**An Introduction to Point-Set-Topology Part (II)**
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**Lecture No. 45**
**Partially Ordered Sets**

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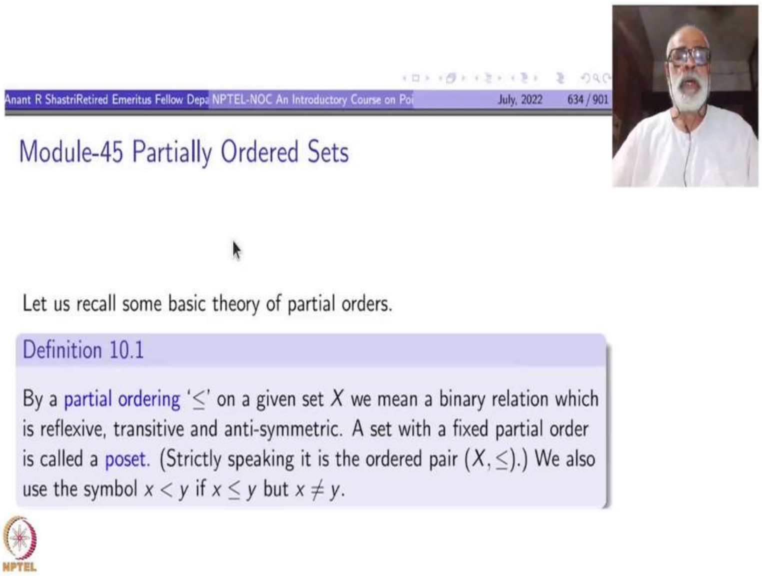


Hello. Welcome to NPTEL NOC, an introductory course on Point-Set-Topology, Part II. Today, we shall start a new topic, module 45, Partially Ordered Sets. Let us recall some basic theory of partial orders. Some of them must be already familiar to you. So, we should be somewhat quick here.

By a partial ordering, which is usually written less than or equal to, on a given set $X$, we mean a binary relation which is reflexive, transitive and anti-symmetric (unlike the case of equivalence relation, which is reflexive, transitive and symmetric). So that is the big difference here, anti-symmetric. Anti-symmetry just means that if $x$ is less than or equal to $y$ and $y$ is less than $x$, then $x$ must be equal to $y$.

A set with a fixed partial order is called a poset. A partial ordered set. It has been shortened to poset. Strictly speaking, it is the ordered pair $(X, \leq)$. But as usual in topology, in metric spaces, just the way we say, let $X$ be a metric space, let $X$ be a topological space etc., similarly, we will say `let $X$ be a partially ordered set, let $X$ be a poset. But then we may immediately mention what the partial order there is whenever we are dealing with it, specifically.

We also use the symbol $x < y$, to mean that $x$ is less than equal to $y$ but $x$ is not equal to $y$. Less than equal to $y$ will include $x$ equal to $y$ case also because if $x$ is less than $y$, $y$ is less than or $x$, then $x$ is equal to $y$, that is what we have seen. And reflexivity just means that $x$ is always related to $x$. So, that is already there.

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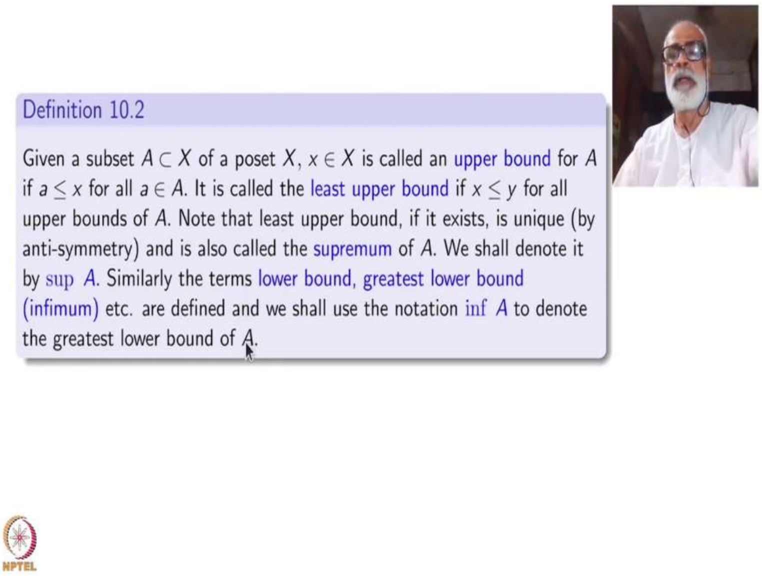


Given a subset $A$ of a partially ordered set, an element $x$ in $X$ is called an upper bound for $A$ if $a$ is less than equal to $x$ for all $a$ inside $A$. So, you may say that $x$ is the biggest amongest all elements of $a$, but $x$ itself may not be an element of $A$. So, that is called an upper bound. There may be many upper bounds. There may not be any upper bound either. So, it is just a definition, no assertion of existence or uniqueness. It is called a least upper bound, if $x$ is less than $y$ for all upper bounds $y$ of $A$. For all upper bounds $y$, $x$ must be less than or equal to $y$. So then it is called a least upper bound.

Note that least upper bound, if it exists, is unique by anti-symmetry, because if $x$ and $y$ are both least upper bounds then $x$ is less than $y$, and $y$ will be less than equal to $x$, so $x$ is equal to $y$. That is anti-symmetry is used here. Then least upper bound would be unique. So, we will call it the supremum of $A$, and denote it by $\sup\ A$. Similarly, the terms `lower bound' `greatest lower bound' i.e., infimum etc are defined.

So, all these concepts we are using here and we have been using in the study of real numbers, the same definition, same thing. Only thing is, in real numbers, there will be addition, subtraction, multiplication. Nothing of that sort will be used here. They are just arbitrary sets. How far you can go, what analysis you can do, what topology you can do with just putting an order on a set? That is the topic here today.

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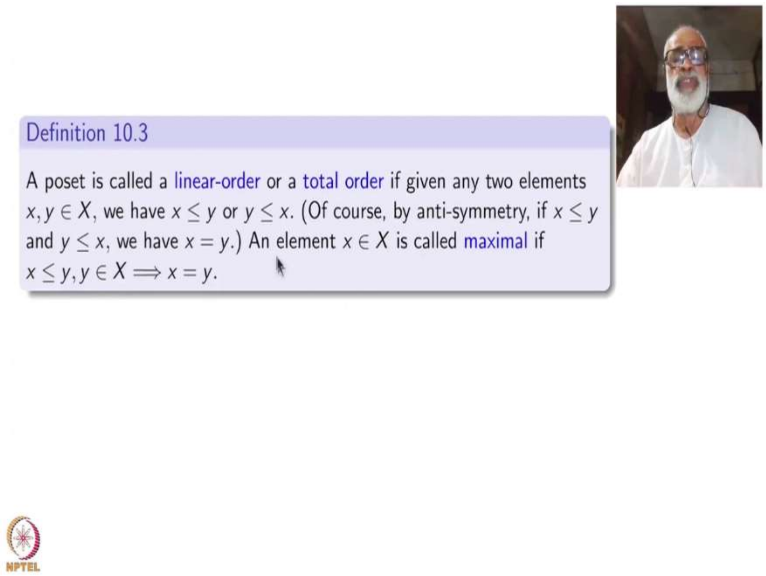


A poset is called a linear ordered set or a total ordered set, both the terms are used, if given any two elements $x$ and $y$ inside $X$, we must have $x$ is less than equal to $y$ or $y$ is less than $x$ or equal to $x$. Of course $x$ equal to $y$ is also allowed. So you can say there is a Law of trichotomy. Of course, by anti-symmetry, if both $x$ is less than equal to $y$ and $y$ less than equal to $x$, then we have $x$ equal to $y$.

An element $x$ inside $X$ is called maximal if $x$ is less than equal to $y$, for some $y$ inside $X$ implies $x$ is equal to $y$.

So, take any partial order set, maybe you take a partial order set and take a subset of that with the induced partial order, restricted partial order. Then you can talk about maximal elements inside that. So, this is just similar to supremum and so on. Maximal elements here, by the way, in an arbitrary partial order need not be unique.

Of course, with a linear order or a total order, maximal element is unique. That is a different thing. A maximal element means what? If anything is bigger than that, it must be equal to that.

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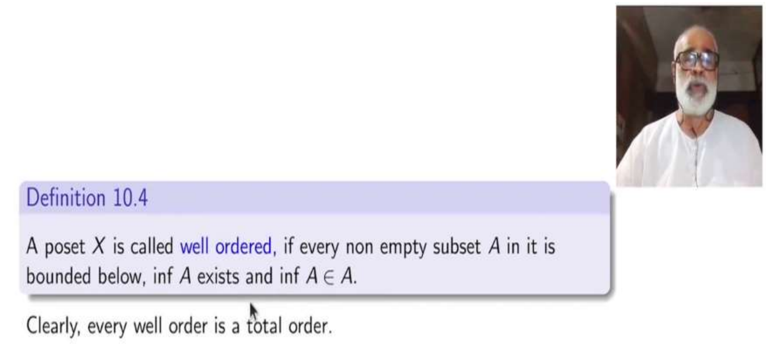


A poset $X$ is called well-ordered if every non empty subset $A$ in it is bounded below, (suppose, you take any non empty subset which is bounded below, that is not the case, every non empty subset must be bounded below) and the infimum of $A$ exists, and that infimum must be inside $A$.

So, this is the definition of well order. This a very strong condition. Simple example of well ordered set (apart from any totally ordered finite set) is the set of natural number with the standard order.

For given any subset of natural numbers, has one number which is the smallest. So, that property has been generalized here. So, natural numbers are also totally ordered, but being well ordered is something more. Automatically, it will be total ordered. Why? Take any two elements $x$ and $y$. That is a subset. That subset must have an infimum. That infimum must be inside that. That means infimum is one of the elements, $x$ or $y$.

If it is $x$, then $x$ will be less than or equal to $y$. If it is $y$, then $y$ is less than equal to $x$. So, automatically a well order is a total order. A total order may not be well ordered. Just, you can take the example of all integers. Subsets may not have an an infimum, which we know.

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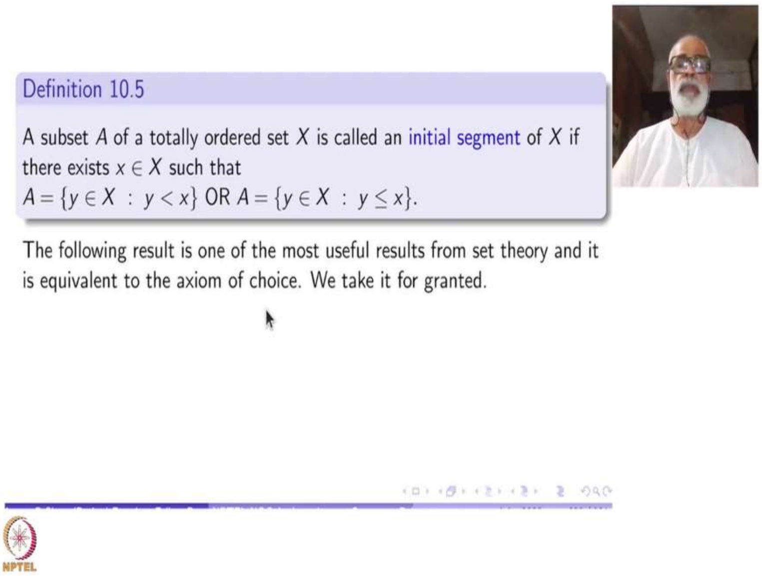


A subset $A$ of a totally ordered set $X$ is called an initial segment (so just pay attention to this definition) in $X$, if you have $x$ inside $X$ such that $A$ is the set of all elements $y$ in $X$ which are less than $x$ or $A$ is the set of all $y$ in $X$, such that $y$ less than or equal to $x$. So, see, this may not include $x$, but it is possible that this $A$ includes $x$ also. So, there are the two different cases. So, both of them are called initial segment the first one may be called open initial segment and the second one a closed initial segment. Just like the closed ray $(-\infty, 0]$ or the open ray $(-\infty, 0)$ inside $\mathbb{R}$. So, these are the standard examples. Following result is one of the most useful results from set theory, and it is equivalent to axioms choice. We take it for granted. What is this? Zorn's Lemma.

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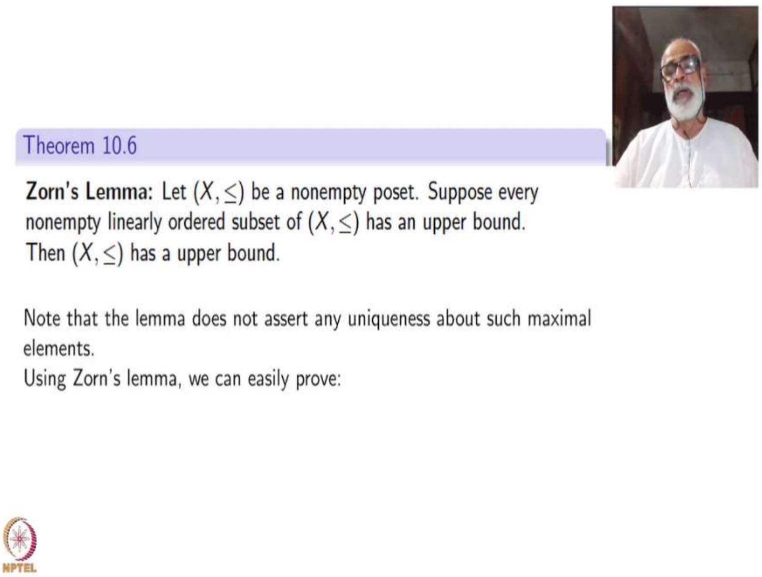

So, Zorn’s Lemma states the following.

Take any partially ordered set $X$, non empty. Suppose every linearly ordered subset of $X$ has an upper bound in $X$. Then, $X$ has a maximal element. Every linearly ordered subset, every totally ordered subset of $X$, means what? The subset with the restricted order that must be linearly ordered.

If each a linear ordered subset has an upper bound, then $X$ has a maximal element. There may be many maximal elements. $X$ is just a poset. A totally ordered set may have only one maximal element it at all. Note that the lemma does not assert any uniqueness about such maximal elements. It is very important.

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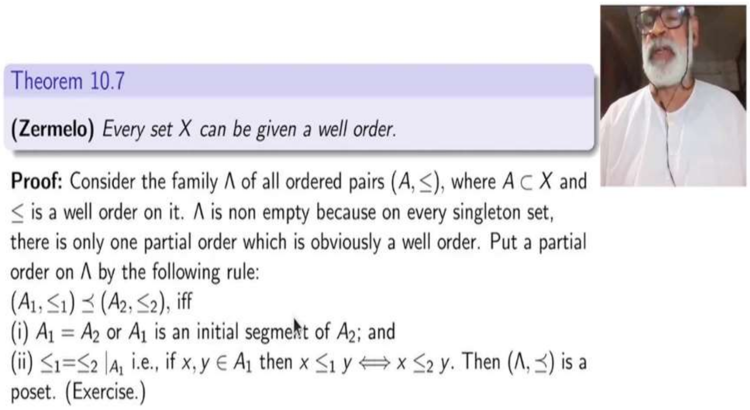


Using Zorn’s Lemma, we can easily prove, Zermelo’s, axiom, another very important theorem in set theory. We call it an axiom:

Every set $X$ can be well ordered.

We have just seen the set of integers is not well ordered, if you take the usual order. So, what does Zermelo's axiom says? There is another order on the integers in which it will be well ordered, it assures the existence of some partial order which will be a total order that is the meaning of this one.

To prove this one, you can do it independently of Zorn's lemma, but you will have to use axiom of choice. That has to be there. There can be no really `independent proof' because these statements Zorn's Lemma and Zermelo's axiom, each of them is equivalent to the axiom of choice. So, we are assuming this one which just means that we are assuming axiom of choice in the background.

But now, what we will do? We will prove this one, we will use Zorn's lemma to prove Zermelo. So, that way, our task will be simpler. How do we do that? In order to apply Zorn's Lemma, you have to have some family of ordered sets and so on. Then you say something is maximal and that maximal element may satisfy whatever you wanted.

So, I start with a family $\Lambda$ of all ordered pairs ($A$, a partial order), where $A$ is a non empty subset of $X$ where $X$ itself is non empty. Two elements of $\Lambda$ may have the same subset of $X$ but if the well order on them are different they will be different elements of $\Lambda$.

Why this $\Lambda$ is non empty? Because singleton sets can be given only one partial order and that partial order is automatically well ordered. And they are all elements of $\Lambda$. Therefore, $\Lambda$ is non empty. (Note that here, I am assuming $X$ non empty. I do not worry about non empty subsets of an empty set.) Now on $\Lambda$, we will put a partial order as follows.

What is that partial order? $(A_1, \leq_1) \preceq (A_2, \leq_2)$ (I will read it as $\preceq$, or we can read it as $A_1$ precedes $A_2$) if and only if

(i) $A_1$ maybe equal to $A_2$ along with, along with what? The partial orders are also the same, $\leq_1 = \leq_2$ Or

(ii) $A_1$ is an initial segment of $(A2, \leq_2)$, with $\leq_1 = \leq_2$ when restricted to $A_1$.

So, that is the condition.

First we show that $(\Lambda, \preceq)$ is a poset. What are the things that you have to verify? What are the members of $\Lambda$?

So, reflexivity is there because of (i). Transitivity is also obvious because if we have $A_1$ precedes $A_2$ precedes $A_3$ then there may an equality somewhere, that case is fine, or if they are initial segments, then the element is $A_2$ which defines $A_1$ as an initial segment will also define $A_1$ as an initial element in $A_3$. Details are left to you as an exercise. (Do that one, you have to spend some time in doing this exercise by yourself so that you get familiar with these definitions.)

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Suppose $\Gamma$ is a linearly ordered subset of $\Lambda$. Put $B = \cup\{A : A \in \Gamma\} \subset X$. Given $x, y \in B$, it follows that there exists $(A_1, \leq) \in \Gamma$ such that $x, y \in A_1$. Define $x \leq' y$ iff $x \leq_1 y$ in $A_1$. It follows that $\leq'$ is well defined on $B$ and $(B, \leq')$ is a totally ordered set.

Next, I have to show that the condition in Zorn's lemma is satisfied. Take a linearly ordered subset $\Gamma$ of this $\Lambda$. I will show that this has an upper bound in $\Lambda$. $T$ After that, we can conclude that there is a maximal element here in $\Lambda$ and that maximal element is going to give you whatever we want, namely a well order on $X$.

So, let us see how. Start with a linear ordered subset $\Gamma$. Put $B$ equal to union of all these members inside this $\Gamma$. Remember what the members of $\Gamma$, $(A, \leq)$, where $A$ is a subset of $X$ and $\leq$ is a total order, a linear order on $A$. You take $B$ to be the union of all $A$, such that $(A, \leq)$ belongs to $\Gamma$. Then $B$ is a subset of $X$. Now, if you take any two elements $x, y$ in $B$, there will be one $A_1$ belonging to $\Gamma$ such that both of them are in $A_1$. Why? Because $x$ may be in $A$ and $y$ may be inside some $A'$. But because $\Gamma$ is linearly ordered, we have $A$ is a subset of $A'$ or the other way round. Therefore, you can take the bigger one that will contain both $x$ and $y$.

So there is $A_1$ for which both $x, y$ are inside $A_1$. Now, you define the new relation on $B$, which we denote by $\leq'$, as follows. How I am going to define? $x \leq' y$ if and only if $x$ is less than or equal to $y$ inside $A_1$. This makes sense because $x$ and $y$ are members of $A_1$. Now you have to see that this is well- defined, viz., it does not depend upon what $A_1$ has been chosen. If I have chosen some $A_2$, such that $x, y$ are in $A_2$, then $A_1$ is an initial segment of $A_2$ or $A_2$ is an initial segment of $A_1$ which means this relation will be the same as far as $x$ and $y$ are concerned. That is why this is well defined. Automatically, this new realtion $\leq'$ is a partial order on $B$.

So, what we have to show is that it is well order on $B$. (That it is a total order is obvious from the way we have just defined it.) Then it would follow that $B$ is member of $\Lambda$. And then we have to show that this member is an upper bound for all the elements inside this $\Gamma$. So, quite a bit of work to do.

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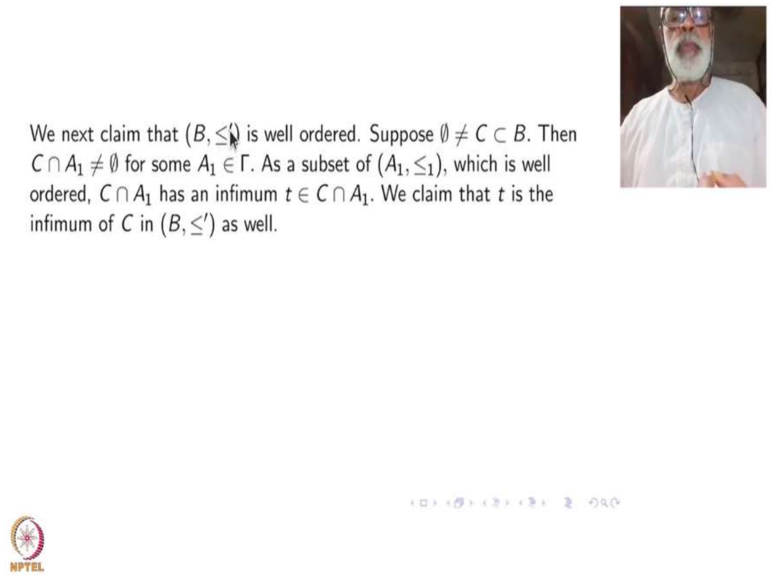


So, we claim that $(B, \leq')$ is a well order. What does that mean? Take any non empty subset $C$ of $B$, show that it has an infimum, a minimum. $C$ must have a non empty intersection with one of the members of $\Gamma$ because $B$ is a union of all members of $\Gamma$ as a subset of $X$. S0, there is some $A_1$ in $\Gamma$ such that $C \cap A_1$ is non empty. Then $C \cap A_1$ must have minimum. Let us say it is $t$. This $t$ is an infimum of $C \cap A_1$ inside $A_1$ and it belongs to $C \cap A_1$. We claim that $t$ is the infimum for the entire $C$ inside $B$. So, that is what we want to show. Once we show that, well orderness of $(B, \leq')$ is proved.

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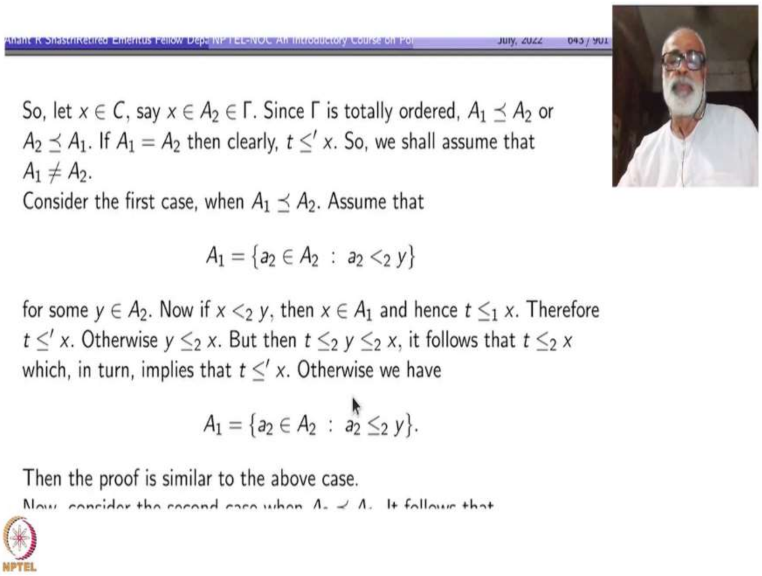


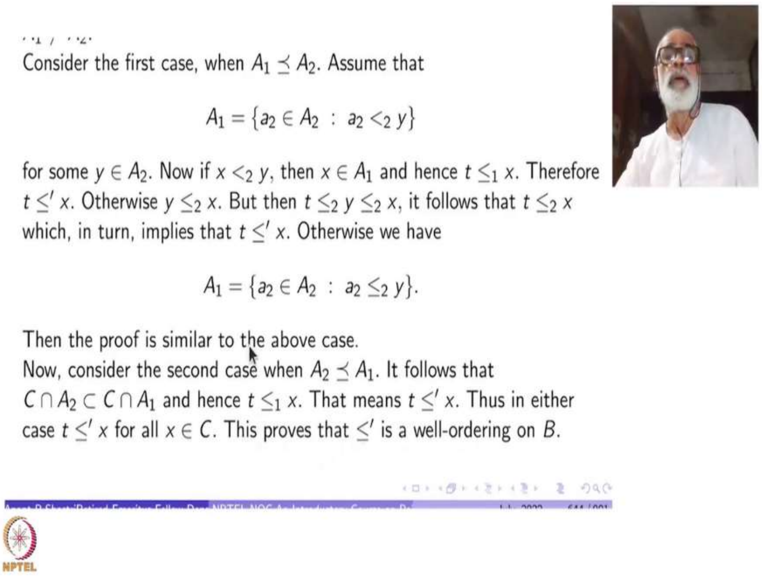


So, here is the proof. Take any element $x$ inside $C$. Say $x$ must be inside some $A_2$ belonging to $\Gamma$. Because all elements of $C$ are, all members of $B$. Since $\Gamma$ is totally ordered, we have $A_1$ is precedes $A_2$ or $A_2$ is precedes $A_1$, or $A_1$ maybe equal to $A_2$. In the last case, obviously, we have $t$ precedes $x$.

So, we shall assume that $A_1$ is not equal to $A_2$. Now, there are two different cases, either $A_1$ precedes $A_2$ or $A_2$ precedes $A_1$.

Suppose $A_1$ precedes $A_2$. Now, there are two subcases again. Because $A_1$ is not equal to $A_2$, either $A_1$ is an open ray in $A_2$, open initial segment or a closed initial segment, so there is a $y$ in $A_2$ which defines $A_1$ as an initial segement. Now, if $x$ is less than $y$, (or less than or equal to $y$, respectively) in $A_2$, then by the definition, $x$ is inside $A_1$. t is least amongst all elements of $A_1$, therefore $t$ is less than (or less than or equal, respectively) to $x$ in $A_1$. In either case, (viz., whether $A_1$ is an open ray or a closed ray), it follows that $t \leq' x$.

Now, in second case is $y$ is less than or equal to $x$ because $y$ is less than equal to $x$, but then $t$ will be less than equal to $y$ less than equal to $x$. So it follows that $t$ is less than $x$.

Now, the second case is the other way around viz., $A_2$ precedes $A_1$. Then it follows that $C \cap A_2$ is a subset of $C \cap A_1$. It may equal also, I do not care.

And hence $t$ will be less than or equal to $x$ also, because for all these elements, $t$ is smallest. But again, $t \leq' x$.

So, in all these cases, you have shown that $t \leq' x$ for all $x \in C$, no restrictions on $C \cap A_1$ etc. now. So, this means that $B$ is well ordered.

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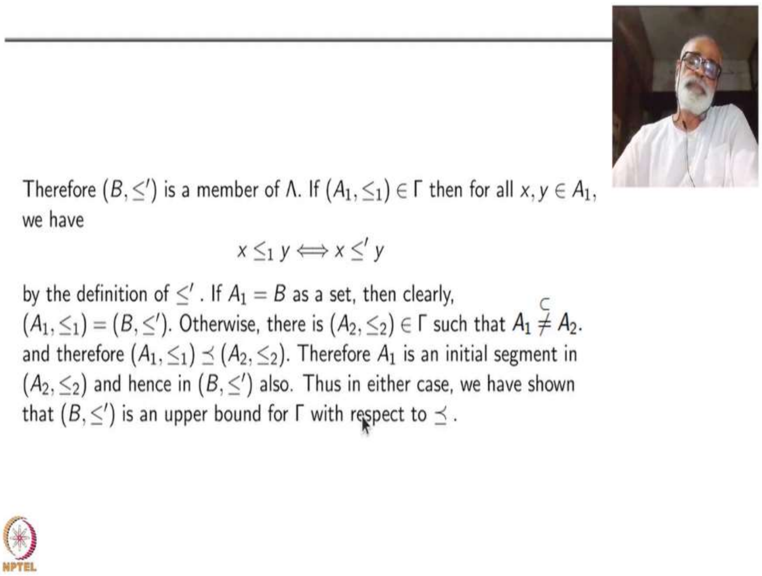


Now, we shall prove that this $B$ is an upper bound for $\Gamma$.

So far it is well ordered, and is a member of $\Lambda$. Now, why is it the upper bound for all members of $\Gamma$? Start with a member $A_1$ of $\Gamma$. Then for $x$ and $y$ inside $A_1$, we have $x$ less than or equal to $y$ in $A_1$, if and only if $x$ is less than or equal to $y$ by the definition of this $\leq'$. Therefore, one thing is clear. Namely, the order on each subset $A_1$ is fine. We have to show that this $A_1$ less than equal to $B$.

First, suppose $A_1$ is the whole of $B$ as a set. Then clearly, these two are orders are also equal and so $A_1$ is equal to $B$ as posets.

Now suppose $A_1$ is not the whole of $B$, it is a proper subset of $B$. That means there must be $A_2$ inside this $\Gamma$, such that $A_1$ is not equal to $A_2$ but contained inside $A_2$. If everything is contained inside $A_1$, then the union will be $A_1$ which is $B$. So, that is not the case now. So, $A_1$ is contained inside $A_2$, but not equal.

Therefore, $A_1$ precedes $A_2$, because Gamma is totally ordered. See $A_2$ cannot be a subset of $A_1$. So, we must have $A_1$ precedes $A_2$. Therefore, $A_1$ is an initial segment, either open or closed, does not matter, it is the initial segment of $A_2$. Therefore, it is initial segment in $B$ also. Same element $y$ will have the required property. So, we have proved that every $A_1$ in Gamma is an initial segment of $B$ and hence $A_1$ precede $B$.

So, in either case $B$ is an upper bound for $\Gamma$.

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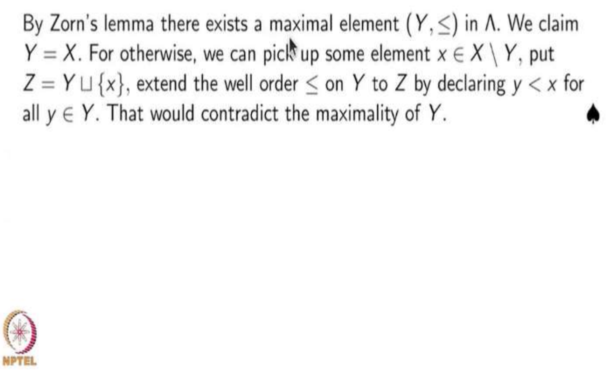


So, far what we have proved that conditions for the Zorn's Lemma are satisfied. Therefore, there is a maximal element $(Y, \leq)$ in $\Lambda$. What is the meaning of this? This means $Y$ is a subset of $X$ and this $\leq$ is a well order on $Y$ and if there is another $(Y', \leq')$ in $\Lambda$ then that cannot be follow this one, $Y$ cannot precede $Y'$. This is a maximal element of $\Lambda$. So, that is the meaning of maximal element.

So, now, we claim that this, the underlying set $Y$ is the whole of $X$. That will complete the proof of the theorem, viz., we have found a partial order on the given set $X$ which is a well order.

Suppose $Y$ is a proper subset of $X$. We can pick up some element $x$ inside the complement of $Y$, put $Z$ equal to $Y$ disjoint union $\{x\}$. That means one extra element. And extend the well order on $Y$ to $Z$ by declaring the only thing that we have to have, viz., how $x$ is related with elements of $Y$, so declare y less than $x$ for all $y$ inside $Y$. This way, $Z$ becomes a poset. It is automatically well ordered because if a subset $A$ is already inside $Y$, it is well ordered by itself. If not it will contain $x$, but $x$ is largest. So the minimal element of $A \cap Y$ gives you the minimal element for $A$, if this in nonempty. Finally, if you have just singleton $x$, then that singleton $x$ itself is the minimal element.

Therefore, this is a well order. Now, what we have got? $Z$ is a well ordered set and hence is a member of $\Lambda$. But $Y$ precedes $Z$ since $Y$ is an open initial segment of $Z$. And that is a contradiction, because we assumed $Y$ is a maximal element of $\Lambda$. This contradiction arose because we assumed $Y$ is a proper subset of $X$. If $Y$ is $X$, there is no problem.

So, that completes the proof of Zermelo's axiom. Here we have made it a theorem, namely, every set can be well ordered.

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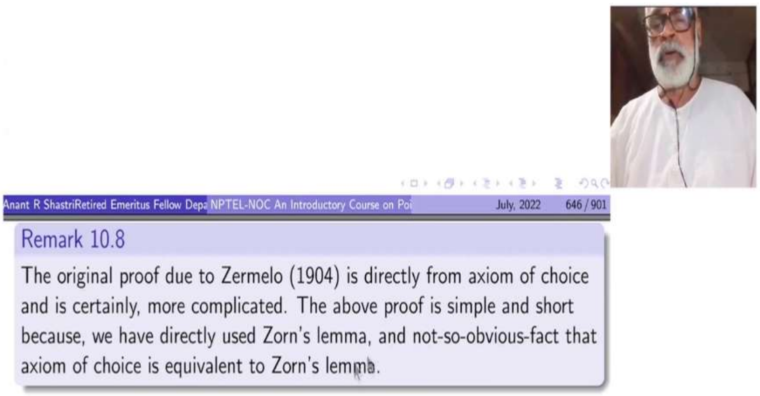


As I told you the original proof of Zermelo's which was proved 1904, more than 100 years now, 120 years, is directly from axiom of choice and is certainly more complicated. The above proof is simple and short, because we have directly used Zorn's Lemma and not so obvious fact that axiom of choice is equivalent to Zorn's Lemma. If you try to prove Zorn's Lemma using the axiom of choice that will be again a horrendous task. So, we have avoided that.

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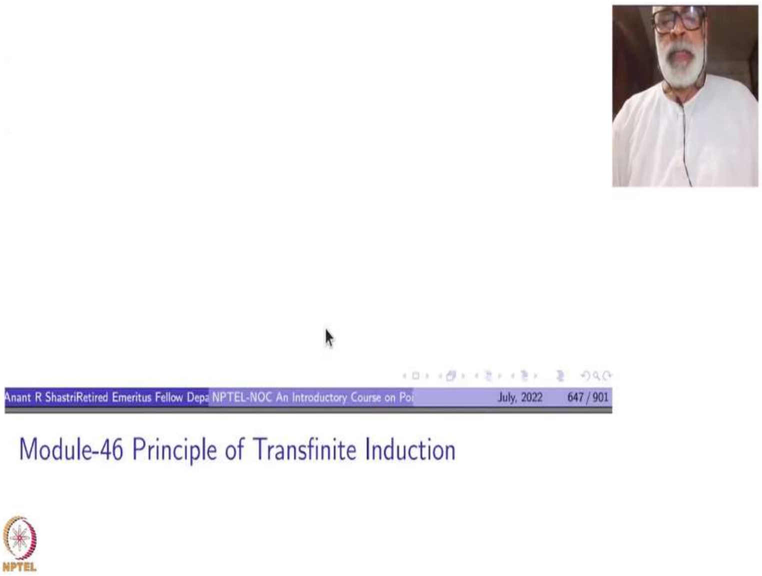


Next time, we will do another important landmark result in set theory, principle of transfinite induction. Thank you.