An introduction to Point-Set-Topology Part-II
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Lecture 10
Compactly Generated Spaces

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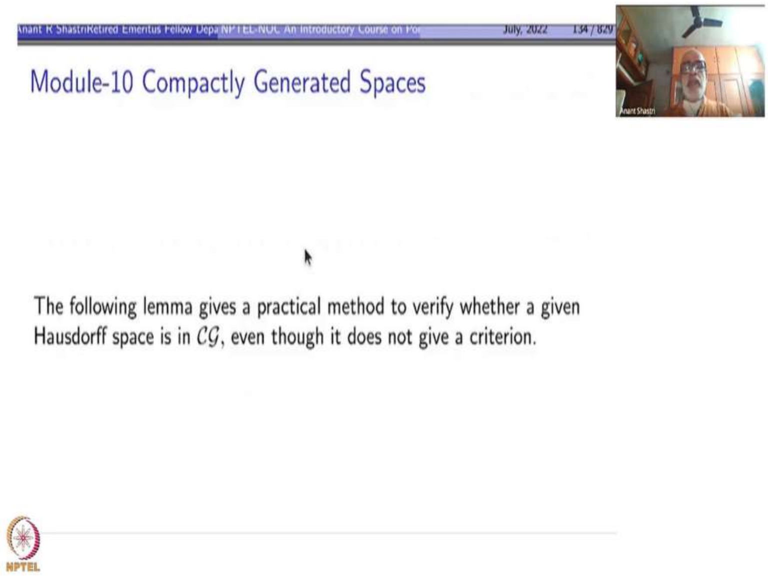

Hello, welcome to NPTEL NOC course on point-set-topology part 2. Today, we shall do compactly generated spaces, further. Last time, I already told you that it is not easy to find examples of spaces that are not compactly generated. So, now, we will see a lot of examples of compactly generated spaces, they come from a wild spectrum, quite unexpectedly. So, let us go through these things carefully.

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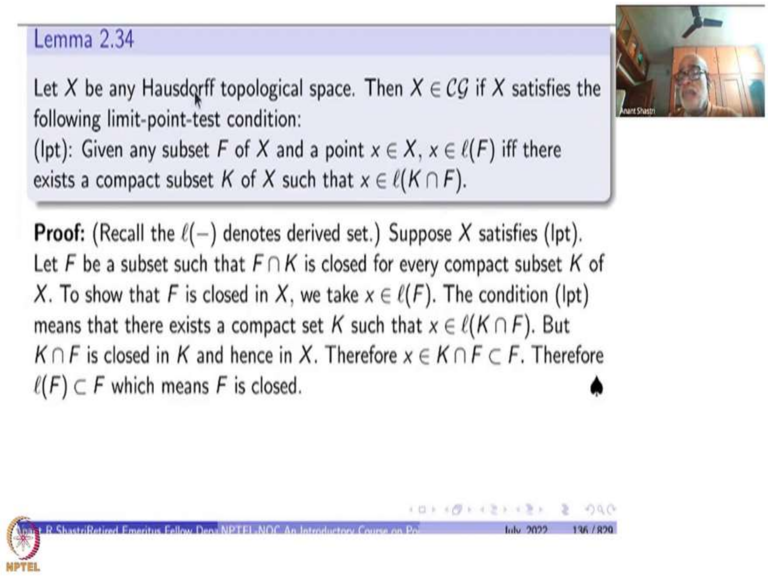

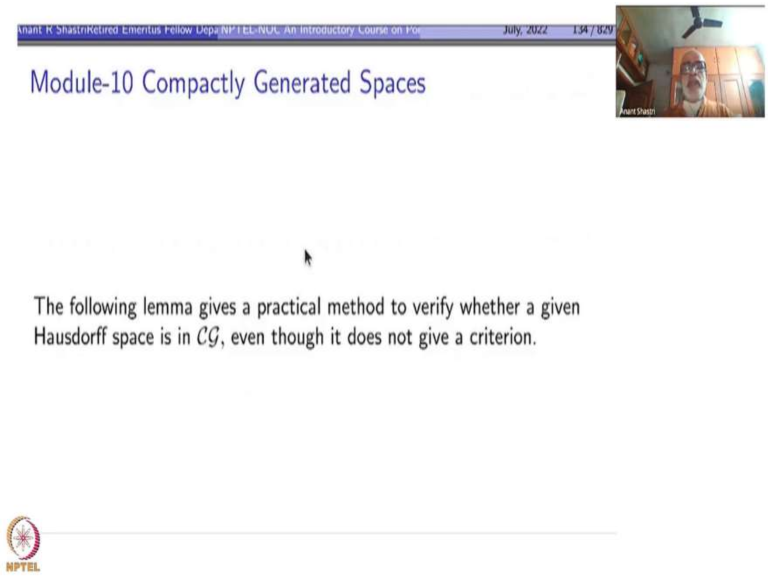

Let X be any Hausdorff topological space. We are deliberately restricting ourselves to Hausdorff spaces. If not, we actually can get many more examples but from practical point of view, they are not much important. So, let us restrict ourselves to Hausdorff spaces.

Then, X is in $\mathcal{CG}$, if X satisfies the following limit-point-test condition, I am calling it a limit-point-test condition, you will see why. So, I will just use the short form lpt. So, what is this condition?

Given any subset F of X, and a point x inside X, x is a limit point of F, (which I denoted by x in $\ell(F)$, remember, we have introduced this notation, for the set of limit points of F. It is also called the derived set of F), if and only if, there exists a compact subset K of X, such that x is a limit point of $K \cap F$.

So, pay attention to the clause `there exist a compact subset'. And not `for every compact subset', which will be totally incorrect. Given a point $x \in X$, the point is inside $\ell(F)$, if there exist some K, such that $x \in \ell(K \cap F)$. This `if' part is easy and is always true. The `only if' part is the real extra condition here. Of course, it now becomes `if and only if'. That is important. So, if this condition is satisfied then, X is $\mathcal{CG}$, that is the lemma. This lemma therefore gives you a nice sufficient condition. But this is not a criterion. This condition lpt may not hold and yet X may be in $\mathcal{CG}$. However, I do not know any example of a space in $\mathcal{CG}$ which does not satisfy this condition. I am sure that there are such examples, otherwise this will not be stated like this. It would have been stated as a criterion. So, such an example may be quite complicated. So, that is the story.

So, let us go through the proof of this lemma. Let F be a subset such that $F \cap K$ is closed for every compact subset K of X. Then, we have to show that F is closed in X. That is what X belonging to $\mathcal{CG}$ means.

So, to show that F is closed in X, we take a point $x \in \ell(F)$ and show that this x is inside F itself. Then we are through, why? We know that if F is closed, then $\ell(F)$ must be inside F. That is easy. But also, $\ell(F)$ is inside F implies F is closed, because what are the closure points of F? They are either points of F, or there limit points of F. Therefore, limit points of F is contains inside F, means $\bar{F}$ is contained inside F. Therefore, $\bar{F}$ is equal to F.

So, I start with a point x inside $\ell(F)$ and I want to show that it is inside F. Now, condition lpt comes into picture. x is in $\ell(F)$ means there is a compact subset K of X such that x is inside $\ell(F \cap K)$. But as soon as I take a compact subset K, F satisfies the condition that $F \cap K$ is closed in K.

So, x is a limit point of $F \cap K$, but $F \cap K$ is closed in K Therefore, it is closed in X also. (Remember, we started with a Hausdorff space X, and hence compact subsets of X are closed in X. This is why Hausdorffness is important here.) Something is closed in K, therefore, it is closed in X. Therefore, the limit point x is inside $F \cap K$ and hence inside F.

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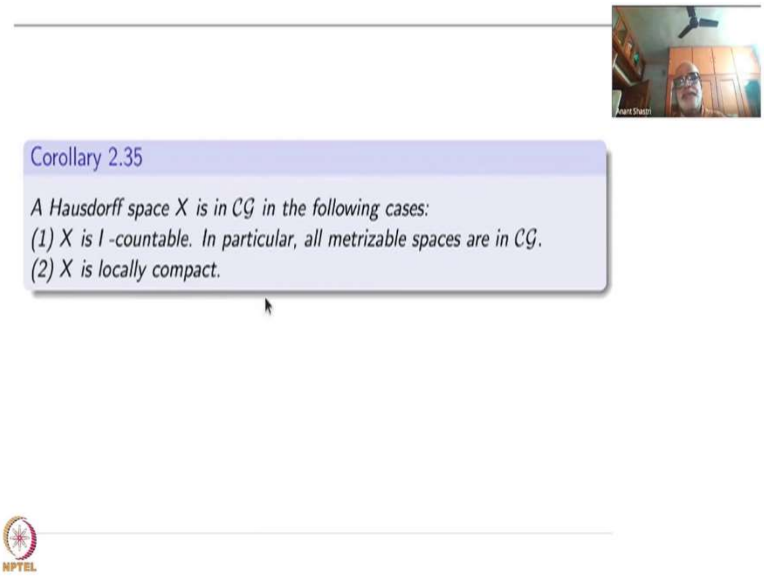

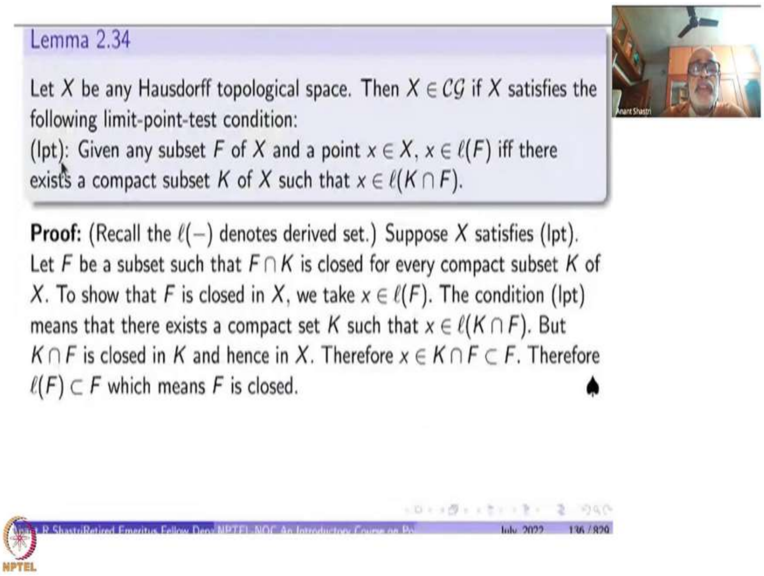

As a corollary, we have two classes of compactly generated spaces. What are they?

(i) Any first countable space is compactly generated.

(ii) Any locally compact space is (that is of current interest to us) also compactly generated.

In the back of my mind, I am also assuming that they are Hausdorff. A Hausdorff space X will be in $\mathcal{CG}$ if one of these two conditions is satisfied, viz., Hausdorff plus first countable or Hausdorff plus locally compact. See two are completely diverse fields. We have also seen something like this happening earlier. Metric spaces and any locally compact spaces are quite different things, but both of them are Baire spaces. Now, here is another example, where-in first countability of metric spaces and local compactness are coming together in a different way. So, this is just for your observation and see every metric space is first countable. This time, you do not have metric space, it is first countability only. Local compactness and I countability are completely different concepts, but both of them along with Hausdorffness will give you compactly generatedness. Why? So, I have put this as a corollary. To prove it, all that you have to do is to verify that both of them satisfy lpt, and then apply this lemma.

So, how do you see that they are satisfying lpt? As soon as you are in a Hausdorff first countable space, a point is a limit point if and only if you have a sequence converging to that point. Along with that sequence, take that point also. Suppose x_n converges to x, then take the set of all x_n together with x is always a compact set. So, actually x will be a limit point of that compact set. That verifies lpt. For local compactness, it is easier, because take a point x belong to limit point of some set, there will be a neighbourhood around that point which is compact. You do not have to go outside that neighbourhood at all to see whether it is a limit point, you can restrict yourself to that compact set already. So, local compactness immediately gives you that lpt, very easy, the first countability also gives you because of this a sequence converging to a point union with the limit point is always compact set.

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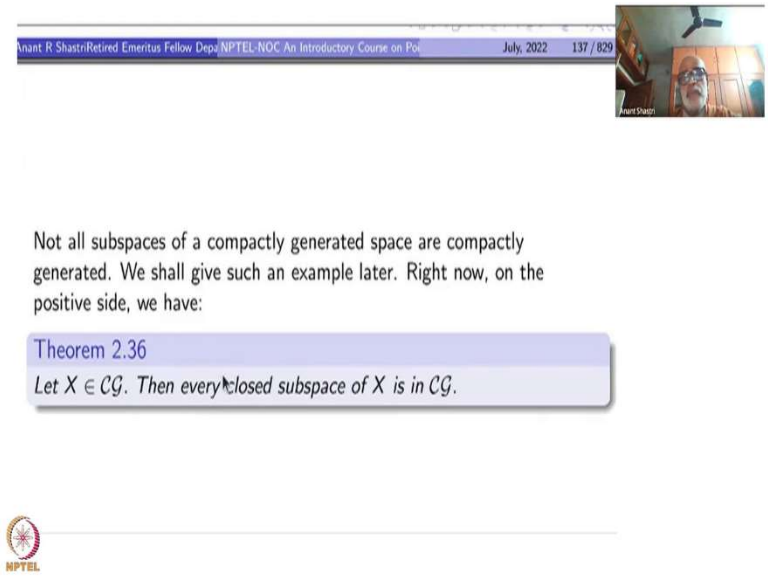
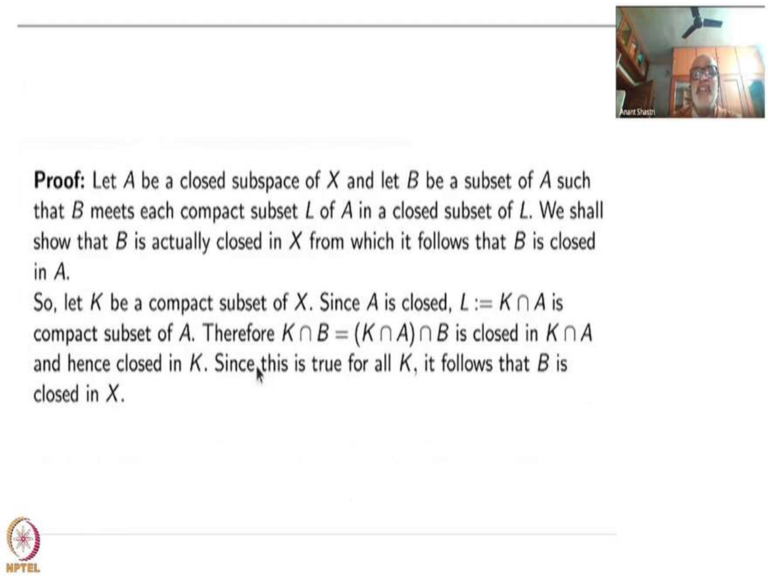


Not all subspaces of a compactly generated space are compactly generated. By this I mean that, compactly generatedness is not hereditary. To see such an example, we have to wait a little bit, it is not coming so easily. Right now, on the positive side, we have the following. Take X belonging to $\mathcal{CG}$, Then every closed subspace of X is in $\mathcal{CG}$. That is, $\mathcal{CG}$ weakly hereditary, in one sense only but not globally weakly hereditary, does not hold for open subsets. X is on $\mathcal{CG}$ every closed subspace is compactly generated. (Refer Slide Time: 11:26)



The proof is not very difficult. A be a closed subspace of X, B be a subset of A such that B meets each compact set L of A inside a closed subset of L. We want to show that, B is actually closed in X from which it follows that B is closed in A. See, X is compactly generated A is a closed subspace, I want to show that A is compactly generated. So, what should I do? Start with a subset B of A which has this property, B meets each compact subset L of A, in a closed subset of L. From this I have to show that, this B is closed in A, but I will actually show the B is closed in X.

So, let K be a compact subset of X. Since A is closed in X, $L = K \cap A$ is a compact subset of A. A is closed, so $K \cap A$ will be a closed subset of the whole space. So, it is a closed subset of a compact subset, and therefore a compact subset of A also. Therefore, $K \cap B$, which is $K \cap A \cap B$, because B is already a subspace of A, is a closed in $L(= K \cap A)$, and hence closed inside K. Since this is true for all K, and X is in $\mathcal{CG}$, it follows that B is closed in X.

Let us make a definition.

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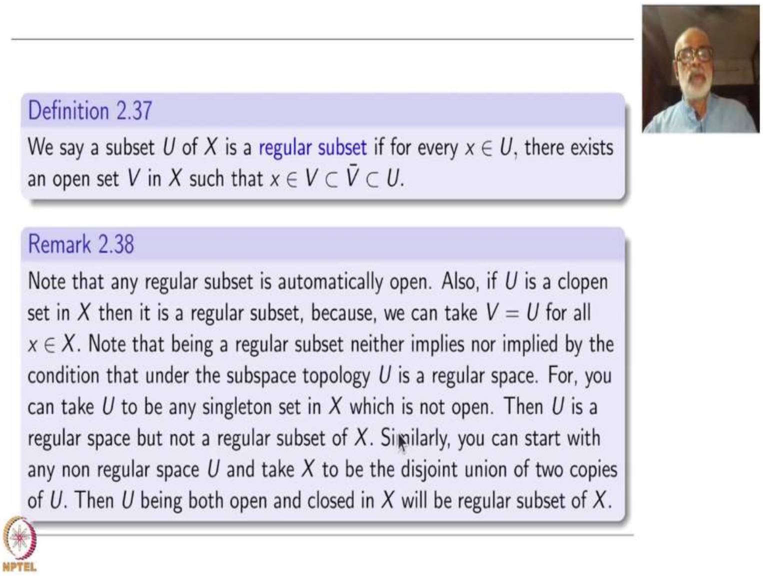

We say a subset U of X is a regular subset, if for every x inside U, there exist an open set V in X, such that x belongs to V contained in the $\bar{V}$, $\overline{V}$ contained inside U. To begin with, though we have used the usual notation for open set, there is no assumption that U should be open. But however, this condition for every x is inside U is an open set V in X. So, x belongs to V which is contained inside U, automatically says that any regular subset U has to be open. Also, if U is a clopen set, that is closed and open, then it is automatically regular because we can then take V equal to U for all x, x belongs to U, U contained inside $\bar{U}$ because U is clopen, and $\bar{U}$ is U. So a clopen set is regular.

So, why the word `regular' is used? Only because of the resemblance of this condition to regularity. In that you have to be very careful. Note that being a regular subset neither implies nor implied by the condition that under the subspace topology, U is regular space. So, these two notions are quite different. For you can take U to be any singleton set in X which is not an open set. There are plenty of subspaces, then U as a singleton space is already a regular space, but not a regular subset, because it is not open inside X.

Similarly, you can start with any non-regular space U, (you can allow it to be Hausdorff also, and there are such spaces which you have seen) and then take X to be disjoint union of two copies of U. Then each copy of U will be open in X as well as closed. Therefore it will be regular subset of X. But as a space on its own, it will not be regular, because we started with a non-regular space. So, that these two examples should convince you that this is just an adhoc definition. This is going towards what? to ensure that certain types of open subspaces of a compactly generated space are again compactly generated.

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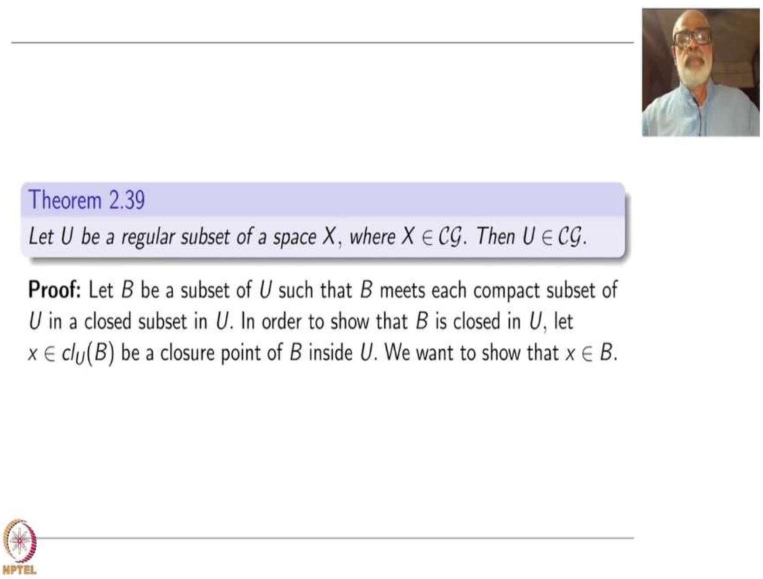

So, this is a theorem. Let U be a regular subset of a space X. Then X is in $\mathcal{CG}$ implies U is in $\mathcal{CG}$. Let B be a subset of U such that B meets each compact subset of U in a closed subset. In order to show that B is closed in U, let x belong to U be a closure point of B. We want to show that x is inside B.

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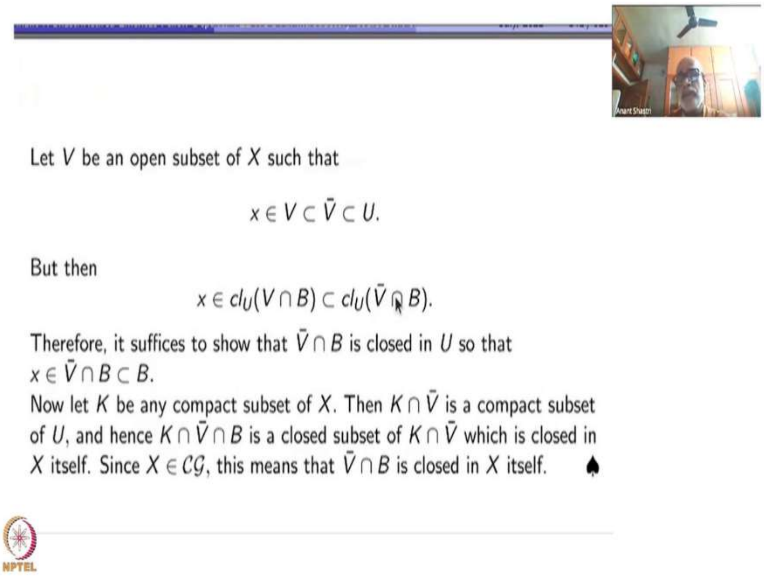

So, let V be an open subset of X such that x belongs to V which is contained in $\bar{V}$ contained in U. This is by the regularity of the regular subset U. But once we have an open subset of the whole space X containing x, this will imply that x is in the closure of $V \cap B$ itself. That means, it is contained inside the closure of $\bar{V} \cap B$, because it is the larger set, that is all. Therefore, it suffices to show that $\bar{V} \cap B$ is closed in U, so that x will be inside $\bar{V} \cap B$, but $\bar{V} \cap B$ is contained inside B. Our aim is to show that x is inside B.

Therefore, we have come to the point of showing that $\bar{V} \cap B$ is closed inside $\bar{V} \cap B$, everything is happening inside $\bar{V} \cap B$. So, take a compact subset of X. Then $K \cap \bar{V}$ is a compact and it is contained inside U. Therefore it is a compact subset U. And hence, $K \cap \bar{V} \cap B$ is a closed subset of $K \cap \bar{V}$. This is from the hypothesis on B. We have started with a B which has this property. So, it is closed in $K \cap \bar{V}$ which is closed inside X itself.

So, $K \cap \bar{V} \cap B$ is a closed subset of X. But X is $\mathcal{CG}$ and we have verified that for arbitrary K , $K \cap \bar{V} \cap B$ is closed in K. Therefore, $\bar{V} \cap B$ is closed in X. That means its closure is itself. Therefore x is in $\bar{V} \cap B$ and hence in B, as required.

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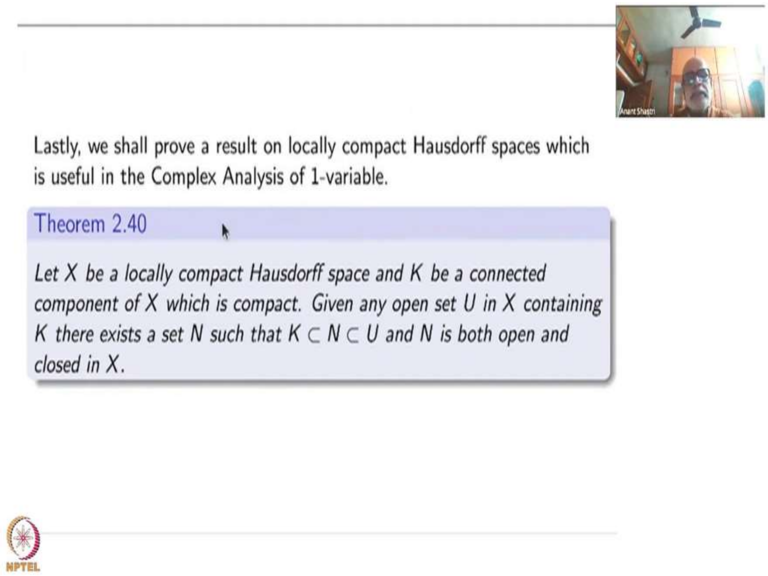

Before ending up today's lecture, let me give you one very important application of this one in complex analysis. Why I say complex analysis? I do not know personally any application of this in other analysis, but that does not mean that it is not applicable. So, as far as complex analysis is concerned, I have used it and I know and I have taken this theorem from a very famous book viz., Raghavan Narasimhan's complex analysis of one variable.

So, what does it say? It is a peculiar statement here, this kind of study we will do later on in the course, in a different context altogether.

Right now, take a locally compact Hausdorff space X, and a connected component K of X which is compact. Given any open set U in X containing K, there exists a set N such that K is inside N contained inside U, and N is both open and closed in X.

Think of this for the case when K is singleton. (Single point could be a component also. And, single point is automatically compact). What does it mean? This means inside every neighbourhood U of that point, we have another neighbourhood which is clopen set which both open and closed. Not just an open neighbourhood whose closure is contained in U.

So, this is happening to every compact component of X. So, I suspect that this will be useful in all these complex dynamics also, not just complex analysis of one variable, wherein you have to study Julia sets and such things. Let us see the proof. Proof is not all that easy at all. Indeed, as a habbit, when I read a new result like this, I try to prove it myself. But with this one, I could not prove it myself.

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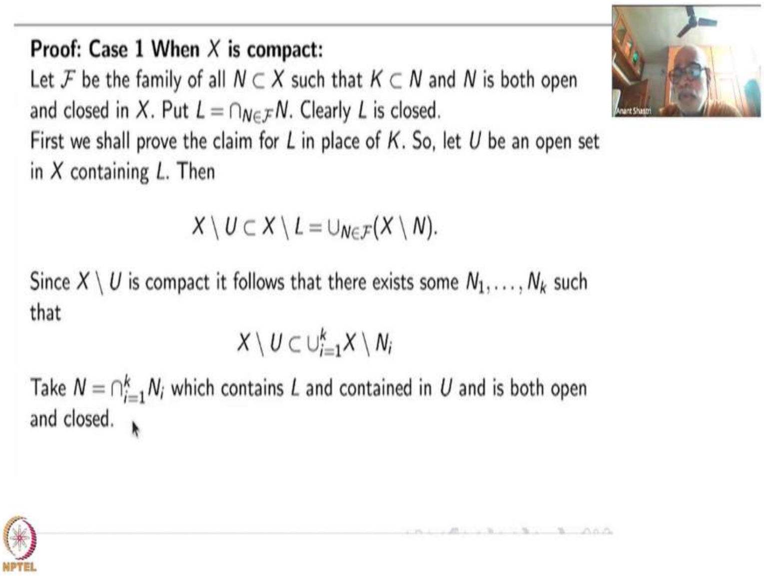

So, first consider the case when X itself is compact. So, we are going step by step here. Assume X itself is compact. That does not mean that it is connected, you will have to take a component, it will be compact as well, fine. So, connected component are compact. For each compact component something is happening, that is what we want to show. Let $\mathcal{F}$ be the family of all N contained inside X, such that K is contained inside N, N is both open and closed. We do not know whether this family $\mathcal{F}$ is non-empty one. We have just taken $\mathcal{F}$ like this. Let L be the intersection of all members N of $\mathcal{F}$. Since each member N is closed, therefore L is closed.

Clearly all members contain K. Therefore, this L will contain K.

First, we shall prove the claim for L in a place of K. So, we are trying to prove something for K, but we have taken a set slightly larger than K, maybe very large, I do not know, but it contains K. So we shall prove it for this L. L itself is not assumed to be connected or anything. We do not know that. But what we will show is that given any open neighbourhood U of L, there exist a closed and open subset (like N) containing L and inside set at open subset U.

So, let U be an open subset of X containing L. Then look at $X \setminus U$. It is contained in $X \setminus L$, which is the union of all $X \setminus N$, N ranging over $\mathcal{F}$. This is by De Morgan law, because L is the intersection of all N's. Now, $X \setminus U$ is compact, because U is open and X is compact. (So, that is where this assumption that X is compact being used.) It follows that you have finitely many $N_1, N_2, \ldots, N_k$, such that $X \setminus U$ is contained inside the union of $X \setminus N_i$, i ranging from 1 to k.

Now, take N to be the intersection of these finitely many N_i's. Finite intersection of clopen sets is clopen. Again by DeMorgan law, this N contains L and contained in U.

So, we have already proved this for L.

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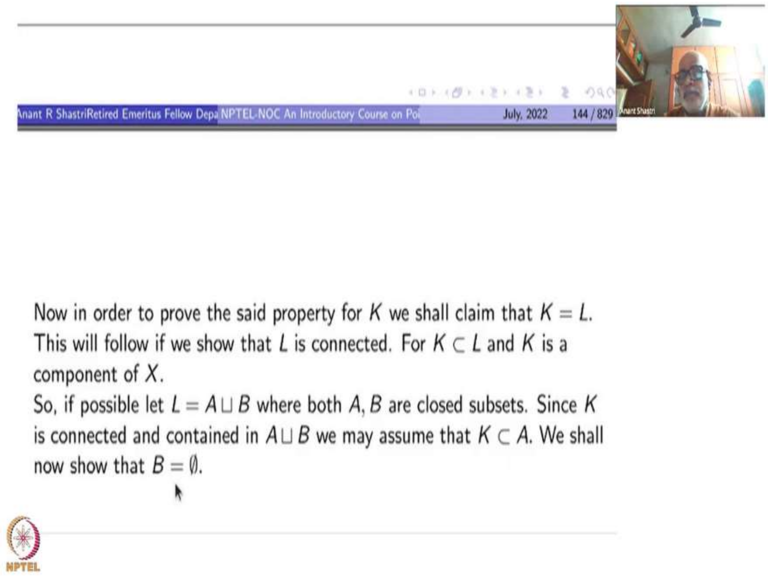

Now, in order to prove the property for K, we shall actually prove that K is L. So, intersection of all such neighbourhoods is actually K is what we shall prove. This will follow if we show that L itself is connected. Any connected subset larger than a connected component has to coincide with it. K is a component of X. So, if we show that L is connected, K must be equal to L. Alright?

If possible, let L be the disjoint union of two closed subsets, non empty closed subsets, A and B say. Then we will get a contradiction. K is connected, and contained in the union. Therefore, it must be contained in one of them, only one of them. So, K is contained inside A let us say.

So, we shall actually prove that B is empty. So, we are assuming that A disjoint union B is L, and we want to show that B is empty, under the assumption that K is inside A, alright?

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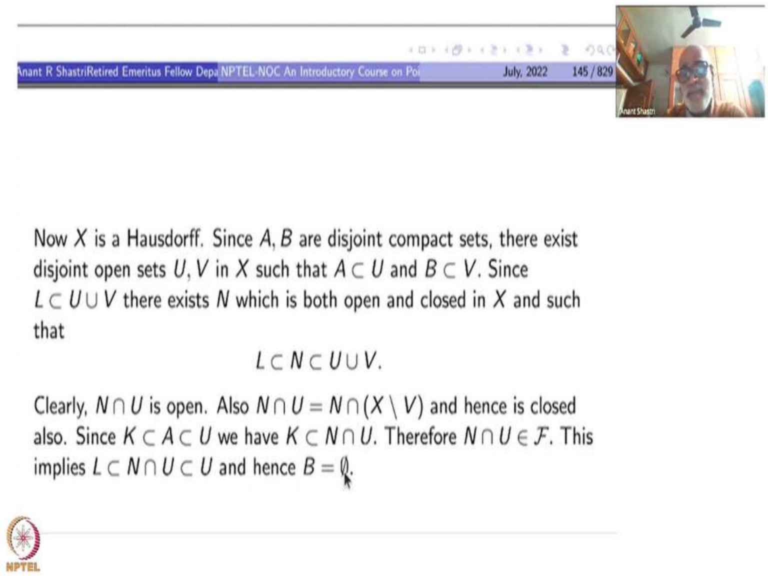

Now, use the fact that X is Hausdorff. Since A and B are disjoint compact subspaces, there exist disjoint open subsets U and V in X such that, A is inside U, B is inside V. Remember this theorem, in a Hausdorff space two disjoint compact subsets can be separated by open subsets. This we have proved, first by proving it for a compact set and point outside, and then improving this for two disjoint compact sets. I am using that theorem here. A and B are disjoint closed subsets of X, therefore they are compact. (So again, I am using the hypothesis that X is compact here, alright.)

So we get two disjoint open subsets U and V around A and B respectively. But now L is contained inside $U \cup V$. Therefore, there exist N, which is both open and closed in X, and such that L is contained in N which contained $U \cup V$, because the latter is an open subset.

Clearly, $N \cap U$ is open, because N is open and U is open. Also, $N \cap U$ is $N \cap (X \setminus V)$, hence it is closed also, because N is closed and V is open so XV is closed. Since K is contained inside A, and A is contained inside U, we have K is contained inside $N \cap U$.

Therefore, $N \cap U$, which is both open and closed and contains K is a member of $\mathcal{F}$. This means that L is contained is $N \cap U$, which is contained inside U, but L is $A \cup B$, therefore B be must be empty.

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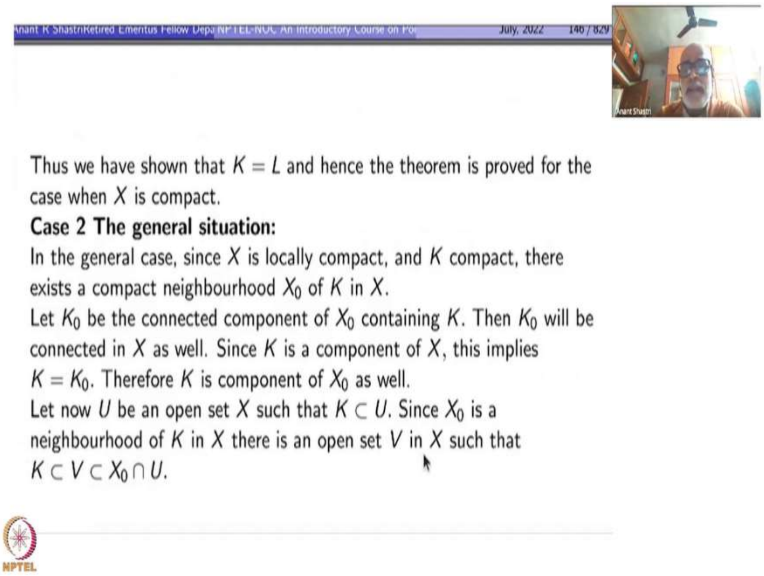

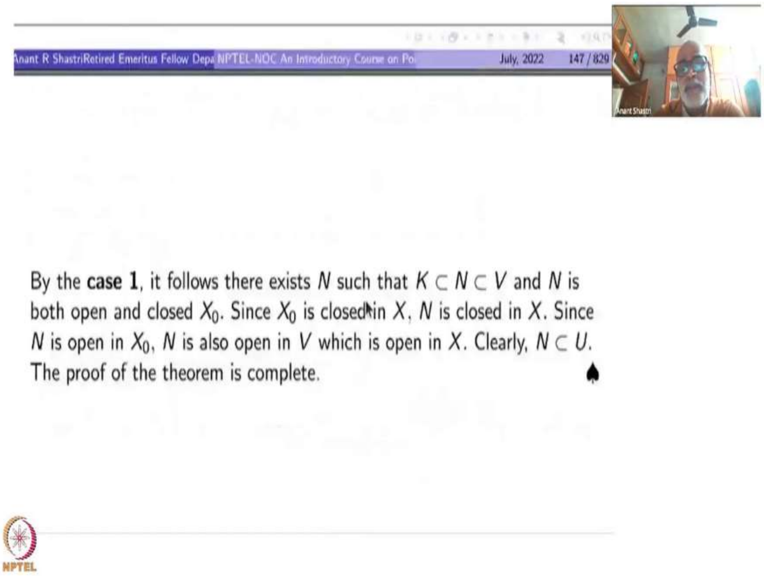

Thus, we have shown that K itself is$L$L, and L have satisfies the said property. So, K satisfies this property under the assumption that X is compact. So, that completes the case 1.

The general case is much simpler now. X is locally compact, and K is compact. Therefore, there exists a compact neighbourhood X_0 of K in X. Each point of K has a compact neighbourhood, you take the union that will be a cover K. You can extract a finite cover out of it. Their union will give you a compact neighbourhood. Compact neighbourhood means what? An open subset containing K with its closure compact, that much you can say. So, there exists a compact neighbourhood X_0 of K inside X.

Let K_0 be the connected component of X_0 containing K. (K is connected remember that. So K is contained in one of the components of X_0. So, take K_0 to be the connected component of X_0 containing K. But then K_0 is connected subset of X as well. Since K is a component of X, this implies $K = K_0$.

So, what I am saying is that in passing from X to X_0, a smaller space, the advantage is now that X_0 is compact. And this is what you have to see, viz., K is a component of X_0 and hence K is equal to K_0 that is all. Now, let U be an open subset of X, such that K is contained inside in U. This X_0 is a neighbourhood of K, there is an open subset V of X such that K is contained in V which is contained in $X_0 \cap U$.

So, by case one applied to X_0, because X_0 is compact, there exist N such that K is contained in N which is contained in V and N is both open and closed inside X_0. Everything is happening in X_0. Since X_0 is closed in X, see X_0 is compact and hence closed in X, N is also closed in X because N is closed in X_0. Since N is also open in V, which is open in X, N is open also in X. See, for openness of N in X, you cannot go via X_0, but via V you can see that this N must be open inside X.

Therefore, the proof of theorem is complete. Later on, when we are studying one point compactification, I will give you some relevance of this result to compactifications of $\mathbb{R}^n$. In complex analysis, $n = 2$, the 1-pt compactification of that is nothing but the extended complex plane, and that is how it is important in complex analysis. Thank you.