An Introduction to Point Set Topology
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Lecture 33
Ultra-closed Filters

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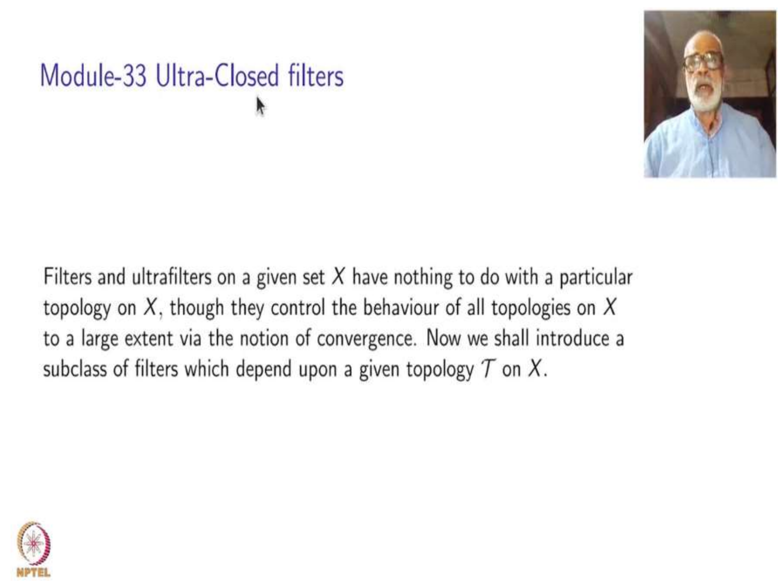

Welcome to NPTEL-NOC, an introductory course on Point-Set-Topology Part II, continuing with the study of filters today Module 33 ultra-closed filters. Filters and ultra-filters on a given set have nothing to do with a particular topology on X, though they control the behavior of all the topologies on X to a large extent, namely, via the notion of convergence.

Now, we shall introduce a subclass of filters which depend on the given topology $\mathcal{T}$ on X. So, that is a difference between these filters and ultra-filters and ultra-closed filters. As the name says that closedness comes because we are referring to a particular topology $\mathcal{T}$ on X.

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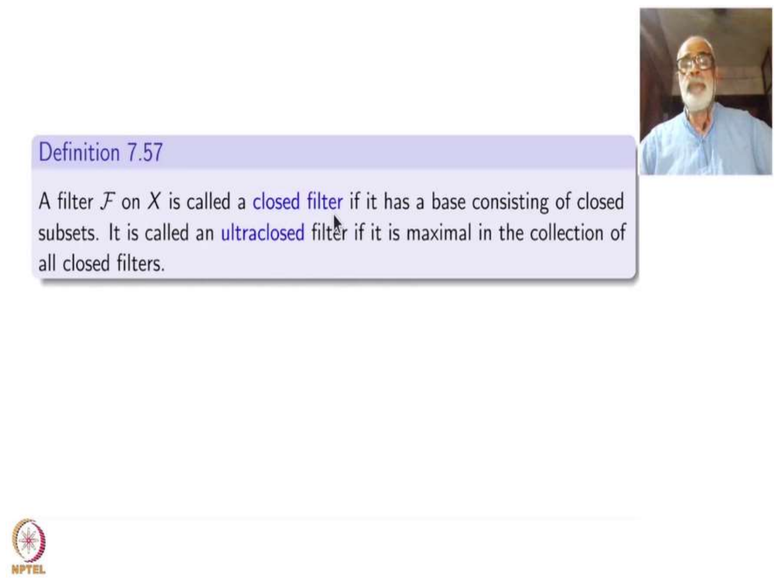

A filter $\mathcal{F}$ on X is called a closed filter if it has a base consisting of closed subsets of X. (So, I started with a topological space X, so closed subsets makes sense.) It is called an ultra-closed filter if it is maximal in the collection of closed filters on X. So, that is the definition of closed filter and ultra-closed filters.

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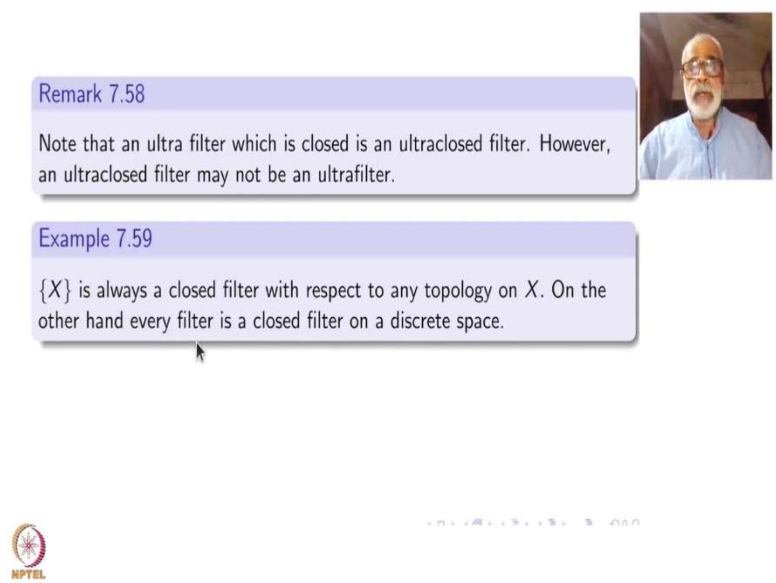

Now, if an ultra-filter is closed filter then it is ultra-closed filter, because it is already maximal in the collection of all filters. On the other hand, you start with an ultra-closed filter then this may not be an ultra-filter, because it is maximal only in a subclass. So, there may be a larger ultra-filter which are not closed. So, this difference you have to keep in mind, that is all.

Now, let me have some examples. Singleton X is always a closed filter, because X is always closed in whatever topology you take on X. On the other hand, every filter is a closed filter on a discrete space, because every subset is closed also. So, you can take the filter itself as a base if you like, no problem. So, these are some easy examples of closed filters.

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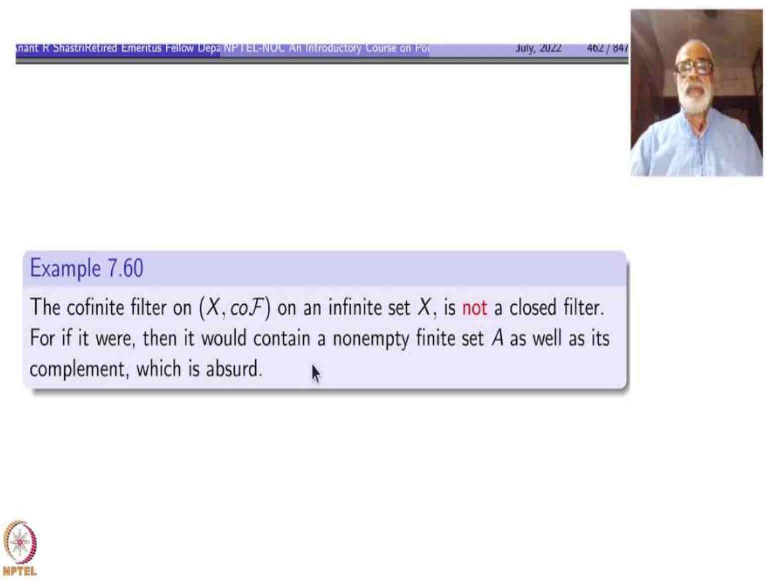

The cofinite filter on an infinite set is not a closed filter. For if it were, then it would contain a non-empty finite subset because the non-empty closed subsets are the whole space or the finite subsets. If $\{X\}$ is a base, then X is the only element and hence it cannot be the cofinite filter.

But the moment there is a non-empty finite subset, that is a contradiction, because its complement will be also there. So, conclusion is that this cofinite filter which consists of all open subsets other than the empty set in the cofinite topology is not a closed filter. So, I have given you both examples, easy examples of closed filters as well as not closed filters.

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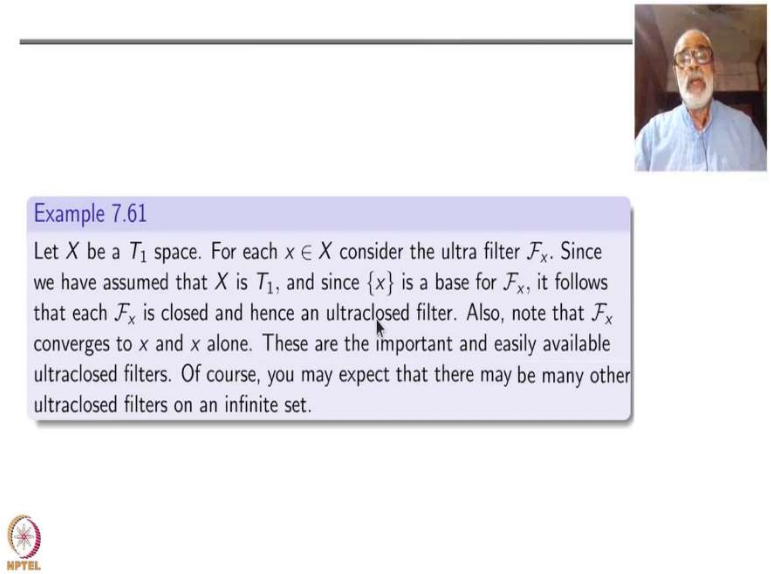

Let X be a T_1 space. Then I can give you more examples. For each x belonging to X, consider the ultra-filter $\mathcal{F}_x$. Singleton {x} being a base for it and singleton $\{x\}$ being closed, this $\mathcal{F}_x$ is a closed filter. Already it is ultra-filter, therefore it must of ultra-closed filter. Also note that, $\mathcal{F}_x$ converges to x and x alone. Why, because $\mathcal{F}_x$ contains $\{x\}$.

So, these are the important ones and easily available ultra-closed filters. Of course, you may expect that there are many other ultra-closed filters on an infinite subsets and you are right.

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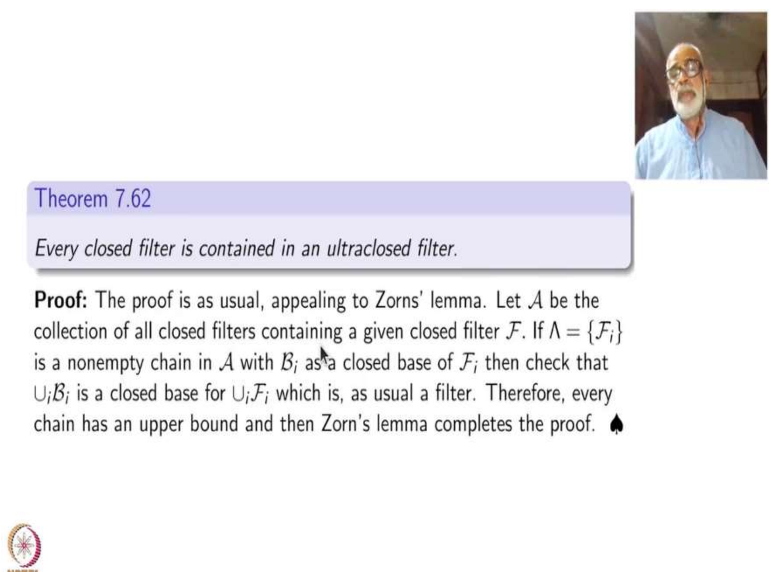

Every closed filter is contained in an ultra-closed filter. So, this is similar to the result that every filter is contained in ultra-filter. Of course, again the proof is using Zorn's lemma. You have to verify something. The verification is very easy here also. The proof is as usual appealing to Zorn's lemma.

Take A to be the collection of all closed filters containing a given closed filter $\mathcal{F}$. Automatically it is non-empty.

If Lambda is a chain inside this family A, each element of this chain has a closed base, $\{B_i\}$ is a closed base for F_i, where $\{F_i\}$ is a chain. Then take the union of all the B_i's, that family will be a closed base for the union of F_i's. This is what you have to verify. Because it is a chain, this is possible.

Again, union of F_i's is a filter is easy just like in the earlier case. So, it is a closed filter. Therefore, it is a member of this A and it is an upper bound for Λ. So, that means condition for Zorn's lemma is satisfied. Therefore, conclusion of Zorn's lemma says there is a maximal element in A. Check that it is maxil in the set of all closed filters as well. So, every filter is contained in an ultra-closed filter.

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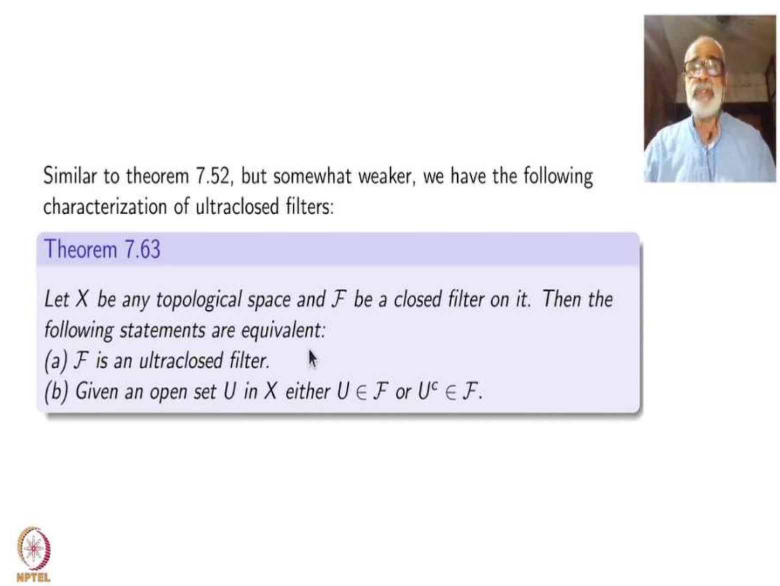

Here is a theorem similar to the theorem 7.52. It is nothing but characterization of ultra-filters, the three characterizations we had given, we have the following characterization. It is not so strong, a little weaker because we have only two of them here, (a) and (b). Exactly similar. Let $\mathcal{F}$ be a closed filter on X. Then the following two conditions are equivalent.

(a) $\mathcal{F}$ is an ultra-closed filter. (The second one is not with arbitrary subset but only for open subsets.)

(b) For every open subset U in X either U or U^c must be inside $\mathcal{F}$.

If this condition is satisfied, it must be ultra-closed filter. Started the closed filter of course, you do not prove the disclosed filter. Only ultraness is proved. The proof is more or less same, but you have to see why the openness is coming here. This condition is obviously weaker condition. It is not for every subset. After all maximality is also inside the family of closed filters. So that is the trick. So, let us see. Let us go through the proof correctly.

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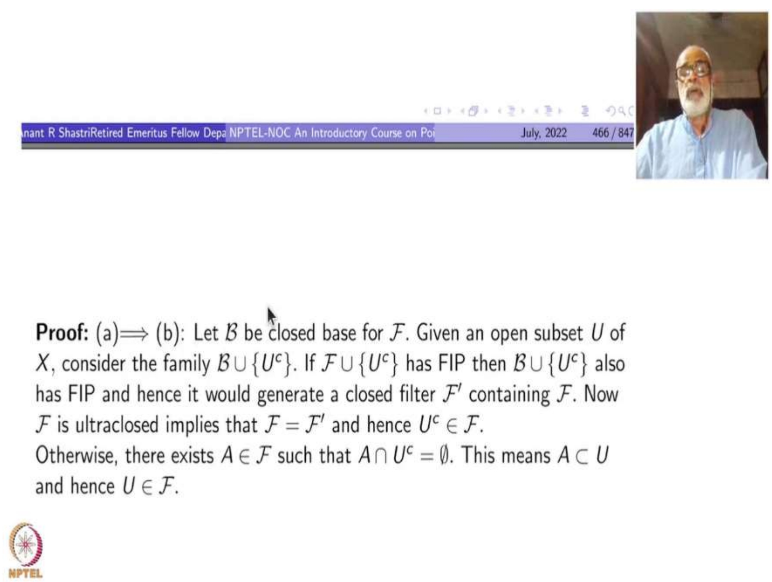

(a) implies (b) is not very easy or not very difficult. Start with an ultra-closed filter $\mathcal{F}$, let $\mathcal{B}$ be a closed base for that. $\mathcal{B}$ consisting of only closed subsets and it is a base for $\mathcal{F}$. Given an open subset U of X, consider the family $\mathcal{B} \cup \{U^c\}$. That is a closed set. So we have family of closed subsets.

There are two different cases to be considered. If $\mathcal{B} \cup \{U^c\}$ has finite intersection property then it would form a closed base for a closed filter $\mathcal{F}'$. Clearly $\mathcal{F}'$ contains $\mathcal{F}$. Since $\mathcal{F}$ is ultra-closed filter, it follows that $\mathcal{F}'$ is $\mathcal{F}$ and therefore, U^c is in $\mathcal{F}$.

The second case is that $\mathcal{B} \cup \{U^c\}$ does not satisfy finite intersection property. It does not have finite intersection property. Then there exist A inside $\mathcal{B}$ such that $A \cap U^c$ is empty. That is the finite intersection property is violated. This just means that A is contained inside U. Therefore, A is inside $\mathcal{F}$ also. So, either U is there or U^c is there. That is what we have proved.

Now, let us prove (b) implies (a).

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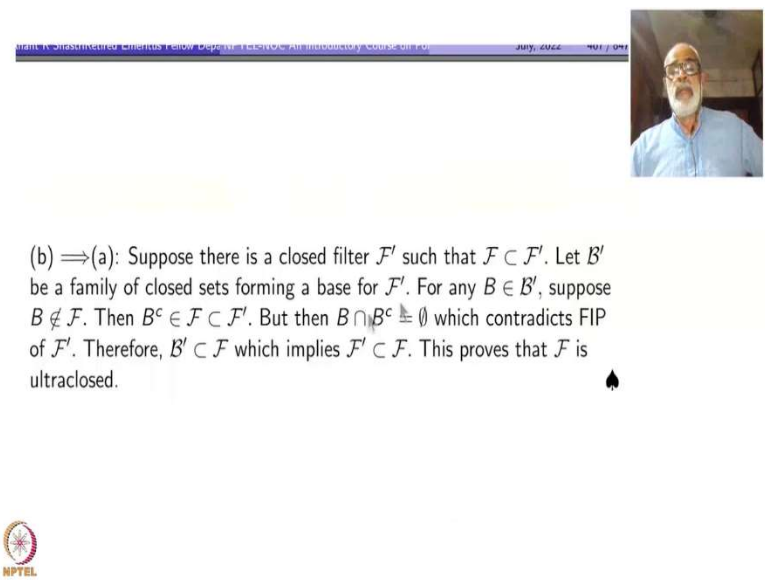

Suppose there is a closed filter $\mathcal{F}'$ such that $\mathcal{F}$ is contained in $\mathcal{F}'$. We want to show that $\mathcal{F}$ is equal to $\mathcal{F}'$. Let $\mathcal{B}'$ be a family of closed sets forming a base for $\mathcal{F}'$. There is one, because $\mathcal{F}'$ is a closed filter. For any B inside $\mathcal{B}'$ suppose this B is not in $\mathcal{F}$, I am just supposing, this B is not in $\mathcal{F}$. Then the complement of B must be in $\mathcal{F}$ by condition (b). But $\mathcal{F}$ is contained in $\mathcal{F}'$. So, B complement is in $\mathcal{F}'$.

B and B^c both are in $\mathcal{F}'$ that is a contradiction. Therefore, this B must be inside $\mathcal{F}$. Since this is true for all B in $\mathcal{B}'$, the $\mathcal{B}'$ is contained in F. But B prime generates $\mathcal{F}'$, so $\mathcal{F}'$ is inside $\mathcal{F}$. So, therefore, equality holds. That proves (a).

So, you see that the proof is more or less the same. But the flavor is different because we have to use the openness and closeness and so on here. That is all.

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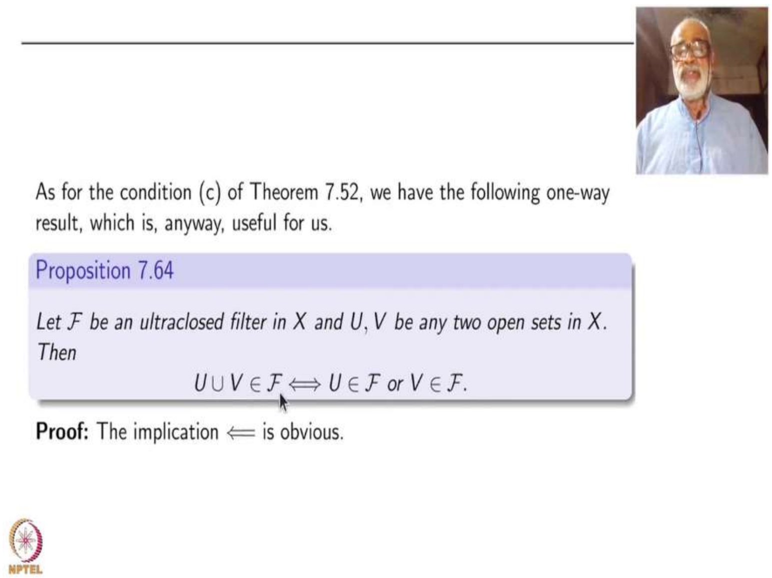

But condition (c) is missing here. See now you understand why. Last time we put this condition at the last and then proved (a) implies (c), (c) implies (b) and (b) implies (a). This condition (c) is nothing but given $A \cup B$ inside $\mathcal{F}$, A is there or B is there in the case of ultra-filters. Now, similar condition, but slightly different can be expected, but no, this only one way implication is there. The corresponding condition (c) is not equivalent to being ultra-closed filter.

So, let me state it separately. Do not get confused with the earlier theorem. This proposition says that suppose $\mathcal{F}$ is an ultra-closed filter, then given U and V two open subsets of X this condition holds. What is this condition? It is an `if and only if' statement. $U \cup V$ is inside $\mathcal{F}$ if and only if U is inside $\mathcal{F}$ or V is inside $\mathcal{F}$.

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Proof of $\Longrightarrow$: Assume $\mathcal{F}$ is an ultraclosed filter. Suppose U, V are open and $U \cup V \in \mathcal{F}$. We have to show that $U \in \mathcal{F}$ or $V \in \mathcal{F}$. If this is not true, by the previous theorem, it follows that $U^c, V^c \in \mathcal{F}$. Therefore, $(U \cup V)^c = U^c \cap V^c \in \mathcal{F}$. But then $(U \cup V) \cap (U^c \cap V^c) = \emptyset$ which contradicts FIP of $\mathcal{F}$. ♠

Assume that $\mathcal{F}$ is an ultra-closed filter and U and V are open subsets such as that the union is inside $\mathcal{F}$. We have to show one of them is inside $\mathcal{F}$. This is not true means what? Let us examine that. Suppose this is not true. Then it follows that the complements U^c and V^c must be inside $\mathcal{F}$. This not true means what, neither U is there nor V is there. But then the complements must be there because $\mathcal{F}$ is an ultra-closed filter and we can use the previous result.

Therefore, $(U \cup V)^c$ which is nothing but $U^c \cap V^c$ that must be in $\mathcal{F}$. But now you have a problem. We already assumed that $U \cup V$ is inside $\mathcal{F}$, but you are assuming that intersection of U^c and V^c is also there. The intersection of these two is empty. Its complement and this one both of them cannot be there. That is all.

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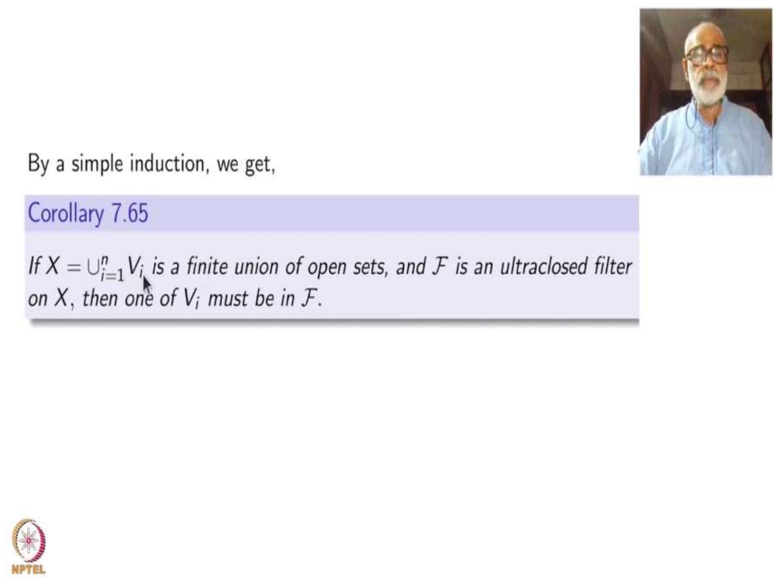
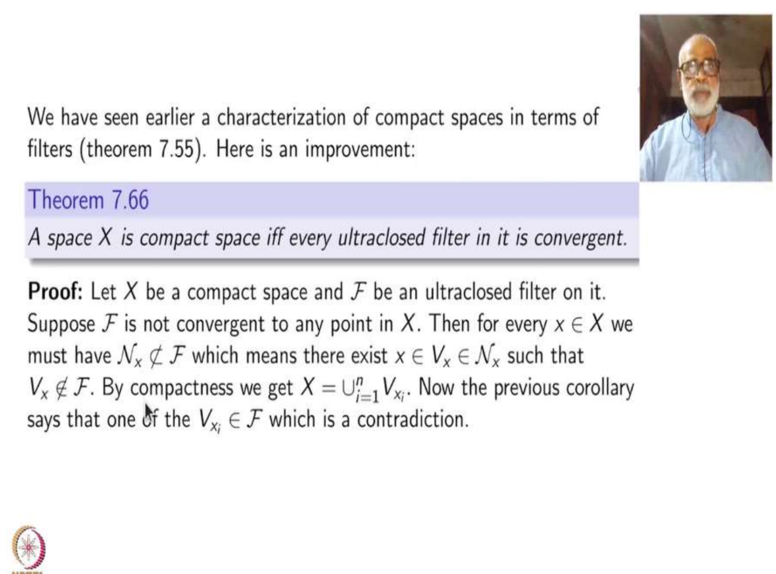
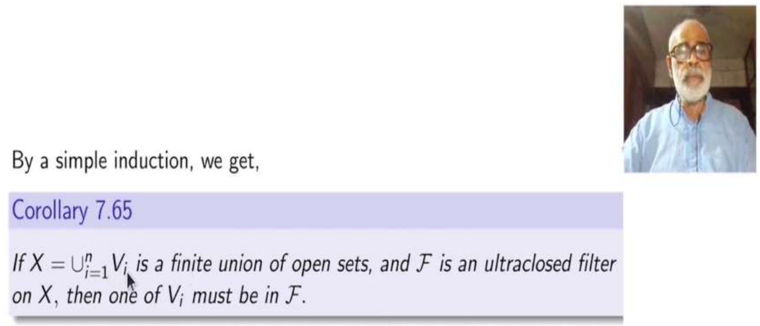


By a simple induction, we get the following important result. If X is written as a union of finitely many open subsets V_i and $\mathcal{F}$ is an ultra-closed filter on X, then one of the V_i's must be in $\mathcal{F}$. So, this is very easy, because what we have proved is that if U and V are open subsets and $U \cup V$ is inside $\mathcal{F}$ then one of the U and V must be inside $\mathcal{F}$. To begin with, every filter contains the entire set X.

Therefore, you can just write X as the union of V_n and the union of V_i's; i ranging from 1 to $n-1$ Apply this criterion. If V_n is there in $\mathcal{F}$, we are done. Or union of V_i; i ranging from 1 to $n-1$ is inside $\mathcal{F}$. Now applying induction, one of the V_i' s must be inside $\mathcal{F}$.(Refer Slide Time: 17:34)





So, from this one, let us deduce another important result here which is a characterization of compact spaces in terms of ultra-closed filters. Remember in an earlier theorem, we had characterized the compact spaces in terms of ultra-filters. Every ultra-filter is convergent that is a condition which ensures that X is compact and conversely. So, exactly same result is now with ultra-closed filters.

A space X is compact if and only if every ultra-closed filter in it is convergent. So, what is the idea? Let X be a compact space and $\mathcal{F}$ be an ultra-closed filter in it. Suppose $\mathcal{F}$ is not convergent to any point. Then you will get a contradiction very easily. Namely, it does not converge to any point means none of these $\mathcal{N}_x$, the neighborhood systems is contained inside $\mathcal{F}$. That is the meaning of that $\mathcal{F}$ does not converge at all.

What does this mean? For each point x inside X we have a neighborhood V_x of x such that this V_x is not in $\mathcal{F}$, one neighborhood for each point. But when you vary these points, you get an open covering for X. But now X is compact, so you get a finite covering. As soon as you have finite covering, this corollary says that one of those open sets must be inside $\mathcal{F}$, but that is a contradiction. So, compactness implies that every ultra-closed filter converges to some point. It may converge to more than one point also, nobody ensures the uniqueness.

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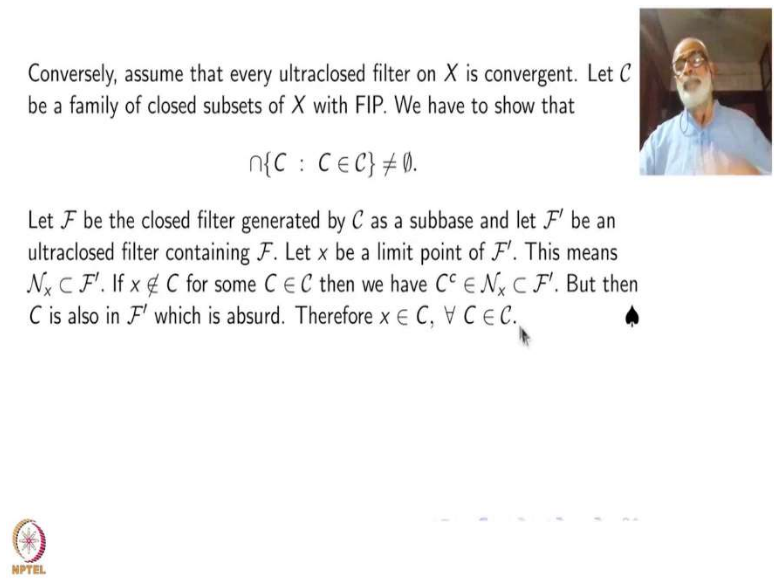

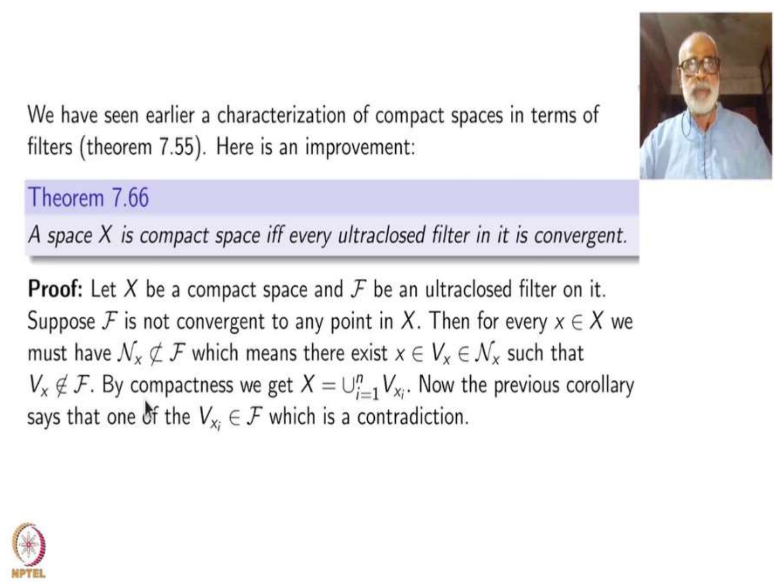

Now, let us do the converse. Assume that every ultra-closed filter on X is convergent. Let $\mathcal{C}$ be a family of closed subsets of X with finite intersection property. We have to show that the entire intersection of members of $\mathcal{C}$ is non-empty. So, that is enough to show that X is compact. So, let $\mathcal{F}$ be the closed filter generated by $\mathcal{C}$, because any family which has finite intersection property generates a filter, but that will a closed filter because this will be members of all this they are all closed subsets.

So, we get a closed filter contained in an ultra-closed filter. So, let $\mathcal{F}'$ be an ultra-closed filter containing $\mathcal{F}'$. Let be a limit point of this $\mathcal{F}'$, because we are assuming that every ultra-closed filter is convergent. So, let x be a limit point of $\mathcal{F}'$. This means that $\mathcal{N}_x$ is contained inside $\mathcal{F}'$.

If x is not in $\mathcal{C}$ for some C inside $\mathcal{C}$ that would have meant that C^c is a neighborhood of x and hence inside $\mathcal{F}'$. But all the members of $\mathcal{C}$ are inside $\mathcal{F}$ and hence inside $\mathcal{F}'$. That is a contradiction. Therefore, x must be inside C for every C inside $\mathcal{C}$. That just means that the intersection of members of $\mathcal{C}$ is non-empty.

So, characterization of a compact space in terms of ultra-closed filters comes out. See, we have to prove this afresh both ways, because if every ultra-closed filter is convergent this does not mean immediately that every ultra-filter converges, because there are many more of them.

On the other hand, if X is compact space, we have shown that every ultra-filter on X is convergent, but none of the ultra-closed filters maybe an ultra-filter or some of them maybe, some of them may not be. Ultra-closed filter does not mean that it is ultra-filter. So, either way it is not true, but, so we have to prove it fresh. But on the other hand, if you look at the proof, it is not all that different. Of course, this is a little harder. You have to use closed base etc.

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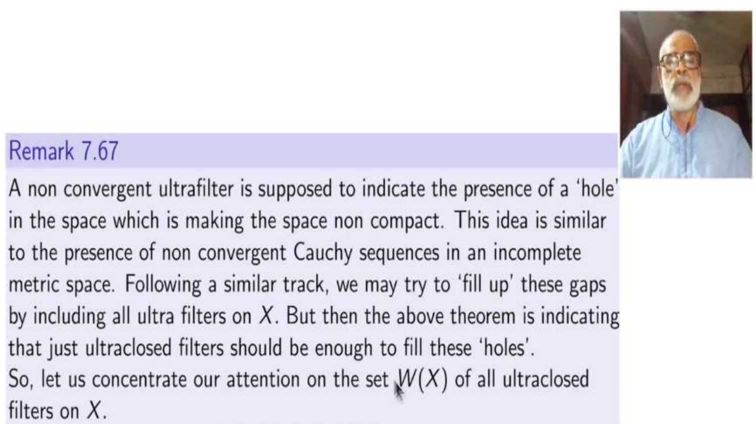

A non-convergent ultra-filter is supposed to indicate the presence of a hole in the space which is making the space non-compact, because if it were compact then every ultra-closed filter would have been convergent. So, for a non convergent ultra-closed filter on X, there is no space for it to be convergent, that indicates there is a missing point in the space. So and that must be making it non-compact.

So, this idea is similar to the presence of non-convergent Cauchy sequences in an incomplete metric space. A metric space incomplete means there is a Cauchy sequence which is not convergent. So, non convergent Cauchy sequence indicates there are holes inside a metric space.

Following a similar track as in the case of completion of a metric space, we may try to fill up these gaps by including all ultra-filters on X along with points of X. So that could be one way of looking at it. But then you are warned already by the above theorem that you do not need all ultra-filters. If all ultra-closed filters are convergent, then the space will be compact. Therefore, it should be possible to get a compactification by just taking care of all the ultra-closed filters.

So, just the ultra closed filters should be enough to fill up all the holes. These are all just surmising, I mean we are just loud thinking, this may be true and that may be true etc. Finally, you have to prove all this.

Let us concentrate our attention on the set $W(X)$ of all ultra-closed filters on X. Let us not give up this idea. That is all. We have to work of course. So, $W(X)$ is nothing but set of all ultra-closed filters on X.

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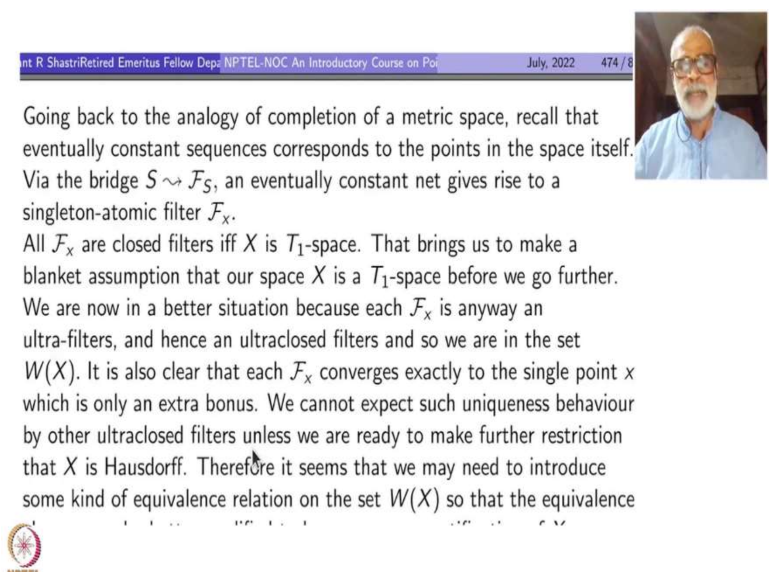

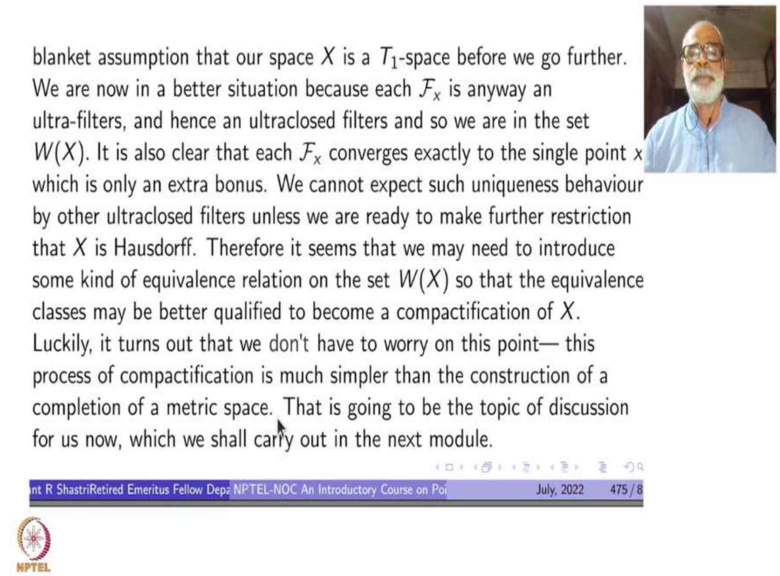

Next, going back to the analogy again of completion of a metric space, recall that eventually constant sequences correspond to points in the space itself. If you have some sequence $(a_1, a_2, \ldots, a_n, \ldots)$ and all $a_{n+k} = x$ for all k, that sequence is automatically convergent to x. What will you do in the completion? The element x was identified with this Cauchy sequence or the other way around, whichever.

So there was a map from the space X to the collection of all sequences. And then of course, we introduced an equivalence relation on this collection, and so on. So, that was the idea.

So, we can try to do something similar here. Instead of sequences first of all you have can pass onto nets. Then eventually constant nets is the key word now. On the other hand, we do not want to work with nets, but we will use the bridge from nets to the filters which we have defined earlier.

So, what do we get? If S is an eventually constant net, what is the corresponding $\mathcal{F}$'s? that will be the singleton atomic filter $\mathcal{F}_x$. So, you see we are coming to this atomic filter $\mathcal{F}_x$, it is supposed to represent a prototype of eventually constant sequence that is the whole idea. That is how it is convergent also all the time.

In our definition, it is also an ultra-closed filter provided X is T_1. Till then we do not need T_1-ness. All $\mathcal{F}_x$ are closed filters if and only if X is a T_1 space. That brings us to make a blanket assumption that our space X is a T_1 space, because we just do not want to give up this $W(X)$, the set of ultra-closed filters. So, we better assume X is a T_1 space before we go further. In which case all atomic filters will just correspond to the points of X.

So, this way $W(X)$ can be thought of as a set which contains X via this embedding, via this injective mapping, by this identification x going to $\mathcal{F}_x$. So, we have enlarged our space X. What remains is to put some topology on this one so that it becomes compact. That is the first thing. Then we must examine that the map x going to $\mathcal{F}_x$ which is injective is also continuous and an embedding of X in $W(X)$, so that the image is dense. This is all we have to do.

It is also clear that each $\mathcal{F}_x$ converges exactly to one single point that is very important for us. That is an extra bonus though. We were not bothered about that to begin with. But that is because of some strange reason. There is no T_1-ness or T_2-ness involved in it. We do not want to bring the T_2-ness here at all. See usually a filter will converge to a unique point if you have a Hausdorff space. That is not the reason here. These are very special filters $\mathcal{F}_x$'s; they will converge to only one single point.

We cannot expect such uniqueness behavior by other ultra-filters unless we are ready to make further restrictions that X is Hausdorff. So, therefore, it seems that we may need to introduce some kind of equivalence relation on the set $W(X)$ so that the equivalence classes may be better qualified to become a compactification just like in the case of completion of metric spaces. Luckily, it turns out that we do not have to worry on this point.

This process of compactification is much simpler than the construction of a completion of a metric space. That is going to be the topic of discussion for us now, which we shall carry out in the next module. So, this is the motivation for considering the so called Wallman compactification as we will do next time. Thank you.