An Introduction to Point-Set-Topology (Part II)
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Lecture 47
Order Topology

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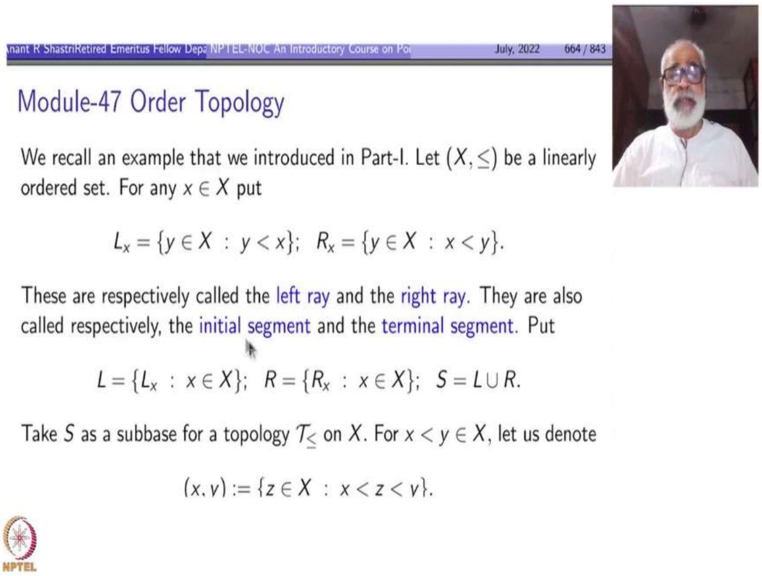

Hello, welcome to Module 47 of NPTEL-NOC an introductory course on Point Set Topology Part 2. So, today we will introduce the Order Topology. So far, we are only studying the partial ordered sets, totally ordered set and so on, only set theoretic aspect. So, today we are bringing topology. In essence, we had already done this in part I. So, some part of it will be just recalling an example, it was an example there of a topology.

So, let us recall that, start with a linearly ordered set or whatever you call totally ordered set. For any point x in X, let us have this notation, L_x is the left ray. So, here I am taking open left ray, the set of all y in X less than x, and R_x is the right open ray, the set of all y in X bigger than x. These are reciprocally called left ray and right ray, they are also called respectively initial segment and terminal segment. So, this terminology also we have used.

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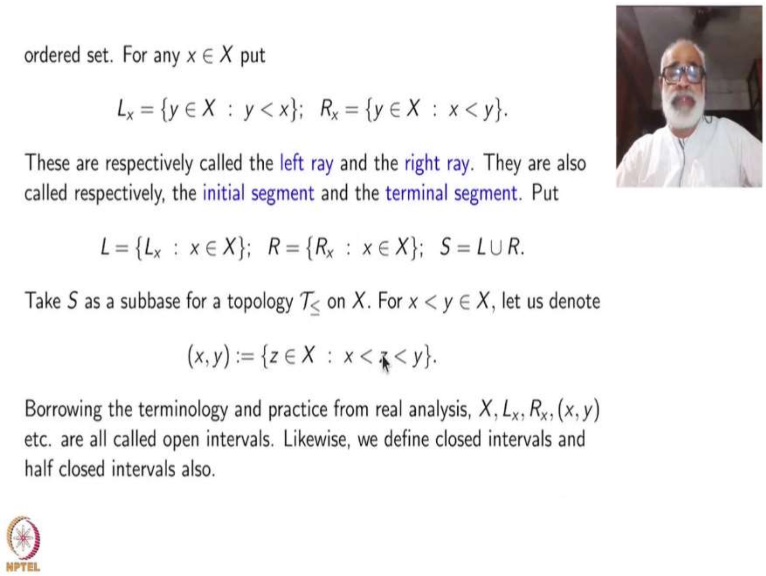

Now, let us define L to be the collection of all L_x, where x ranges over X, this is a family of subsets of X now, the family of all left rays. Similarly let R be the family of all right rays, and S is $L \cup R$. Take this S as a subbase for a topology on X. Any non empty family of subsets of X can be taken as a subbase for a unique topology on X. All that you have to do is take all finite intersections and then take all possible unions that is the topology. So, take S as a subbase for a topology which will denote by $\mathcal{T}_{\leq}$ just to remind you that this topology corresponds to the linear order with which we started.

For x less than y in X let us also have this notation (x, y) (this should not be confused with the ordered pair) to be the set of all z in X such that x is less than z less than y. So all points strictly between x and y. It may be empty also we do not know. Borrowing the terminology and practice from real analysis, L_x, R_x, (x, y), etc are all called open intervals in X.

Likewise, we define closed intervals and half-closed intervals also all that you have to do is your have to put `less than or equal' to here, instead of `less than'. So on I do not want to go into those details, those things are common practice. So, I will also use them, there is no need to again spend 2-3 minutes on it. For example, if I take here if I take less than or equal to, then that will be the call a closed ray, closed left ray and closed right ray and so on.

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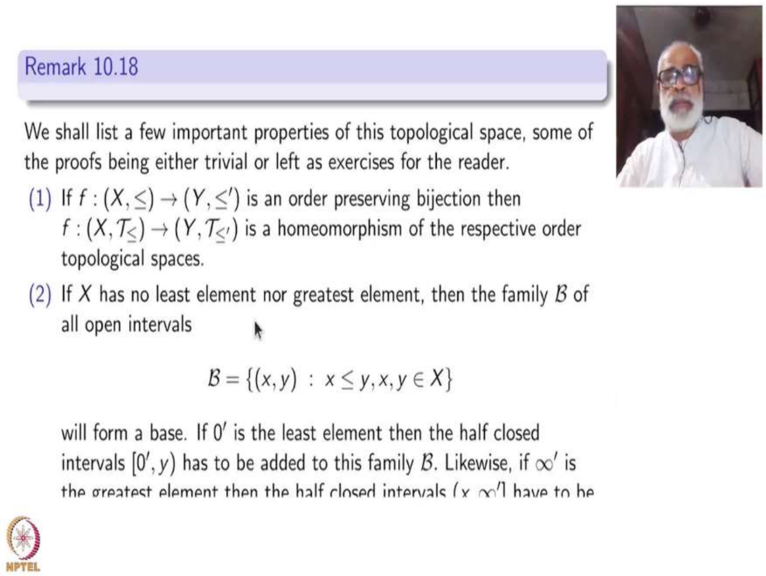

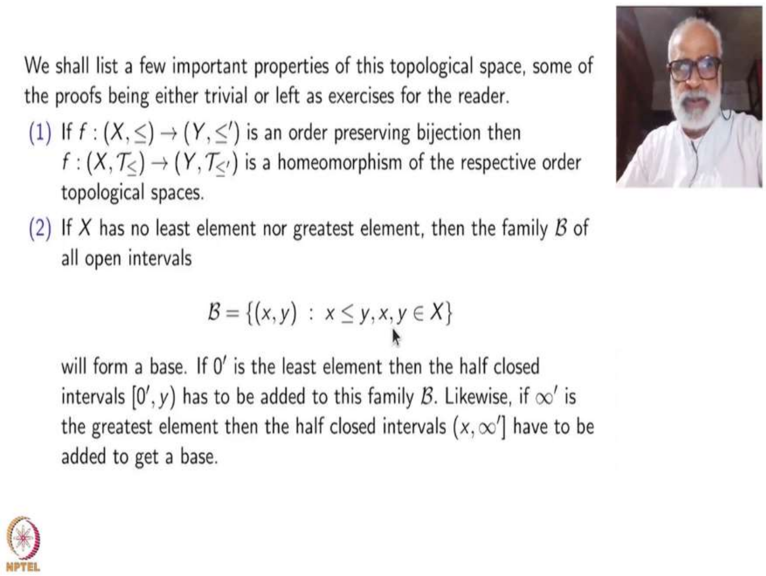

So, we should see at least a few important properties of this topological space, some of the proofs being either trivial or left to you as an exercise because some of these things we have seen before also.

(1) If you have an order preserving bijection from $(X, \leq)$ to $(X', \leq')$ from one linearly order set to another,

Then this thing as a map from topological space with corresponding topologies, is a homeomorphism of the respective order topological spaces. Indeed, it will preserve the subbasic open sets themselves, so it is a very strong homeomorphism.

(2) If X has no least element nor greatest element, then the family $\mathcal{B}$ of all open intervals (x, y) forms a base for $\mathcal{T}_{\leq}$.

If you take intersections of left rays and right rays you will get these intervals.

However, this will not work if there is a least element or a greatest element in X. You know to get a base you will have to include those elements also but the extreme elements will not belong to any open interval. You have to be careful about that one. For example, suppose X is $[0,1]$, the closed interval in $\mathbb{R}$. If we only take open rays, there will be problem because 0 and 1 are not covered.

So, that is why all that you have to do is the following. If $0'$ (I would like to put a prime here not to confuse it with the real number 0) is the least element of X, (if there is one, all that you have to do is to include all half closed intervals, $[0,y)$ also to $\mathcal{B}$. Similarly, if there is a greatest element say infinity prime in X, then include all half closed interval $(x,\infty]$ to $\mathcal{B}$. Then that family will be a base. Whereas, defining the subbase, there is no such problem.

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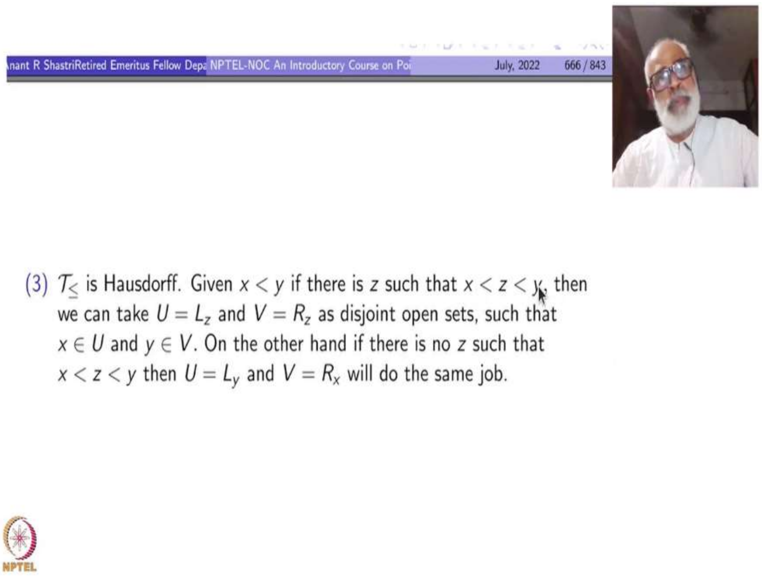

Now, first thing we observe is this topology $\mathcal{T}_\leq$ is Hausdorff. So, given x less than y (of course, given x not equal to y, x is less than y or x is bigger than y, so, I can assume x less than y), if there is z such that x less than z less than y, then we can take U equal to L_z and V equal to R_z as disjoint open left ray and right ray open left ray. Then x will be inside U and y will inside V. (And I am giving you an argument to show that this topology is Hausdorff, I have not completed yet). On the other hand, if there is no z at all between x and y (this can happen for example, when you take natural numbers with the usual order) then what happens? Then take U equal to L_y and V equal to R_x. L_y will contain x and R_x will contain y and there is nothing in between. So, their intersection will be empty.

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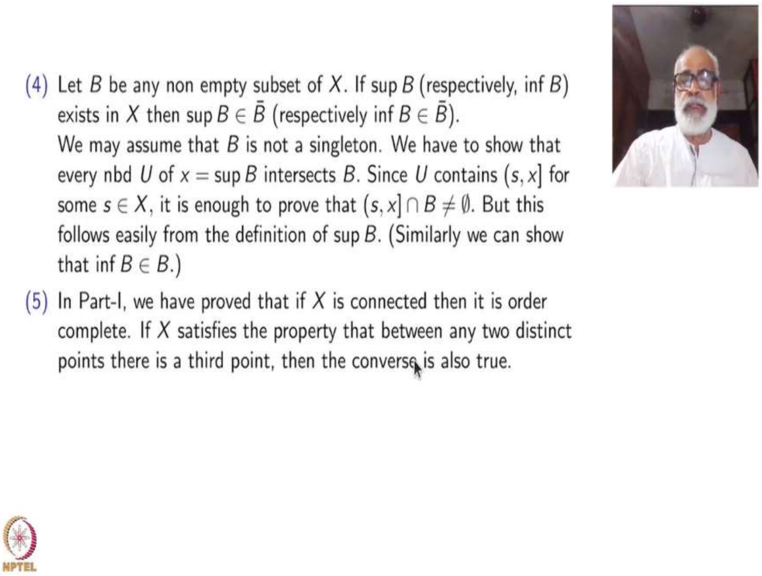

Now, let B be any non-empty subset of X. If supremum of B (respectively infimum of B) exists then supremum of B (respectively $\inf B$) is inside $\overline{B}$. See this closure is with respect to the topology $\mathcal{T}_{\leq}$.

The proof is exactly similar as for real numbers, actually much simpler if you think. Do not use any algebra of real numbers, no need of that. Just the definition of infimum and what is the meaning of open sets here and what is the meaning of closure and so on? You may assume that B is not a singleton, because supremum of A supremum of B will be that singleton itself and it will be already inside the closure. There is nothing to prove then. We can also assume that Sup B is not infinity primes (respectively $\inf B$ is not) because these two cases are easy.

So, we have to show that every neighbourhood U of x (where x denotes the supremum of B) intersects B. That is the meaning of saying x will be in $\overline{B}$. Since U contains an open interval around x there exists s in X such that the half open interval $(s, x]$ is inside U. There is also an open interval on the other side of x, I do not care.

It is enough to prove that $(s, x] \cap B$ is non-empty. I want to show that $U \cap B$ is non-empty. But if this is not the case, what happens? x will not be the supermum of B, because s itself will be an upper bound for B. So, this is a contradiction.

So, similarly infimum of B is also inside closure of B.

So, in part I, we have proved that if X is connected, then it is order complete. I am just recalling that one and not proving it here. If X satisfies the property that between any two distinct elements, there is a third element then the converse is also true. So, these two things we had proved there.

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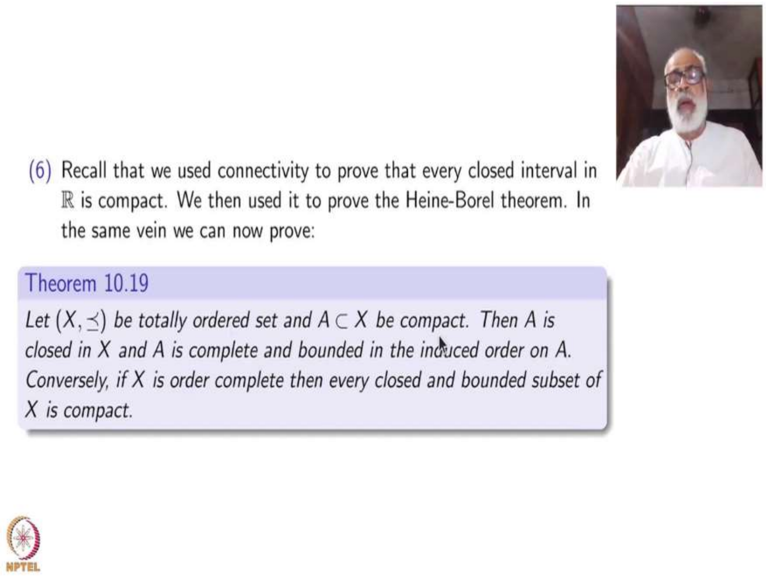

Now, recall that we use connectivity prove that every closed interval in $\mathbb{R}$ is compact. That is why are studying this. We then used this to prove Heine-Borel theorem for $\mathbb{R}^n$. First of all, we proved that closed intervals in $\mathbb{R}$ are compact. And then passed on to $\mathbb{R}^n$ and so on.

Closed and bounded intervals if you like. $(-\infty, a]$ or $[a, \infty]$ are not closed intervals in $\mathbb{R}$. Forget it, open intervals can be bounded or unbounded. $[-\infty, a]$ is not a subset of $\mathbb{R}$ though it looks like a closed interval, it is so in the extended real number system.

The same way we can prove something, similar to Heine-Borel theorem now in the context of an order complete space.

So, start with a totally ordered set X, let A to be a compact subset of X. Then A is closed in X and A is complete and bounded in the induced order topology.

(Bounded means what here? Bounded below and above. In the induced ordered topology on A completeness also comes.)

Conversely, if X is order complete, then every closed and bounded subset of X is compact.

See this order completeness comes by taking the restricted order, the same order on X being restricted to A. that is why I am saying the induced order. So, this result is `if and only if'. Compactness implies closed and bounded close and bounded implies compact.

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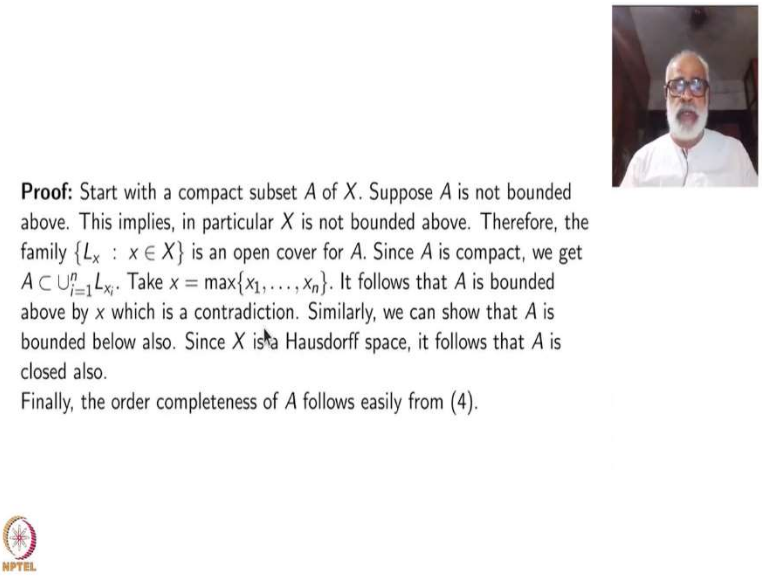

Let us go through it carefully. Start with the compact set A, suppose A is not bounded above, if a subset is not bounded above in your space, the space is also not bounded above. Therefore, the family L_x, x belonging to X becomes an open cover for X. So, in particular, for A also. But since A is compact, we will get a finite cover.

A is contained inside union of L_{x_i} say, for i ranging from 1 to n. Take x to the maximum of $\{x_1, x_2, \ldots, x_n\}$. There are only finite many of them in a totally ordered set, so, you can take the maximum. It follows that A is bounded by x because L_{x_i} are all contained inside L_x. Now we see a contradiction. We just said A is not bounded above.

So A is bounded above. Similarly, we can show that A is bounded below. Instead of L_x, we take R_x's, That is all.

Now, X is Hausdorff space, we have seen just now. It follows that being a compact subset, A is closed.

Finally, the order completeness of A follows easily from what we have just observed, namely, infimum and supremum of this set A which are defined because A is a bounded set, They are both inside A. They are inside $\bar{A}$ but A is $\bar{A}$ because A is closed, so they are inside A over. So that is precisely the meaning of the order completeness.

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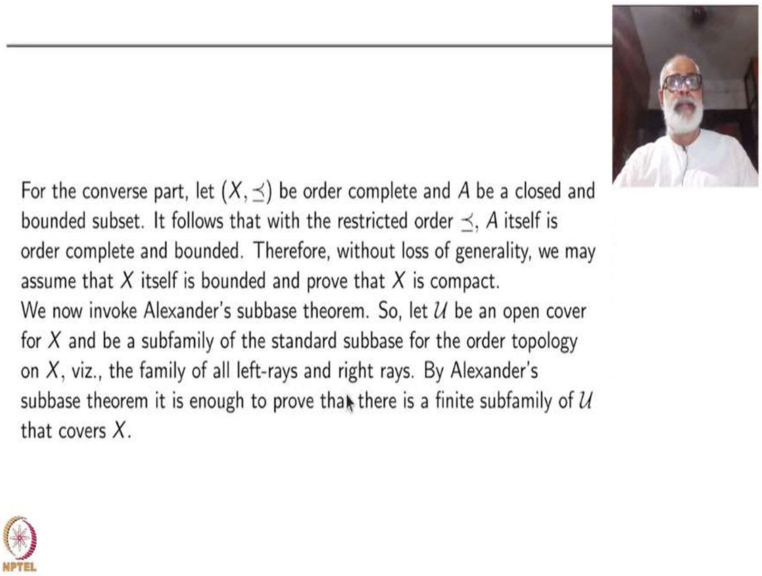

For the converse part, let $(X, \leq)$ be order complete and A be a closed and bounded subset. It follows that with respect to the restricted order A itself is order complete and bounded. Therefore, in order to prove that A is compact, without loss of generality, we may assume that $A = X$ itself is bounded and prove that X is compact.

So, that is how I stated it. If X is order complete and bounded then it is compact. Here we use Alexander's subbase theorem. So, what we will take $\mathcal{U}$ to be an open cover for X by members of this standards subbase. You fix a subbase then take for every subfamily of that which will cover X show that it has a finite subcover, that is enough to prove that X is compact, by Alexander subbase theorem.

So, here what we do? We take the family of all left rays and right rays, viz., $S = L \cup R$, remember that. So, take a subfamily $\mathcal{U}$ of S which covers X. Now get finite subcover. Then Alexander's subbase theorem says the space X must be compact.

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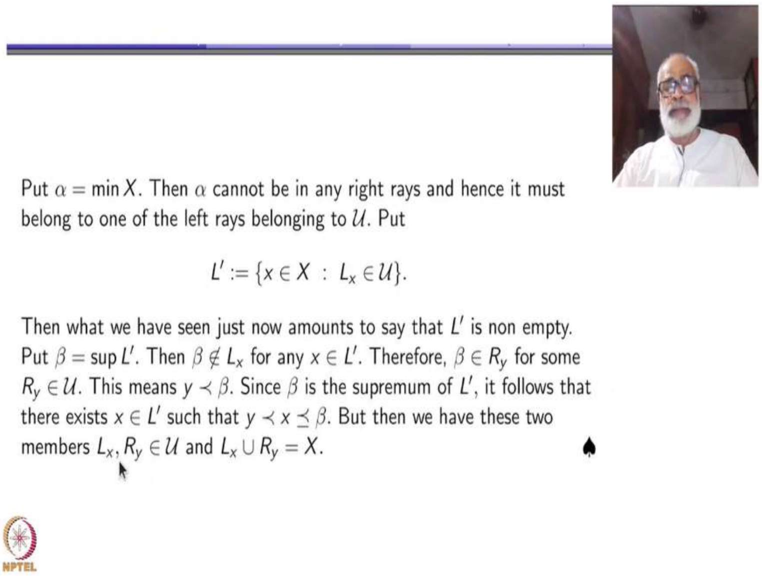

So, put α equal to minimum of X. (We have X is bounded, so immediately we are using that.) Then α cannot be in any right ray because, I am taking only open rays here. Hence, it must belong to one of the left rays, because left rays and right rays in $\mathcal{U}$ are going to cover X I mean I have chosen some family $\mathcal{U}$ of left rays and right rays, that covers X, it is not the set of all the left rays and right rays some left rays and right rays it cover.

So, this α must be in one of the left rays in $\mathcal{U}$. Put L' equal to all those x for which L_x is inside $\mathcal{U}$. So, this L' is a non-empty subset because α must be in one of the L_x.

Now, put β is equal to the supremum of L'. L' being a subset of X is bounded above. So, by order completeness supremum exists. Then this β cannot be in any L_x because it is bigger than all x in L'. Just see here, if β is in L_x for some x then β is strictly less than x right? So, β will not be in L_x for any x inside L'. Therefore, β must be, (all left rays in curly are all taken here), so, what are left out? β must belong to one of the right rays R_y for some R_y inside $\mathcal{U}$. This means that β must be bigger than y. Strictly bigger than y. Since β is the supremum of L' , it follows there is x belong to L' between y and β, y less than x less than less than equal to be β, (actually strictly less than β, because we already saw that β is not in L').

So, I have used that beta is supremum here. So, this is the property of supremum. y less than x less than or equal to β with x in side L'. But then we have these two members L_x and R_y, belonging to $\mathcal{U}$. $L_x \cup R_y$ must be equal to X, because y less than x.

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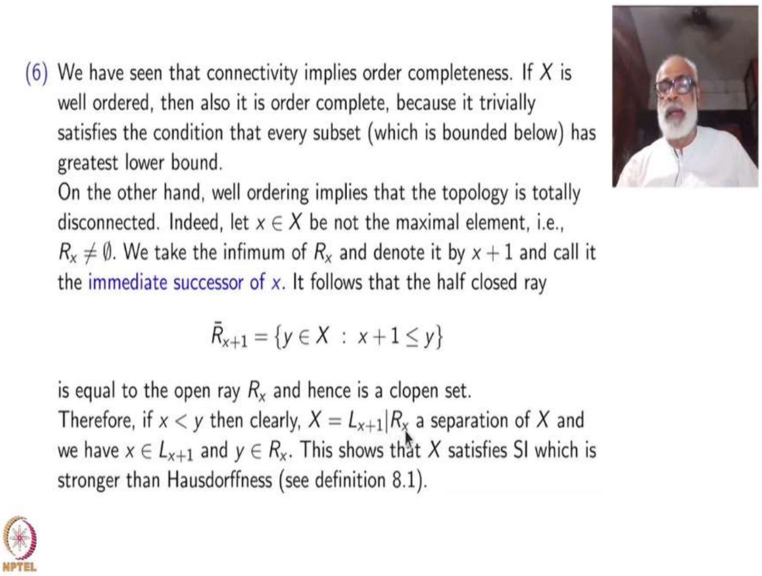

Next we have seen that connectivity implies order completeness. Just recall that this was done in the part I, that is one of the things we have proved. That is one way to get order completeness.

Now if X is well ordered, then also it is order complete, because every subset (bounded or not) has a least element which a greatest lower bound.

On the other end, well ordering implies that the topology is totally disconnected provided it has more than one element. That is what I said is far away from being connectivity. So, how do we see this one, namely, x belong to X be not a maximum, not the maximal element. Pick up some element is not maximal, that is R_x not equal to empty set, there are elements bigger than x. This mean R_x is non empty.

We take the infimum of R_x and denoted by $x + 1$ and call it the immediate successor of x, immediate successor. By the way, this immediate successor is the key to the entire theory involved in Peano's axioms and Zermelo-Frankel's theory and so on. The idea was actually goes back to Grossmann. In modern set theory, perhaps his name does not appear so much but this immediate successor, he pointed out. But, finally, it was Peano whose axioms became the best, there are many many trials in between, several people have tried it, you know, one of the other guy whose name is quite quoted is Lebesgue. So, Peano's axiom is based this. The entire algebra of natural numbers is constructed out of this. This is the first-time that appears this plus sign in algebra, logically, the notation $x + 1$. We are not going to do any algebra, we are stopping here. $x + 1$ is that I have introduced now.

So, if R_x is non-empty for any x, then you take the infimum because X is well ordered. So, infimum exists and it is unique. that infimum you called $x+1$. So, it follows that the half closed ray $\overline{R_{x+1}}$, (this bar denotes the closure also, so, there is no contradiction no clash here, with resepct to the order topology), is just defined to be the set of all y such that $x+1$ is less than or equal to y. Strictly less than would have given you the open ray R_{x+1}.

So, $\overline{R_{x+1}}$ is a closed right ray. But now if you take the open ray R_x what happens? There is no element between x and $x+1$. So, open ray R_x is also equal to the closed ray R_{x+1}. Therefore, this is is both open and closed. Therefore, for all y such that x is less than y, R_x would be a clopen set containing y and not containing x.

So, there is separation, so you can actually write down separation $X = L_{x+1} | R_x$ separation X one containing and another containing y. So, this is stronger than totally disconnectedness, namely, it actually satisfies our SI, what we have studied earlier. So, stronger than Hausdorffness, stronger than total disconnectedness also.

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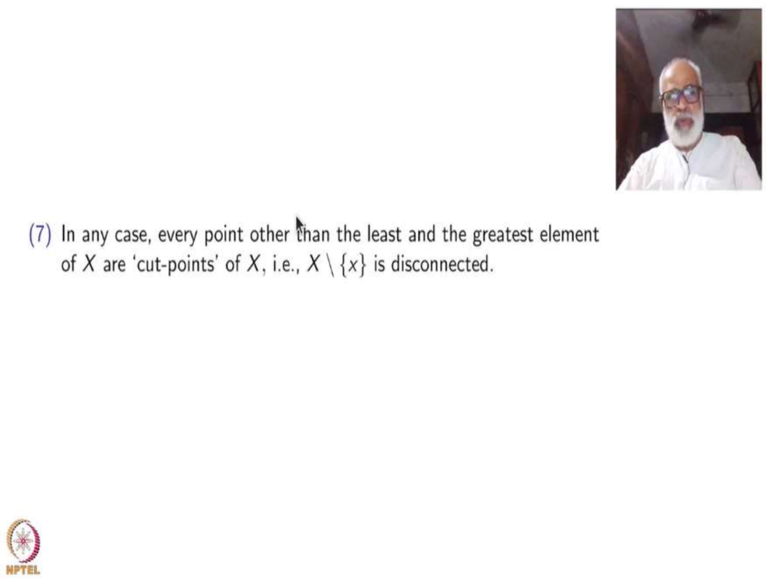

In any case, every point other than the least and the greatest element are `cut points'. This name `cut points is also used in the above situation. It means what? You throw away that point from the space, it becomes disconnected, such points are called cut points.

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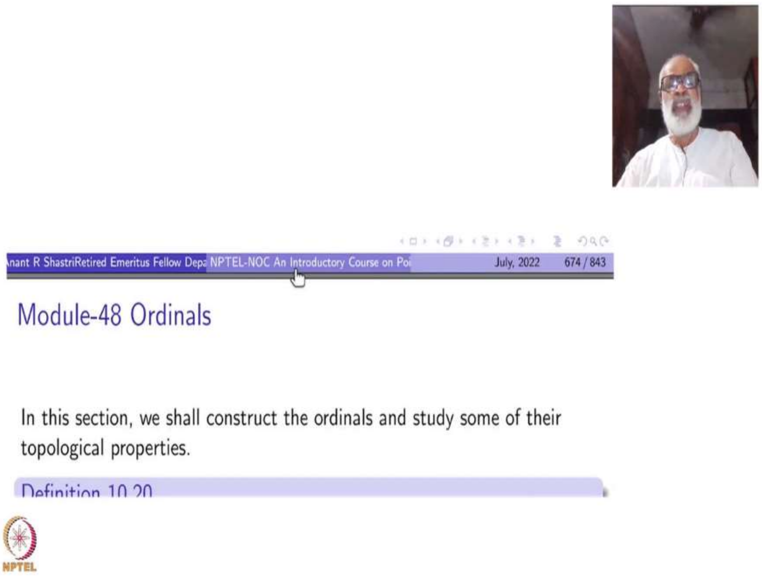

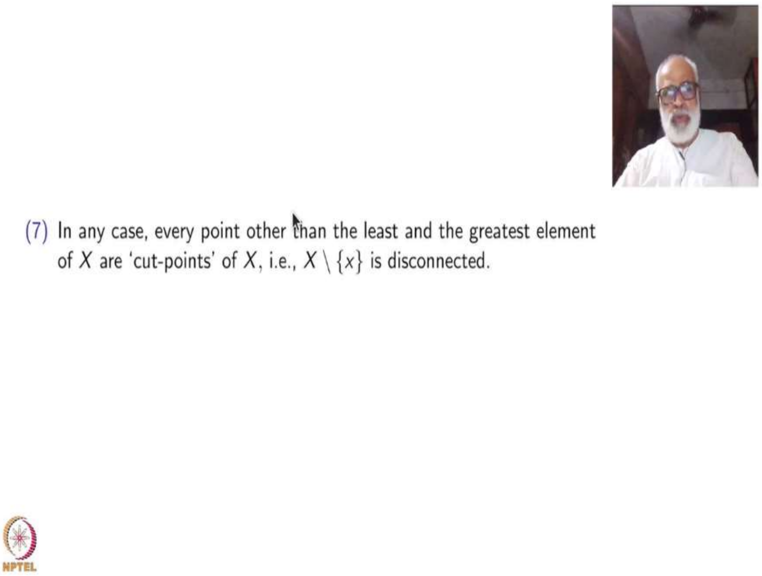

So, I think this much topology is good enough for one day. You will have many other topological aspects of this one because finally, our aim is to produce lots of examples out of one single example. So, we will meet again. Next time we shall construct the ordinals, the example that we are interested in. Thank you.