An introduction to Point-Set-Topology Part-II
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 30
Basics of Filters

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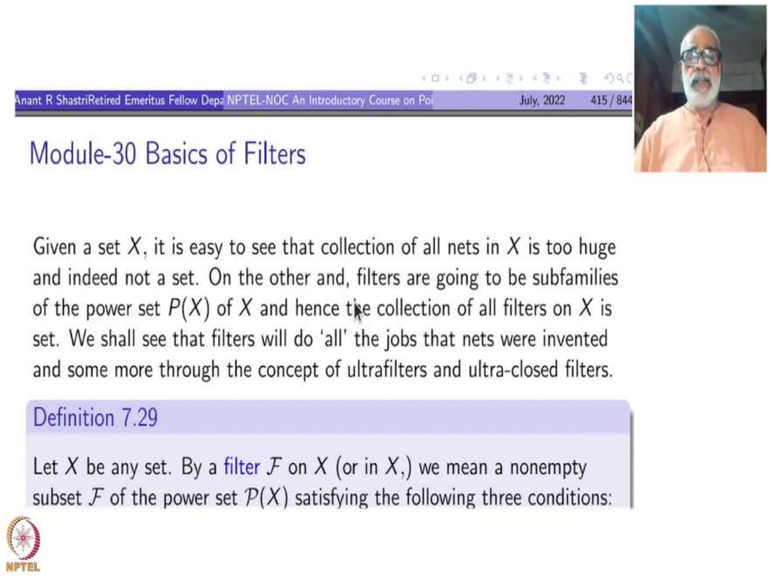

Hello, welcome to NPTEL NOC an introductory course on point set topology part II. So, today we take up another topic in this chapter, namely filters. So, today in model 30 we will just study basics of filters.

Given a set X, it is easy to see that the collection of all nets in X is too large, and indeed not a set. On the other hand, Filters are going to be subfamililes of the power set $P(X)$ of X, and hence the collection of all filters on X is a set.

We shall see that filters will do quote unquote, all the jobs that nets were invented for and some more. Through the concept of ultrafilters and ultra-closed filters. These concepts or anything parallel to them is not available for nets. So, that is one way of looking at nets versus filters. So, filters seems to have an advantage over nets. So, that is one of the, you may say, justification for studying filters now, after having studied nets, quite thoroughly.

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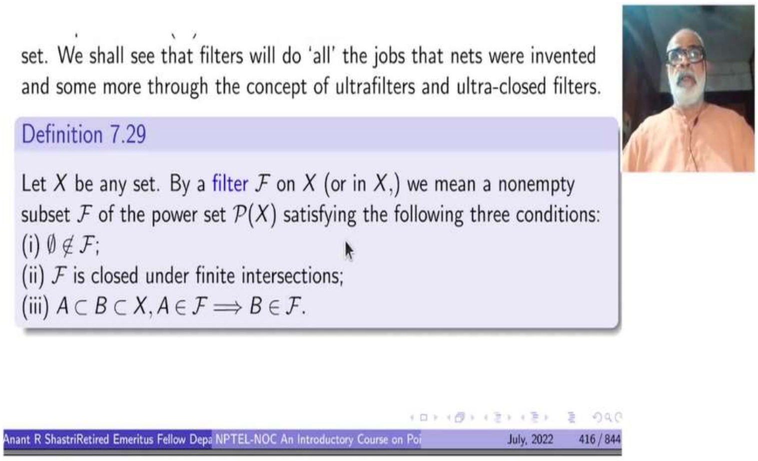

Let X be any set. By a filter $\mathcal{F}$ on X, (I will using this notation $\mathcal{F}$, $\mathcal{F}'$ etc for filters, some people may say filter in X also no problem), we mean a non-empty subfamily of the power set $P(X)$ satisfying the following 3 conditions.

(i) Empty set is not a member of $\mathcal{F}$. (Please note this one. It is very important.)

(ii) $\mathcal{F}$ is closed under finite intersections. (This one is familiar to you, like a property of a topology.)

(iii) A contained inside B contained inside X and A is in $\mathcal{F}$ implies B is in $\mathcal{F}$. (All supersets of a member are also members. Such a condition was not there for a topology at all. These three conditions are similar to the conditions for a topology but only the (ii) one is actually common, being closed under this is finite intersection. For a topology both empty set and the whole X are members X whereas (i) says empty set should not be there. Of course, it follows that X is always a member.

So, this is the important point of definition. Usually, the name filter comes from the property (iii) which is a far generalization of the property (AU) for topology as well as certain feature of a net. Conditions (i) and (ii) seem to have been put there as an afterthought. This property (iii) may occur in other studies such as algebra and geometry and the same word `filter' used in a different sense.

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Let X be any set. By a filter $\mathcal{F}$ on X (or in X,) we mean a nonempty subset $\mathcal{F}$ of the power set $\mathcal{P}(X)$ satisfying the following three conditions:
(i) $\emptyset \notin \mathcal{F}$;
(ii) $\mathcal{F}$ is closed under finite intersections;
(iii) $A \subset B \subset X, A \in \mathcal{F} \Longrightarrow B \in \mathcal{F}$.

Remark 7.30

set. We shall see that filters will do 'all' the jobs that nets were invented and some more through the concept of ultrafilters and ultra-closed filters.

Definition 7.29

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Remark 7.30

Note that (i) and (ii) together imply that $\mathcal{F}$ has finite intersection property(FIP). Since every filter in non empty, condition (iii) implies that $X \in \mathcal{F}$. Also, compare the conditions for a topology and for a filter. A topology always contains both $\emptyset$ as well as the whole set X, whereas Property (i) says $\emptyset \notin \mathcal{F}$. (ii) is common to both of them. (iii) is much stronger than (AU).

Note that (i) and (ii) together implies that $\mathcal{F}$ has finite intersection property. I will be using this one again and again. What does it mean, by finite intersection property? Given any finitely many members of $\mathcal{F}$, their intersection should be non-empty. First of all, (ii) says, their intersection is a member of $\mathcal{F}$. Because it is closed under the finite intersection means take finitely many members here, take the intersection, that also is a member. But then that member is non-empty. So, together, (i) and (ii) imply curly F has finite intersection property.

Since every filter is non-empty, you see I started with a non-empty subfamily $\mathcal{F}$ in the definition of a filter. So, I take some member of $\mathcal{F}$, that is a subset X and hence by (iii), X will be also there. So, I do not have to put that condition in the definition separately,

Compare the conditions for a topology and for a filter. This is what I have done already. I will repeat it. A topology on X always contains both empty set and the whole set X, whereas property (i) says that empty set is not there in a filter. The second one is common to both of them. Topology as well as filter. (iii) for filters is a much stronger than the third property for a topology called (AU) namely, arbitrary union of members of tau inside tau. So, it looks like as if we have replaced (AU) by (iii) but (iii) is much stronger. Even if one member is there in $\mathcal{F}$ their union will be there in curl F. So, this (iii) is much more stronger than (AU) for a topology.

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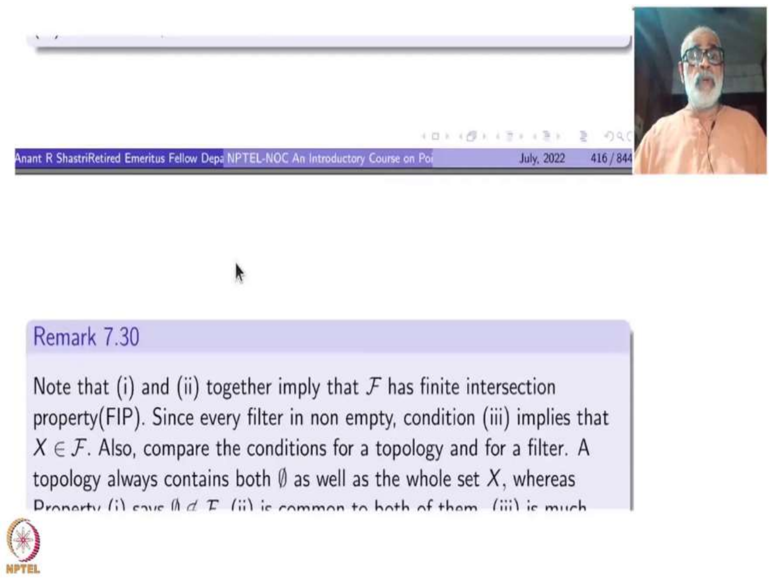
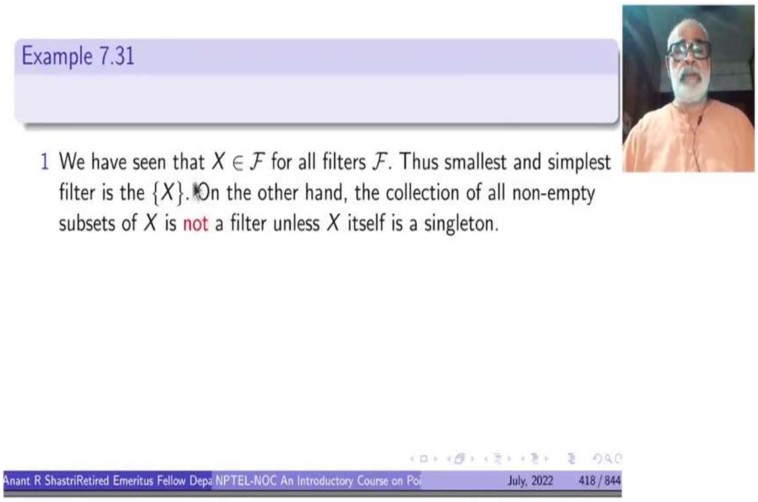
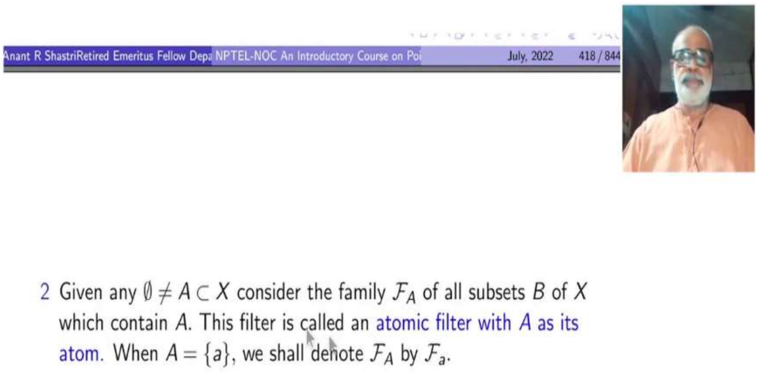




We have seen that X always there, in all filters $\mathcal{F}$ on X. This implies that the smallest and simpler filter is the singleton X. Of course, X itself should not be empty set. Then singleton X is a filter. Because it has finite intersection property and there is nothing larger than X.

On the other hand, the collection of all non-empty subsets of X is not a filter, unless X itself is a singleton. You see, in the case of topology, the collection of subsets of X is allowed and gives you the discrete space. Right? Here, we cannot allow all non empty subsets i.e., even after throwing away the empty set, that will not be a filter unless X is a singleton.

Because as soon as X has two distinct points, you can take $\{x\}, \{y\}$, their intersection is empty, so, that is not allowed. Finite intersection property will not be valid. So, this is one example. Let us see some more useful examples. (Refer Slide Time: 08:46)



Given any non-empty subset A of X, look at the family, $\mathcal{F}_A$, this notation will use again and again, which is the collection of all subsets B of X, which contain A, including A of course. This filter is called an atomic filter with A as its atom. One single member and all its supersets. So, that is a filter obviously and that filter is called an atomic filter with A as its atom. Now this is a notational remark here; If A is a singleton, $A = \{a\}$, I will not put a bracket her and just write $\mathcal{F}_a$ for $\mathcal{F}_{\{a\}}$.

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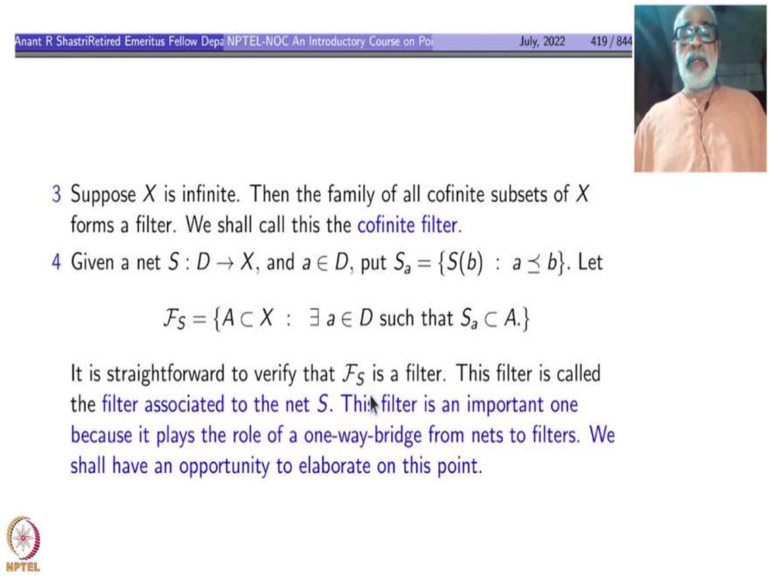

Now, suppose you have an infinite set X. Then just like in the case of topology, there is this co-finite filter. What is it? Do not take the empty set, but take all other members of $P(X)$, such that their complement is finite, co-finite subsets. So, that will be automatically a filter because first of all, if you take any two members intersection has to be nonempty because both of them have a finite complement in an infinite set X. Complement is finite and you are working in an infinite set. That is important. The moment some member is chosen, any subset bigger than that will also have its complement finite and hence (iii) is true. So, this is an example, not very useful though but helps to make the concept of filters a little more clear.

Next one is a useful one. Given a net S from D to X,where D is a directing set, take any element $a \in D$, take the section S_a viz., the set of all $s(b)$ (that is the definition of a section S_a), now look at a subset A of X which contains S_a. Put all such subsets A to get the family, $\mathcal{F}_S$. It is very easy to verify that this is a filter. So, this filter is called the filter associated to the net S. So, I am bringing nets and filters together here, via the association S going to $\mathcal{F}_S$ has. This filter is called filter associated to net S. It is important because it plays the role of a one-way bridge from nets to filters.

Why I am calling it `one-way bridge? I do not know any nice way of going from filters to nets. This one is very nice, very easy work. So, this one-way bridge is there. We shall have an opportunity to elaborate on this point, the importance of this bridge.

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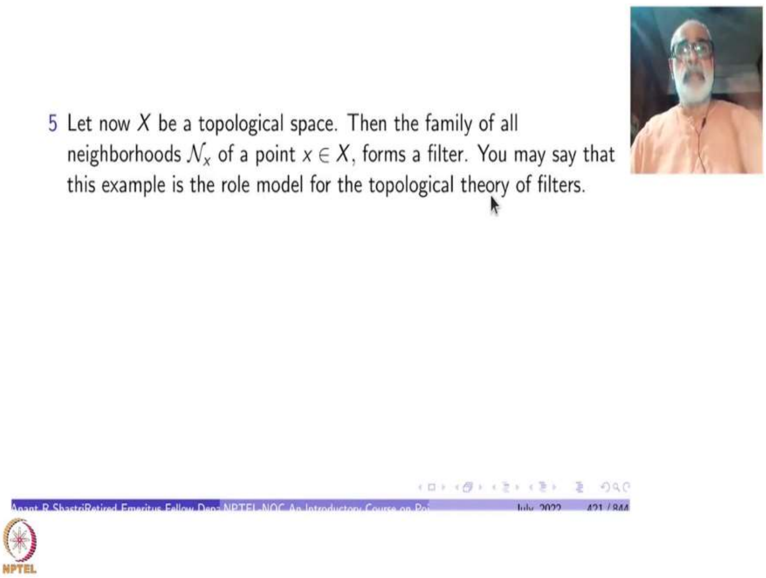

Here is one more example in a topological space X. Take the family of all neighbourhoods $\mathcal{N}_x$ of a point $x \in X$. x is fixed here. $\mathcal{N}_x$ is a filter. Clearly X itself is a neighbourhood of x

and so $\mathcal{N}_x$ is not empty. Intersection of finitely many members definitely contains the point x and so is non empty. Supersets are obviously there. Clearly empty set is not a neighbourhood of x and so is not there.

You may say that this is a simple example. It is a role model for the topological theory or filters. Remember filters were defined without reference to any topology on X. There was no particular topology on X, to begin with. What we did was we compared the concept of a filter with the concept of topology, that, all. So, far there was no topology. This example is the first example wherein we refer to a particular topology on X to obtain a filter. And this filter will guide us as far as the topological theory of filters is concerned, as convergence etc.

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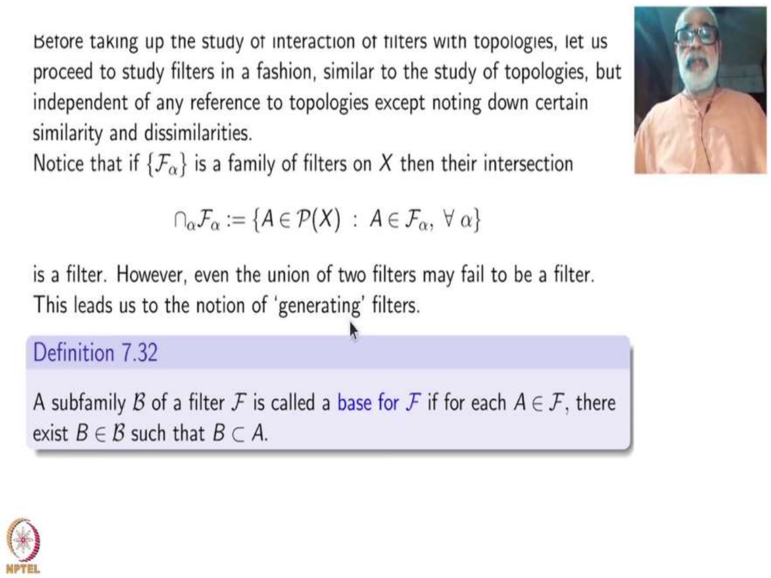

Now before taking up the study of interaction of filters with topologies, let us proceed our study of filters in a fashion, similar to the study of topologies, but independent of any reference to particular topologies on X, except noting down certain similarities and dissimilarities between the two concepts of a filter and a topology, that is all. We will take more examples also later on.

So, first notice that if $\mathcal{F}_\alpha$ is a family of filters on X, then their intersection is a filter. You had a similar theorem for topology as well. That is what I wanted to say. All $\mathcal{F}_\alpha$'s, remember, are subfamilies of $P(X)$. So, their intersection makes sense. What is the meaning of this? Take all subsets A of X where A is inside of is F_α for every α. So, that is the intersection of this family. That is a filter, of course, empty set is not in any of them. So, it will not be here also. So, you can easily verify that it is a filter.

However, even the union of just two filters may have fail to be a filter because one filter may have a member and another one may a member, the two of them begin disjoint. You do not know that there is no hypothesis. This was there, for topologies also. But luckily if we have two topologies, you can take the union and then generate a topology containing that union. Even that may not be possible in the case of filters. So, we would like to do that kind of thing here, namely generating filters, just the way we have done generating topologies.

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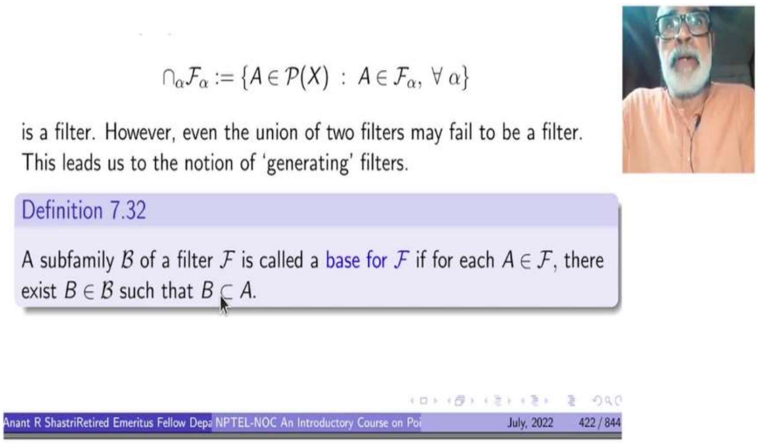

A subfamily, $\mathcal{B}$ of a filter $\mathcal{F}$ is called a base for $\mathcal{F}$ if for each A inside $\mathcal{F}$, there exists B belonging to $\mathcal{B}$ such that B is contained inside A. This is very similar to the definition of a base for a topology. That for each point $x \in X$, and a member of $\mathcal{T}$ containing that, there must be some member $B \in \mathcal{B}$ etc. Note that there is no reference to the point at all in this case, for each $A \in \mathcal{F}$, there must be a smaller B, which is inside $\mathcal{B}$.

In other words, take a member of $\mathcal{B}$ and take all supersets. They all may be in $\mathcal{F}$. That suggests a construction here; it leads us toward something.

So, we have made a definition of a base. The entire of $\mathcal{F}$ is of course a base for itself, just like any topology is a base for itself. So, this definition of a base satisfies similar to the conditions for, what we have done but it is much more general. For example, the base for a topology must satisfy the property that union of all members of it is the whole space X. No such condition is here.

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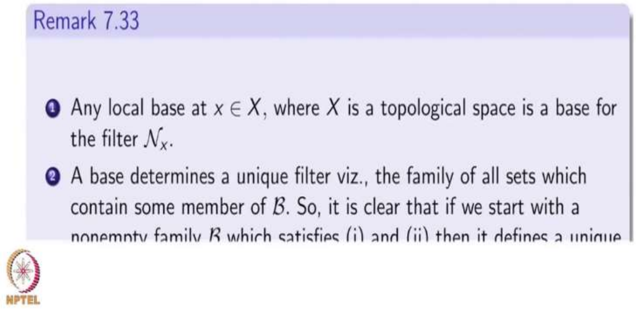

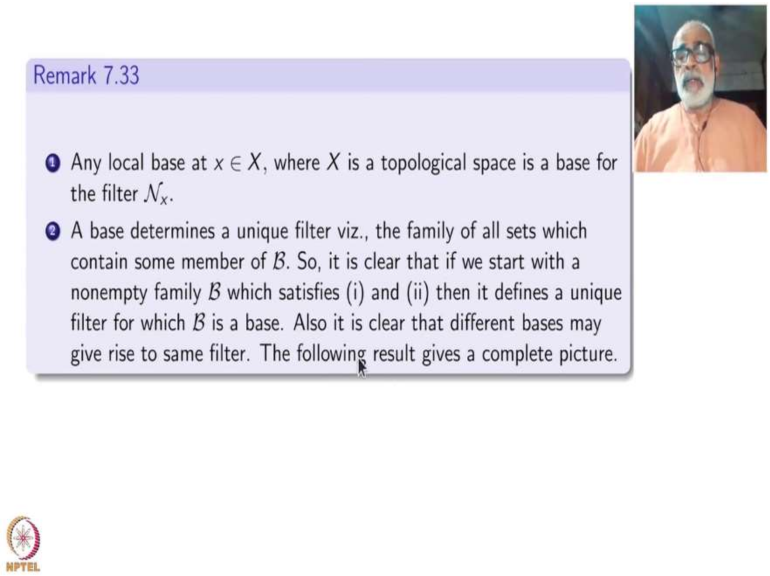

Let us look at some examples. Any local base at a point x belonging to X, where X is topological space is a base for the filter $\mathcal{N}_x$. So, this was one of the important filters. You need not work with all the neighbourhoods. You can take a base in the usual terminology of a local base in the topology. That will be a base in the sense of filters also.

A base for a given filter determines a unique filter (may be a different one) in the following way: The family of all sets, which contain some member of $\mathcal{B}$. You see, this filter is called the filter generated by $\mathcal{B}$ which clearly contains $\mathcal{F}$.

More generally, it is clear that if we start with a non-empty family $\mathcal{B}$, which satisfies (i) and (ii), then it generates a unique filter for which $\mathcal{B}$ is a base. Also, it is clear that different bases may generate the same filter. Generating means what, all subsets and only those subsets of X which contain some member of $\mathcal{B}$. The following result gives a complete picture.

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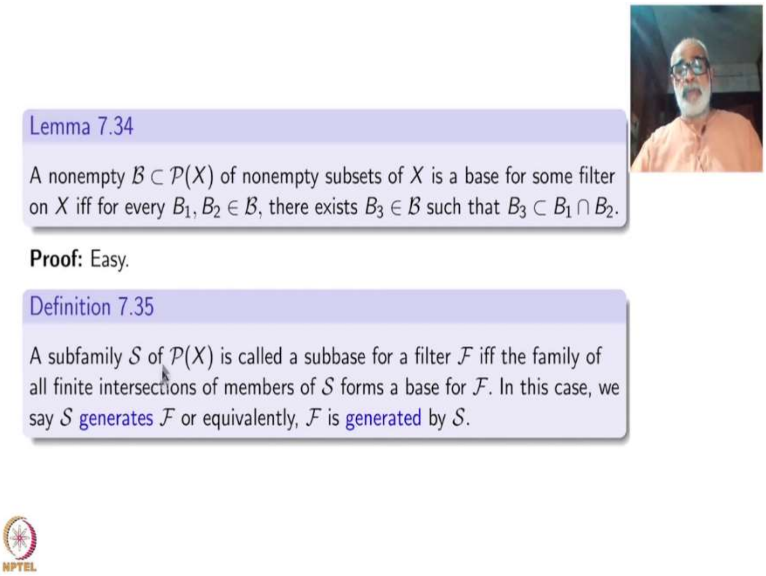

A non-empty family $\mathcal{B}$ contained inside $P(X)$ consisting of non empty subsets of X is a base for some filter on X if and only if for every B_1, B_2 belonging to $\mathcal{B}$, there exists B_3 in $\mathcal{B}$, such that B_3 is contained inside $B_1 \cap B_2$.

You see, this is again similar to a result for topological spaces, except that here there is no pointwise condition.

Proof is easy. You may use induction to prove closed under finite intersection.

Now, we come to the next stage of `generating'. Only after that I should be using the word generating here, to be precise. Similar to what we did in topology. What is this next stage? the notion of a subbase.

What is subbase? Let us define it. A subfamily S of $P(X)$ is called a subbase for a filter $\mathcal{F}$ on X if and only if the family of all finite intersections of members of S forms a base for $\mathcal{F}$.

You see, there is no condition of being closed under finite interactions. That was taken care by this condition on the base. In this case, we say S generates $\mathcal{F}$. Or equivalently, $\mathcal{F}$ is generated by S.

Student: So, this condition like family for intersections from the base, this automatically implied that every, empty set cannot be in S.

Professor Anant R. Shastri: Empty set will not be there, but the family may be empty. Empty set will not be a member of this one at all. Because if empty set is there, then it will be there in the base also. But it cannot be in a base either because once it is in the base, it will be in the filter also. So, that is not allowed.

Student: This S also cannot be empty set?

Professor Anant R. Shastri: S can be empty. Why? Because what is the finite intersection of members of S? Members of an empty set?

Student: That will be complete X.

Professor Anant R. Shastri: Yes. The whole of X will be there. X is the only member now. singleton X forms a base for what? the filter singleton X.

Student: If S is empty. So, we will get our smallest filter?

Professor Anant R. Shastri: Smallest filter. The smallest filter has two subbases, empty set is subbase, singleton X is also a subbase, but it is a base also. Singleton X is the whole filter as well. However, the empty family is not a base for $\{X\}$. Because once it is a base, we are taking only super sets of members in it. We are not going to take finite intersections there.

Therefore, if we start with an empty family, and take supersets, we still get an empty family only. Therefore, for all practical purposes, you can assume that S is non-empty family, that is all.

So, a non empty subfamily family of $P(X)$ is a sub base for a filter, if and only if S has finite intersection property.

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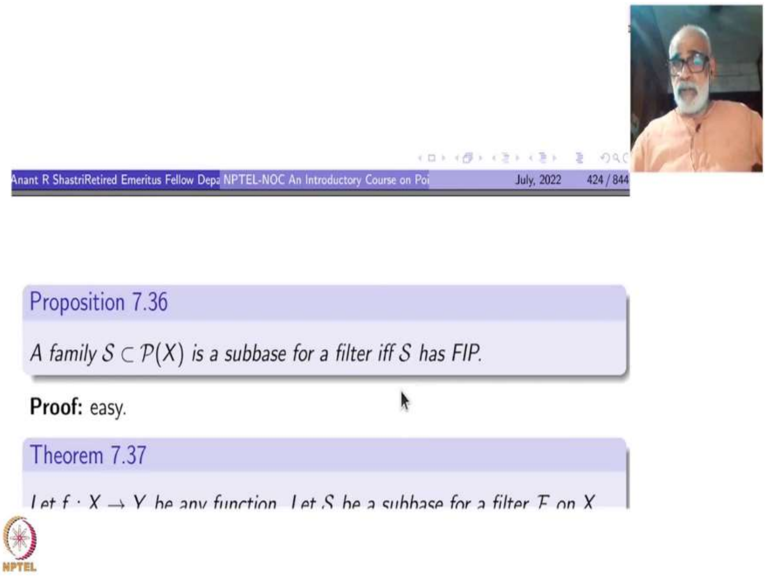

Once again, you may be confused when S is the empty set. It has finite intersection property because the only family which is finite subfamily is the empty family there. So, you can allow an empty family also just for the sake of completeness of the definition.

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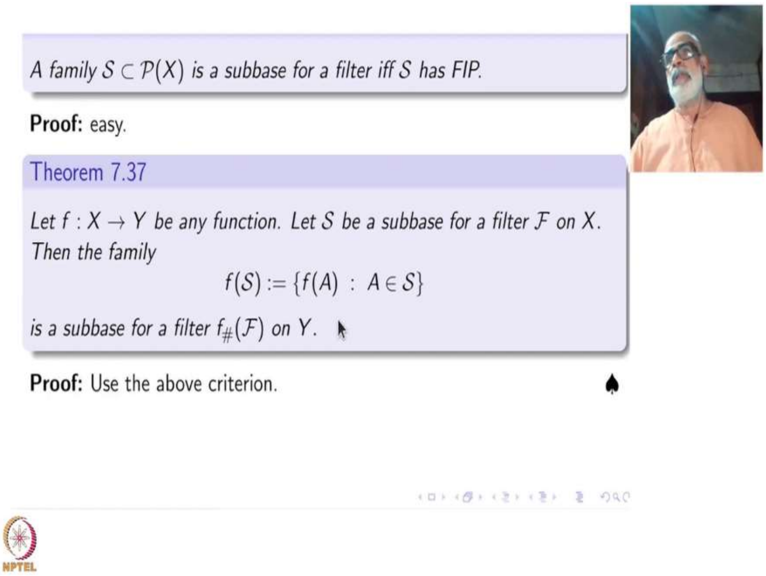

So, now we will do a little more with functions. This is again similar to what we did in topology. Start with any function f from one set X to another set Y, and let S be a sub base for a filter $\mathcal{F}$ on X. Then I want to push it to Y via f. How do I push it? Take $f(\mathcal{S})$ equal to the set of all $f(A)$, where A is inside S. So, this is a subfamily of $P(Y)$. Check that it has finite intersection property. So, this family is a subbase for a unique filter on Y and that filter will be called $f_{\#}(\mathcal{F})$.

So, how do I do that? What is the meaning of sub base for a filter? Take the family of finite intersections of members of the subbase and then take all the supersets. So, that is your $f_{\#}(\mathcal{F})$. So, all that I have to see is that this family has finite intersection properly. You take say, intersection of $f(A_1), \ldots, f(A_n)$. $f(A_1 \cap \cdots \cap A_n)$ will be contained in it. So that is not empty that is all.

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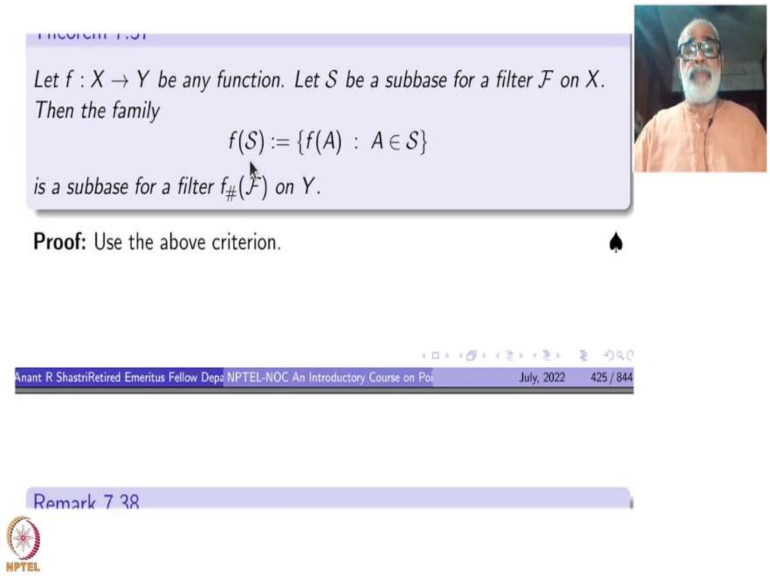

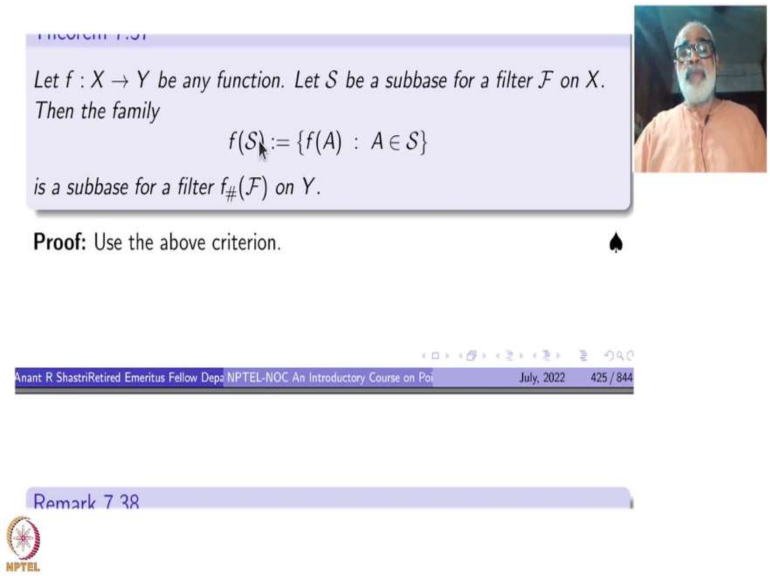

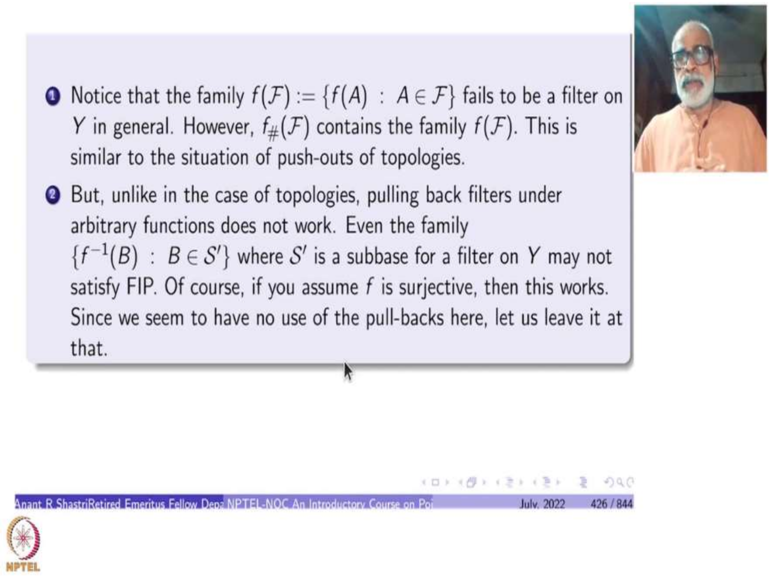

Notice that instead of working with subbases if you directly take the family $\{(f(A) : A \in \mathcal{F}\}$ this may easily fail to be a filter on Y, in general, Y may have bigger subsets, which do come from S or $\mathcal{F}$. However, the above family has finite intersection property and hence will be a subbase for a filter.

Now look at the other round. Start with a subbase S' for a filter $\mathcal{F}'$ on Y.

Unlike in the case of topologies, pulling back filters, going the other way round, pulling back filters under an arbitrary function does not work well for filters. Remember if Y is topological space, X is some set and f from X to Y is some function, then I can take inverse image of open subsets in Y, they themselves form a topology on X. That was a nice situation.

Here, with the filters, it does not work. Not only that, if you take the family of inverse images of members of S prime, that may not have FIP. So, of course, if we assume f is surjective, then this works. So, you have to have more and more hypothesis. So, let us not bother about pulling back filters. Anyway, we are not going to use them in the convergence theory of filters, which is our aim afterall.

So, this is what I meant by saying that basic theory of filters, that is all. Next time we shall study seriously, its relation with topology, namely convergence theory for filters. Thank you.