An Introduction to Point Set Topology (Part 2)
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Lecture No: 18
Generalities on Compactification

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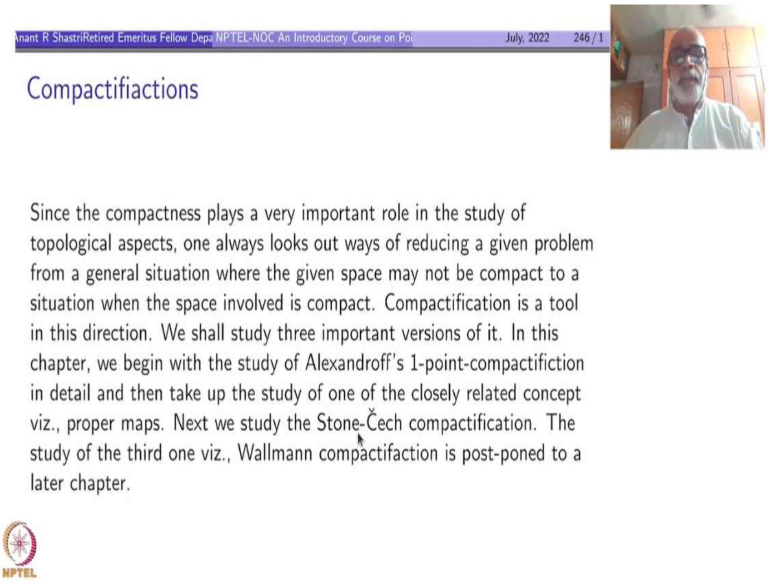

Hello. Welcome to NPTEL NOC an introductory course on point set topology Part II. So, our next chapter is Compactifications. So, compactness plays a very central role, an important role in the study of topological aspects. One always looks out ways of reducing a given problem from a general situation where the given space may not be compact to a situation when the space involved is compact. Compactification is a tool in this direction.

We shall study three important versions of it. In this chapter, we begin with the study of Alexandroff's 1-point-compactification in full detail. Then we take up one of the closely related concept namely proper maps. After that we will study Stone-Cech compactification. The third one is-- see I have mentioned three of them third one which is called Wallmann compactification will be taken later on. It will not be taken in this chapter because that requires some other notions to be developed.

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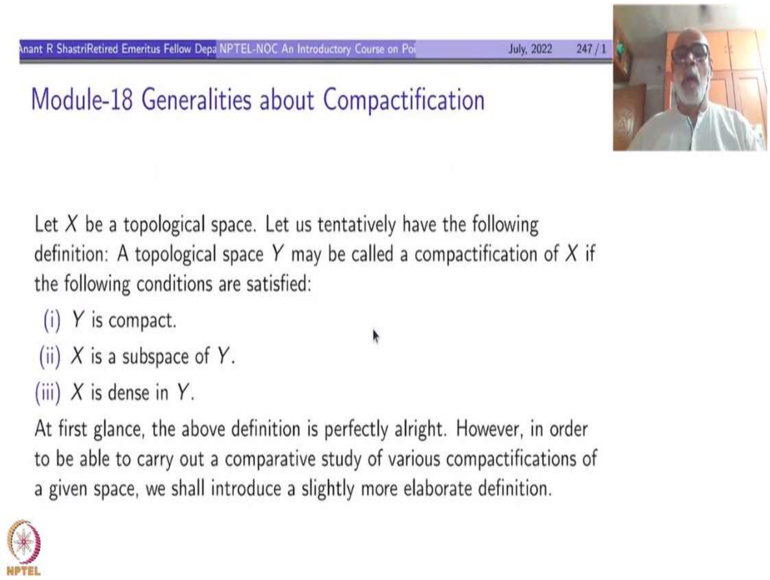

So, welcome to Module 18. First we shall discuss some generalities about compactification. Let X be a topological space. Let us tentatively have the following definition. A topological space Y maybe called a compactification of X if the following conditions are satisfied. The space Y must be compact first of all. That is the first thing.

X must be a subspace of Y. So given space X is an enlarged to another space Y which is compact. So, that is what we want. How mcuh you want to enlarge? Not too large X must be dense in Y. So, these are the three requirements we would like to have. Then you can call Y as a compactification of X.

At the first glance, the above definition is perfectly alright. However, in order to be able to carry out a comparative study of various compactifications and so on, we shall introduce slightly more elaborate definition here.

So, technically we have to be a little more precise, but idea-wise we have to keep just remembering the above these three things. For a compactification these three things are fine, but technically we will put it slightly differently.

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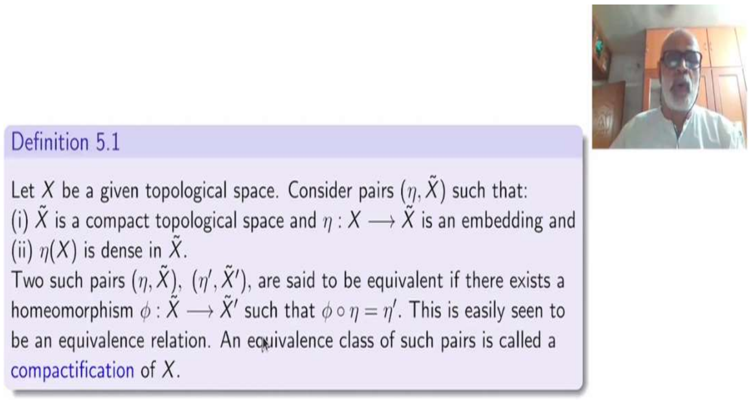

So, here is a technical definition. Start with a topological space X. Consider an ordered pair $(\eta, \tilde{X})$, or say, (η, Y). So, what are these? First of all $\tilde{X}$ is a compact topological space and η is a map from X to $\tilde{X}$ which is an embedding. So, you can use eta to identify X with a subspace of $\tilde{X}$. The second condition is that this subspace $\eta(X)$ is dense in $\tilde{X}$.

So, this way, we are taking care of all the three aspects that we wanted, namely, this Y which has been written as $\tilde{X}$ here now is compact and eta from X to $\tilde{X}$ is an embedding which makes X into a subspace of $\tilde{X}$ and the third condition is that the subspace $\eta(X)$ is dense in $\tilde{X}$.

Now you look at two such pairs $(\eta, \tilde{X})$ and $(\eta', \tilde{X}')$. Suppose both of them are compactifications of X, as defined above.

We will say they are equivalent if there is a homomorphism ϕ from $\tilde{X}$ to $\tilde{X}'$, such that (the subspace $\eta(X)$ and $\eta'(X)$ are the same). $\phi \circ \eta$ is equal to η'. It is not an arbitrary homeomorphism. The condition $\phi \circ \eta$ equal to η' is very important.

So, check that the above relation is an equivalence relation. Every $(\eta, \tilde{X})$ is automatically equivalent to itself; reflexivity is fine. Symmetry is built is the definition, because ϕ is a map from $\tilde{X}$ to $\tilde{X}'$ which is a homeomorphism. and so, ϕ^{-1} will be homeomorphism which will satisfied this property $\phi^{-1} \circ \eta'$ is equal to η. Symmetry is fine. Similarly, transitivity is also easy to verify. So, this is an equivalence relation. So, if you look at the collection of all compactifications of a given space X then the above relation is an equivalence relation on it. And an equivalence class of such pairs is called a compactification of X.

So, this is the final definition. So, what we have brought here is not just some space, not just the embedding, but an equivalence class of them that is one single compactification. In other words, when you are talking about a topological space you have the habit of identifying all spaces which are homomorphic to that space as equal to that, equality means homeomorphic. In the same sense, all compactifications of X, X is fixed here which are equivalent each other in the above sense are treated as one single object.

In practice, whenever you are talking about a class you are always picking up a representative pair of that class, any one member in that class and then you call that itself as the compactification. So, that is the practice that we are following.

So, this much of liberty of language we are taking. But for the rigor, we will stick with above definition.

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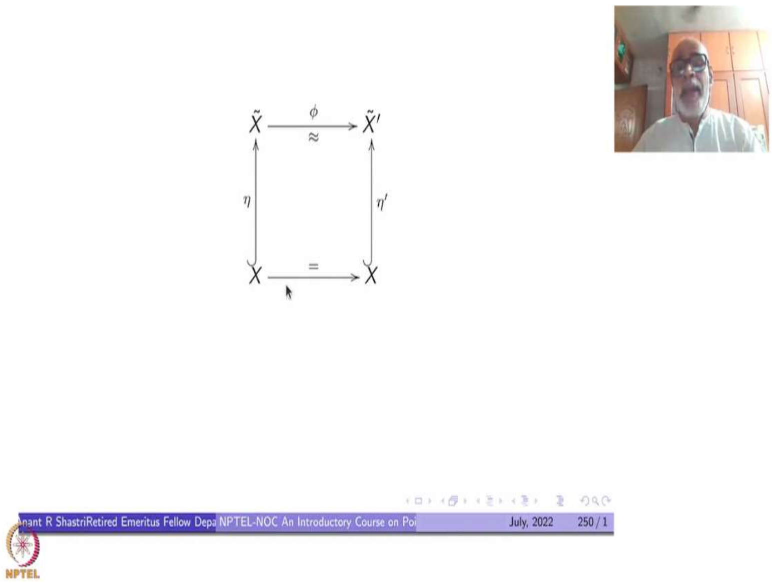

So, this is a picture for what is the meaning of equivalence classes this ϕ is a homeomorphism, but its identity here $\eta \circ \phi$ is η' so this is what you have to remember.

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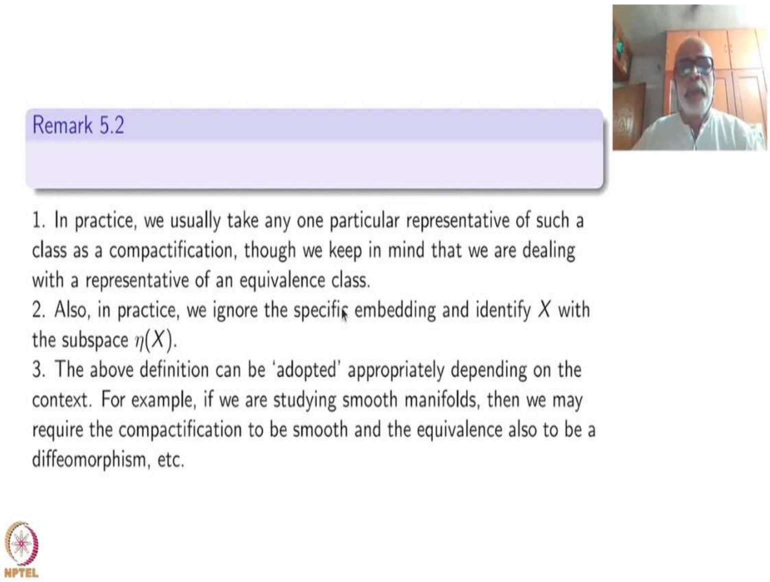

In practice, we usually take any one political representative of such a class as compactification. Though we keep in mind that we are dealing with a representative of an equivalence class. Also in practice we ignore specific embedding of X inside $\tilde{X}$ and identify X with $\eta(X)$. Then this $\eta(X)$ being subspace X can be thought of as a subspace of $\tilde{X}$.

The above definition can be adopted appropriately depending on the context, for example, you can keep adding extra adjectives for your compactification. Suppose, we are dealing with smooth manifolds. There are compactifications which may not be manifolds, but you ignore them you say okay my $\tilde{X}$ must be also be a manifold.

Suppose, we are dealing with only Hausdorff spaces then $\tilde{X}$ chosen as above may not be Hausdorff, in the above definition there is no Hausdorffness condition. But if we do not want to go out of Hausdorffness and then we may put the extra condition that $\tilde{X}$ must be also Hausdorff and so on. That is the meaning of adopting this definition at various context, at different contexts. After that you do not have to keep on saying that my compactification was also Hausdorff and so on. In the beginning of the context, we should make it clear, that is all.

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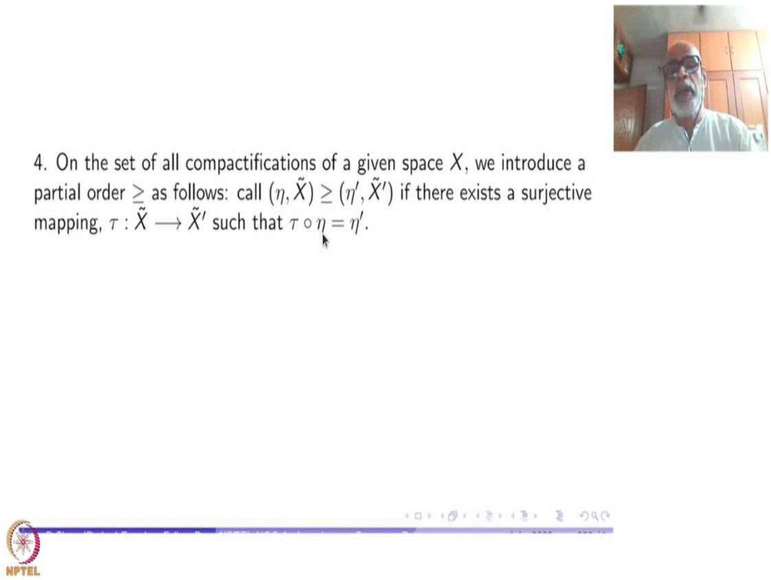

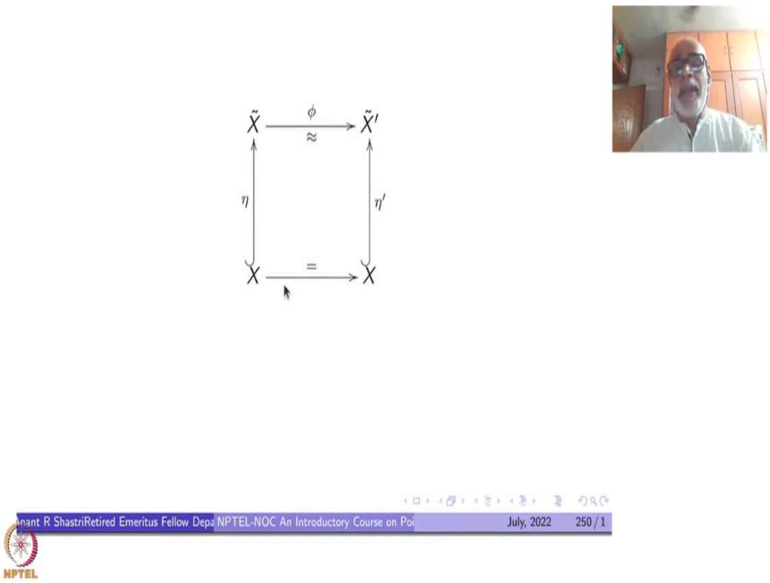

On the collection of all compactifications of a given space X, we can introduce a partial order. So, $(\eta, \tilde{X})$ is bigger than equal to $(\eta', \tilde{X}')$ (I am defining this partial order now,) if there exist a surjective mapping τ from $\tilde{X}$ to $\tilde{X}'$ again satisfying this property, viz., $\tau \circ \eta$ must be η' . You can go back to this picture here. Instead of taking ϕ which is an invertible mapping you just take a surjective mapping that is all; the rest of the things are the same. Then you will call this one is bigger than that.

So, this τ must be a surjective mapping. So, this is the partial ordering. It is easy to see that $(\eta, \tilde{X})$ is bigger than or equal to itself. This relation is anti-symmetric and it is transitive, composite of two surjective maps is surjective and so on. So, this is clearly a partial order alright. On second thought, is it really a partial order?

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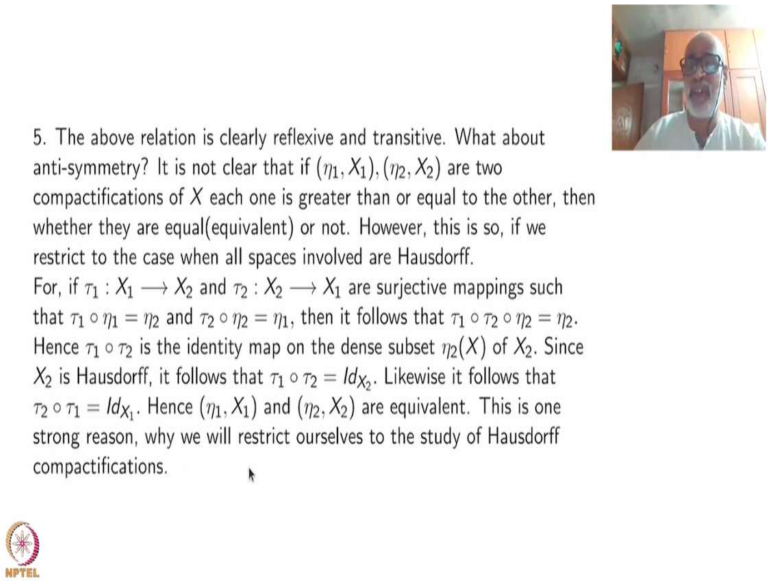

The above relation is clearly reflexive and transitive. What about anti-symmetry? So, I said it is anti-symmetric but I have to be careful here.

It is not clear if(η_1, X_1) and (η_2, X_2) are two compactifications of X each one is greater than or equal to other then whether they are equal or equivalent. See equality here is equivalence, as a class. So, can we prove that? Unless you prove this you cannot really call it a partial ordering. Two given objects, each bigger than the other mus implies that the two are the same.

Indeed, in general I do not know what to do, but let us take a special case. Suppose all the spaces involved are Hausdorff spaces. Then we are in a fine shape, namely, if τ_1 from X_1 to X_2 and τ_2 from X_2 to X_1 are surjective mappings such that $\tau_1 \circ \eta_1$ is η_2 and $\tau_2 \circ \eta_2$ is η_1, then it follows that $\tau_1 \circ \tau_2 \circ \eta_2$ is η_2 and hence $\tau_1 \circ \tau_2$ is the identity map on the dense subset $\eta_2(X)$ of X_2. Since X_2 is Hausdorff and there are two functions $\tau_1 \circ \tau_2$ and the identity map from X_2 to X_2. They are both defined on the whole space X_2 and they are equal on a dense subset. Therefore they must be equal on the whole of X_2, because set of points were two continuous functions into a Hausdorff space are equal is a closed subset. Therefore $\tau_1 \circ \tau_2 = Id_{X_2}$. Likewise, it follows that $\tau_2 \circ \tau_1$ is also identity of X_1. Therefore, τ_1 is homeomorphism, τ_2 is its inverse.

This just means that these two compactifications are equivalent. So, if you use the Hausdorffness then you are in a good shape. So, this is one strong reason why we will restrict ourselves for most of the time to studying of Hausdorff compactifications only. In our mind, we will keep thinking about Hausdorff compactifications, but in the definition we allow ourselves to go out of Hausdorffness because there are situations in which we have to study non Hausdorff compactification also (the third example that I have mentioned).

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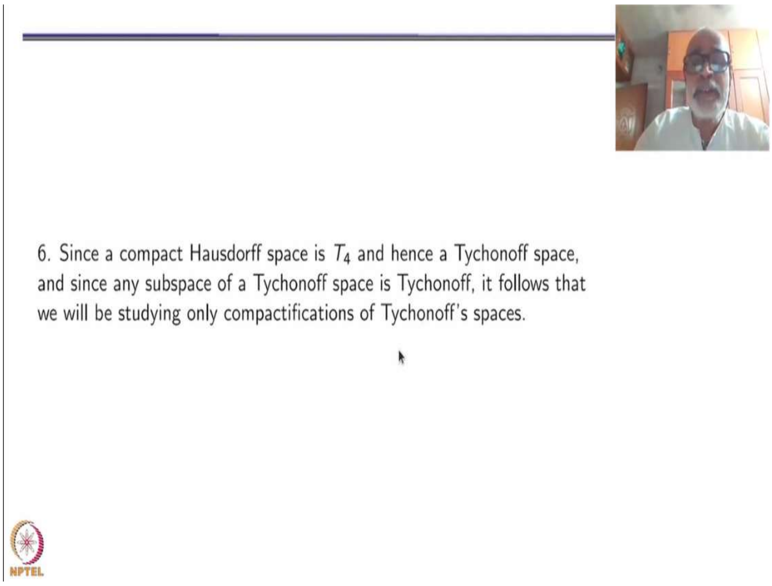

One more remark. A compact Hausdorff space is a T_4 space and hence a Tychonoff space. Tychonoff space is what? $T_{3+1/2}$, completely regular and T_1. And since any subspace of a Tychonoff space is a Tychonoff space, it follows that we will be studying only compactifications of Tychonoff spaces. This is the fall out of restricting ourselves to only Hausdorff spaces. Suppose we start with a Hausdorff space and we have a Hausdorff compactification of it. Then the original space not only just Hausdorff, it must be a Tychonoff space. If it is not, you do not have a Hausdorff compactification. So, there is such a strong restriction here if you want anything other than Hausdorffness. So, this is true whether we buy it or do not buy it. This is what is there, that is it.

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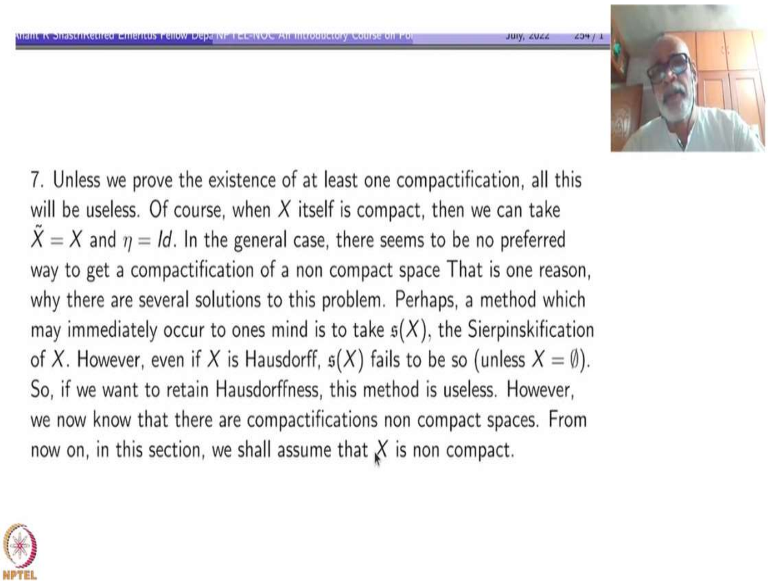

So, here is a general demand. Unless we prove the existence of at least one compactification, all this will be useless, you are talking in the void. It should not happen like that. Of course, when X itself is compact then you can take $\tilde{X}$ equal to X and η to be identity map, there is no problem, but suppose X is non compact then only you want a compactification. There seems to be no preferred way to get a compactification of a non compact space. That is one reason why there are several solutions to this problem.

Perhaps a method which may immediately occur to one's mind is the so called $\mathfrak{s}(X)$, the Sierpinskification of X. However, even if X is Hausdorff $\mathfrak{s}(X)$ may fail to be so. Indeed, $\mathfrak{s}(X)$ is always non Hausdorff unless X is empty and $\mathfrak{s}(X)$ is single point. So, if you want to retain Hausdorffness, taking $\mathfrak{s}(X)$ is useless.

Start with any Hausdorff space you may not get a Hausdorff compactification at all.

However, we now know that there are compactifications of non compact spaces, namely, you can take $\mathfrak{s}(X)$ if you are ready to go out of Hausdorffness. At least we are not working in a void, there are compactification.

From now onwards in this section, we shall assume that X is non compact. There is no point in discussing compactifications of a compact space. So, I may not mentioning X is non compact again and again.

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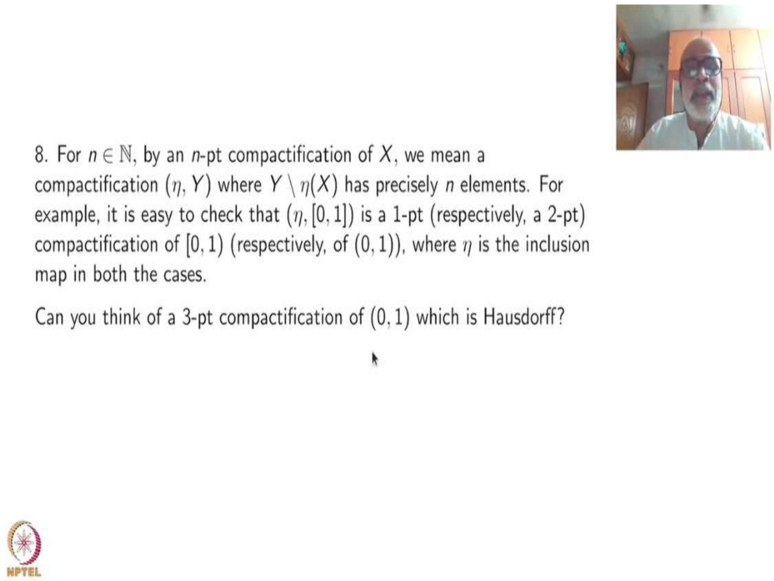

For each $n \in \mathbb{N}$, namely, natural numbers, by an n-point compactification of X, we mean a compactification (η, Y) where $Y \setminus \eta(X)$ has precisely n points. That means I have added exactly n points to the original space X. That is the meaning of this. For example, it is easy to check that $[0,1]$ the closed interval together with the inclusion map from $[0,1)$ (or from $(0,1)$) is respectively a 1-point (or a 2-point) compactification.

I have a question here, an open question for you. Think about it even if you do not get an answer, it is okay. Can you think of a 3-point compactification of $(0,1)$ the open interval which is Hausdorff. So, that is a question so think about it that is all.

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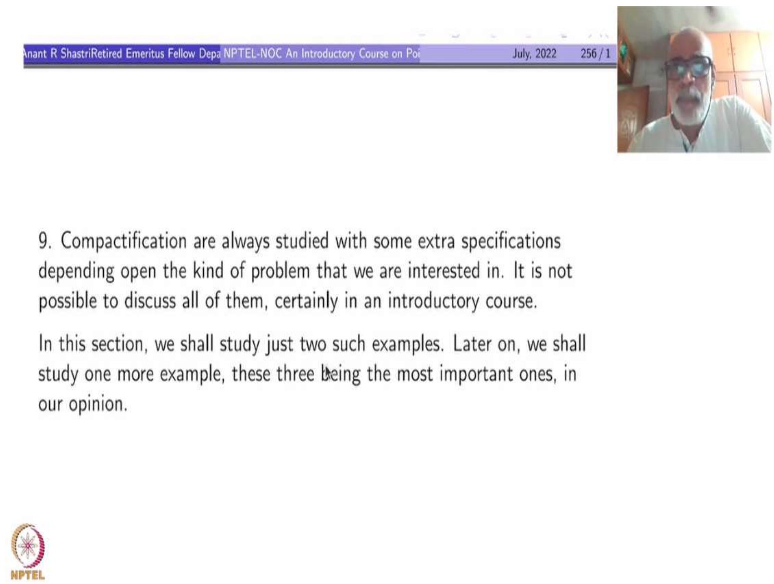

Some more general remarks. Compactifications are always studied with some extra specifications depending upon the kind of problem that we are interested in. It is not possible to discuss all of them, certainly in an introductory course like this. So, in this section, we shall study two such examples. Later on, we shall study one more. So, these three being the most important compactifications in our mind.

So, I have already told you the first two are Alexanderoff's one-point compactification and Stone-Cech compactification. These two things, we will study in this lecture.

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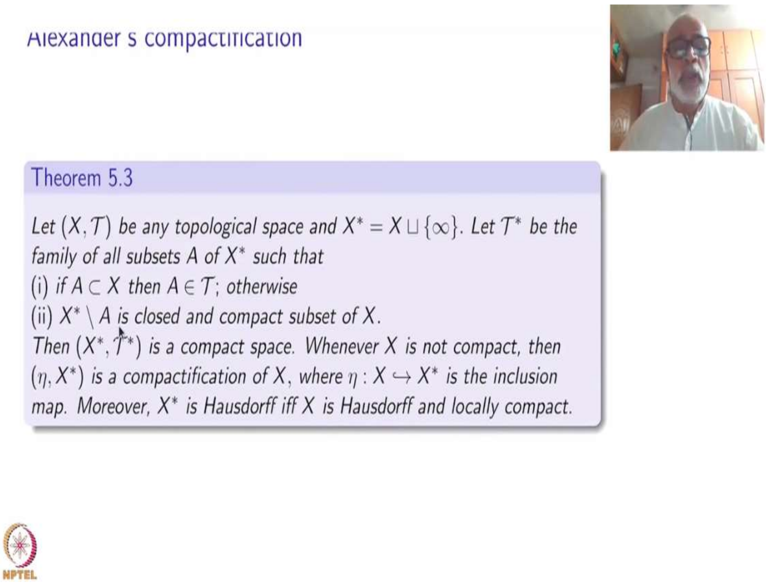

So, let me begin with Alexanderoff's compactification. Let $(X, \mathcal{T})$ be any topological space. Let X^* be the disjoint in union of X with one extra point which I am going to denote by infinity. This is construction is modeled on what you call the extended complex plane. You must be using this notation where X is the complex plane $\mathbb{C}$ and that is infinity. There is no algebraic structures or any partial order etc. here just an extra point you should remember that.

Let $\mathcal{T}^*$ be the family of all subsets A of X^* such that if A inside X then A must be in $\mathcal{T}$. All subsets of X which are inside $\mathcal{T}$, they are allowed inside $\mathcal{T}^*$ that is the first condition. Otherwise what are they? They must be containing infinity. So, the second condition is that $X^* \setminus A$, when you throw away A from X^*, what you get you will get a subspace of X, so, that must be a closed and compact subset of X.

So, this is the condition on the family $\mathcal{T}^*$. Now this $\mathcal{T}^*$ becomes a topology on X^* which makes X^* compact. (X is not compact this the standing assumption I have, but this construction I could have done it even when X is compact. If it is of any use you can use it otherwise you can throw it away, but for logical reasons I could have done this one even when X is a compact space. But it will not be a compactification of X! That is why I am saying `whenever X is not compact'. That is our central theme here.)

What happens to (η, X^*)? It is a compactification of X, where η from X to X^star is inclusion map. See set theoretically X^* is just X union infinity. So η from X to X^* is nothing but the inclusion map.

Further, X^* is Hausdorff if and only if X is Hausdorff and locally compact. Once again we are hitting the notion of local compactness here, in Alexanderoff's compactification.

So, Alexanderoff's compactification is able to achieve Hausdorffness provided we start with Hausdorff and locally compact space X. So, this is the conclusion. So, let us deal with them one by one. One or two things we have to verify here.

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Proof: The verification that $\mathcal{T}^*$ is a topology is straight forward. Condition (i) automatically says that η is an embedding. If $\{U_i\}$ is an open cover for X^*, let us say $\infty \in U_1$. Put $F = X^* \setminus U_1$. Then F is a compact subset of X and $\{U_i \cap X \ : \ i \neq 1\}$ is an open cover for it. If $U_2, \ldots, U_n$ are such that $F \subset (U_2 \cap F) \cup \cdots \cup (U_n \cap F)$, then $X^* = U_1 \cup \cdots \cup U_n$.

First thing is $\mathcal{T}^*$ is a topology. What you have to do. Empty set is there because empty set is subset of X. The whole space X^* is there why because if you throw away the whole space, you get the empty set which closed and compact subset of X. So, that is easy to see. If two subsets are there in $\mathcal{T}^*$ and if they are already in already in $\mathcal{T}$ then their intersection will be also in $\mathcal{T}$ so it will also in $\mathcal{T}^*$. If one of them is a subset of X and another one contains the point infinity, the intersection is again in tau. If both contain the point infinity, then their complements are both closed and compact. On the other hand the complement of their intersection is the union of their complements which is also a closed and compact subset.

Next, arbitrary union of members of $\mathcal{T}^*$ is there because once one of them contains the point infinity, then the entire union also contains and its complement will be the intersection of the complements of each of them which will be closed subset of a compact set.

So, $\mathcal{T}^*$ is a topology. This topology when you restrict it to X becomes precisely $\mathcal{T}$. So, the inclusion map is a homeomorphism, inclusion map X to X^* is homeomorphism because every open subset here becomes open set there, and vice versa under the inclusion map.

Now to see that X^* is compact, if $\{U_i\}$ is an open cover for X^*, I have produce a finite cover. Let us say infinity belongs to U_1. It has to be in one of the open sets anyway. Put $F = X^* \setminus U_1$. U_1 being open in X^* and contains the point infinity, F must be compact and closed subset of X. Members U_i such that $i \neq 1$ cover F and hence there will be finitely many of them which cover F. Together with U_1 we get a finite subcover for X^*.

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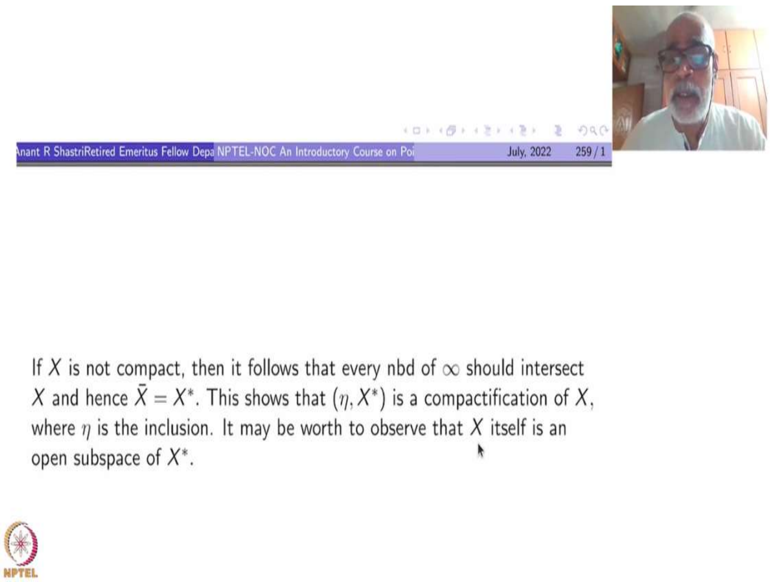

Now come to the hypothesis that X is not compact. Then it follows that every neighborhood of infinity should intersect X. If X is compact the $\{\infty\}$ itself would have been an open subset in X^*, because its complement is compact and closed. X itself is compact and X is closed inside X. So, come back to the case when X is not compact.

Then every non empty open subset of X^* must contain some point of X. That is enough to see that the closure of X is the whole of X^*, which is the same as saying X is dense in X^*. This shows that (η, X^*) is a compactification according to our definition.

It may be worth to note that X itself is an open subspace of X^*, i.e., $\{\infty\}$ is closed why? because its complement in the whole of space X and that belongs to $\mathcal{T}$. So, X itself you want to say $\eta(X)$ itself is an open subset of X^*, open and dense.

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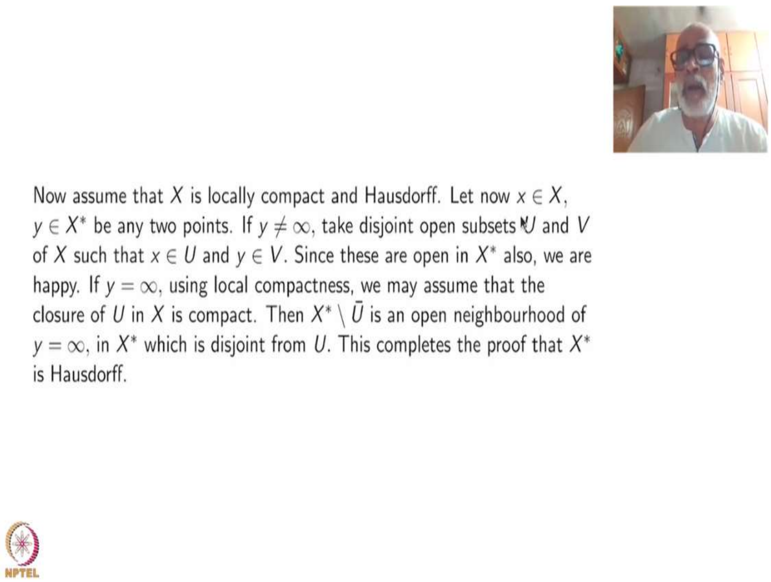

So, finally we want to show that X^* is Hausdorff if and only if X is locally compact and Hausdorff. If X^* is Hausdorff, being a subspace, X will be also Hausdorff that is easy. Why it is locally compact? Let us prove this later.

First assume that X is locally compact and Hausdorff. To show that X^* is Hausdorff, take two distinct point x and y in X^*. If both are in X, by Hausdorffness of X, these two points can be separated by open subsets in X itself. The same subsets will do the job in X^* also, because they are in $\mathcal{T}^*$ also.

So, now suppose one of the point is infinity, say, x belongs to X and $y = \infty$. This is an important case. Using local compactness of X, we may assume that there is an open set U in X such that x is in U and closure of U is compact. Take V to be the complement of $\bar{U}$ in X^*. The V is open in X^* and contains infinity. Clearly $U \cap V$ is empty. So, I have prove X^* is Hausdorff using local compactness and Hausdorffness of X.

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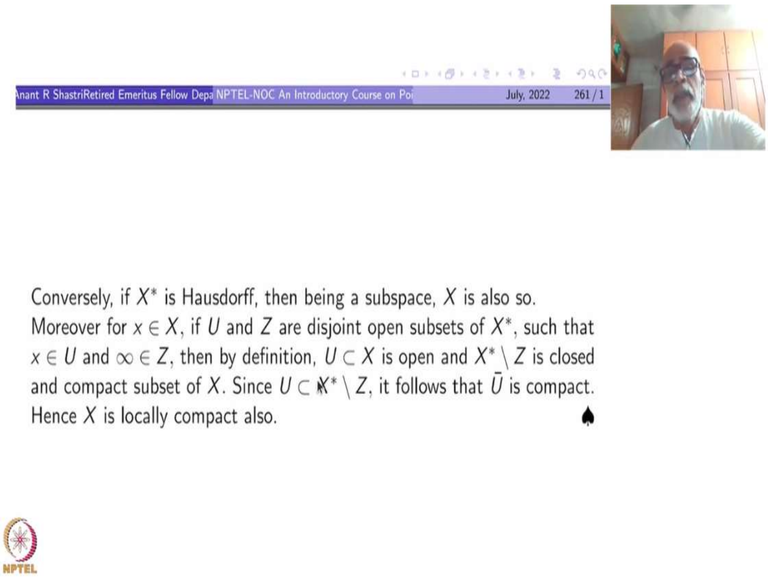

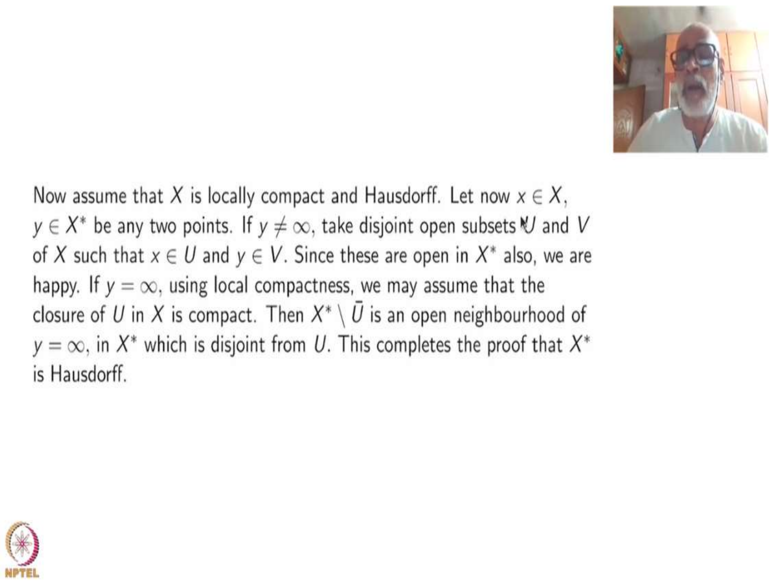

Now the converse. Suppose X^* Hausdorff. Then being a subspace, X is also Hausdorff. Moreover, for x belonging to X if U and Z are disjoint open subsets of X^* such that x is in U and infinity is in Z, then by the very definition of $\mathcal{T}^*$, U is open in X and F equal to $X^* \setminus Z$ is a closed and compact subset of X. Also U is contained in F, because U and Z are disjoint.

It follows that $\bar{U}$ being a closed subset of F is compact. Hence X is locally compact. (For Hausdorff spaces, we have proved that there are several equivalent conditions of local compactness. For each point if we produce a neighborhood with its closure compact, then the space is a locally compact.)

So, what we have got is that Alexanderoff's compactification, which is a one-point compactification, will be Hausdorff if we start with a Hausdorff and locally compact space and that is the most important special case that we are going to study further.

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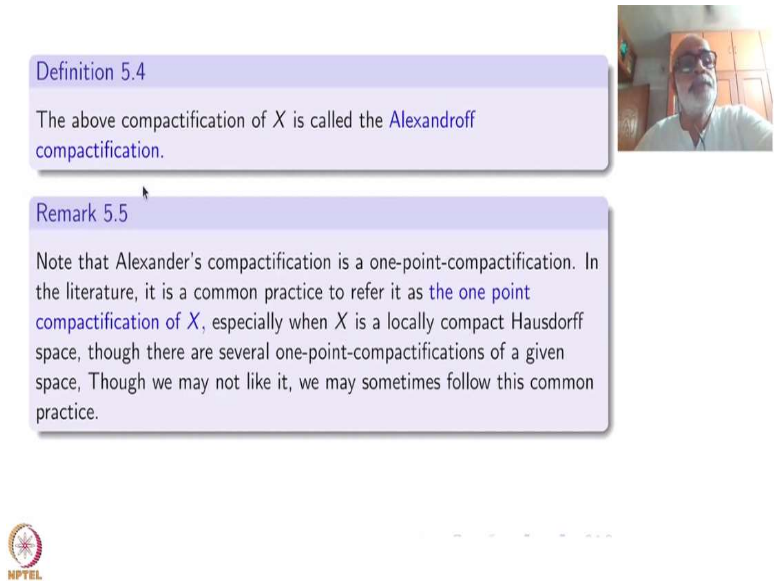

So, I make this definition the above compactification of X is called Alexandroff compactification. Note that Alexanderoff's compactification is a spcific one-point compactification. In the literature, it is common practice to refer to it as the one-point compactification of X. Especially when X is locally compact and Hausdorff space. See there are many one point compactifications.

The compactification that we have constructed namely Alexandroff is a special one, but in the literature, whenever we start with a locally compact Hausdorff space, people always refer to this one as the one point compactification. So, we may also do that if it is not Alexanderoff's compactification, we will specifically mention it, that is all. So, we may sometimes follow this common practice alright, is that clear?

So, let us stop here. We shall continue the study of this one and bring the new concept of properness etc next time. Thank you.