An Introduction to Point-Set-Topology Part (II)
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Lecture No 16

Total Boundedness

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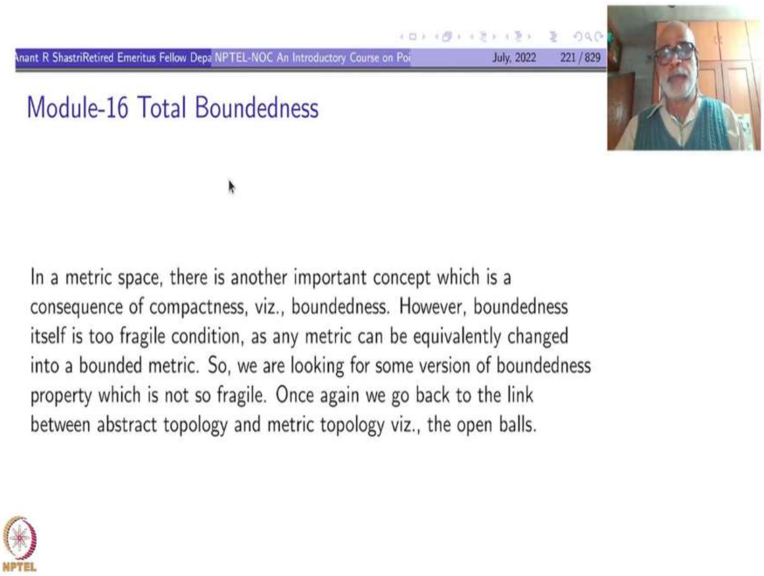
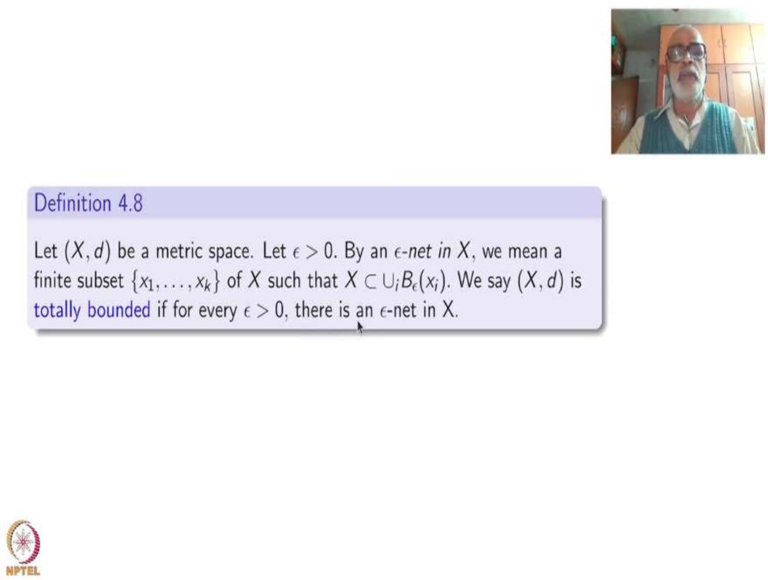


Hello welcome to NPTEL NOC introductory course on Point Set Topology Part (II). Today we will do model 16, a new concept, viz., total boundedness. In a metric space, there is another important concept which is the consequence of compactness, namely, boundedness. However boundedness itself is too fragile condition, because any metric can be equivalently changed to a bounded metric without changing the topology. So, we are looking for some version of boundedness property which is not so fragile. (Refer Slide Time: 01:30)



Once again, we go back to the link between abstract topology and metric spaces, namely, the fundamental open subsets, the open balls. So, here is a definition. Let (X, d) be a metric space and ϵ be positive real number. By an ϵ-net in X, we mean a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that the entire space X is contained inside the finitely many open balls with centres at x_i and radius ϵ.

So, these ϵ-balls are enough to cover the whole thing. And of course, if you take all the points is always possible but finitely many points on. So, we say (X, d) is totally bonded. If for every ϵ, there is ϵ-net. So, obviously, this is much stronger it if you have not taken ϵ-balls that will admit a finite sub cover that is what it means, not all open sub sets are admitting finite. So, this almost comes to very close to compactness.

So, these ϵ- balls are enough to cover the whole X. And of course, if you take all the points as centres, then clearly ϵ-balls will cover X, but here we want only finitely many points. So, we say (X, d) is totally bounded, if for every ϵ-positive, there is an ϵ-net.

So, obviously, this is much stronger than mere boundedness. If you taken arbitrary open sets in place of ϵ, or did not fix ϵ, then admitting finite cover would have been the same as compactness. So, this condition of total boundedness comes to very close to compactness and is implied by compactness, but may not imply compactness.

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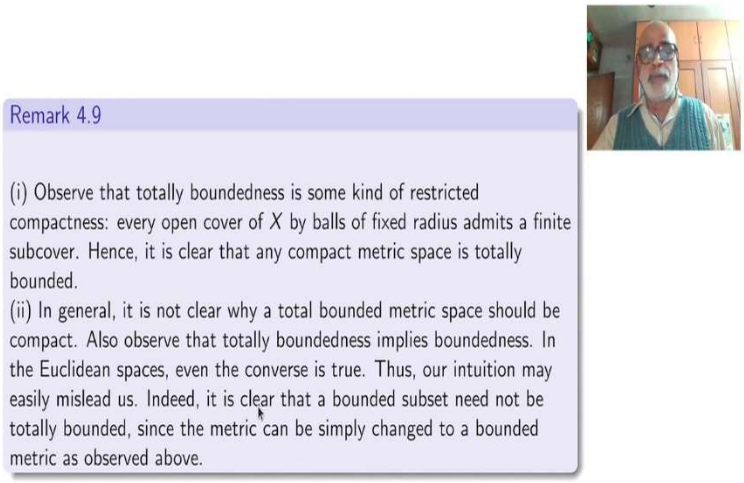

So, total boundedness is some kind of restricted compactness; that is the first thing to notice. Every open cover of X by balls of a fixed radius will admit a finite sub cover. It is clear that any compact metric space is totally bounded.

In general, it is not clear why a total bounded metric space should be compact. Also observe that total boundedness implies boundedness automatically. Because all that you have to take is an epsilon net $\{x_1, x_2, ..., x_n\}$ and look at its diameter and add 2ϵ. That all. Then the diameter of X will be less than this number. So, total boundedness implies boundedness.

In the Euclidean spaces, even the converse is true. Thus, our intuition may easily mislead us. So total boundedness is much stronger than boundedness but in Euclidean spaces, you do not have this problem. Indeed it is expected that the bounded need not imply totally boundedness since the metric can be simply changed to a bounded metric all the time, and hence eery metric space would be equivalent to a totally bounded one.

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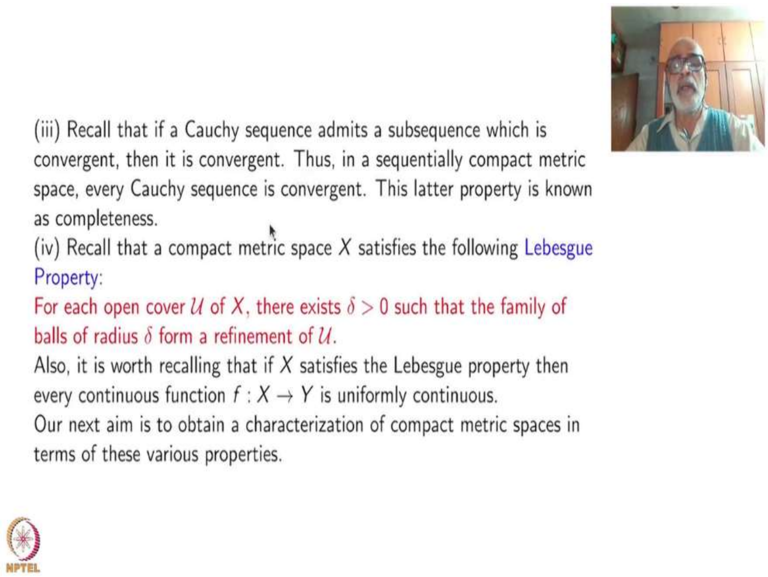

So this is just a guesswork, you know. But unless you see an example, you will not be satisfied.

Recall that if a Cauchy sequence admits a subsequence which is convergent, then the sequence itself is convergent. Thus in a sequentially compact metric space, (sequentially compactness, remember what it is? every sequences has a subsequences which is convergent) every Cauchy sequence will be convergent because it will admit a sub sequence which is convergent and because the original sequence is Cauchy, it is also convergent (to the same limit).

This latter property is known as completeness; every Cauchy sequence is convergent means completeness. So, somehow when studying the sequential compactness etc. we are forced to think about completeness also.

then, there is another important property, all the time used in analysis of metric spaces namely, compact metric spaces have the Lebesgue property. So, I will just recall what is the meaning of Lebesgue property. There is a theorem of Lebesgue and the conclusion of the theorem has been made into a property. Namely, for each open cover $\mathcal{U}$ of X you must find, there exists a positive δ such that the family of balls of radius δ form a refinement of $\mathcal{U}$. If you take any ball of radius δ anywhere in the space, such a ball will be contained in one of the members of $\mathcal{U}$. That is the meaning of refinement.

So, that is the Lebesgue property. So, it is also worth recalling that if X satisfies the Lebesgue property, then every continuous function from X to any space Y is uniformly continuous. Continuity implies uniformly continuous, on a compact space. You know closed intervals, more generally, closed and bounded subset of $\mathbb{R}^n$ and so on that you come across in analysis. You see that all that you need is the Lebesgue property for this uniform continuity. The full force of compactness is not necessary.

So, how far can away are these properties from compactness? Our next aim is to obtain a characterization of compact metric spaces in terms of these properties, Lebesgue property, sequential compactness, total boundedness, and so on.

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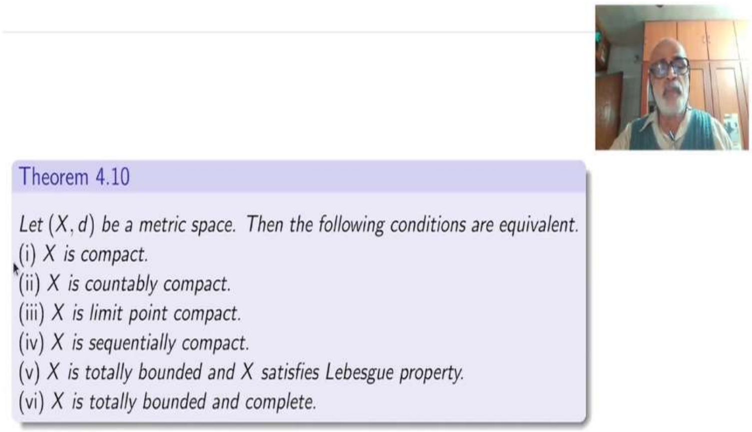

So, that is the theorem here. There are six equivalent statements, giving five criteria for compactness for a metric space. You start with a metric space. Then the following conditions are equivalent.

(i) X is compact,.

(ii) X is countably compact.

(iii) X is limit point compact.

(iv) X is sequentially compact.

(v) X is totally bounded and has the Lebesgue property.

(vi) X totally bounded and complete.

So, these first three are the ones which we have studied last time. These two new criteria are there now. Lebesgue property and completeness are old. Total boundedness is the new concept. So, let us go through the proof of these equivalences. Proving (i) implies (ii) implies (iii) implies (iv) implies (v) implies (vi) implies (i) that would have been ideal and easiest way.

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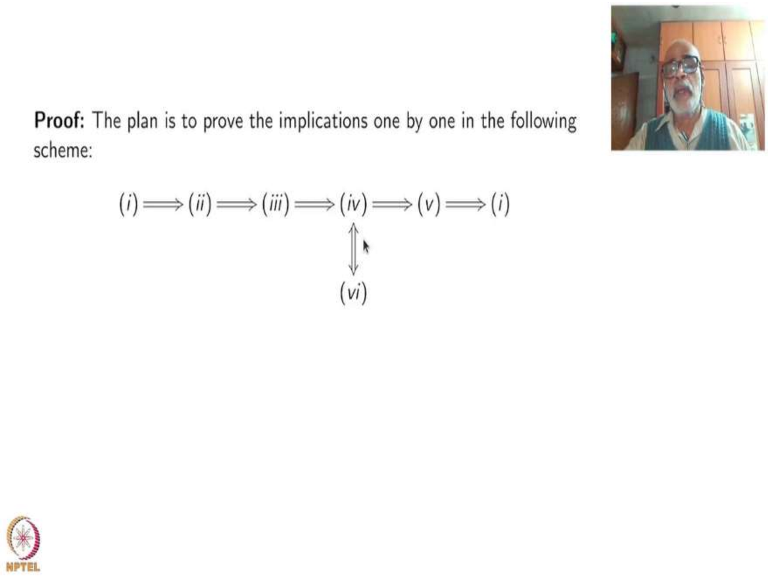

Somehow I am not able to arrange it in that way. So, I will just follow a slightly different approach here. The implications (i) through (v) and back to (i) will be done. But equivalence of (iv) and (vi) will be done separately.

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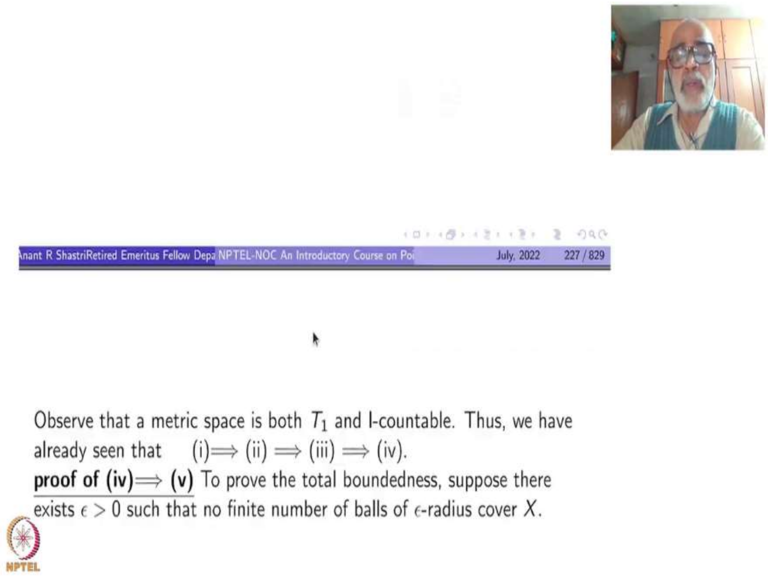

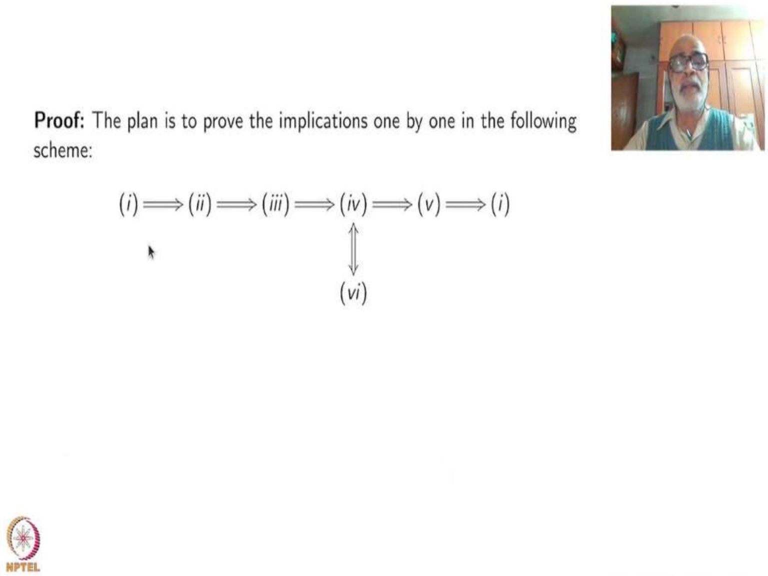

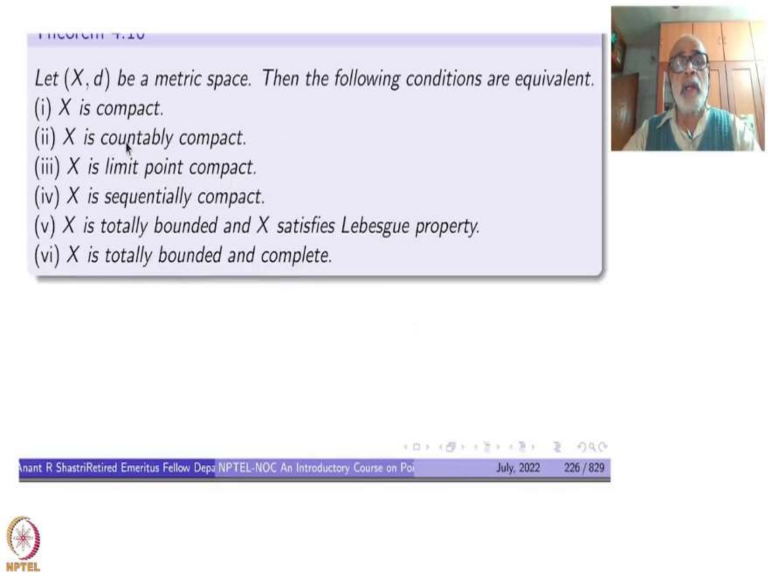

So, observe that a metric space is both T_1 and I-countable. This is what I have already told you last time using which, we have seen that (i) implies (ii) implies (iii) implies (iv). Last time we have seen more than that, but right now, concentrate upon, these statements.

So, these two implications and these two implications we are left out right now. So, let us prove (iv) implies (v) namely, this is sequential compactness sequentially compact metric space. So, we want to show that it is totally bounded and satisfies the Lebesgue property that is what we have to show.

So, let us now prove (iv) implies (v), namely, assume that X is sequential compactness metric space. So, we want to show that it is totally bounded and satisfies the Lebesgue property.

To prove total boundedness. Suppose on the contrary, that there exists ϵ positive such that no finite number of balls of radius ϵ cover the whole of X. (There is some ϵ for which there is no ϵ-net, that is the negation of total boundedness). Choose a point $x_0 \in X$, any point, does not matter. Having chosen $x_1, x_2, x_3, \ldots, x_{n-1}$, inductively, how do you choose x_n? x_n will be chosen in X setminus union of all balls of radius ϵ around x_i, for $i < n$.

So, we have proved total boundedness, without any problem. Things are quite straightforward. In the second part what we have to prove?

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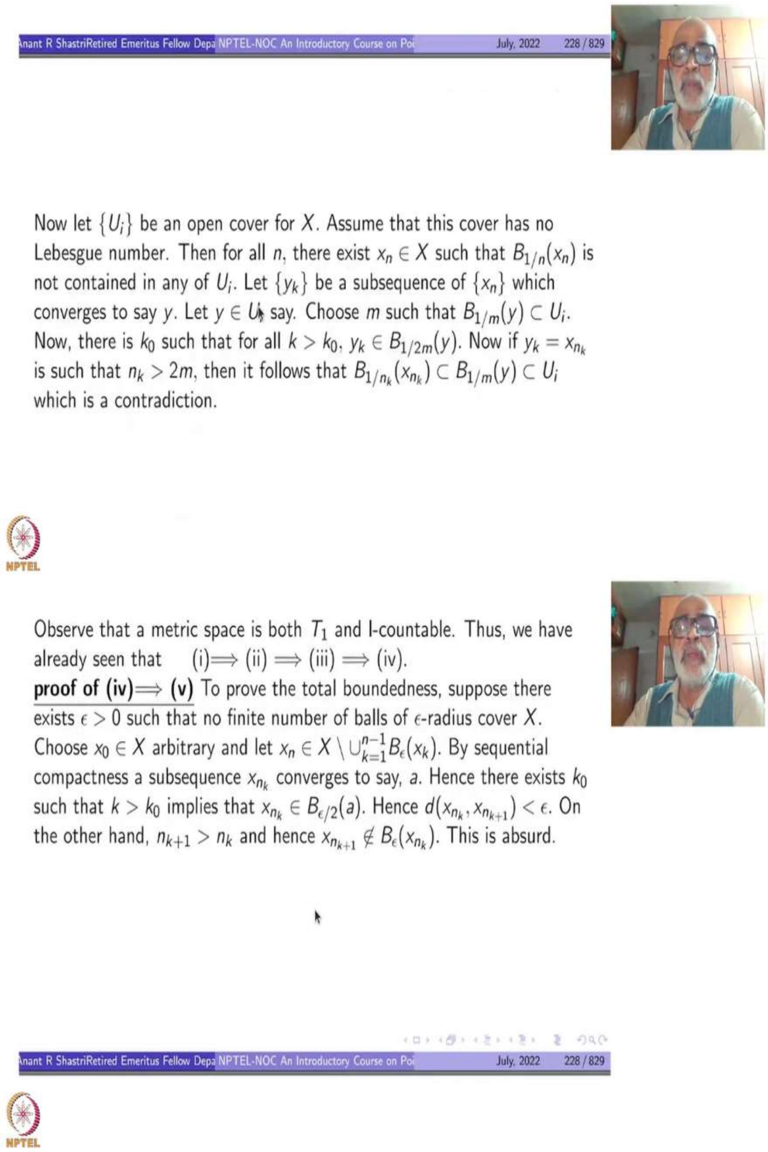

Lebesgue property. Lebesgue is also very easy here. Suppose on the contrary X does not satisfy the Lebesgue property. This means that we have an open cover $\mathcal{U}$ which has no Lebesgue number. What is the meaning of that? No $\delta > 0$ is a Lebesgue number for $\mathcal{U}$.

That means for every positive integer n, (instead of δ, I use $1/n$) there exist x_n inside X, such that the ball of radius $1/n$ around x_n is not contained in any member of $\mathcal{U}$. For instance for $n = 1$, you get $B_{1/2}(x_1)$ will not be contained in any U and so on.

By sequential compactness, there is a subsequence $\{y_k\}$ of $\{x_n\}$ which is convergent to some point $y \in X$. Let y belonging to U_y; y must be in one of the open subsets because $\mathcal{U}$ is an open cover. Choose m such that $B_{1/m}(y)$ is contained inside U. By convergence of $\{y_k\}$, there exists k_0 such that for all for all $k > k_0$, all the y_k must be in $B_{1/2m}(y)$, the open ball of radius $1/2m$ around y.

Now if $k > 2m$ also, then $B_{1/k}(y_k)$ is contained in $B_1/m(y)$ which is contained in U. But $\{y_k\}$ is a subsequence of $\{x_n\}$ means that $k = n_i$ for some i and so $y_k = x_{n_i}, B_{1/n_i}(x_{n_i})$ is contained in U which is a contradiction.

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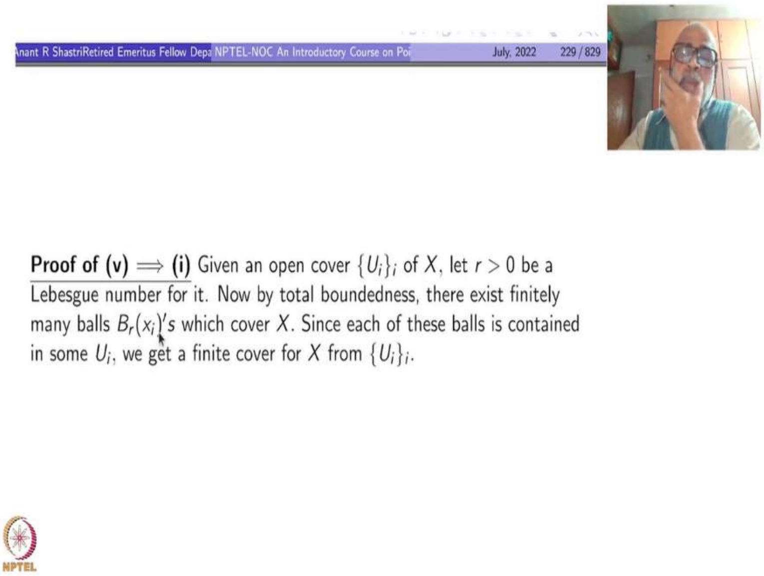

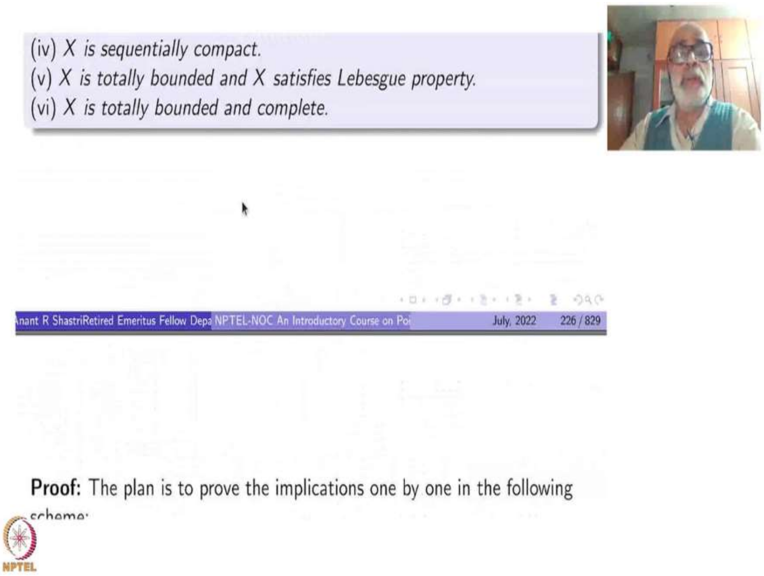

So, now, I will prove (v) implies (i) namely, total boundedness and Lebesgue property together imply compactness, just like we observe that countably compact and Lindelof implies compact. So, the proof is that simple. So (v) implies (i) is also not difficult.

So, let us go through the proof. Start with any open cover $\mathcal{U}$ and let r positive be its Lebesgue number. Now by total boundedness, there exist finitely many points $\{x_i\}$ such that balls of radius r arond them cover X. Since each of these balls is contained in some U_i in $\mathcal{U}$, we finitely many U_i will cover X. So that is all.

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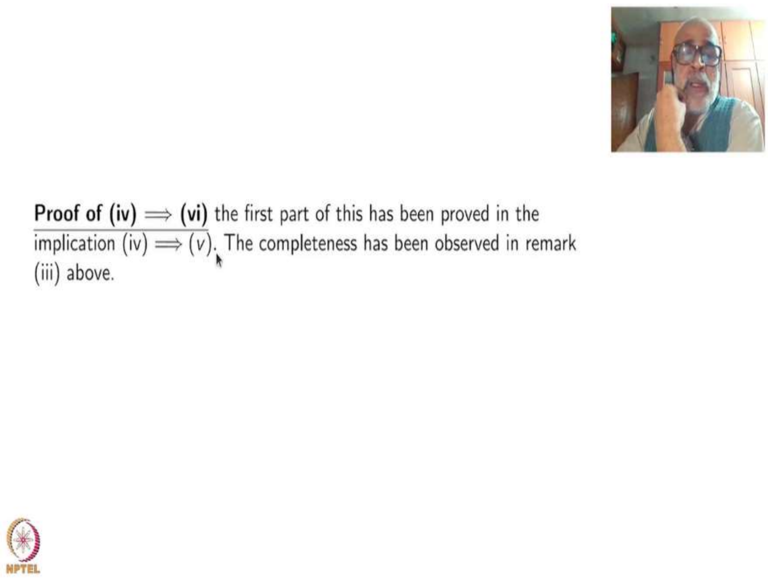

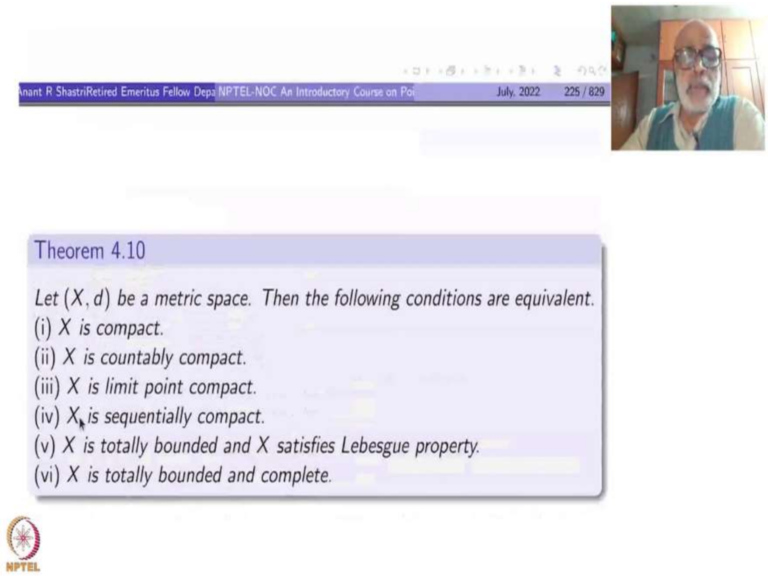

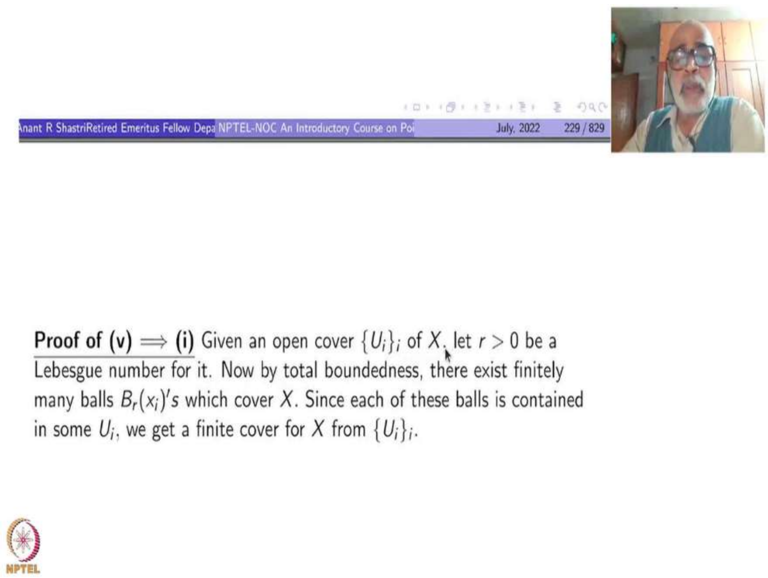

So, we are left with the tasks of proving (iv) implies (vi) and (vi) implies (iv). (iv) is sequentially compactness and (v) is total boundedness and completeness. So, these are also not difficult.

For (iv) implies (vi), already we have SC implies total boundedness, while proving (iv) implies (v). In the remark (iii) above, we have also proved that SC implies completeness, namely, start with a Cauchy sequence, it has a subsequence which is convergent, therefore it is convergent. So, (iv) imply (vi) is done. (That is why I have chosen to prove that instead of other conditions.)

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proof of (vi) $\Longrightarrow$ (iv) Let $\{x_n\}_n$ be any sequence in X. The idea is to show that it has a subsequence which is Cauchy. (Then by completeness, this subsequence will converge.) From the totally boundedness, for each $k \geq 1$, let us first get a finite subset A_k such that $X = \cup_{a \in A_k} B_{1/k}(a)$. Now by pigeon-hole principle, there exists an infinite subset $N_1 \subset \mathbb{N}$ and $a_1 \in A_1$ such that for all $n \in N_1$ we have $x_n \in B_1(a_1)$. Inductively, for each k, there exists an infinite subset $N_k \subset N_{k-1}$ and $a_k \in A_k$ such that for all $n \in N_k$, we have $x_n \in B_{1/k}(a_k)$.

Now, finally, I have to show that (vi) implies (iv). Let $\{x_n\}$ be any sequence in X. The idea is to show that it has a subsequence which is Cauchy. By completeness, the Cauchy sequence will converge. You want to show that the sequence has a subsequence which is convergent. Instead you just show that it is Cauchy.

Now, from total boundedness for each integer $k \geq 1$, let us first get a finite subset A_k of X such that X is covered by balls of radius $1/k$ around the points of A_k.

What is A_k? A_k is $1/k$-net. Total boundedness of X gives you this by taking $\epsilon = 1/k$. By the pigeon-hole principle, there exists an infinite subset N_1 of $\mathbb{N}$, and a point $a_1 \in A_1$, n is in N_1 implies $x_n \in B_1(a_1)$. That is because there are only finitely many balls covering $\{x_n\}$.

For the same reason, there exists an infinite subset N_2 of N_1 such that n in N_2 implies $x_n \in B_{1/2}(a_2)$ for some $a_2 \in A_2$. Inductively, having choose N_{k-1}, there exists an infinite subset N_k of N_{k-1} and a point $a_k \in A_k$, such that $n \in N_k$ implies $x_n \in B_{1/k}(a_k)$.

Now we can construct a subsequence of $\{x_n\}$ which is Cauchy. Choose n_1 to be any number in N_1. After choosing n_{k-1}, let n_k be any number in N_k which is bigger than n_{k-1}. This is possible because each N_k is infinite.

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Let n_k be the least number in N_k which is bigger than n_{k-1}. Then we claim that $\{x_{n_k}\}$ is a Cauchy sequence. For, given $\epsilon > 0$ choose k_0 such that $1/k_0 < \epsilon/2$. Then $k, l > k_0$ implies that $x_{n_k}, x_{n_l} \in B_{1/k_0}(a_{k_0})$. Hence $d(x_{n_k}, x_{n_l}) < \epsilon$. This completes the proof of the theorem. ♠

by pigeon-hole principle, there exists an infinite subset $N_1 \subset \mathbb{N}$ and $a_1 \in A_1$ such that for all $n \in N_1$ we have $x_n \in B_1(a_1)$. Inductively, for each k, there exists an infinite subset $N_k \subset N_{k-1}$ and $a_k \in A_k$ such that for all $n \in N_k$, we have $x_n \in B_{1/k}(a_k)$.

Let (X, d) be a metric space. Then the following conditions are equivalent.
(i) *X is compact.*
(ii) *X is countably compact.*
(iii) *X is limit point compact.*
(iv) *X is sequentially compact.*
(v) *X is totally bounded and X satisfies Lebesgue property.*
(vi) *X is totally bounded and complete.*

Proof: The plan is to prove the implications one by one in the following scheme:

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (v) \Longrightarrow (i)$$
$$(iv) \Updownarrow (vi)$$

Then we claim that this sequence $\{x_{n_k}\}$ is a Cauchy sequence. So, this construction was not all that obvious. We have done even more complicated constructions elsewhere anyway.

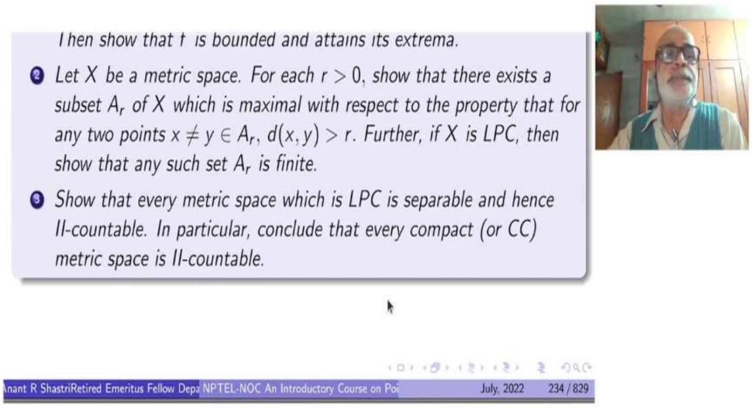
So, why $\{x_{n_k}\}$ is a Cauchy? Given $\epsilon > 0$, take k_0 such that $1/k_0$ is less than $\epsilon/2$. Then if k and 1 are greater than k_0, x_{n_k} and x_{n_l} will be inside $B_{k_0}(a_{k_0})$, both of them will be here.

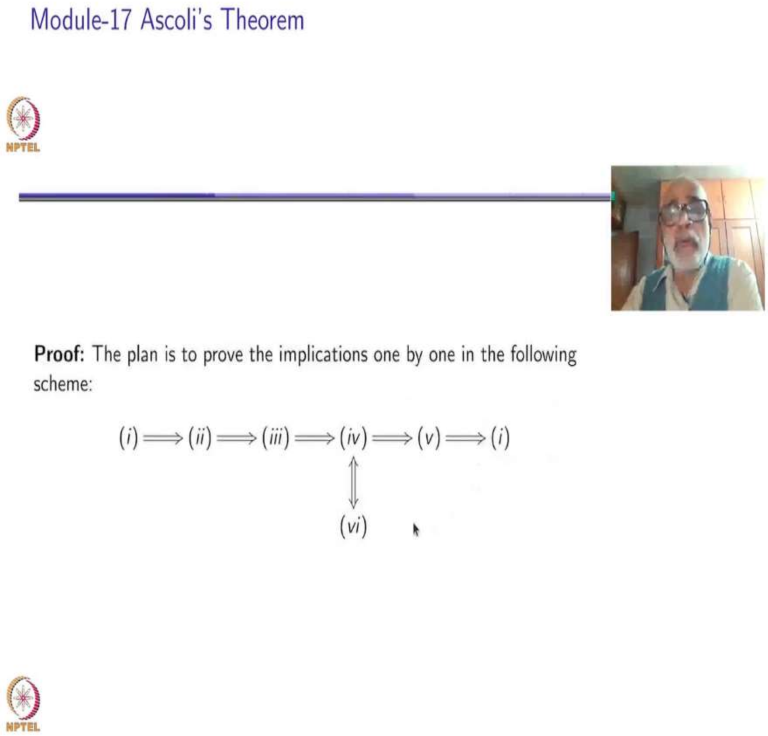
What the meaning of that? The distance between them is at most $2/k_0$ which less than ϵ. So, given every ϵ, I have given you some k_0 such that beyond that the distance between any two members of the subsequence is less than ϵ. That means that the subsequence is Cauchy.

So, that completes proof for this big theorem, at least it looks big. So, we have these implications that we have shown. (Refer Slide Time: 23:30)

Exercise 4.11

1. *Here is an easy application of theorem 4.10: Let $f : X \to \mathbb{R}$ be a continuous real valued function, where X is CC. Then show that f is bounded and attains its extrema.*
2. *Let X be a metric space. For each $r > 0$, show that there exists a subset A_r of X which is maximal with respect to the property that for any two points $x \neq y \in A_r$, $d(x, y) > r$. Further, if X is LPC, then show that any such set A_r is finite.*
3. *Show that every metric space which is LPC is separable and hence II-countable. In particular, conclude that every compact (or CC) metric space is II-countable.*





So, here is an exercise which I will just go through. You are welcome to solve them and get your answers checked. The first one is an easy application of theorem 4.10.

Take X to $\mathbb{R}$, a continuous real valued function, where X is countably compact. Then show that f is bounded and attains its extrema. So, this is similar to what we remarked namely Lebesgue property implies uniform continuity. So, under compactness, this is the standard Weierstrass's theorem: any continuous function on a compact set to real numbers is bounded and attains its extrema, both the maximum and minimum. This exercise shows no need for compactness, just countably compactness is enough. So, try your hand.

Next, let X be a metric space. For each r positive show that there exists a subset A_r of X which is maximal with respect to the property that for any two distinct points $x, y \in A_r$ the distance between x and y is bigger than r.

For example, for $r = 1$, you can begin with any point x_1. The second point should be at a distance atleast one from x_1 and so on. However, the third point should be such that it distance from both the earlier points must at least one. (It can easily happen that you cannot have more than a single point also, for instance, if the diameter of X itself is less than one.)

So, that is that exists a maximal subset that is what you have to show. Apply Zorn's lemma). Further you have to show that if X is limit point compact then any such A_r is finite. In a compact metric space, you have proved such things. Now, you are asked to prove the same for limit point compact metric spaces.

Next, show that every metric space which is limit point compact is separable and hence, second countable. In particular, conclude that every compact or countably compact metric space is second countable which you might have proved in a different way elsewhere.

So, next time we will prove the very important functional result namely, Ascoli's theorem. Thank you.