An Introduction to Point - Set - Toplogy (Part II)
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Lecture No: 19
Alexandroff's compactification

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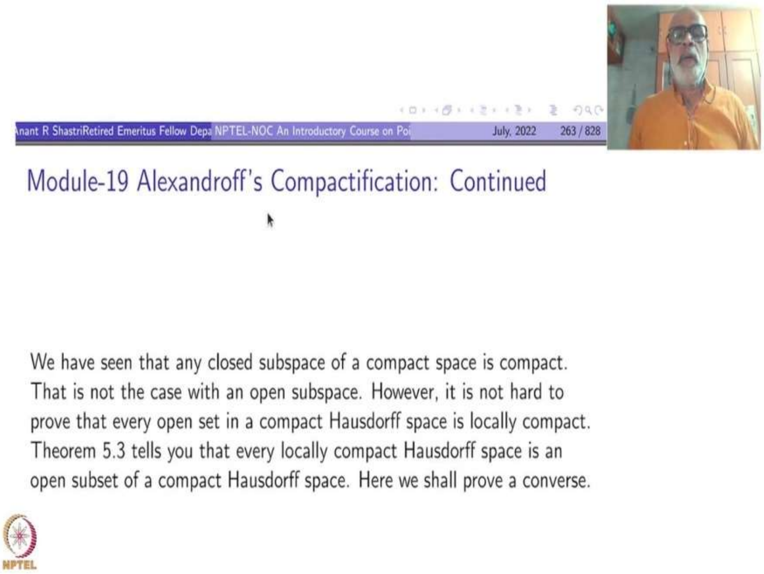

Hello welcome to NPTEL NOC on introductory course on Point-Set-Topology part II. So, we continue with the study of compactification, today module 19, Alexandroff's compactification continued. Last time we introduced this special one point compactification of a locally compact Hausdorff space which will be automatically compact Hausdorff space that is what we have seen.

We have seen that any closed subspace of a compact space is compact. This is not the case with an open subspace. However, it is not hard to prove that every open set in a compact Hausdorff space is locally compact. This you can do directly. Whatever theorem we proved tells you that every locally compact Hausdorff space is an open subset of a compact Hausdorff space. That is a corollary to the Alexandroff's compactification that we produced the other day. Here we shall prove the converse now.

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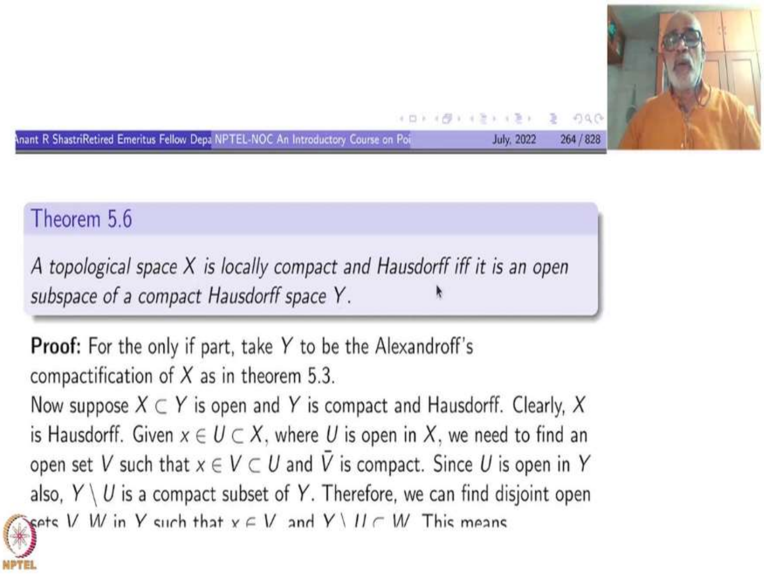

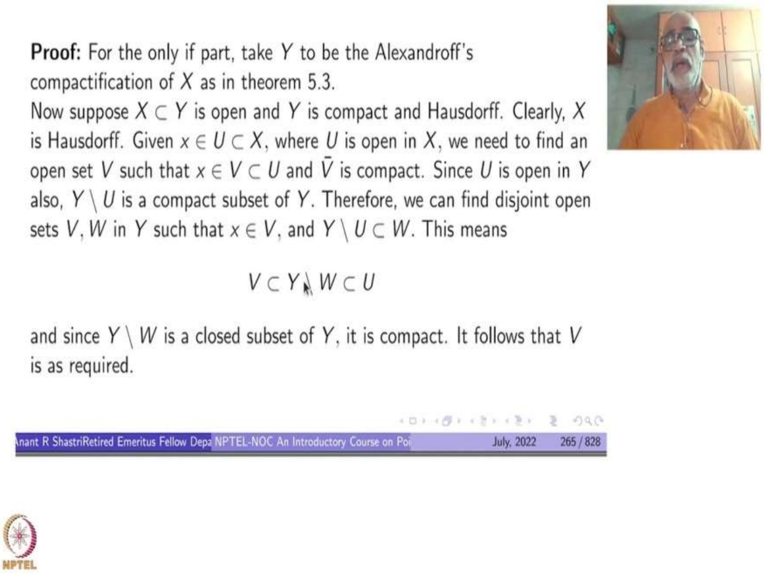

A topological space X is locally compact and Hausdorff if and only if it is an open subspace of a compact Hausdorff space. So, this is what we want to proof. The `only if' part all that you have to do is to take one point compactification namely Alexandroff's compactification of $X$ as in the previous theorem. Now, let us come to the converse part, `if' part.

Suppose X is an open subspace of a compact and Hausdorff space Y. Clearly X is Hausdorff; every subspace of a Hausdorff space is Hausdoff. So, we are only left with proving local compactness. So, given $x \in X$ belonging to an open set U in X, we need to find an open subset V such that x is inside V, $\bar{V}$ is contained inside U and $\bar{V}$ compact. Since U is open in Y , $Y \setminus U$ is a closed and hence a compact subset of Y.

Therefore, we can find disjoint open subsets V and W in X such that x in V and $Y \setminus U$ is contained inside W. Here I am using that Y is compact Hausdorff space and hence is regular. This means that this V is contained is a $Y \setminus W$ which is contained inside of U because V and W are disjoint open subsets. Since $Y \setminus W$ is a closed subset of Y, because W is open, so, $Y \setminus W$ is compact. Therefore, $\bar{V}$ will be compact. It follows that V is as required. That is all.

So, it is easy the part but we wanted to record this one, viz., what happens to an open subsets of a compact Hausdorff space and vice versa.

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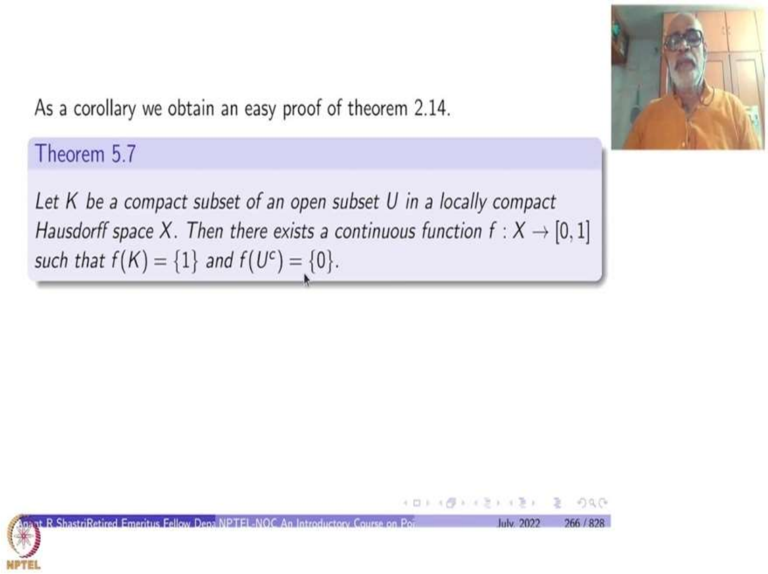

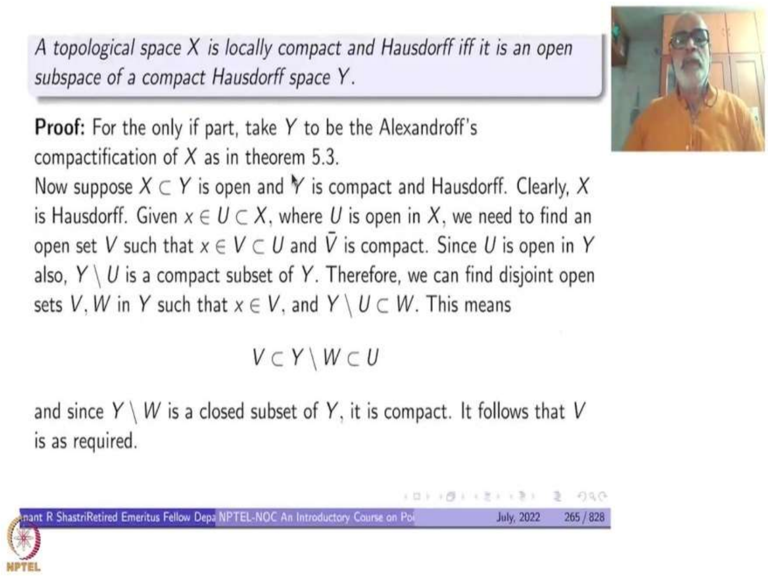

Here is a corollary to this theorem. Let K be a compact subset of an open set U in a locally compact Hausdorff space. Then there exists a continuous function f from X to $[0,1]$ such that $f(K)$ equals $\{1\}$ and $f(U^c)$ equals $\{0\}$.

This theorem we have proved earlier. At that time also I had indicated that we will have a different proof of it.

Now, this becomes an easy consequence. Because starting with a K compact subset of an open subset U of a locally compact Hausdorff space, (K contained inside locally compact Hausdorff space is the situation but you pass on to one point compactification, you are transferring the whole situation into a compact Hausdorff space, then this becomes an easy corollary that is what it is.

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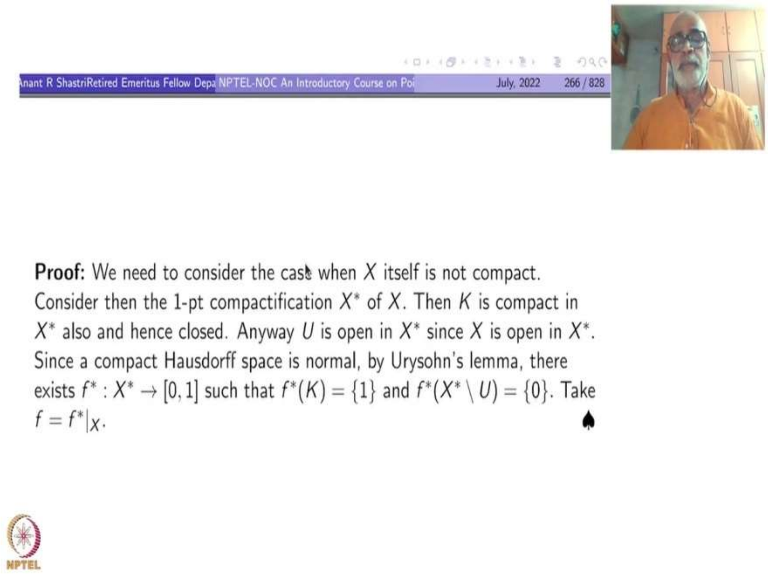

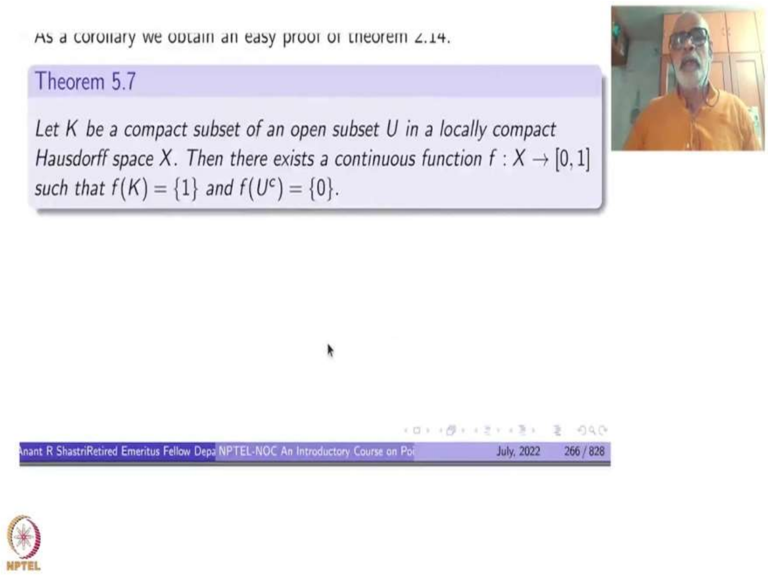

So, I repeat we need to consider the case when X itself is compact. If X itself is compact, then there is nothing to prove. Otherwise, consider the one-point-compactification X^* of X (which is actually the Alexandroff's compactification). Then K is compact in X^* also because K is already compact subset of X, and hence closed in X^* because X^* is Hausdorff. Anyway,

U is open in X^*, since X is open in X^*. Since a compact Hausdorff space is normal, by Urysohn's lemma we will get the required function f.

So, instead of local compactness, we are able to convert the situation into compact space, by compactification.

So, this was one of the reasons, I had told you why people study compact spaces and compactifications. The things which cannot be solved inside an arbitrary space you can solve it by going to the compact space and then solve it there and come back. So, this is just a small illustration of that. In any case, this theorem itself we have proved directly also in to 2.14.

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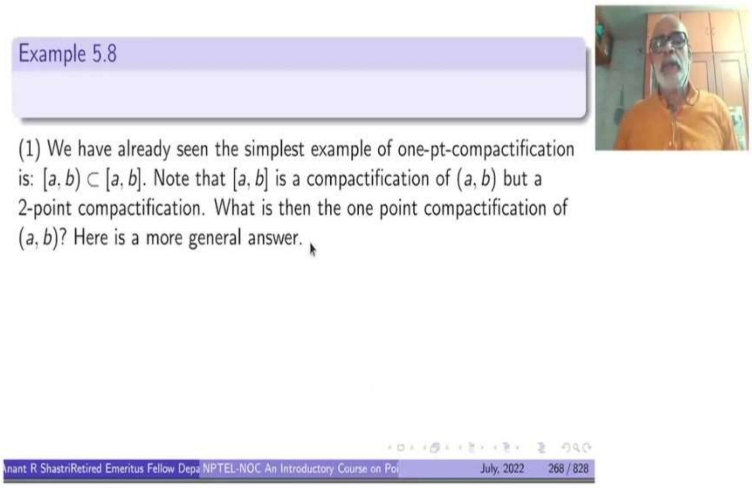

So, let us examine a few simple examples. First, we have already seen that the simplest example of one-point-compactification is any half open interval contained in the closed interval. Note that $[a, b]$ is a compactification of the full open interval (a, b) also but that is 2-point compactification. It is not one-point compactification. What is then the one-point-compactification of this open interval? In order to answer this, we will actually answer a much more general question. Remember any open interval is $\mathbb{R}$ is homeomorphic to $\mathbb{R}$ itself. So, in general, what I am asking is what are the compactifications of $\mathbb{R}^n$, one-point-computation in particular. Since $\mathbb{R}^n$'s are locally compact Hausdorff, we are actually talking about Alexandroff's compactification here.

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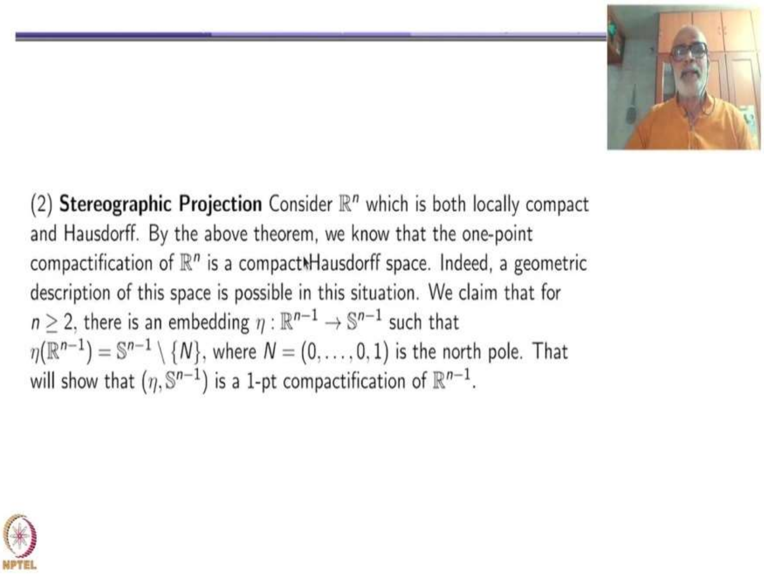

So, let us have a clear geometric understanding of these compactifications. Consider $\mathbb{R}^n$ which is both locally compact and Hausdorff. By the above theorem, we know that Alexadroff's one-point-compactification of $\mathbb{R}^n$ is a compact Hausdorff space. Indeed, a geometric description of this space is possible in this situation.

We claim that for n greater than equal to 2, there is an embedding η of $\mathbb{R}^{n-1}$ into $\mathbb{S}^{n-1}$ such that the image is an open set with its complement being a single point. I have chosen a very specific point here namely the north pole $N = (0, 0, \ldots, 0, 1)$ the last coordinate 1. It is usually called the north pole. The point S on $\mathbb{S}^{n-1}$ with its last coordinate equal to -1 is called the south pole.

Clearly, $\mathbb{S}^{n-1} \setminus \{N\}$ is an open dense subset of $\mathbb{S}^{n-1}$. This is an open subset. So, all that we have to do is to construct this η. Later, we will actually show that $(\eta, \mathbb{S}^{n-1})$ is the Alexandroff's compactification.

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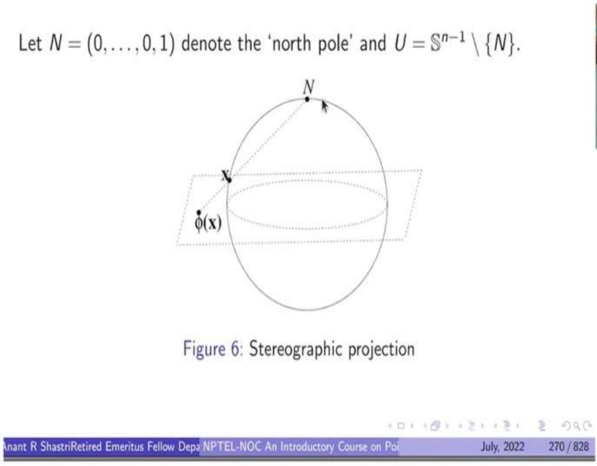

So, recall some notation here. $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$: all points $x = (x_1, \ldots, x_n)$ such that $\sum x_i^2 = 1$. And N is a north pole and U is the complement of the singleton $\{N\}$. So, what I am going to do here? Geometrically I will explain what I am doing here. Look at this North pole N. Take any point x of $\mathbb{S}^{n-1}$ other than N, that is a point of U. It just means that this line segment $[x, N]$ extended to a full line, call it L_x, will hit the subspace $\mathbb{R}^{n-1} \times 0$, i.e., the subspace of all points with their n^th coordinate being 0. Why? Because this line will never be parallel to $\mathbb{R}^{n-1} \times 0$, because the n^th coordinate of x will be less than 1. So the extended line will hit $\mathbb{R}^{n-1} \times 0$ exactly at one point.

So, this is clear from this picture. Denote the point of intersection of the line and $\mathbb{R}^{n-1} \times 0$ by $\phi(x)$. Then ϕ defines a mapping of U onto $\mathbb{R}^{n-1} \times 0$. It is called the stereo graphic projection from N. (Instead the north pole, you can take any point y on $\mathbb{S}^{n-1}$ and define a stereograpphic projection onto the hyperplane perpendicular to the vector y.)

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Given any point $x \in U$, there is a unique line L_x passing through x and N and this line is not parallel to the hyperplane $\mathbb{R}^{n-1} \times \{0\}$. Therefore L_x intersects $\mathbb{R}^{n-1} \times \{0\}$ in a unique point which we shall denote by $\phi(x)$. The function $\phi : U \to \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\}$ is called a stereographic projection. Let us compute this explicitly.
The line L_x can be parameterized as $tx + (1-t)N, t \in \mathbb{R}$. We want to find the value of t for which this point belongs to $\mathbb{R}^{n-1} \times \{0\}$. Therefore, we put the last coordinate equal to zero to obtain the equation $tx_n + 1 - t = 0$, i.e., $t = \frac{1}{1-x_n}$. (Note that $x \neq N$ implies $x_n \neq 1$ and hence this makes sense.)

Let $N = (0, \ldots, 0, 1)$ denote the 'north pole' and $U = \mathbb{S}^{n-1} \setminus \{N\}$.

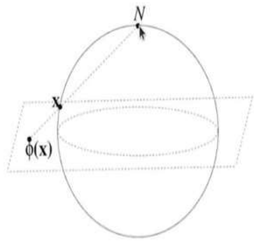

Figure 6: Stereographic projection

So, let us now work out a formula for ϕ. So, how to write down the formula? You start writing down the parameterization of the line passing through x and N, specialize to the case when the last coordinate becomes zero, that is all.

So, given any two distinct points in $\mathbb{R}^n$, t times this into $1-t$ times that will give you the entire line as t ranging from $-\infty$ to plus ∞.

So, here specifically we have, $tx + (1-t)N$, as t varies inside $\mathbb{R}$, gives you all the points on the line. We want this point to have its n-th coordinate 0. That is the point of intersection of this line L_x and the plane $\mathbb{R}^{n-1} \times 0$. So, what is the n^th coordinate of this? It is $tx_n + (1-t)$. n^th coordinate of N is 1 here is multiplied by $1-t$, so $tx_n + (1-t) = 0$. That will give you a unique t which is equal to $1/(1-x_n)$. Remember this $x_n \neq 1$. That is very important here. So, this is a valid solution. Go back here, put t equal to $1/(1-x_n)$, in the general formula for the points of L_x, you precisely get $\phi(x)$.

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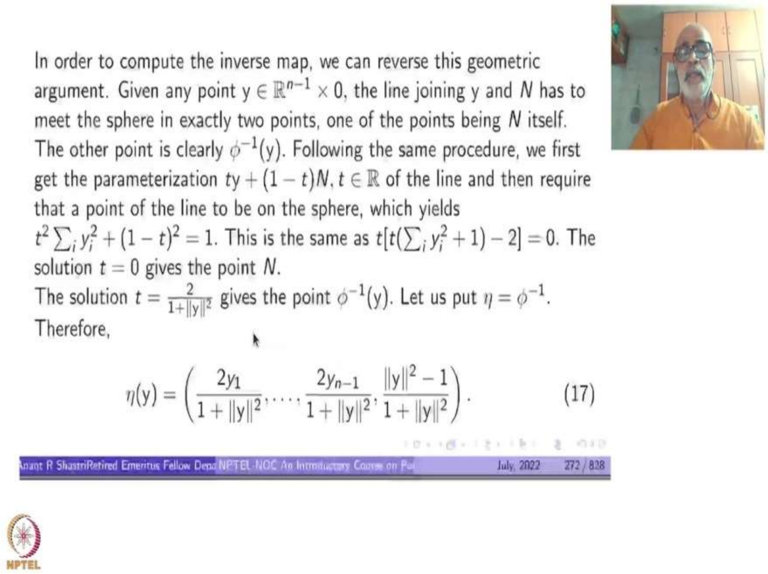

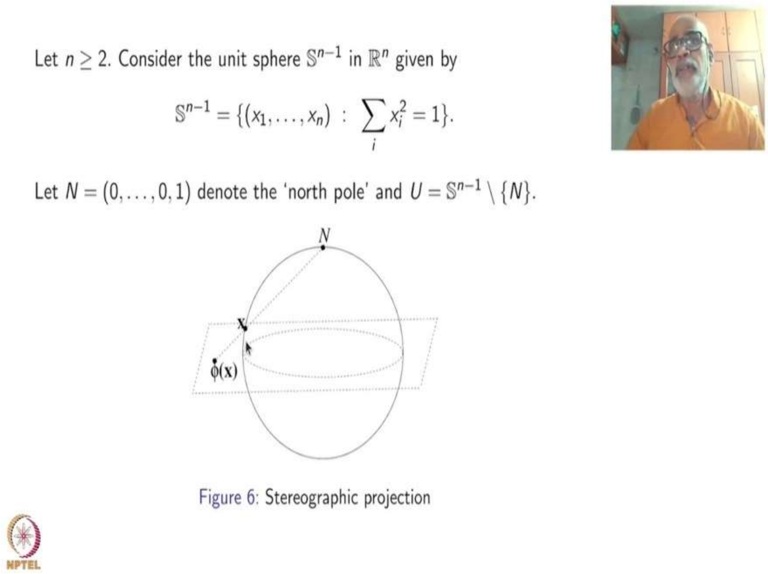

Therefore, $\phi(x)$ has the first coordinate divided by $1-x_n$, second coordinate divided by $1-x_n$ and so on. We can ignore the last coordinate which zero and write ϕ as a function from U to $\mathbb{R}^{n-1}$.

In order to compute the inverse map, we can reverse this geometric argument. So, go back here, start with the point y on this hyperplane $\mathbb{R}^{n-1} \times 0$. How to get $\phi^{-1}(y)$? Very easy, namely, join $y = \phi(x)$ and N, the line intersects the sphere $\mathbb{S}^{n-1}$ exactly in one point other than N and that is proecisely the point x. Note N is already a point on the line and the sphere. So to obtain this point x? In the parametetric for of the line you have to put the condition that the point x whatever we want lies on the sphere, summation x_i^2 is 1. So that will lead to a quadratic equation in t giving the two solutions. We already know one solution N. So, the other solution is very easy to determine. So, that is the geometric way of describing the inverse of ϕ as well as getting a formula.

So again, I am taking y is an element of $\mathbb{R}^{n-1} \times 0$ as thought of as an n^th vector with the last coordinate being 0. $ty + (1-t)N$ where t runs over $\mathbb{R}$ is the line. When you take coordinates, take their square take their sum, and equate it to 1. That is the condition that I want this point to be on the sphere. That gives you $t^2 \sum y_i + (1-t)^2 = 1$. Note that the n ^th coordinate of y is zero whereas all other coordinates of N are 0.

The constant term cancel out. So, you get $t\|y\|^2 + 1 - 2$ and the entire thing mutliplied by t equal to zero. So, the solution corresponding to t equal to 0 is the point N. So, I do not want that. The other solution I want, namely cut down this t what you get is t equal to 2 divided by $1 + \|y\|^2$.

For this value of t, if you plug in here, what you get is the point x where y is $\phi(x)$. So, this is $\phi^{-1}(y)$. So, I am denoting it by η. This is a notation now. $\eta(y) = 2y_1/(1 + \|y\|^2), \ldots,$ etc. $2y_{n-1}/(1 + \|y]\|^2)$ and the n^{th} coordinate is $1 - t$ which is equal to $\|y\|^2 - 1/(\|y\|^2 + 1)$.

Geometrically there is no need to verify that these two function ϕ and η are inverses of each other. However, if you are not satisfied, you can plug this formula for x inside here and see that $\phi \circ \theta$ as well as $\eta \circ \phi$ will be identity maps.

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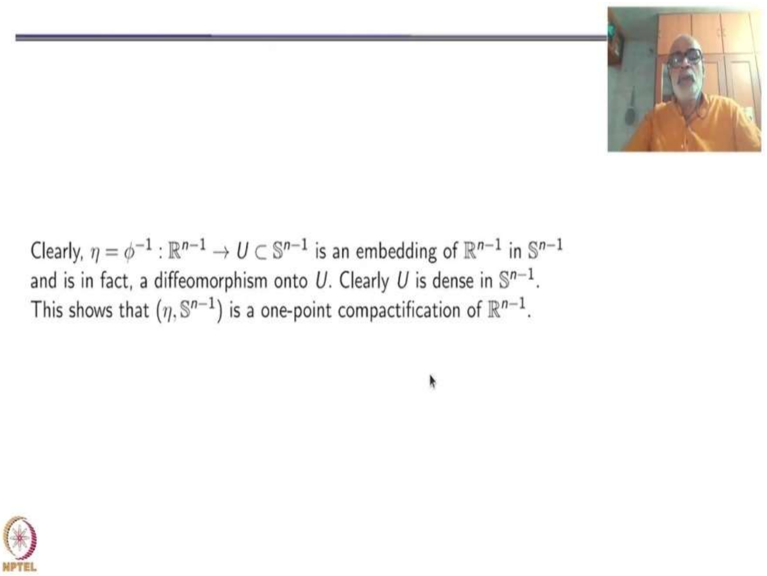

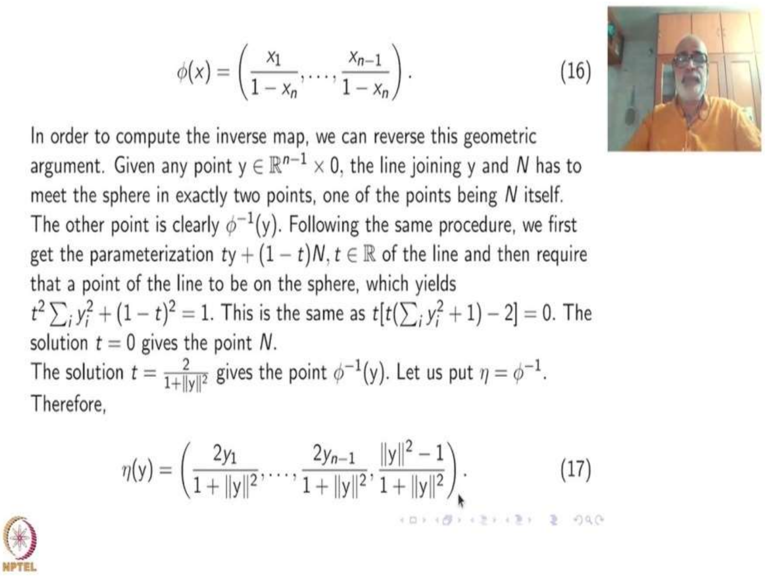

So, look at the formula, they are not only continuous, they are actually differentiable as many times as you want. Therefore, what we have is that both η and ϕ are diffeomorphisms in their respective domain. In particular, η from $\mathbb{R}^{n-1}$ to U is a diffeomorphism onto U, so, it is an embedding of $\mathbb{R}^{n-1}$ in $\mathbb{S}^{n-1}$. Thus what we have proved so far, is that $(\eta, \mathbb{S}^{n-1})$ is a one-point-compactification of $\mathbb{R}^{n-1}$.

So, let us also verify that this one-point-compactification is actually Alexandroff's compactification. There are several ways of doing it.

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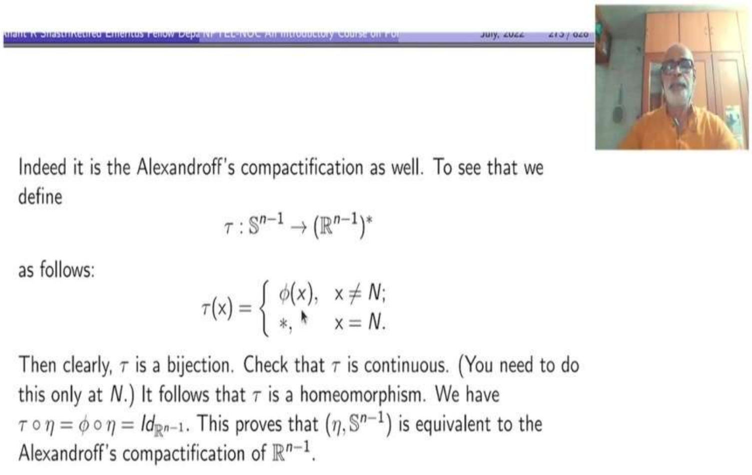

One simple way is to get a homeomorphism τ from $\mathbb{S}^{n-1}$ to $(\mathbb{R}^{n-1})^*$, the this Alexandroff's compactification which preserves the two embeddings. All that I do is to take this ϕ which is already defined on this open subset of $\mathbb{S}^{n-1}$ to equal to τ and then extend this to τ by sending the point N to the point infinity in $(\mathbb{R}^{n-1})^*$, or star whatever the notation is.

Whenever x is equal not to N, $\tau(x)$ is equal to $\phi(x)$. So, ϕ has been extended to the function tau on the whole of $\mathbb{S}^n$. So, this function τ is well defined and is a bijection, no problem there. The only thing we need to verify is continuity of τ at N, which is very easy. I will leave it to you as an exercise. Once you check τ is continuous, it follows that this τ is a homeomorphism because it is a bijection from a compact space $\mathbb{S}^{n-1}$ to a Hausdorff space $(\mathbb{R}^{n-1})^*$. Moreover, $\tau \circ \eta$ is what? $\eta(y)$ is inside U and therefore, $\tau(\eta(y)) = \phi(\eta(y))$. But ϕ is the inverse of η and so that is equal to y. So $\tau \circ \eta$ is identity on $\mathbb{R}^{n-1}$.

So, this proves that $(\eta, \mathbb{S}^{n-1})$ is equivalent to the Alexandroff's compactification of $\mathbb{R}^{n-1}$.

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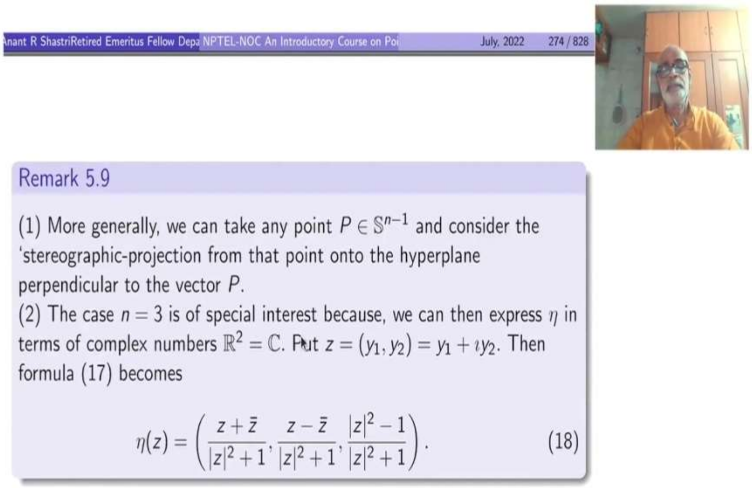

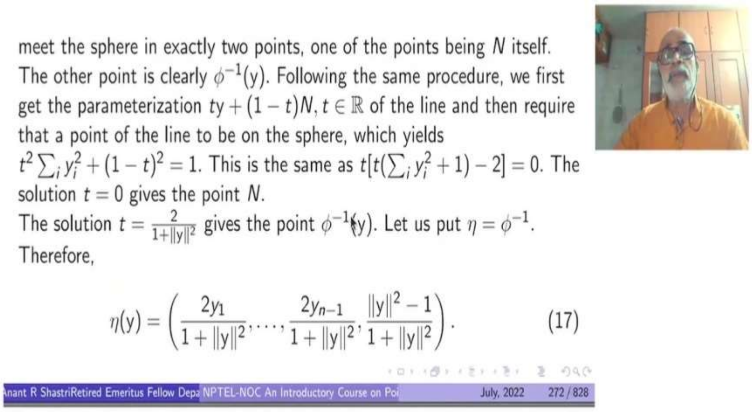

More generally, instead of taking the North Pole you could have taken any point P on $\mathbb{S}^{n-1}$. The geometric argument will be exactly the same. The only thing is now you should not take $\mathbb{R}^{n-1} \times 0$ but you should take the vector P and take the subspace which is perpendicular to this P to project the entire of $\mathbb{S}^{n-1} \setminus \{P\}$. That is all the modification. However, the formulae will get more complicated, though the geometric argument is as good as in this special case, there is no change there. So, all of them would have given you embeddings of $\mathbb{R}^{n-1}$ inside $\mathbb{S}^{n-1}$.

The case $n = 3$ is of special interest, because, we can then express eta in terms of complex numbers also. First identify $\mathbb{R}^2$ with complex numbers $\mathbb{C}$. Then η can be written in a different way in terms of z. Suppose $z = (y_1, y_2)$ or you write $z = y_1 + iy_2$, then our formula for η will be much simpler. Actually all this norm etc can be written nicely, $\eta(z)$ is equal to $(z+\bar{z})/(|z|^2+1), (z-zbar)/(|z|^2+1), (|z|^2-1)/(|z|^2+1))$.

So, we are working now with $n = 3$. So, this picture will actually give you the so called extended complex plane. $\mathbb{C}$ union infinity, will be identified by this map with unit sphere inside $\mathbb{R}^3$.

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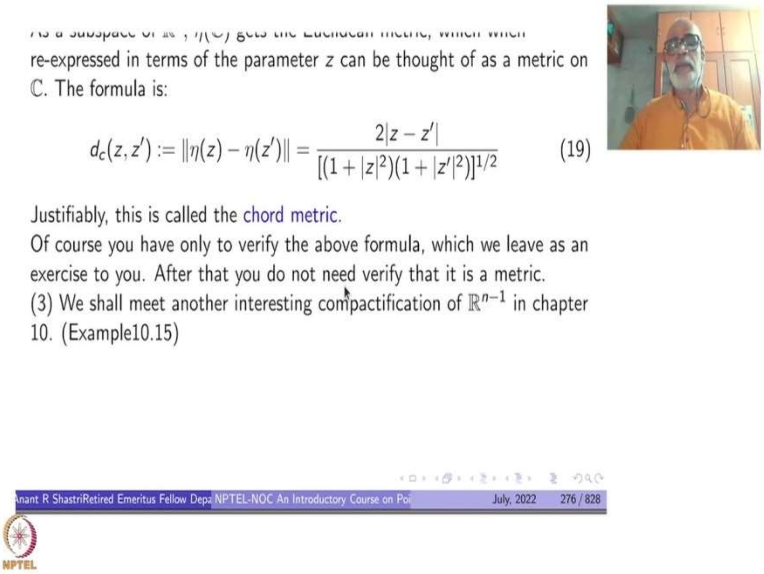

As a subspace of $\mathbb{R}^3$, $\eta(\mathbb{C})$ gets the Euclidean metric from $\mathbb{R}^3$ which when expressed in terms of the parameter z can be thought of as a metric on $\mathbb{C}$ itself. So, you are metrizing the complex numbers in a different way. So, what is the formula? So I am writing this new distance function as $d_c(z,z')$ equal to $\|\eta(z)-\eta(z')\|$. If you compute it using the formula for η, it will be equal to$2|z-z'|/[(1+|z|^2)(1+|z'|^2)]^{1/2}$. So this is called chord-metric and that is why I put a suffix `c' here.

So, when you take two points of $\mathbb{C}$ sitting inside $\mathbb{S}^2$, all that you have to do is to look at the chord $[z,z']$ and its length. That is $d_C(z,z')$. So, it is called the chord metric. In particular, this metric on $\mathbb{C}$ is a bounded metric. Topologically it will give you the same space.

The same thing you can do for any $\mathbb{R}^n$, but you will not get this nice formula. Because here we have used specific algebra of complex numbers, that is all.

We shall meet another interesting compactification of $\mathbb{R}^{n-1}$ in chapter 10, namely, example 10.15. There are, as I have told you, many compactifications of a given non compact space. So, it is not possible to discuss all of them.

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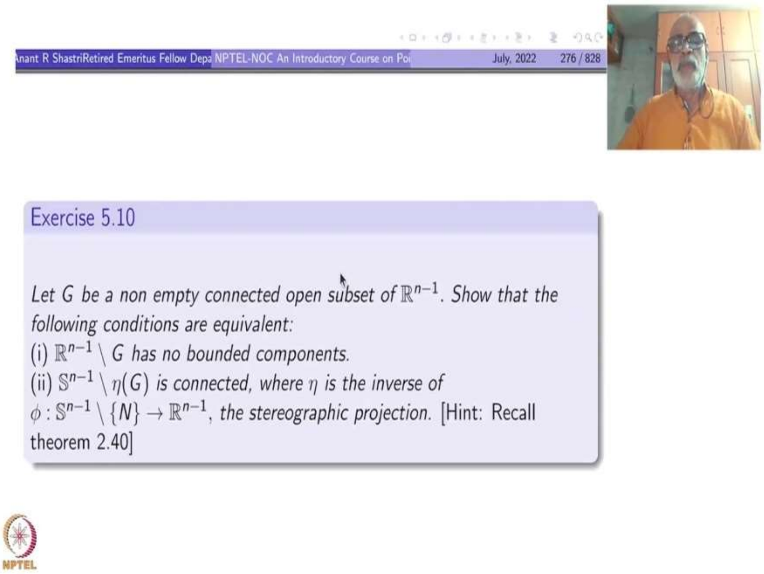

So, here is an exercise. The hint is that I want to draw your attention to namely, we proved a big theorem about locally compact Hausdorff spaces in 2.40 about the neighbourhood of connected components of such a space. So, if you use that one, you can solve this problem.

And when $n = 3$, this has a special significance, in the case of complex numbers. So, it will give you a characterization of simply connected domains in $\mathbb{C}$ purely in terms of $\mathbb{C}$ itself, viz., without going to the extended complex plane. Namely, a domain G in $\mathbb{C}$ is simply connected iff $\mathbb{C} \setminus G$ has no bounded components. So, this is the characterization which is purely in terms of the topology of $\mathbb{C}$. So, solve this exercise and enjoy it. Thank you. Next time we will study proper maps.