**An Introduction to Point-Set-Topology Part 2**
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**Lecture No: 21**
**Stone-Cech Compactification**

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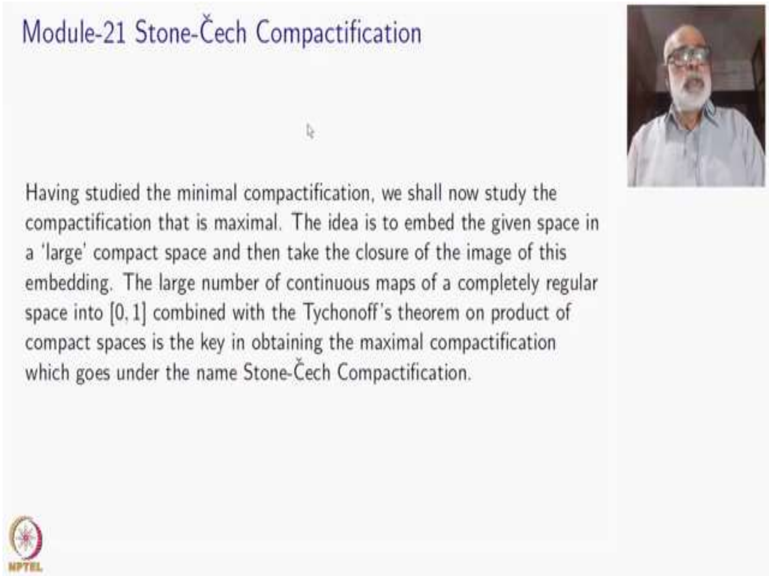


Hello, welcome to module 21 of NPTEL Point-Set-Topology course, part II. Today's topic is Stone-Cech compactification. Having studied the minimal compactifications namely the Alexandroff's compactifications, we shall now study the compactification that is maximal.

The idea is to embed the given space in a large compact space and then take the closure of the image of this embedding. The large number of continuous maps of a completely regular space into the closed interval $[0, 1]$ combined with the Tychonoff's theorem on product of compact spaces is the key in obtaining this maximal compactification which goes under the name Stone-cech compactification. You should remember that all these we are doing only for Hausdroff spaces.

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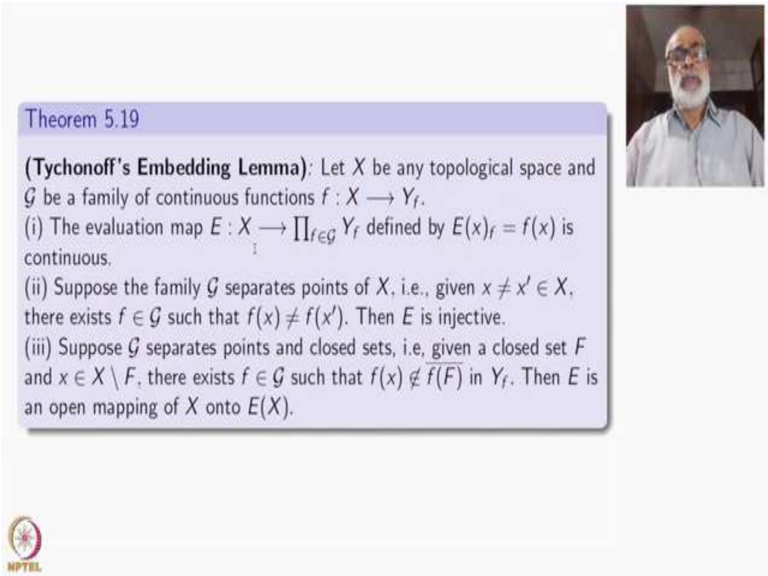


So, our first lemma which you can call Tychonoff's embedding lemma is the following: Start with a topological space, any topological space and any family of continuous functions, which I will denote by $\mathcal{G}$. The domain of each function $f$ always $X$, but co-domain may change from function to function so, I denote it with $Y_f$. So, each member of $f \in \mathcal{G}$ is a continuous function from $X$ to $Y_f$; that is all.

(i) Let the evaluation map $E$ from $X$ into the product of all the spaces $Y_f$'s, as $f$ ranges over $\mathcal{G}$ , be defined as follows: for each $x \in X$, the f-coordinate of $E(x)$ equal to $f(x)$. (Remember that the points of the product space are completely described by describing their coordinates.) The $E$ is continuous.

The second part (ii): Suppose the family $\mathcal{G}$ separates points of $X$. (This just means that if you are given two distinct points $x$ and $x'$ inside $X$, then there exists a function $f$ inside this family $\mathcal{G}$ such that $f(x)$ is not equal to $f(x')$, that $x$ and $x'$ have been separated by $f$ that is the whole idea.) So, if that condition is satisfied then the evaluation map $E$ (defined as in (i)) is injective. This is the second part.

The third statement (iii): Suppose the family $\mathcal{G}$ separates points and closed sets. (That just mans that given a closed set $F$ in $X$ and a point $x$ not in the closed set, there must be a function $f$ inside $\mathcal{G}$ such that $f(x)$ is not inside the closure of $f(F)$. See $F$ is a closed set but $f(F)$ may not be a closed set, you take the closure of $f(F)$ in $X$, this $x$ must be outside this closure. That is the meaning of this separation). Then $E$ is an open mapping of $X$ onto its image $E(X)$.

Now, an open mapping injective map will be an embedding. That is why the name embedding in the lemma has come. So there are three distinct things here. So, one by one let us have a proof which are all straight forward.

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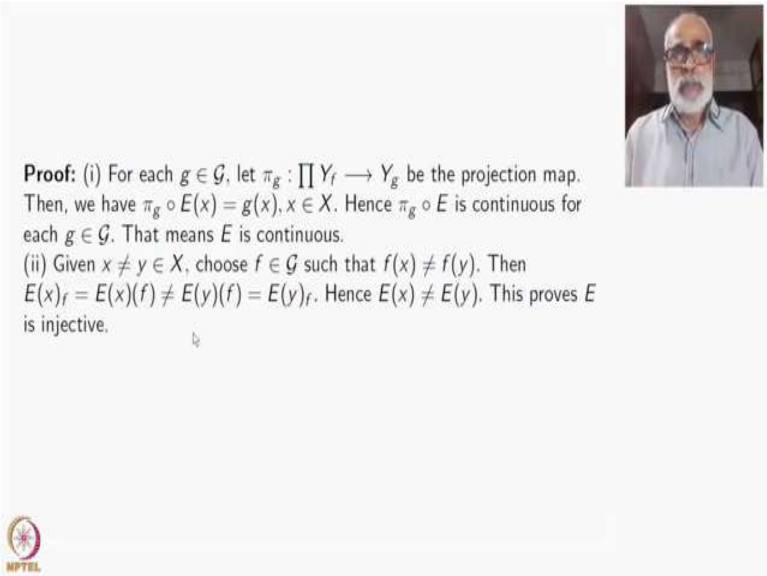


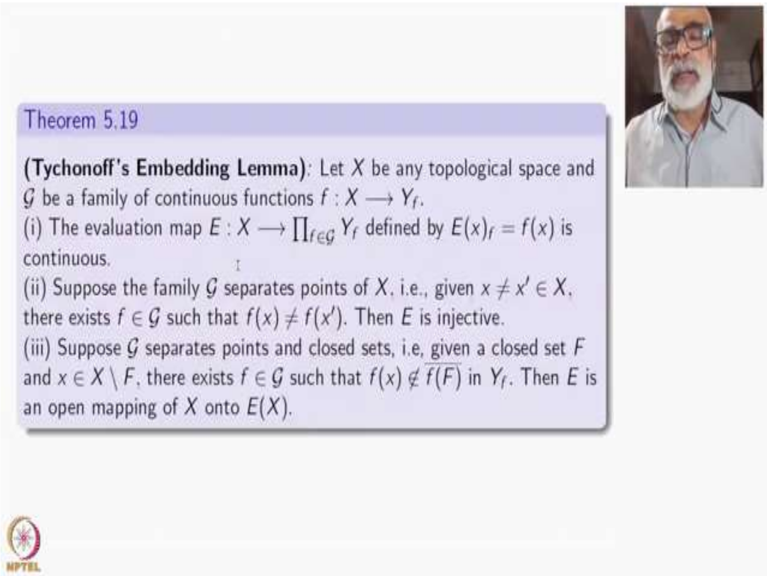


The first thing is continuity of the function $E$. Where is it? It is on some space $X$ into the product space. A function into a product space, we know, is continuous if and only if all its coordinate functions are continuous, viz., composed with each projection map is continuous.

Let us take this $\pi_g$ to be the projection map from the product $Y_f$ to $Y_g$. Then $\pi_g \circ E$ operating upon any point $x$ is nothing but $g(x)$ by definition of $E$, $E(x)(f) = f(x)$. So, you have to put $f = g$ here. What does that mean? That $\pi_g \circ E$ is the function $g$ itself. That $g$ is continuous because we are taking all functions inside this $\mathcal{G}$ to be continuous functions. So, this just completes a proof that $E$ itself is continuous.

The second part (ii). Start with two points $x$ and $y$ not equal to each other inside $X$. As soon as you have that, there will be some function $f$ inside $\mathcal{G}$ such that $f(x) \neq f(y)$. So, that is the meaning of the property that $\mathcal{G}$ separates points of $X$. that is the condition that we are using. Remember $f(x)$ is nothing but the $f$-coordinate of $E(x)$ and $f(y)$ is the $f$-coordinate of $E(y)$. We know that these two are different, $f(x) \neq f(y)$. So, one of the coordinates of the points $E(x)$ and $E(y)$ are different. Therefore, $E(x)$ is not equal to $E(y)$. So, statement (ii) is proved.

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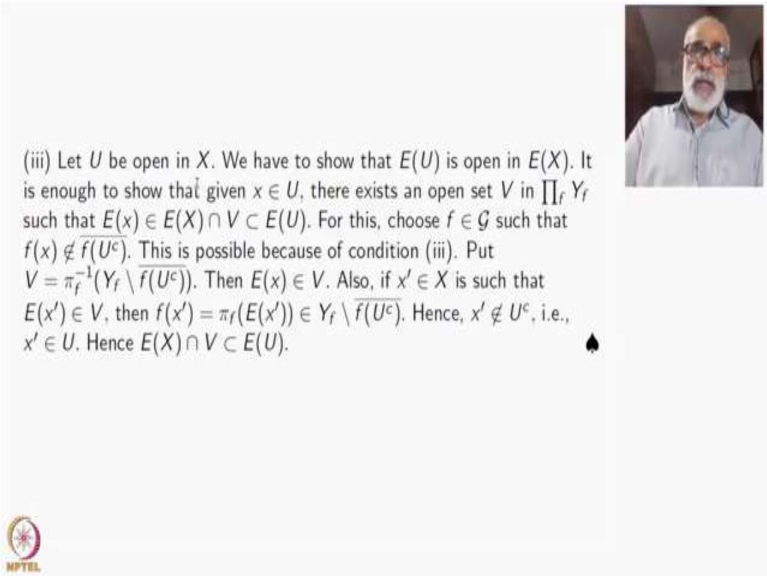


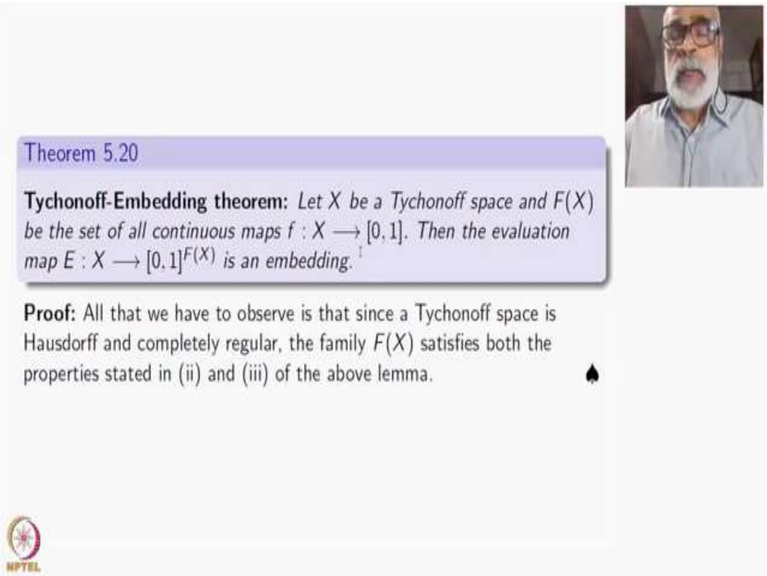

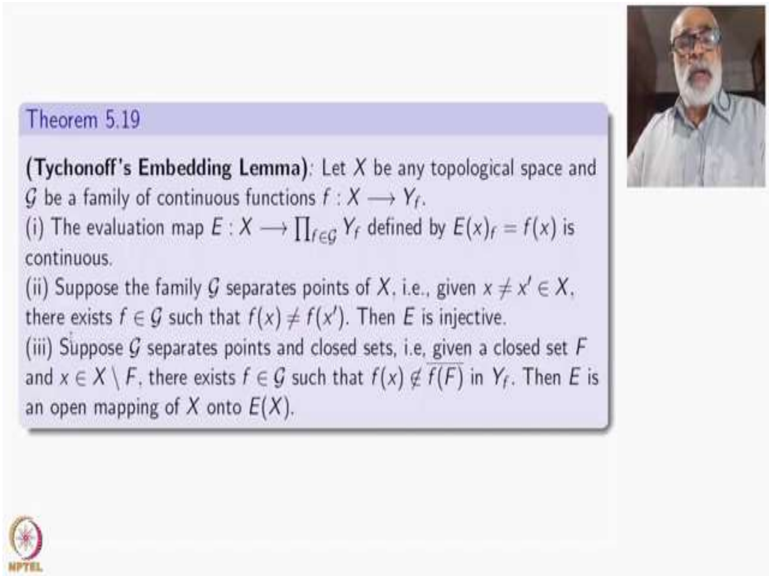


Now, the third statement (iii). So, what we have to show? We have to show that starting with an open subset $U$ of $X$, $E(U)$ is open inside $E(X)$ is what we have to show. Where is $E(X)$? $E(X)$ is a subspace of the product space. So, how do we ensure that given some subset is an open subset in the subspace? If you get an open subset of the product space and then you intersect it with the subspace, and show that that intersection with $E(X)$ is given set.

Or it is enough to show that given any point $x \in U$, there exists an open set $V$ in the whole product space $Y_f, f$ running inside $\mathcal{G}$ such that this way we have the property that $E(x)$ belongs to $V \cap E(X)$ which is contained in $E(U)$.

So, let us see how to construct this $V$. For this, all that I do is choose $f$ such that $f(x)$ is not in the closure of $f(U^c)$. Remember $U$ is an open set. So its complement is closed in $X$ and does not contain $x$. By condition in (iii) such a function $f$ exists.

Now you take $V$ equal to $\pi_f^{-1}(Y_f \setminus \overline{f(U^c)})$. Remember $f(U^c)$ is a subset of $Y_f$, because $f$ is a map from $X$ to $Y_f$. So, if I take the complement of the closure that is an open subset of $Y_f$. Therefore $\pi_f$ inverse of that will be an open subset and that is what I am taking to be $V$.

So, this $V$ will be an open subset of now the entire product space because $\pi_f$ is after all a projection map from the entire product space into $Y_f$. So, this $V$ is open. Obviously $E(x)$ belongs to $V$ because $\pi_f(E(x)) = f(x)$ and $f(x)$ is not in the closure of $f(U^c)$.

Now suppose $x'$ is any other point such that $E(x')$ is inside $V$. That is $E(x')$ is a point of $E(X) \cap V$. Then what happens? $\pi_f(E(x'))$ is nothing but $f(x')$ is in the complement of the closure of $f(U^c)$. So, it is in $Y_f \setminus \overline{f(U^c)}$. That means that $x'$ in not inside $U^c$ which is same thing saying that $x'$ is in $U$. Therefore, $E(X) \cap V$ is contained inside $U$. So, that shows that $E(U)$ is open inside $E(X)$. So, lemma is proved. Now, we can read some important conclusions here. The first one, I am calling Tychonoff's embedding theorem.

I start with a Tychonoff space. Tychonoff space means what? Completely regular and a Hausdorff or $T_1$ space. So, start with a Tychonoff space and let $F(X)$ be the set of all continuous maps $f$ from $X$ to $[0,1]$. So, here I am have a specific choice of these $\mathcal{G}$ in the lemma. So, $\mathcal{G}$ is $F(X)$, the entire space of all continuous functions from $X$ to $[0,1]$. Then look at the evaluation map $E$ from $X$ to this product space $[0,1]^{F(X)}$, copies of the closed interval $[0,1]$ taken $F(X)$, times. (For each member of $F(X)$,, take a copy of $[0,1]$ and take the product. Earlier in the lemma, we had arbitrary space $Y_f$. Now each $Y_f$ is equal to the closed interval $[0,1]$. So, that is the special case of this previous lemma.) This $E$ is an embedding.

That is the statement which is an immediate consequence of lemma; all that you have to show is that the condition (ii) and (iii) are automatically satisfied, if X is a Tychonoff space. A Tychonoff space is Hausdorff space then we know that each point is closed and any closed set and point outside are separated by $F(X)$ because of complete regularity.

So, (i) and (iii) will be automatically satisfied. Of course, $E$, the evaluation map is always continuous. For that no extra assumption on this family $\mathcal{G}$ is necessary. So, conditions in (ii) and (iii) come only because we have assumed $X$ is a Tychonoff space. So, what we already saw in the final conclusion is that for every Tychonoff space $X$, $E$ is the evaluation map from $X$ to the product of $[0,1]$ taken $F(X)$, times namely, the space of all continuous functions into $[0,1]$ is an embedding.

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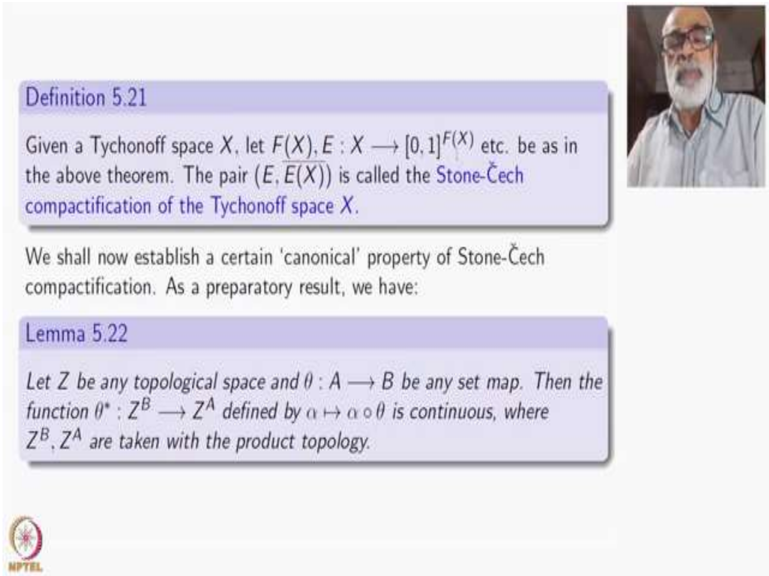


Now we can make the definition of this Stone-Cech compactification. Look at any Tychonoff space $X$, take $F(X)$ and the evaluation map $E$ from $X$ to $[0,1]^{F(X)}$. Then the pair $(E, \overline{E(X)})$ (closure taken in $[0,1]^{F(X)}$), is called the Stone-Cech compactification of the Tychonoff space $X$.

Of course, the product space is compact since every closed interval is compact. That is why $\overline{E(X)}$ will be compact. Each factor in the product is also a Hausdorff space and hence the product is a Hausdorff space. So, the subspace $\overline{E(X)}$ is also Hausdorff. By the very definition, $E(X)$ is dense in $\overline{E(X)}$. So, these are Hausdorff compactificatios. Now $X$ can be identified with $E(X)$ via the embedding $E$, then this $\overline{E(X)}$ can be thought of as an extension of the space $X$ itself. That is the way to think about compactificatios. However, while dealing with technical aspects, elaborate definition is needed and so we just do not forget the actual embedding $E$. For instance, if the embedding is chosen differently by chance, then we do not call it Stone-Cech compactification.

You shall now establish a certain canonical property of Stone-Cech compactification. I will explain the use of the word `canonical' in this context, a little bit later. As a preparatory result, we have this lemma. This itself partly explains the word `canonical' implictly. So, pay attention to this lemma.

Let $Z$ be any topological space and theta from $A$ to $B$ be any set map, just set theoretic functions. map, there is no topology here. $Z$ is a topological space.

Then consider the function $\theta^*$ from the $Z^B$ to $Z^A$, i.e., product of copies of $Z$ taken $B$ times once and $A$ times next. So, what is $\theta^*$? Look at any element of $Z^B$, that is a function from $B$ to $Z$. Pre-compose it with $\theta$ to get a function from $A$ to $Z$. Thus $\theta^*(\alpha) = \alpha \circ \theta$.

This map $\theta^*$ itself is a continuous function where this $Z^B$ and $Z^A$ are given the product topology and the product topology. Note that $\theta$ is any function, there is no continuity condition, there is no topology there. But the domain and codomain of $\theta^*$ are topological spaces and $\theta^*$ is continuous.

This statement looks somewhat strange. But the proof is just one line here. Because what you have to do to prove the continuity of a function into a product space? You have to check the continuity of the function composed with each coordinate function. That is all.

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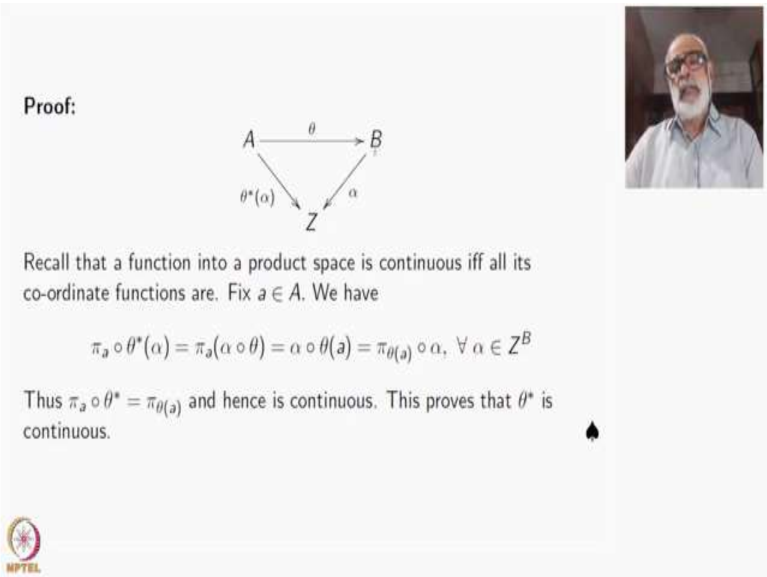


So, that we have to so here is a diagrammatic representation of $\theta^*$, $\theta$ is this function here. For each function from $\alpha$ from $B$ to $Z$, these are the set theoretic functions being elements of $Z^B$, you compose it with $\theta$ then $\alpha \circ \theta$ is a function from $A$ to $Z$. That is the definition of $\theta^*(\alpha)$.

All that I have to do is, for each point $a \in A$, to show that $\pi_a \circ \theta^*$ is continuous. These $\pi_a$ are coordinate projections. Operated on any $\beta$, it is by the very definition, it is nothing $\beta(a)$. Therefore, $\pi_a(\theta^*(\alpha)) = \pi_a(\alpha \circ \theta) = \alpha \circ \theta(a) = \pi_{\theta(a)}(\alpha)$. This just means that $\pi_a \circ \theta^*$ is nothing but $\pi_{\theta(a)}$ a coordinate projection from the product space $Z^B$. All coordinate projection are continuous. So, that completes the proof that $\theta^*$ is continuous.

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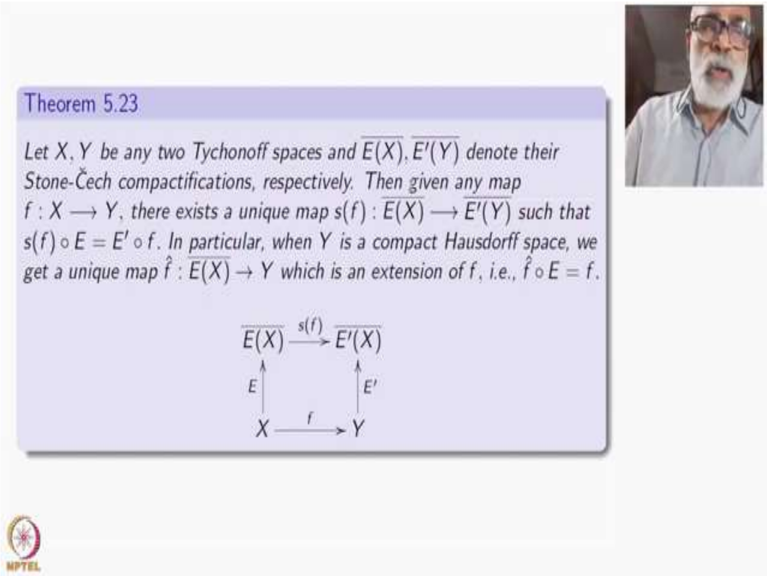


Now, I am making the statement of theorem 5.23. Let $X$ and $Y$ be any two Tychonoff spaces, and $\overline{E(X)}$ and $\overline{E'(Y)}$ be short notations for our standard Stone-Cech compactifications of $X$ and $Y$ respectively. (But just to be careful, I am taking $E'(Y)$ here so that there is no confusion since for both we have the evaluation maps, which are obviously not the same maps, their domains and codomains are both different.

Now, given any map $f$ from $X$ to $Y$, there is a unique map $s(f)$ from $\overline{E(X)}$ to $\overline{E'(Y)}$ such that (this $s(f)$ is a new function, the existence and the uniqueness is asserted) $s(f)$ has the following property, viz., that $s(f) \circ E$ is equal to $E' \circ f$. So, look at the diagram. $f$ is here, $E$ is going into $\overline{E(X)}$ here, $E'$ is going into $E'(Y)$ there. These are two Stone-Cech compactifications. There will be a continuous function sf here, which makes this diagram commutative.

There may be many such functions? No, there is only one such map like this, a unique map $s(f)$. That is conclusion of the first part. The second part says: In particular, if $Y$ itself is compact and Hausdroff space, we get a unique map, this time denoted by $\hat{f}$ from $\overline{E(X)}$ to $Y$ which is an extension of $f$. Again extension means what, now $\hat{f} \circ E$ is equal to $f$. The function is from of $\overline{E(X)}$ to $Y$. You restrict it to the subspace $E(X)$, which is the copy of $X$ that is the function f itself. So, that is why this $\hat{f}$ is called an extension that is all. So that is the second part here.

I will explain this one once we complete the proof of the first part.

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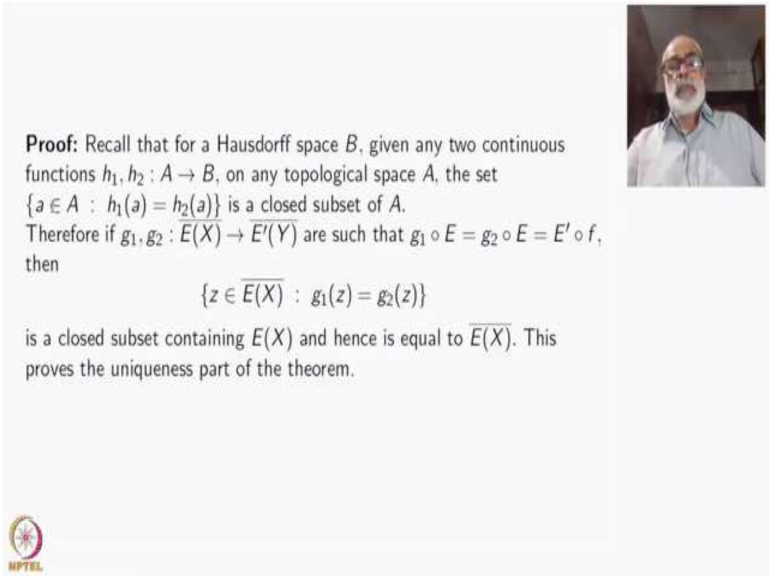


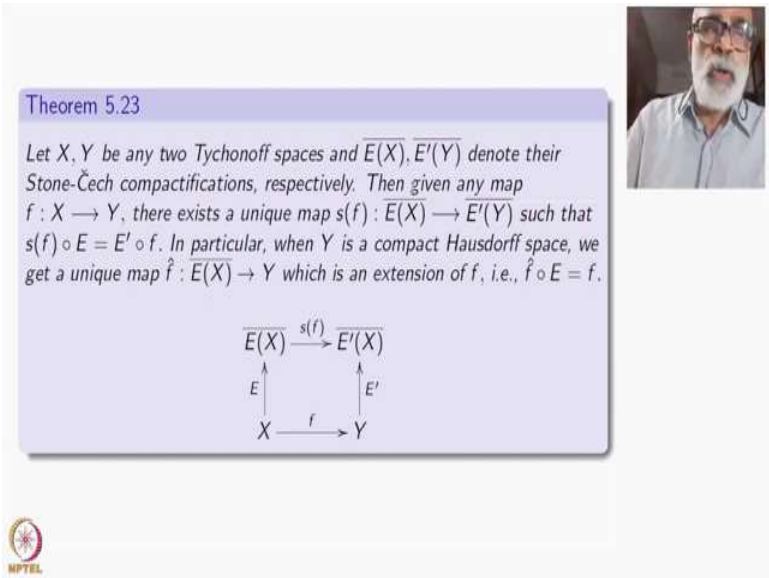


Let us start with the uniqueness part of $s(f)$. Suppose you have any topological space $A$ and any Hausdroff space $B$ and two continuous function $h_1, h_2$ from $A$ into $B$, such that if you look at the set of all $a$ belonging to $A$ with $h_1(a) = h_2(a)$, that set is always a closed subset of $A$. So, that is a property of the Hausdroffness of $B$, of the co-domain. So, this is one thing which we have used several times. So, here also we are using it. I am going to use it for the uniqueness part. I am going to use the fact that the Stone-Cech compactification $\overline{E'(Y)}$ is a Hausdroff space. So, let us apply this conclusion here to $g_1$ and $g_2$ two functions from $\overline{E(X)}$ to $\overline{E'(Y)}$, such that $g_1 \circ E$ is equal to $g_2 \circ E$. Both of them are what? By the very definition and by the very condition they must be equal to $E' \circ f$.

Therefore, $g_1(E) = g_2(E)$. That just means that set of all points of $\overline{E(X)}$, let us call it $z$, wherein $g_1(z) = g_2(z)$ which is a closed subset contains the space $E(X).E(X)$ is contained is this closed set that means its closure is also contained the this closed set. Some subset is contained in a closed set implies the closure of it is also contained there. So, what is the meaning of that? $g_1 = g_2$ that is all. So, that is the proof of the uniqueness.

Now, I have to show the existence of $s(f)$. Because of uniqueness, we can write a notation for it such as $s(f)$, $s$ stands for Stone-Cech compactification, for every continuous function $f$ from $X$ to $Y$. So, let us go to the existence part. Given a function $f$ from $X$ to $Y$, first we define $f^*$ from $F(Y)$ to $F(X)$ as in the lemma. Remember $F(Y)$ denotes all continuous functions from $Y$ into $[0,1]$. Similarly, $F(X)$ denotes all continuous functions from $X$ into $[0,1]$. Starting with a function $\alpha$ from $Y$ to $[0,1]$, you pre-composite it with $f$ to get a function from $X$ to $[0,1]$. So $f^* \circ \alpha$ is just $\alpha \circ f$.

Next thing is to take one more composition here. Now to $f^{**}$ from the product space $[0,1]^{F(X)}$ into the product space $[0,1]^{F(Y)}$, just the way we have defined $\theta^*$ in the lemma. So, $\theta$ is replaced by a $f^*$ here. Here is a diagram which shows the two stage construction of $(f^*)^*$. Starting with $\alpha$ you composite it with $f$ to get $f^*(\alpha)$. So the codomain is always $[0,1]$ here. Then apply the same construction for $f^*$: starting with any function $\phi$ here, you compose it with $f^*$ to get $f^{**}$. This $f^{**}$ is now from the set of all functions from $F(X)$ to $\mathbb{I}$, namely the product space, $\mathbb{I}$ raised to $F(X)$ to $I^{F(Y)}$, these are both products. This $f^{**}$ which is continuous from the statement of this lemma 5.22.

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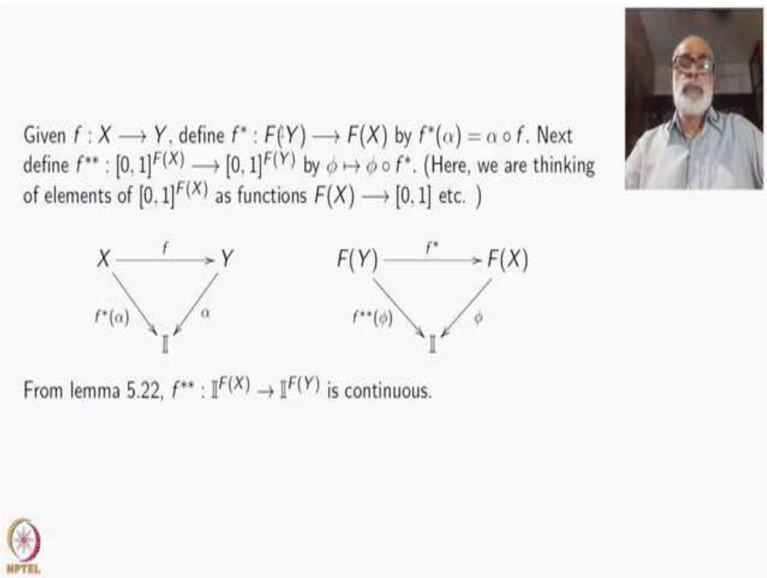

All that I have to do now is that this same $f^{**}$ will restrict itself to the Stone-Cech compactifications $X$ and $Y$ respectively, inside these product spaces. So, that is the next step we want to do.

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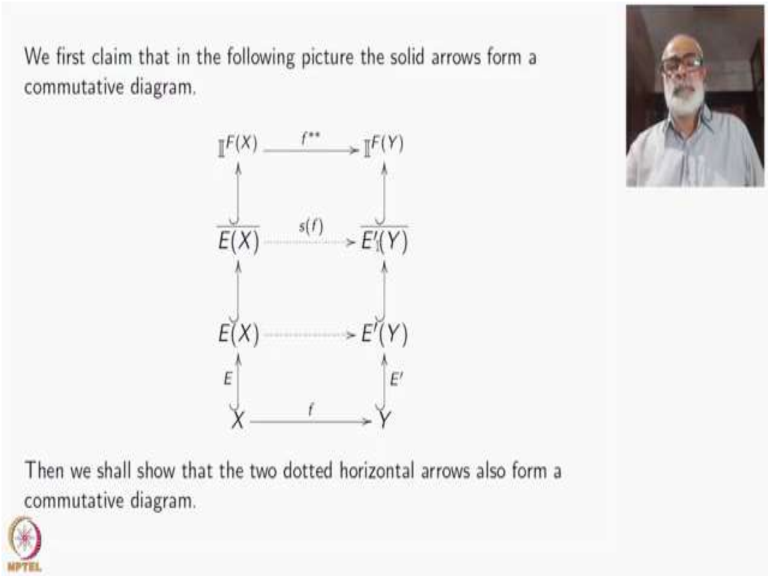


In the following picture, you start with $f$ here and you have got $f^{**}$ here, $\overline{E(X)}$ is sitting here $\overline{E'(Y)}$ is sitting here. These are the Stone-Cech compactifications of $X$ and $Y$ respectively. First I want to show that all the solid arrows form a commutative diagram, viz., the outer-most ones, excluding the two dotted arrows in the figure.

Once you have this commutative diagram, it follows that $f^{**}$ of $E(X)$ goes inside $E'(Y)$. That would automatically implies, by continuity of $f^{**}$ that $f^{**}$ of closure of $E(X)$ goes inside closure of $E'(Y)$. Therefore, the entire diagram including the two dotted arrows will be commutative. So, automatically these dotted arrows are nothing but just the restrictions of $f^{**}$ to the corresponding domains. If I take $s(f)$ to be the restriction of $f^{**}$ to $\overline{S(X)}$, then it follows that $s(f) \circ E$ is equal to $E' \circ f$.

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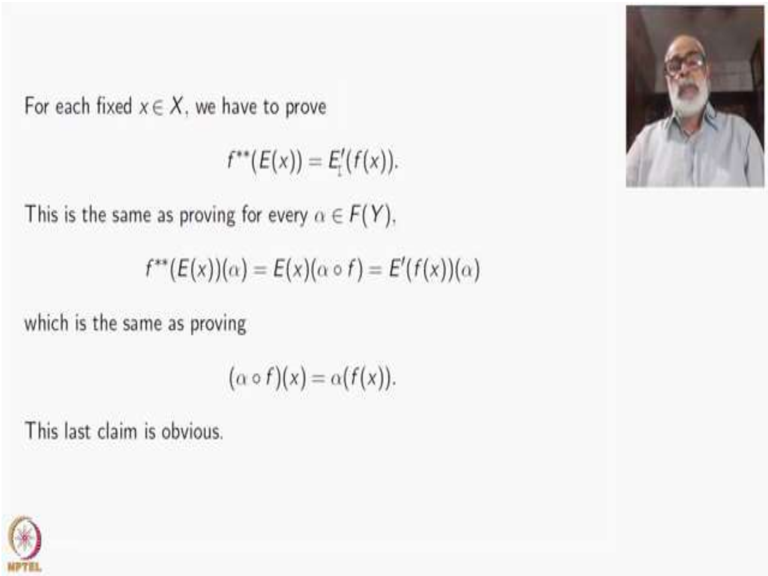


So, let show that for each $x \in X$, $f^{**}(E(x)) = E'(f(x))$. Remember that $E$ (and respectively $E'$) is the evaluation map from $X$ into the product space $\mathbb{I}^{F(X)}$ ($\mathbb{I}^{F(Y)}$ respectively.) This claim is the same thing as, by the definition of $f^{**}$, $f^{**}(E(x))$ operating upon $\alpha$ is equal to $E(x)$ operating upon($\alpha \circ f$), for $\alpha \in F(Y)$. So, we have to show that this latter thing is equal to $E'(f(x))$ operating upon $\alpha$. But this is the same thing as proving that $\alpha \circ f$ operating upon $x$ is the same as $\alpha(f(x))$, which is obvious, since $E'$ is another evaluation map.

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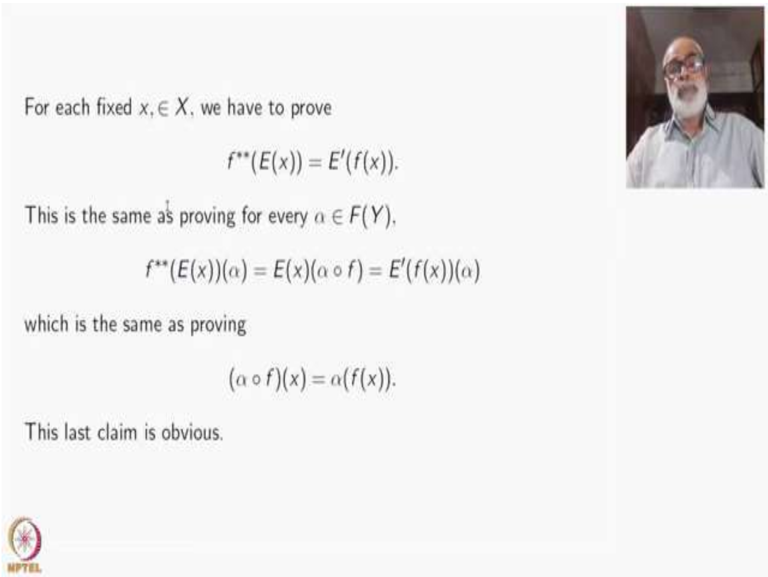

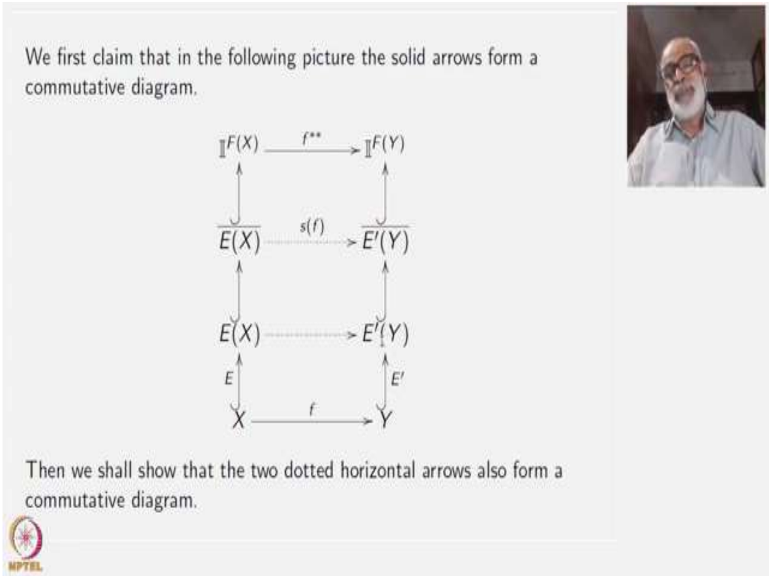


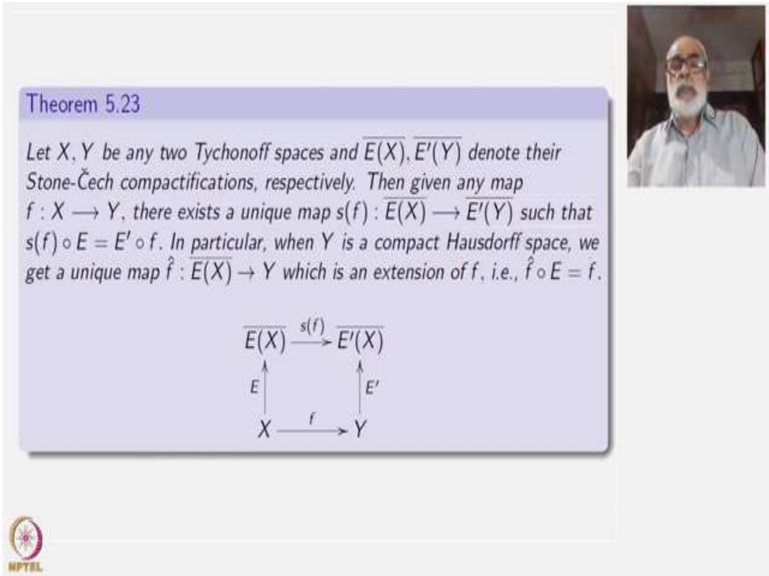


So, I repeat now. Having proved that this solid diagram is commutative, all that I have to observe is that $f^{**}E(X)$ is contained inside $E'(Y)$. That is purely a set theoretic observation. Any point of $E(X)$ is nothing but one coming from $X$. The evaluation maps on either side have been factored through the Stone-Cech compatifications, which are subspaces of the corresponding codomains.  So that is purely an observation. Thus $s(f)$ makes sense and has the required properties.

The last part remember what is the last part here? In particular when $Y$ is a compact Hausdroff space, what is $E'(Y)$? $E'(Y)$ will be a compact subset of a Hausdroff space. Therefore, it is closed already. So, $\overline{E'(Y)}$ will be equal to $E'(Y)$ which is a copy of $Y$. So, the Stone-Cech compactification of a compact space Hausdorff space is itself. `Itself' means that $E(Y)$ is being identified with $Y$ via the homeomorphism $E'$.

So, the map $s(f)$ is actually from $\overline{E(X)}$ to $Y$ itself. This $\overline{E'(Y)}$ is just $Y$ itself. When you say $Y$ itself, you are identifying $E'(Y)$ with $Y$, and treating $E'$ as the identity map. That is why you have a simpler notation here. If you write $s(f)$ equal to $\hat{f}$ here, treat it as a function into $Y$, then what you have is that $\hat{f} \circ E$ which is nothing but $s(f) \circ E$ is just $f$ because $E'$ is treated as an identity map.

So, second part just follows because compact Hausdroff space has itself as its Stone-Cech compactification. That is all.

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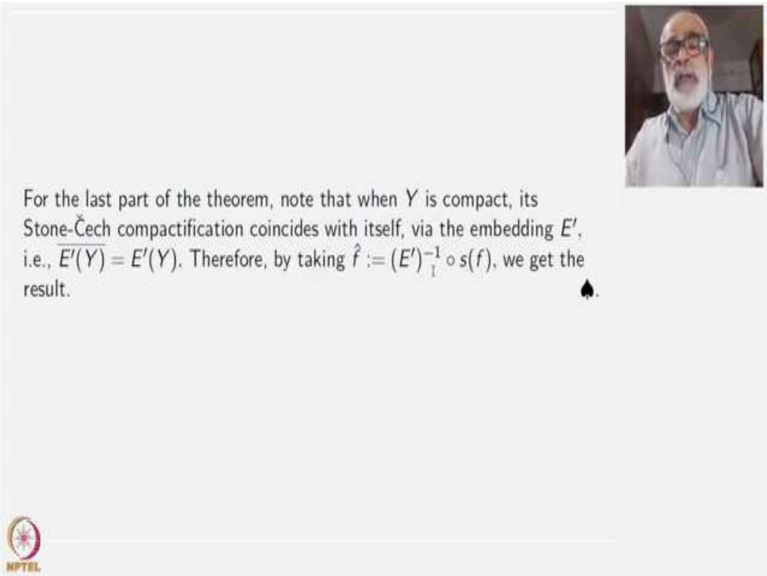


So this is obtained very easily all that you have to say $\hat{f}$, it is $E'^{-1}(s(f))$ or if you do not want to do all that you think of $E'$ is identity map with inclusion map, that is all.

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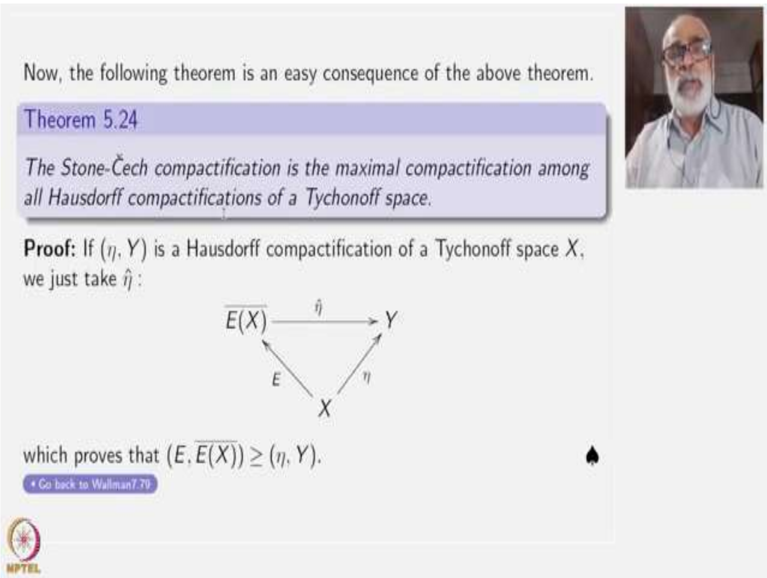

This theorem has many interpretations. So, Stone-Cech compactification is the maximal compactification among all Hausdroff compactifications of a Tychonoff space. Only when we have Tychonoff space, we can talk about Stone-Cech compactification. And that is a Hausdroff compactification.

Look at all other Hausdroff compactifications, this will be larger than all of them. Remember, we have a partial ordering defined on all compactification of a given space. If we restrict to the collection of only Hausdroff compactifications, amongst all of them, Stone-Cech is the largest. If $(\eta, Y)$ is a Hausdroff compactification of a Tychonoff space $X$, then we just take $\hat{\eta}$ as in the last part of the previuos theorem.

Remember that any continuous function $f$ from $X$ into a compact Hausdorff space $Y$ admits an extension $\hat{f}$ from $E(X)$ to $Y$. This $\hat{\eta} \circ E$ is precisely $\eta$. That is precisely the meaning that this compactification is bigger than that compactification, $(E, \overline{E(X)})$ is bigger than $(\eta, Y)$. So, there is just a restatement of the previous second part of the previous theorem you may say. But that has more content than this one, this is just a consequence of that theorem.

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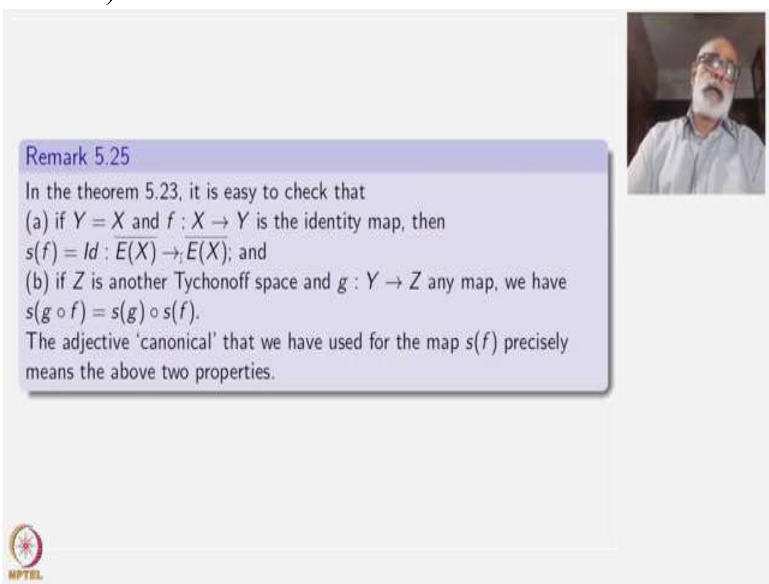


So, I make a remark here. In theorem 5.23, it is easy to check that if $Y$ is $X$ itself and $f$ from $X$ to $Y$ is identity map then this $s(f)$ is nothing but the identity map of $\overline{E(X)}$ to $\overline{E(X)}$. The second comment: if $Z$ is another Tychonoff space, $g$ from $Y$ to $Z$ another continuous map then you can look at $s(g \circ f)$. The map $g \circ f$ will be from where to where? From $X$ to $Z$, so $s(g \circ f)$ nothing but $s(g) \circ s(f)$ from $\overline{E(X)}$ to $\overline{E(Z)}$. So, in between you have this $\overline{E(Y)}$, so,

$\overline{E(X)}$ to $\overline{E(Y)}$ to $\overline{E(Z)}$, you have $s(g) \circ s(f)$. So, I will leave verification of this to you. It is just purely set theoretic verification to you.

The importance of these two properties is that these two properties actually have a very good name. They are called canonical properties or functoriality properties. The adjective `canonical' that we are using for the map $s(f)$ is precisely for this reason.

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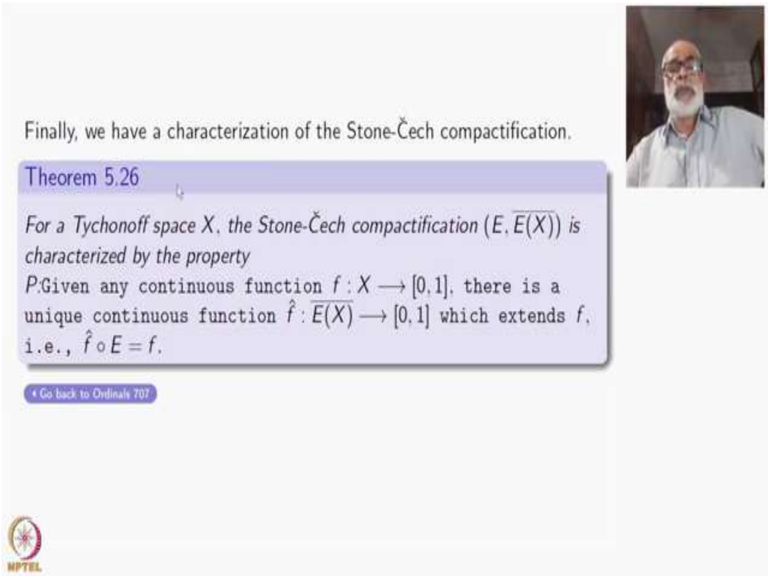


Finally, we shall end this topic today by giving a characterization of Stone-Cech compactification. You can also call this the universal property of the Stone-Cech Compacgtification. For a Tychonoff space $X$, Stone-Cech compactification $(E, \overline{E(X)})$ is characterized by the following property $P$: given any continuous function $f$ from $X$ to the closed interval $[0,1]$, there is a unique continuous function $\hat{f}$ from $\overline{E(X)}$ to $[0,1]$ which extends $f$. Again `extends $f$' in the sense that $\hat{f} \circ E$ is $f$.

In the theorem, we have prove that the Stone-Cech compactification has this stronger property, viz., instead of taking $[0,1]$ we could take any compact space. That is the second part of previous theorem. In the characterization, we are restricting it to only $[0,1]$, the co-domain is always just $[0,1]$. So, that is the beauty of this characterization.

The characterization has its own use. Often, you do not have to use the fact that $\overline{E(X)}$ is sitting in that product space. Remember that $\overline{E(X)}$ is taken as a subspace of certain product space. You do not have to use that product structure or anything you can just use this property $P$. So that will be automatically give you many properties of Stone-Cech compactification.

Take any compactification of $(\eta, Y)$ of a Tychonoff space $X$, which has this property $P$; it has to be equivalent to Stone-Cech compactification. That is the whole idea. So, proof is not all that difficult.

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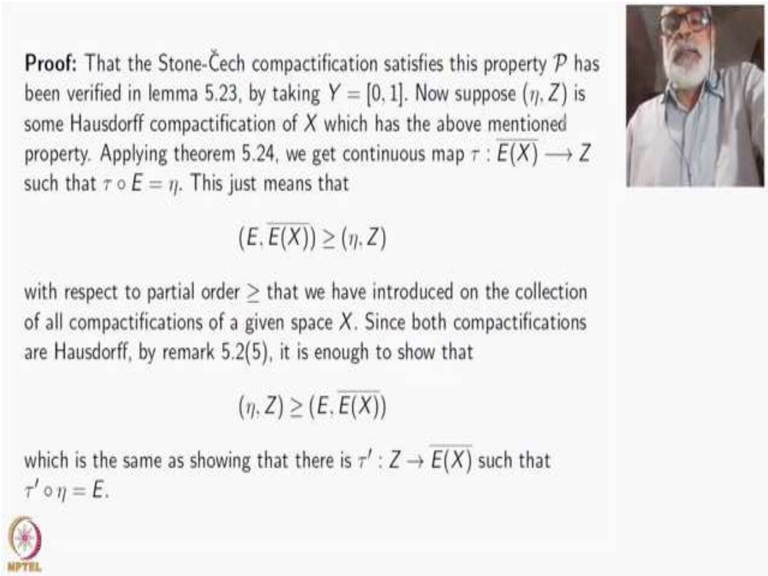


First of all, the Stone-Cech compactification itself has this property. As a special case of 5.23 which I have just told you, by taking $Y = [0, 1]$. Now, I want to prove the converse. Suppose $(\eta, Z)$ is some Hausdroff compactification of $X$, which has the above mentioned property $P$.

Applying 5.24, we get a continuous tau from $\overline{E(X)}$ to $Z$ such that $\tau \circ E = \eta$. So, this just means that $(E, \overline{E(X)})$ is bigger than or equal to $(\eta, Z)$. You have already proved it. I am just repeating this part. Stone-Cech compactification is the largest. So, what we have to prove is this $(\eta, Z)$ is all the larger than $(E, \overline{E(X)})$. Then because all are Hausdorff spaces, as seen before, the two will be equivalent. So, it remains to prove that $(\eta, Z)$ is bigger than equal to $(E, \overline{E(X)})$ which is the same as finding a map tau prime from $Z$ to $\overline{E(X)}$ such that $\tau' \circ \eta = E$. So, we have to reverse the arrows here. This also not very difficult, but it is something new. So that is why I have to give you a complete proof.

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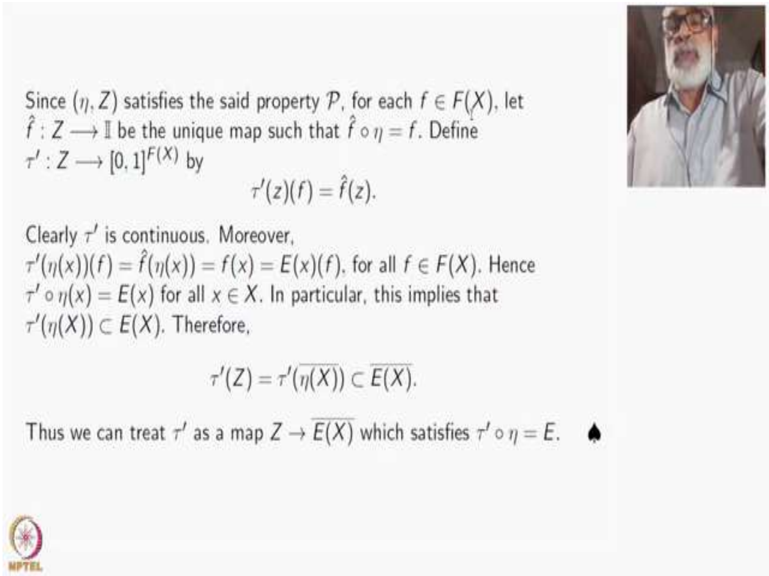


So, what is the assumption on $(\eta, Z)$? It satisfies the said property $P$. Therefore, for each $f$ inside $F(X)$, what is $F(X)$? Space of all continuous functions from $X$ to the closed interval $[0,1]$, let us take $\hat{f}$ from $Z$ to $\mathbb{I}$, the unique map such as $\hat{f} \circ \eta = f$. So, this is the property $P$.

Once you have these $\hat{f}$s, define $\tau'$ from $Z$ to the product space $[0,1]^{F(X)}$ by the formula $\tau'(z)$ is that element which has its $f$-th coordinate equal to $\hat{f}(z)$. So, the $\tau'(z)$ is defined by this equation. So that defines a point here for each $z$. I want to say that first of all this is $\tau'$ continuous. Why? Because its $f$-th coordinate is $\hat{f}$. So, each coordinate function is continuous. Therefore, the function $\tau'$ is continuous.

So, continuity of this function is fine. Moreover, take $\eta(x)$ and look at $\tau'(\eta(x))$. What is it? Operating upon $f$, it is $\hat{f}(\eta(x))$ is nothing but $f(x)$. But, what is $f(x)$? It is $E(x)(f)$ by definition of evaluation map. This is true for every $f$. Therefore, $\tau'(\eta(x))$ is nothing but $E(x)$. So that is true for every $x$. It just means that $\tau' \circ \eta$ is $E$. In particular, this implies that $(\tau' \circ \eta)(X)$ is contained inside $E(X)$. Therefore, $\tau'(Z)$ (what is $Z$? $\overline{\eta(X)}$) is the same as $\overline{\tau'(\eta(X))}$. So that is contained inside the closure of $\tau'(\eta(X))$, closure can be pulled out. So, but that is contained in $\overline{E(X)}$.

So, tau prime factors through $\overline{E(X)}$. Just starting with a map from $Z$ to $[0,1]^{F(X)}$, we show that it is actually taking values inside the Stone-Cech compactification. So, take that map tau prime as a function from $Z$ into $\overline{E(X)}$. So, we have already shown that $\tau' \circ \eta = E$. Of course that completes the proof of the characterization also.

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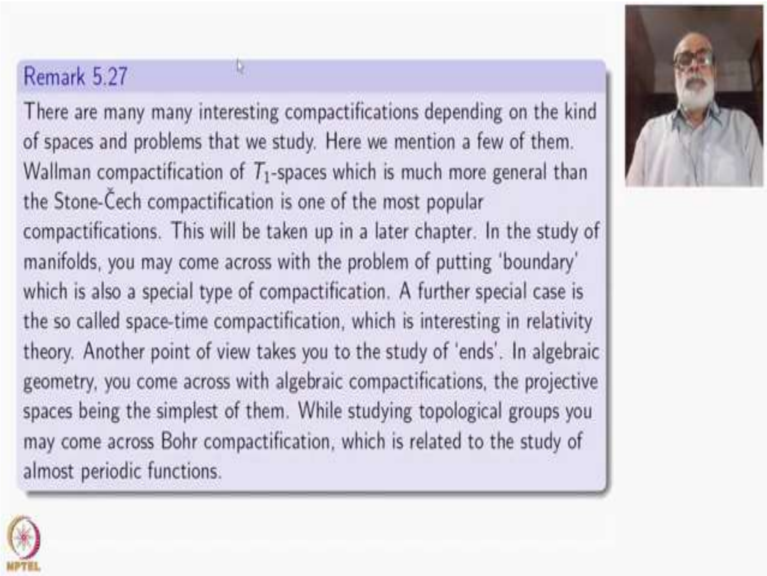


So, I will end this talk today with a general remark now. There are many many interesting compactifications depending on the kind of spaces and the kind of problems that we are studying, the problems whatever you are interested in. So, for each kind of problems, there may be some compactifications to consider, so as to simplify the problem and try to get the answers and then come back and so on. That is the game.

Here we mentioned a few of them, other than the Alexandroff's one point compactification and Stone-Cech compactification. The smallest one and the largest ones we have discussed. So, Wallman compactification is another important one which is much more general than these two compactification. It works for all $T_1$ spaces. Stone-Cech compactification is one of the most popular compactifications. But, Wallman compactification is also equally popular. That is what I wanted to say. The study of this will be taken in a later chapter.

In the study of manifolds you may come across with problems of putting a boundary to the manifold. Since you do not know much about manifolds I cannot explain this more than this. This is only for information, which if you remember this is what was told to you, that will suffice. So, `putting a boundary' is actually some kind of a compactification. A further special case is the so called space-time compactification which is interesting in the relativity theory. Another point of view of this takes you to the study of `ends'. For example, you will see that the real number system, the real line has two ends, whereas the complex plane has only one end. I do not want to elaborate anything more than that here. These are all, you know, part and parcel of various types of compactifications.

In algebraic geometry, you come across with algebraic compactifications. Projective spaces are standard examples an algebraic compactification of the affine space $\mathbb{C}^{\mathbb{N}}$. $\mathbb{C}^{\mathbb{N}}$ has may more algebraic compactifications.

While studying topological groups, you may come across what is called Bohr compactification. Maybe when you compactify you would like retain the topological group structure itself there see the group structure. That should also extend and so on. So, this is related to the study of what is called almost periodic functions. So, when you are studying that you will come across Bohr compactifications. With these many little bit of remarks, let us end today's talk. So, next time we will take up a different topic. Thank you.