this will still be empty. That is all I have been telling so far.

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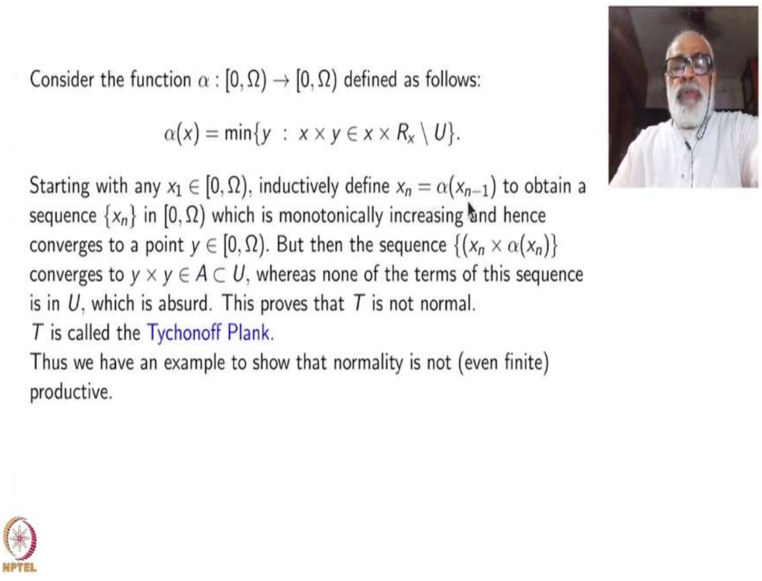

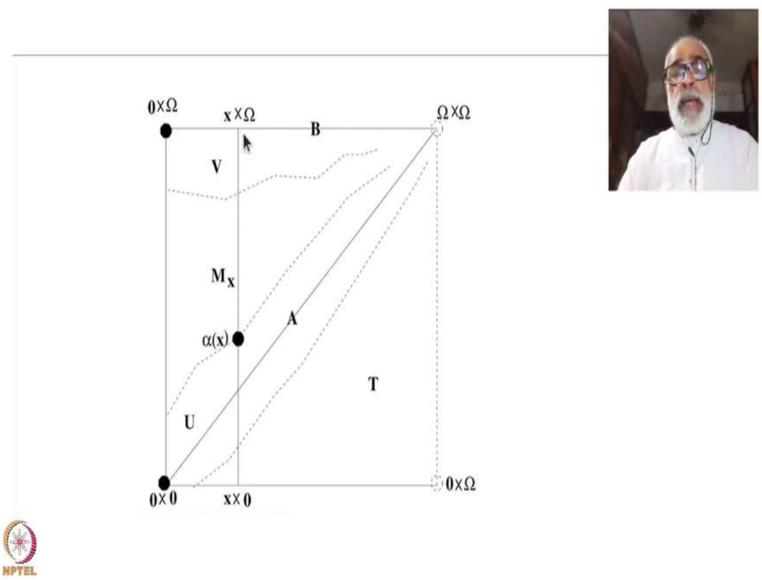

So, we can define a function here namely, take the minimum of y such that $x \times y$ is inside this non-empty set $(x \times R_x) \setminus U$. Denote it by $\alpha(x)$ as shown in the picture. So, that is the definition of α which is a function from $[0,\Omega)$ to $[0,\Omega)$.

We shall now define a sequence $\{x_n\}$ inductively. Start with any $x_1 \in [0,\Omega)$ and put $x_2 = \alpha(x_1)$. Inductively having defined x_n put $x_{n+1} = \alpha(x_n)$.

Now, α of this 1 will be some point here. So, that point will be somewhere here then you take alpha of that will be some point here that point will be somewhere here, now α of that you will get that point will be somewhere here, you take α of that and so on. So, this is the inductive process here namely starting x_1 belong to 0 my anything wherever you want to start inductively define x_2 equal to $\alpha(x_1)$ and $x_3 = \alpha(x_2)$ and so on.

So, we have obtained a sequence $\{x_n\}$ which is monotonically increasing. Why? Look at this one this. $\alpha(x)$ always is bigger than (x) because $x \times x$ is inside U. So, no chance that this $\alpha(x)$ which is the minimum of all y such that $x \times Y$ is in the complement of U inside M_x, that cannot be less or equal to x.

So, this is a strictly monotonically increasing sequence converging to some point in $[0, \Omega)$. Any sequences in $[0, \Omega)$ which is monotonic has to converge to y inside $[0, \Omega)$. But then the sequence $\{x_n \times \alpha(x_n)\}$ which is the same as $\{x_n \times x_{n+1}\}$, where does it converge?

$\alpha(x_n)$ is nothing but x_{n+1} so it is the same sequence $\{x_n\}$ except the indexing is changed. So, both $\{x_n\}$ and $\{\alpha(x_n)\}$ converge to same point. Therefore, $\{x_n \times \alpha(x_n)\}$ converges to $y \times y$ which is inside A and hence contained inside U by our choice.

Whereas, none of the terms of this sequence is inside U. Look at this point $x \times \alpha(x)$. This point is not in U is the claim. That is same as saying that $\alpha(x)$ is not in $M_x \cap U$. And that follows because $\alpha(x)$ in the minimum of all y such that $x \times y$ is not in U.

So that is a contradiction.

So, finally, why I have used this notation T for this sapce? Because this is called Tychonoff's Plank. This example is called Tychonoff's Plank.

In conclusion, we say normality is not finite productive.

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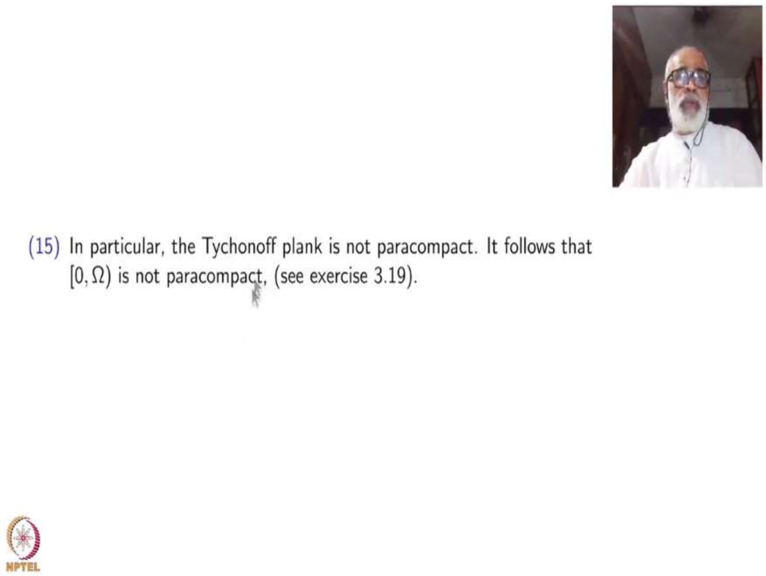

As a corollary, we can say or that Tychonoff's Plank is not paracompact. Paracompact Hausdorff spaces or paracompact regular spaces will be normal. That is what we have seen already. It follows that $[0, \Omega)$ is not paracompact because paracompact cross paracompact would have been paracompact. Of course, $[0, \Omega)$ is not compact. So, in one single example you have so many counter examples here.

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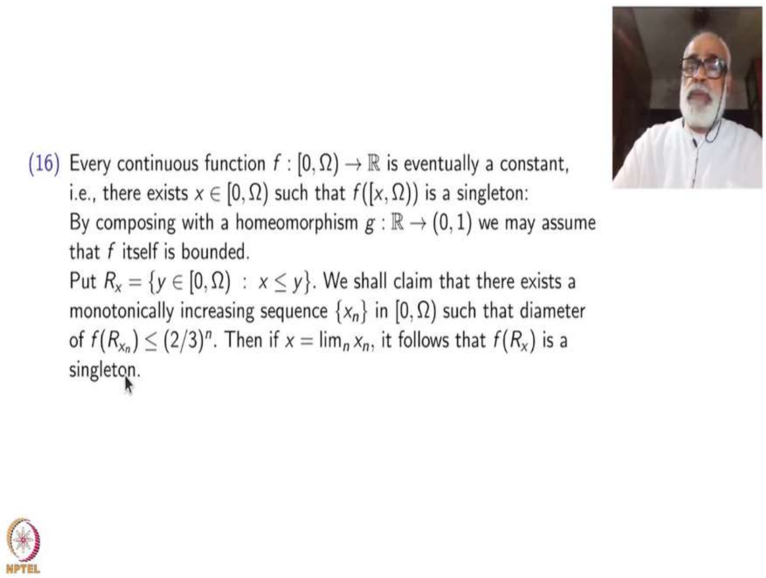

There will be more to come. This is another wonderful property of $[0, \Omega)$.

Every continuous real valued function on $[0, \Omega)$ is eventually a constant. See the role is reversed here. Usually, if you have a continuous function from $\mathbb{R}$ into a disconnected space connected space or a discrete space, then it is a constant. That is the kind of result we have been familiar with and coming across all the time.

But here, this is highly disconnected, and $\mathbb{R}$ is connected space. So this is eventually constant, so what is the meaning of that? There will be some x inside $[0, \Omega)$ such that f of the right ray R_x is a singleton.

By composing with a homeomorphism g from $\mathbb{R}$ to $(0, 1)$ (there are many homeomorphisms like this, for example you can take tan^{-1}) we may assume that the function is bounded and indeed we can assume that f itself is taking values inside $(0, 1)$. Put R_x equal to the closed right ray (I am changing the notation here, instead of $\overline{R_x}$.) We shall claim that there exists a monotonically increasing sequence $\{x_n\}$ in $[0, \Omega)$ such that the diameter of $f(R_{x_n})$ (as a subset of $(0, 1)$ the diameter makes sense with respect to the standard metric on $(0, 1)$) is less than two thirds raise to n. Some number raised to n is good enough, that number must be less than 1 that is important and crucial for us.

Then if you take x as limit of x_n, in $[0, \Omega)$, it will follow that $f(R_x)$ is a singleton. This is due to Cantor's theorem. Balls of radius smaller and smaller converging to 0. So the intersection must be a singleton.

So, that is the way we are going to prove this theorem just by constructing a monotonically increasing sequence in $[0, \Omega)$ which will automatically converge and that convergent point will be our x. Then $f(R_{x_n})$ will be a nested sequence of non empty closed sets with their diameter tending to 0. So, we have to construct this sequence.

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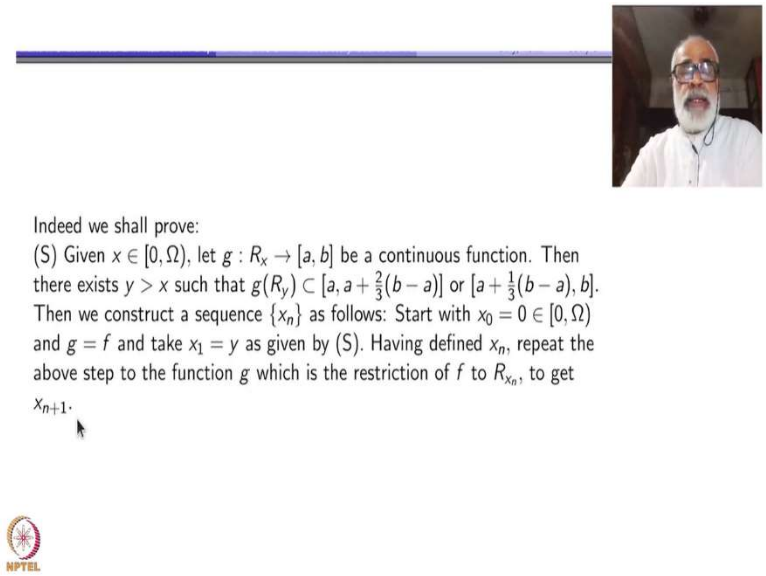

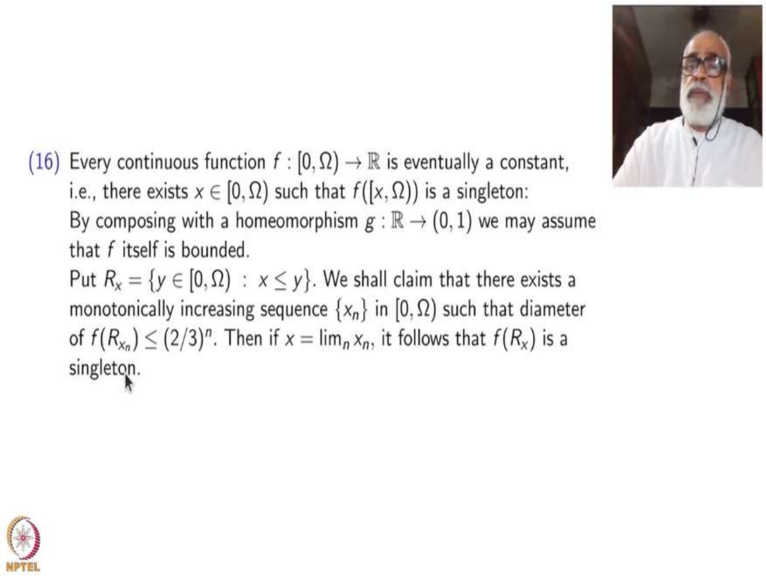

However, we are going to prove slightly different statement (perhaps s lightly stronger one). An inductive statement here and then apply it. So, this statement (S) is as follows:

(S) Given x belonging to $[0,\Omega)$, let g from R_x to $[a,b]$ be a continuous function (right now, recall that R_x denotes the closed right ray). Then there exists y bigger than x such that $g(R_y)$ is contained inside the first two third of $[a,b]$ or this second two third of $[a,b]$.

See first two third means what? $[a, a+2/3(b-a)]$. $b-a$ is the length of $[a,b]$. So the first $2/3$ is the interval from a to a plus that much. Similarly, the second $2/3$ is the interval starting from a plus one third of $(b-a)$ all the way to b. I want to say $f(R_x)$ is either here or here. It may be in the intersection also that is good enough for me, no problem. It is contained in one of them is important. That is what we have to prove.

Assuming (S), we construct a sequence $\{x_n\}$ as follows. See, first I had a, b then if this ab is 1 then what happens? The diameter of $g(R_y)$ will be at most two third. apply this g, apply the same thing to this function repeat this process you will get another y such that that diameter of the new thing will be two third of that, and that is what we are going to do.

So, then we construct a sequence $\{x_n\}$ as follows. Start with any x_0 inside $[0, \Omega)$. Take g equal to f restricted to R_x, take x_1 equal to y as given in (S). Inductively having defined x_n, repeat the above step to the function g equal to f restricted to R_{x_n} and then take x_{n+1} to be the corresponding y given by (S).

This (S) is a general of statement, I am starting with this hypothesis that f is a continuous function from $[0, \Omega)$ to $[0, 1]$. So, to begin with the diameter of R_x is less than 1. At very first stage, I will be getting diameter less than two third, and next one two third square, two third cube and so on. So, we have to prove this statement (S) once we prove this one, this inductive step is over, Then the proof is over.

So, let us prove this statement (S) now.

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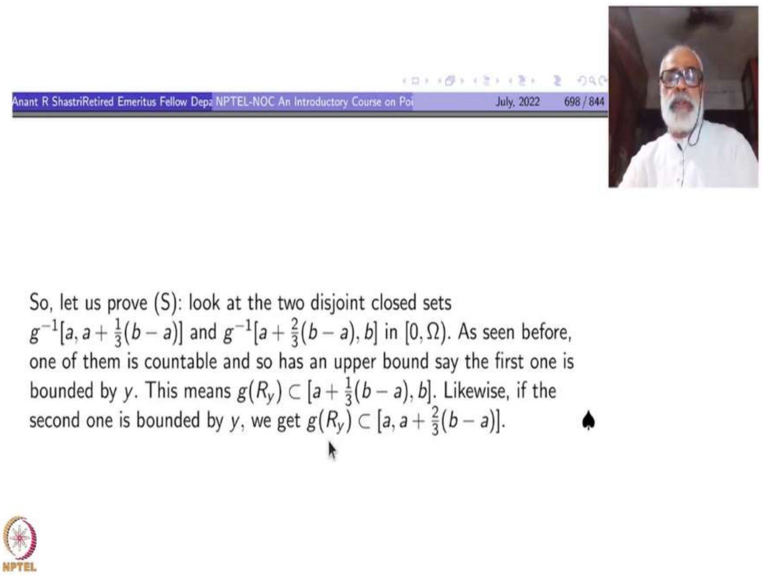

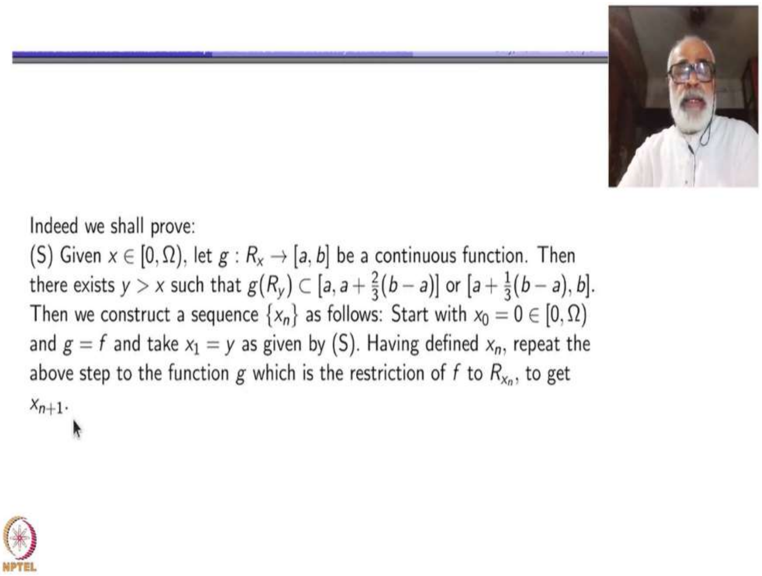

Look two disjoint closed subsets $g^{-1}[a, a + 1/3(b - a)]$ and $g^{-1}[a + 2/3(b - a), b]$. These two disjoint closed intervals are got by deleting the middle one third open interval from $[a, b]$, like in the Cantor set construction.

If you take g^{-1} of these two disjoint closed intervals, they will be disjoint closed subsets of $[0, \Omega)$. What do we know about the disjoint closed subsets of $[0, \Omega)$, we have done something on them, you see. So, that can be used now. As seen before, one of them is countable and so, has an upper bound say the first one is bounded by y. It could be the other one, that one does not matter, it is symmetric.

So let us assume that the first one is bounded by y, y is an element of one element of g^{-1} of sorry, y element the domain of g viz., R_x, so y is bigger than or equal to x, and we have $g(R_y)$ will be contained in the complement of the first interval.

Likewise, if the second one is bounded by y, then you will that $g(R_y)$ will be contained in the complement of $[a + 1/3(b - a), b]$. So, that is the conclusion of (S).

So, that is why I separated out this (12) while proving normality of $[0, \Omega)$, instead of proving directly that disjoint closed subsets can be separated by disjoint open sets. Pairs of disjoint close of subsets have themselves some interesting property namely one of them has to be countable and that is used here.

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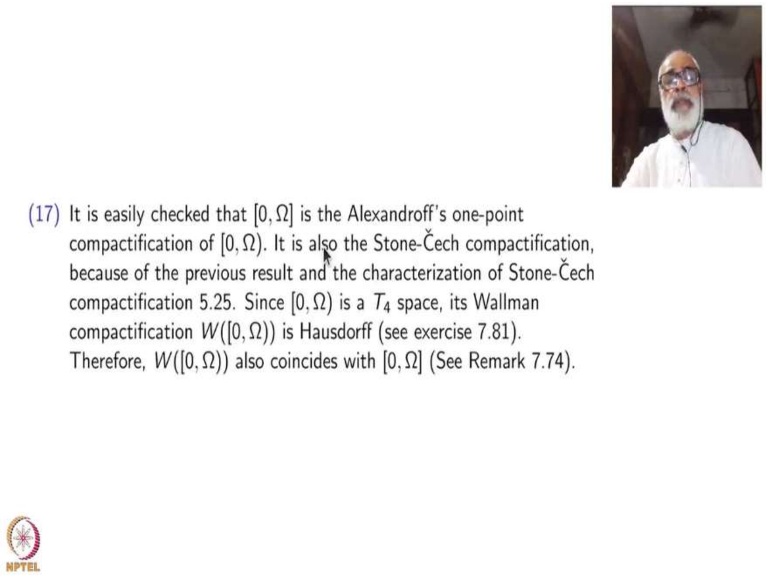

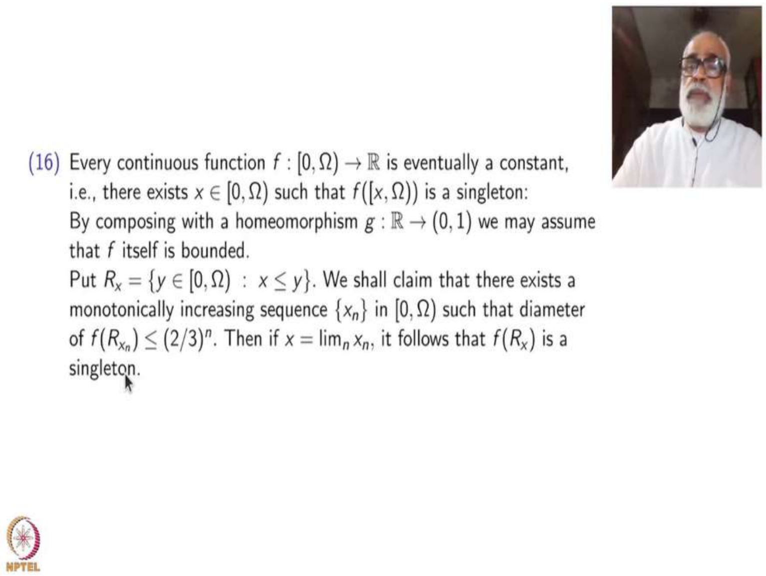

So, next thing is a strong conclusion. The first thing is $[0,\Omega]$ is the Alexandroff's one-point compactification of $[0,\Omega)$. This is very easy to see, by the very definition. There is only one extra point Ω in the larger space. And what are the neighbourhoods of this Ω? Something like $(\alpha,\Omega]$. The complement of this will be $[0,\alpha]$ is closed and compact. And conversely. Any compact subset of $[0,\Omega)$ is a closed subset of $[0,\Omega]$ also and hence its complement is a neighbourhood of Ω.

Also note that $[0,\Omega]$ is a compact Hausdorff space. So, it follows that $[0,\Omega)$ is locally compact also. So, Alexandroff, one-point complication makes sense. If the extra point here is denoted by infinity, that infinity can be sent to this capital Ω to get a homeomorphism that is all, So whichever way you want to see it is easy to see that this is the Alexandroff's one-point compactification of $[0,\Omega)$.

It is also the Stone-Cech compactification of $[0,\Omega)$. Because of this property that every continuous function f from $[0,\Omega)$ to $\mathbb{R}$ is eventually a constant. Therefore, you can extend it continuously to $[0,\Omega]$ by defining $f(\Omega)$ equal to that constant.

So, every continuous function from $[0,\Omega)$ to $[0,1]$ can be extended to $[0,\Omega]$, uniquely. That is the characterization of Stone-Cech compactification. So, that is what we have to remember now. Any compact space X which contains $[0,\Omega)$ and having this property of unique extension of continuous functions into $[0,1]$ must be equivalent to the Stone-Cech compactification of $[0,\Omega)$. That the meaning of the characterization of Stone-Cech compactification.

There is a final thing here. Since $[0,\Omega)$ is a T_4 space, (T_1 and regular, that is enough), its Wallman compactification W exists and is normal. We have seen that. Perhaps it was an exercise. If you have T_4 space, then its Wallman compactification is Hausdorff.

So, we have got $W([0,\Omega))$ is a Hausdorff space. Then we have made a remark that whenever the Wallman compactification is a Hausdorff space, it is the same as Stone-Cech compactification. So that was another remark, which you have studied. In particular, $W([0,\Omega))$ is equal to $[0,\Omega]$.

Using all these things, we conclude that $[0,\Omega]$ is three different compactifications of $[0,\Omega)$ simultaneously. I do not know any other example with such a property.

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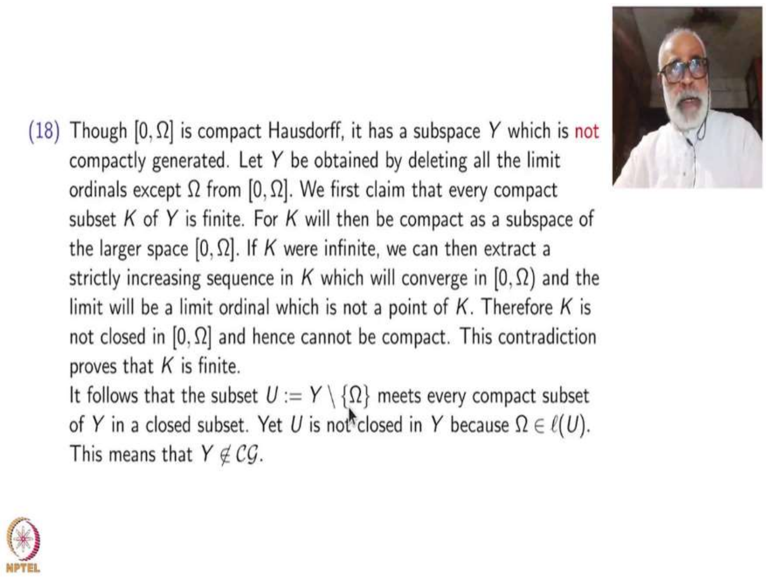

So here is one more concluding remark. Though [0, Omega] is a compact Hausdorff space, it has a subspace Y, which is not compactly generated. See I wanted to have an example of a subspace of a compactly generated space which is not compactly generated. So, this example seems to be a simplest example have you studied. All these are now easy examples for us. The original space is compact Hausdorff. Obviously it is compactly generated, but the subspace we have here is not compactly generated.

So, what is that subspace? Let us see. Take Y to be obtained by [0, Omega] as a subspace of this, by deleting all the limit ordinals except the top one, capital Omega. For instance, the first thing I will omit is little omega, then I will omit twice omega, then I omit three times omega, and so on, then I will omit omega square and so on, all those limit ordinals are omitted. But the last one viz, Omega I will keep.

We first claim that every compact subset K of Y is finite now. Every compact subset of $[0,\Omega)$ is countable, that we have seen. Now, we are claiming something stronger: every compact subset of this subset Y, is automatically compact in $[0,\Omega)$ also but it is a subset of Y, then it is finite. Let K be a compact subset of Y. Then K will then be a compact of $[0,\Omega]$.

If K were infinite, first of all it is countable. If it is infinite countable, we can then extract a strictly increasing sequence in K which will converge to a point s in $[0,\Omega)$. This limit point s will a limit ordinal. Being a limit of a sequence in K which is closed s will be inside K. But K is inside Y and Y does not contain any limit ordinals inside $[0,\Omega)$. So, that is a contradiction.

So, K has to be finite.

Now take $U = Y \setminus \{\Omega\}$, i.e., you throw away Ω also. Then U will meet every compact subset of Y in a closed subset K because K is just a finite. Yet U is not closed because Ω is a closure point of U and yet is not in U. This contradiction proves that the topology on Y is not compactly generated.

Remember compactly generated means what? For every compact subset K, if $U \cap K$ is a closed subset inside K then U itself must be closed inside the original space. And that property is violated here.

I think you must have been satisfied by now that this ordinal space $[0,\Omega)$ has so many properties, wonderful topological properties, so that justifies our efforts in studying these ordinals. So, next time we will do one more example using the ordinals. But we will construct something more, some more interesting things, from a topologist's point of view. Thank you.