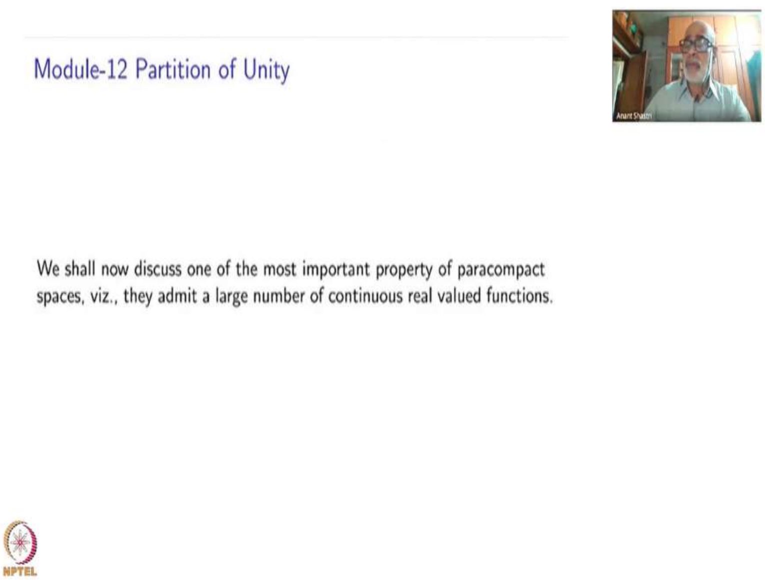
An Introduction to Point-Set-Topology (Part II)

Professor Anant R Shastri
Department of Mathematics
Indian Institute of Technology Bombay

Lecture 12

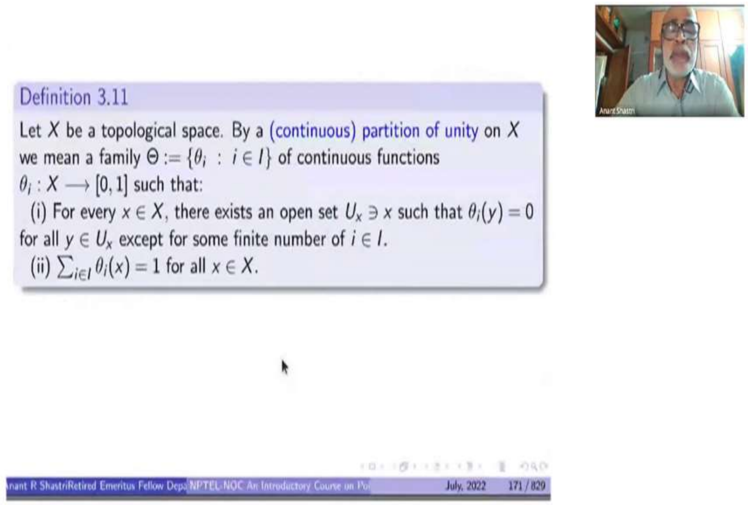
Partition of Unity

(Refer Slide Time: 00:16)



Hello, welcome to NPTEL NOC introductory course on point set topology part II. So, module 12 today. We continue with the study of paracompact spaces. This time partition of unity. So, we shall now discuss one of the most important properties of paracompact spaces, namely that they admit a large number of continuous real valued functions.

(Refer Slide Time: 01:02)



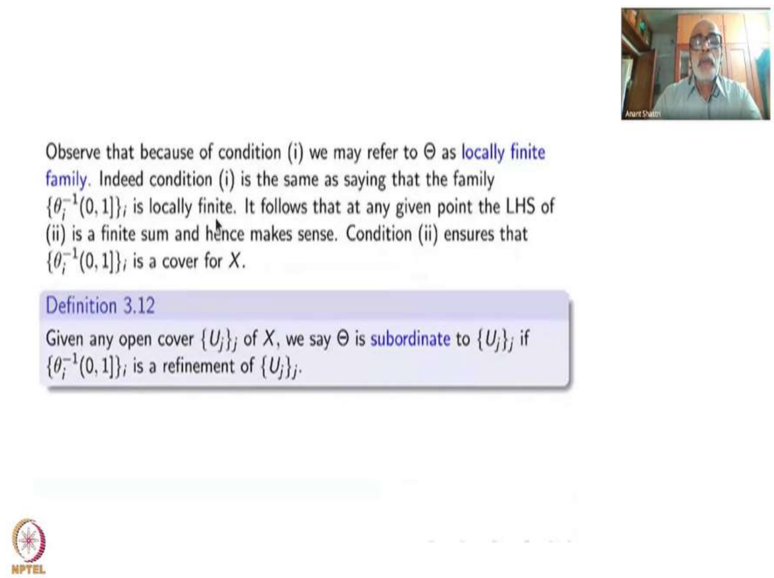
As usual, we shall begin with a definition. Take any topological space X. By a (continuous) partition of unity on X, continuity is assumed without specifically mentioning it, so, I have put it in a bracket, partition of unity on X, we mean a family of continuous functions θ_i, indexed by $i \in I$, all these functions are defined on the whole of X to the closed interval $[0,1]$, first of satisfying the condition (i) for every x in X, you have a neighborhood U_x of x such that $\theta_i(y) = 0$ for all y belonging to U_x, except for finitely many $i \in I$.

So, that is like saying that, the family is locally finite, i.e., the family of support of θ_i's is locally finite. In particular, for every $x \in X$, since there are only finite many of i such that $\theta_i(x)$ not zero, their sum makes sense. Even if I write the total summation θ_i as i ranges over all of I, this will be a finite sum for each x. The second condition is that (ii) that sum must be equal to 1. So, this is the second condition.

The second condition gives the name `partition of unity'. The constant function 1 on the whole of X, is broken up into a family of functions, each of them continuous and the sum total is equal to 1. What is the meaning of a sum of arbitrary family of functions? This local finiteness which automatically implies point finiteness ensures that the sum makes sense. But we will see that local finiteness is important here not just point finiteness.

Instead of an arbitrary space X, if I have some subset of $\mathbb{R}^n$, then I can talk about partitions unity which are smooth, C^1 or C^2 and so on. So, you can put those adjectives there for the members of a partition of unity. However, because we are studying them on arbitrary topological spaces, there is no notion of differentiability. That is all.

(Refer Slide Time: 04:01)



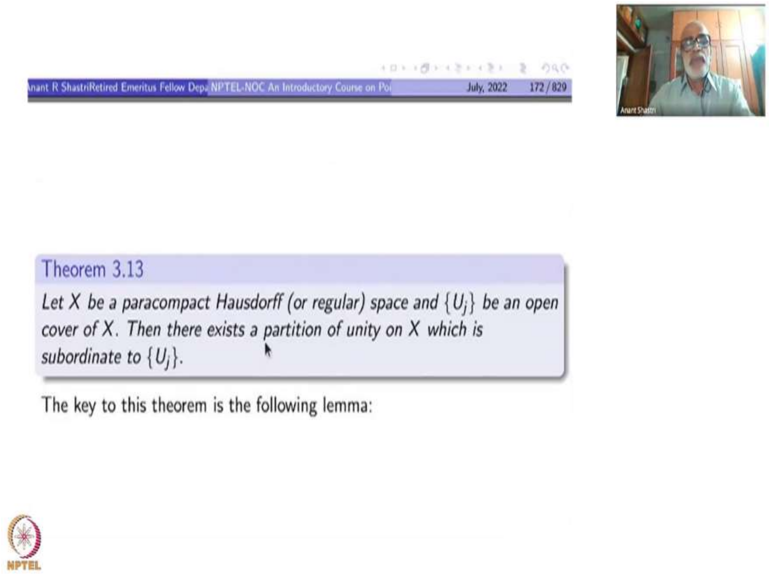
The condition (i) we may be called locally finiteness. It is the same as saying that family $\theta_i^{-1}(0,1]$, of open subsets of X is locally finite. It follows that at any given point x in X, the LHS of this second sum here this sum here is a finite sum and hence makes sense.

So, condition (ii) which ensure that the sum total is equal to 1 means, in particular, that this family of open subsets is a cover for X. Every x must belong to $\theta_i^{-1}(0,1]$ for some i. If it is not true, then the sum total would be 0 at x.

Given any open cover $\{U_j\}$ of X, we say that the family Theta is subordinate to $\{U_j\}$, if this family $\{\theta_i^{-1}\}$ is a refinement of $\{U_j\}$. See this family is indexed by J and this is indexed by I.

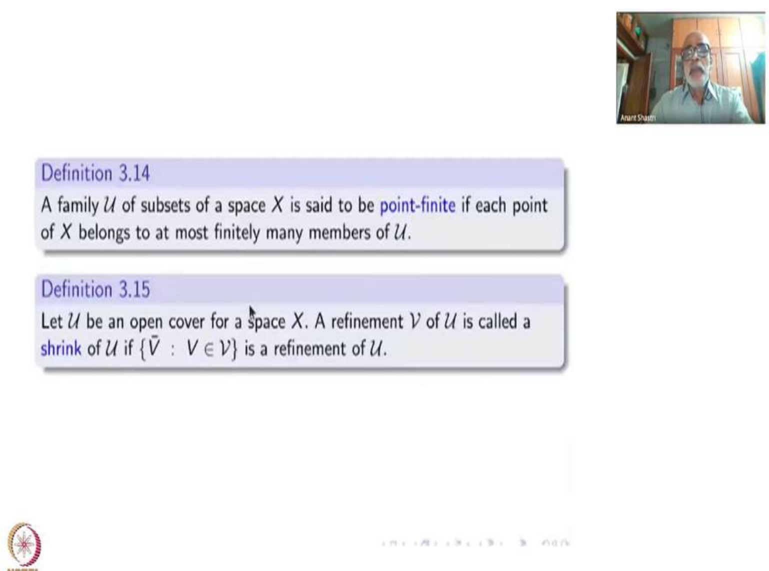
So, indexing sets are different; they may be different or they may be the same. What is the meaning of refinement? For each member here there is a member there which contains it. That itself gives an association, a function on the indexing sets, called the refinement function. We are not writing all that elaborately right now. When theses things are crucial you may have to write down those things also, refinement functions and so on.

(Refer Slide Time: 06:09)



So, here is a theorem. Take a paracompact and Hausdorff or regular space. Let $\{U_j\}$ be an open cover for X. Then there exists a partition of unity on it which is subordinate to $\{U_j\}$. Recall that support of a real or complex valued function is the closure of the set of all points at which the function is non zero. Here we assume that $\theta_i^{-1}(0,1]$ is contained in some U_j. Actually, you will see later, that the family of supports of θ_i itself will be a refinement $\{U_j\}$. The key to this theorem is the following concept and a lemma.

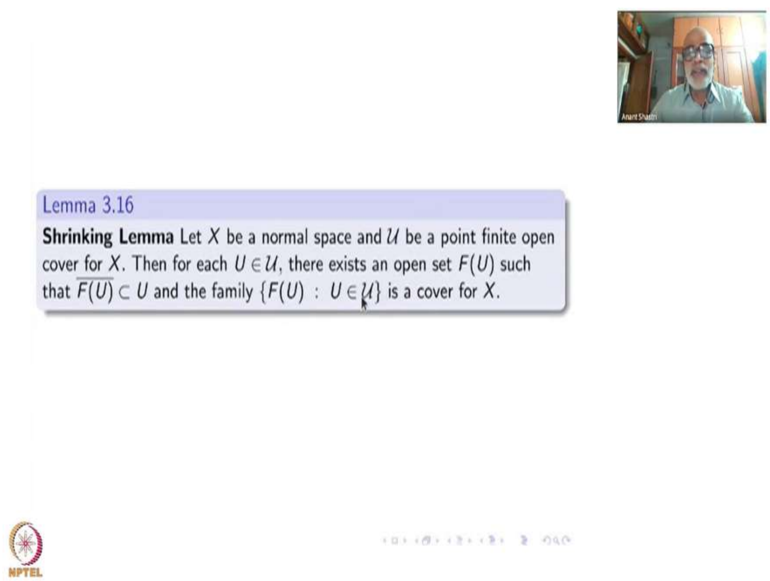
(Refer Slide Time: 06:58)



So, this is where we are going to use Zorn's Lemma also later on. So, first of all we have a couple of definitions. A family $\mathcal{U}$ of subsets of X is said to be a point finite if each point of X belongs to at most finitely many members of $\mathcal{U}$. Here, what I already told the last time, but we repeat it here because it is necessary. A locally finite family will be automatically point finite. But here is another definition which is new and needed for us to proceed with.

Take $\mathcal{U}$, an open cover for X. A refinement $\mathcal{V}$ of $\mathcal{U}$ is called a shrink of $\mathcal{U}$ if the closures of members of $\mathcal{V}$ is a refinement of $\mathcal{U}$. Of course, $\mathcal{V}$ itself must be an open cover for X. So, this is what we would like to have just not just arbitrary refinement, such a thing is called a shrink. If the family of closures of each V inside $\mathcal{V}$ is a refinement of $\mathcal{U}$. If V is contained inside U for each V inside $\mathcal{V}$, that is the order refinement. This is slightly stronger than just a refinement.

(Refer Slide Time: 08:38)

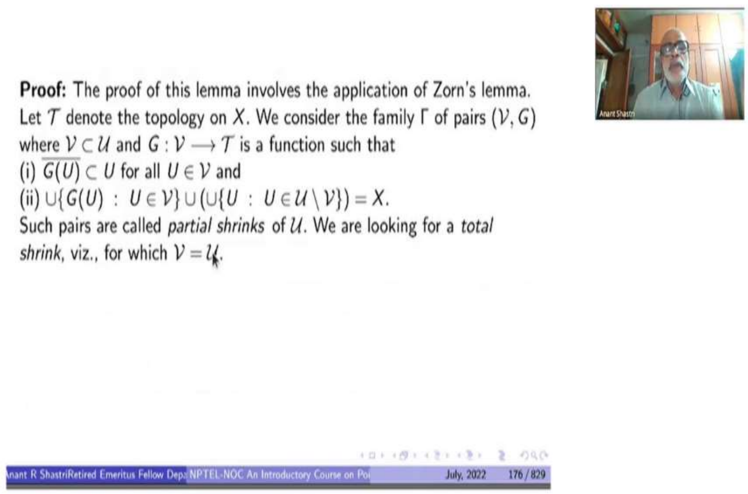


Next, the Shrinking Lemma is something about just a normal space. Here, we do not bring paracompactness, the paracompact Hausdroff spaces are normal. So, some kind of normality is built in there and hidden inside paracompactness. You add Hausdroffness then it comes out. So, though there are possibilities of proving a result without the shrinking lemma, but we will go through this one so that the shrinking property corresponding to normality is brought out separately. So, that is the whole idea of putting this lemma separately.

Take a normal space, $\mathcal{U}$ be a point finite cover. Then for each U belonging to $\mathcal{U}$, there exists open set $F(U)$ such that $\overline{F(U)}$ contained inside U and the family $\{F(U), U \in \mathcal{U}\}$ is a cover for X. So, if you call this new family $\mathcal{V} = \{F(U), U \in \mathcal{U}$, then $\mathcal{V}$ is a shrink of $\mathcal{U}$.

Therefore, this will be called a shrink you see, so that is the whole idea, so $\overline{F(U)}$ is a shrink for this one. The family $F(U)$ is a cover because in the definition of shrink we are not putting the cover. So, this shrinking lemma tells you that there is a cover which is a shrink of the given family, indexing will be the same here for each U you have $F(U)$, that $F(U)$ comes back to you in fact $\overline{F(U)}$ is inside. The statement must be clear here..

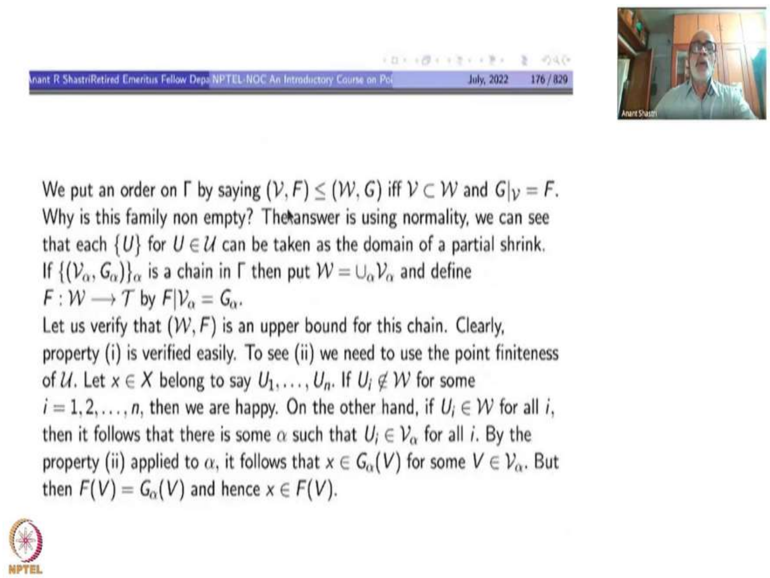
(Refer Slide Time: 10:38)

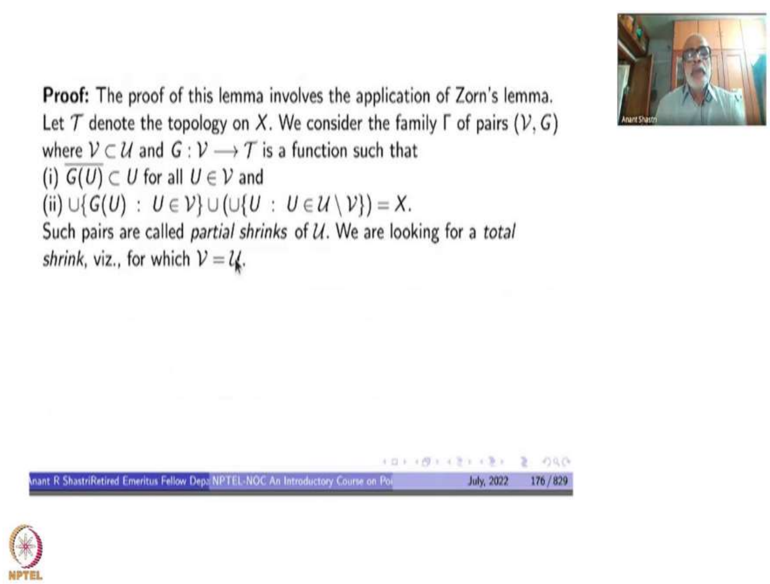


But the proof depends upon using Zorn's lemma. So, how do we use Zorn's lemma? We start with a family Gamma of pairs $(\mathcal{V}, G)$, where $\mathcal{V}$ is a subfamily of U and G is a function from $\mathcal{V}$ to tau the family of open subsets inside $(X, \mathcal{T})$ is the topology on X, G is a function now. For each member here you will get an open subset here with the property that the closure of $\overline{G(U)}$ is contained inside U for all U inside the sub family $\mathcal{V}$. The second thing is that the those $G(U)$ were U is inside $\mathcal{V}$ and all those U which are not inside $\mathcal{V}$, they cover the whole of X.

See that if $\mathcal{V}$ is the whole of $\mathcal{U}$ then this would be a total shrink. Since in general this not the case, we use the word partial shrink, for members of Γ. What we are looking for is a member of Γ in which $\mathcal{V}$ is the whole of $\mathcal{U}$, so nothing is left here. So, that is the statement of the lemma.

(Refer Slide Time: 12:49)





So, how do we prove it? On this family Γ, we put a partial order. Say $(\mathcal{V}, F)$ is less than or equal to the other one $(\mathcal{W}, G)$, if and only if the family $\mathcal{V}$ is a sub family of this $\mathcal{W}$ and the function F is the restriction of the function G to $\mathcal{V}$. Why this family Gamma is non empty? Answer is using normality, we can see that each singleton U for U belonging to $\mathcal{U}$ can be taken as a domain for a partial shrink. U and rest of the members of $\mathcal{U}$ they cover.

So, if you take the union of all $V \neq U$ that is one open set use another open set, these two together they cover the whole of X. The complements will be disjoint closed sets with normality you can take as much slightly open subsets containing side the closure of all these open sets, union of all this open set that will be contained inside this U and so on. So, you get a partial cover. So, this is easy part.

(Added by the reviewer: If the rest of the members of $\mathcal{U}$ cover the whole of X, then define $G(U) = \emptyset$. Otherwise. the complement of the union of all these other members is a closed set C contained in U. By normality, there is an open set call it $G(U)$ such that C is contained in $G(U)$ contained in $\overline{G(U)}$ contained in U. This G will do the job.)

Now, if you have a chain inside Γ, chain means what, a totally ordered subset, indexed by a some totally ordered set A, say $\{(\mathcal{V}_\alpha, G_\alpha), \alpha \in A\}$. I must show that the chain has an upper bound inside Γ. So, for that I take $\mathcal{W}$ as union of all these $\mathcal{V}_\alpha$'s that will be some subfamily of $\mathcal{U}$, that is fine. But now, we want a function on $\mathcal{W}$.

So, define F from $\mathcal{W}$ to $\mathcal{T}$ to be such that restricted to $\mathcal{V}_\alpha$ to be G_α. Now A is a total ordered set, if α and β are given, either α is less than β or β is less than α that is the meaning of total order, say α is less than β. Once that is a case, $\mathcal{V}_\alpha$ is contained in $\mathcal{V}_\beta$ and G_β restricted to $\mathcal{V}_\alpha$ is G_α. Therefore, this definition of F makes sense on the whole of $\mathcal{W}$. there is no ambiguity here.

So, we have got a pair $(\mathcal{W}, F)$, which is not yet a member of Γ. We have to verify that it is a partial shrink, then it will be a member of Γ. Automatically it will be the upper bound for the chain.

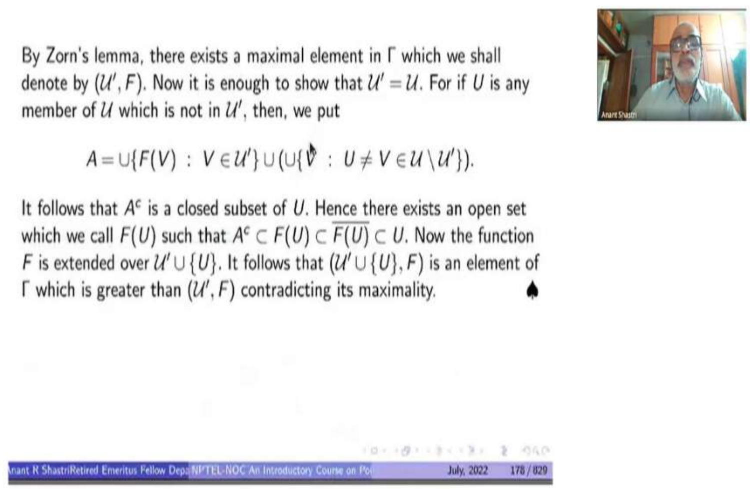
So, we have to show that this pair is a partial shrink. So, property (i) is verified easily, $\overline{F(U)}$ is contained in U, by the very definition here because they are all G_α's. See property (ii) is the important thing. How do we see (ii)? Namely, why all the $G(U)$, U inside $\mathcal{W}$ together with those members of $\mathcal{U}$ not in $\mathcal{W}$ cover the whole of X? So, to see we need to use the point finiteness of $\mathcal{U}$. I have no other way to prove this one.

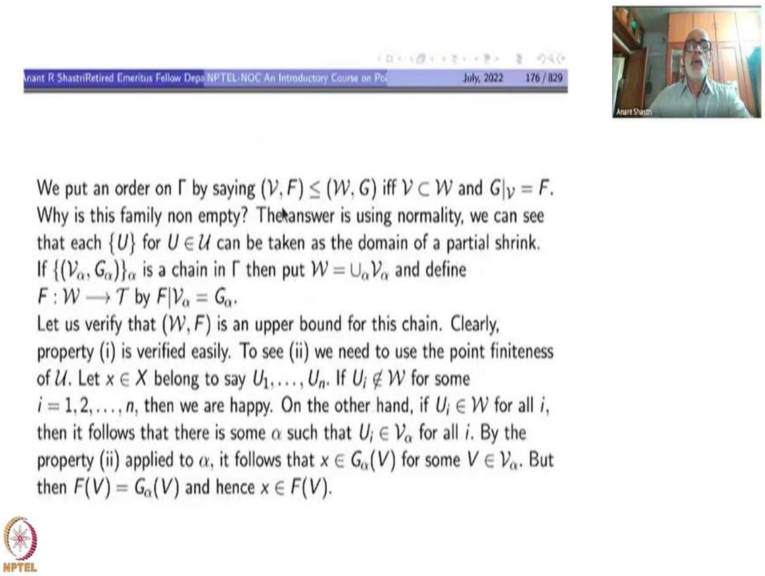
Take x belonging to X. Then x belongs to only finitely many members $U_1, U_2, \ldots, U_n$ of $\mathcal{U}$. If U_i s are not in $\mathcal{W}$, even if one of the U_i is not in $\mathcal{W}$, it will be in this part and therefore, the point x is there, no problem. It is the other case, viz., when all of U_i are in $\mathcal{W}$, which gives you problem because you have shrunk these members, you have made them smaller, so the point maybe left out. So, you have to worry about that.

So, if one of the U_i's is not inside $\mathcal{W}$ you are happy. On the other hand, suppose if all the U_i's are inside $\mathcal{W}$, $U_1, U_2, \ldots, U_n$ are in $\mathcal{W}$. Then it follows that there are indexes $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that U_i belongs to V_{α_i} for each i. Therefore, you can take the maximum of these α_i's, say α, so that there will be one single V_α to which all U_i's will belong.

So, this is where the property of being a chain is used. By the property (ii) applied to this α, it follows that x must be in $G_\alpha(V)$ for some V inside V_α because it is not in other part, that is all. So, all that I wanted to show you is that x is covered by $F(V)$ for some member V of $\mathcal{W}$. But for this V in $\mathcal{V}_\alpha$ we have $F(V) = G_\alpha(V)$ by the definition of F. So, $(\mathcal{W}, F)$ is an upper bound of the chain.

(Refer Slide Time: 19:35)





By Zorn's lemma, we have a maximal element in Γ. Remember this itself may not be maximal element. This is an upper bound for the chain. There is a maximal element in Γ. We do not know what is that maximal element. But that is good enough for us. So, we shall denote it by $(\mathcal{U}', F)$. Remember $\mathcal{U}'$ is a subfamily of $\mathcal{U}$, F is a shrink function. Now, it is enough to show that $\mathcal{U}'$ itself is equal to $\mathcal{U}$.

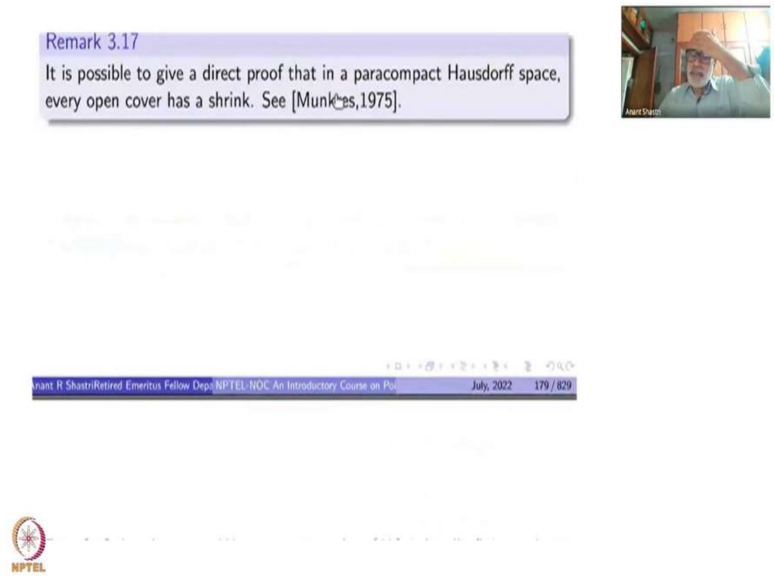
But if U is any member of $\mathcal{U}$, and not in $\mathcal{U}'$, then look at this set A which is union of all $F(V)$, V inside $\mathcal{U}'$ and all V so set V is not equal to U, and inside $\mathcal{U} \setminus \mathcal{U}'$. A is an open subset of X such that together with U it covers the whole of X.

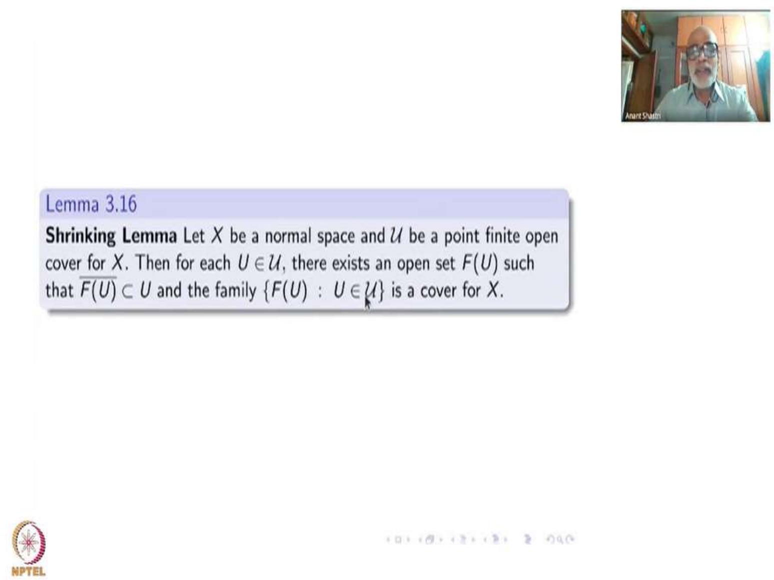
It follows that the complement of this open set is is a closed subset of U. Therefore there exists an open set which we call $F(U)$ such that the closed set A^c is contained inside $F(U)$ contained in $\overline{F(U)}$ contained inside U. Normality is used again now. Normality was used to show that Γ is non-empty that is all. So, right in the beginning right at the end.

(Added by Reviewer: It may happen that $A = X$, in which case, we can take $F(U) = \emptyset$.)

Now, the function F extend to $\mathcal{U}' \cup \{U\}$ as above, (for all members of $\mathcal{U}'$ other than U, this is the old F itself) gives you a member of Γ this is a larger than the member $(\mathcal{U}', F)$. That is a contradiction. Why is the contradiction? Because we assumed that $\mathcal{U}'$ is not $\mathcal{U}$. That completes the proof of the shrinking lemma.

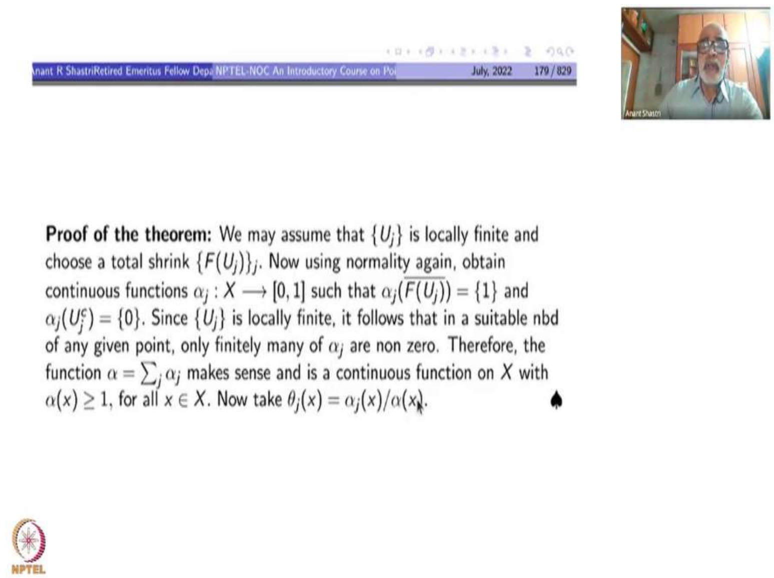
(Refer Slide Time: 22:29)

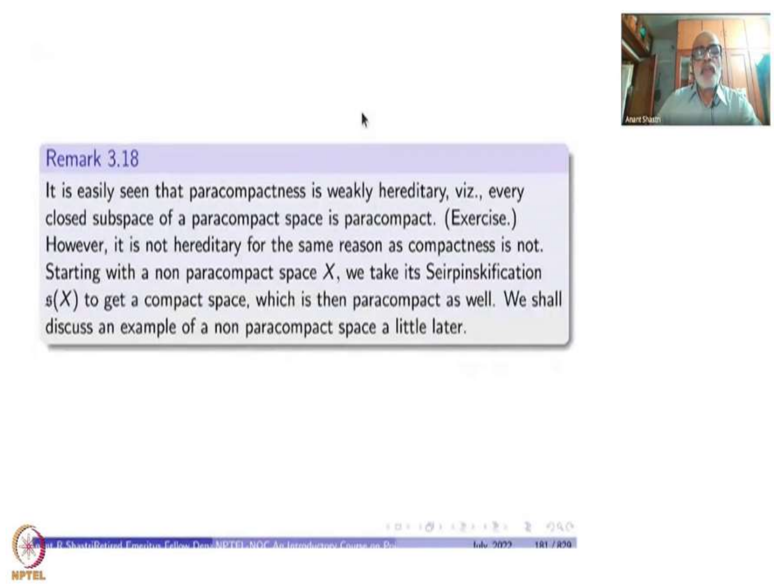




So, I have already made this remark. For relevant things you can read from Munkres book that will give you a proof of the fact that for a paracompact Hausdroff space, every open cover has a shrink, no assumption of local finiteness on the cover, local point finiteness on the cover. But do not confuse it with the statement above. Just on a normal space if you have an open cover, it may not admit a shrink.

(Refer Slide Time: 23:20)





So, here is a comment and then we will stop here. The actual proof of the theorem is now very easy. So, first let us go through that proof.

We may assume that $\{U_j\}$ is locally finite. Because by paracompactness, so start with the open cover, replace it by locally finite refinement. So, you can assume the locally finiteness for the given cover itself. Now choose a total shrinking function F, which is possible because we have proved that a paracompact Hausdorff space is normal, and this local finite implies point finiteness.

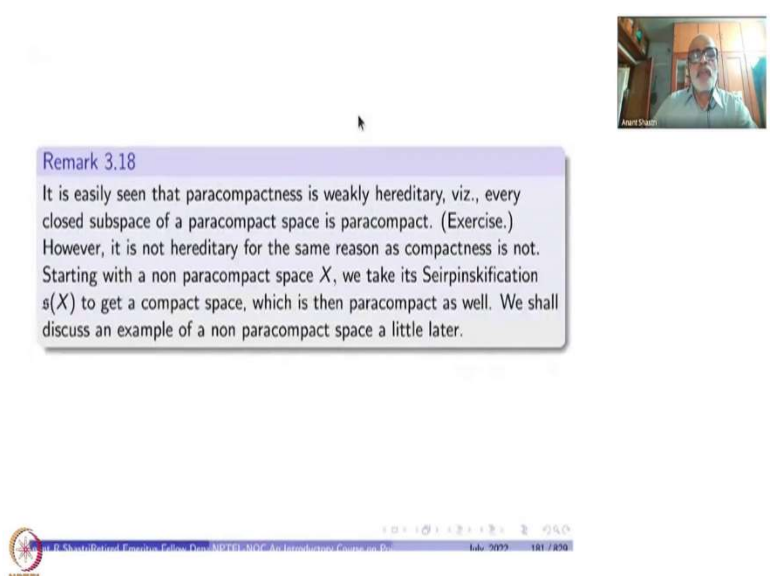
Using normality again, obtain continuous functions α_j from X to $[0,1]$ such that the closures of $F(U_j)$'s are taken to 1 and U_j^c's are taken to 0. This is by normality. Whenever you have two disjoint closed subsets you have such functions. This $\{U_j\}$ is locally finite, local finiteness of the family $\{\alpha_j\}$ follows.

Therefore, the function α which is sum of all α_j's make sense. In a small neighborhood of every point, only some finitely many of them will be non zero. Since each α_j is continuous on such a neighborhood, it follows that the sum α is a continuous function on the whole of X.

Moreover, since we have total shrink, given any $x \in X$, it belongs some $F(U_j)$ and that that $\alpha_j(x) = 1$. At the rest of the indexes it will be either 0 or positive. Therefore, the sum total is always bigger than or equal to 1. Now, all that you have to do is take $\theta_j(x) = \alpha_j(x)/\alpha(x)$. You have a continuous function which is never 0, therefore, you can divide by that function. It is still continuous. Now, summation θ_j will be summation α_j/α_x which is equal to 1.

What is the zero set of θ_j? It is the same as the zero set of α_j. Therefore, the support of θ_j will be contained inside the corresponding U_j. So, the proof existence of partition of unity, finally. It is very simple, very easy. So, major work went in proving the shrinking lemma thing here, namely, using Zorn's Lemma.

(Refer Slide Time: 27:12)



So, here is a comment. It is easily seen that paracompactness is weekly hereditary, namely every closed subspace of paracompact space is paracompact. Just for the sake of clarity write it down as an exercise. However, it is not hereditary for the same reason as compactness is not hereditary, weakly hereditary, yes. For closed subspace, fine, but for arbitrary subspaces, may not be.

How to see that? Starting with a non-paracompact space if there is one (we will show that there are such things), you can always add an extra point, take the Sierpinskification to get a compact space which is then paracompact as well. We shall discuss an example of a non-paracompact space later. Thank you, that is enough for today.