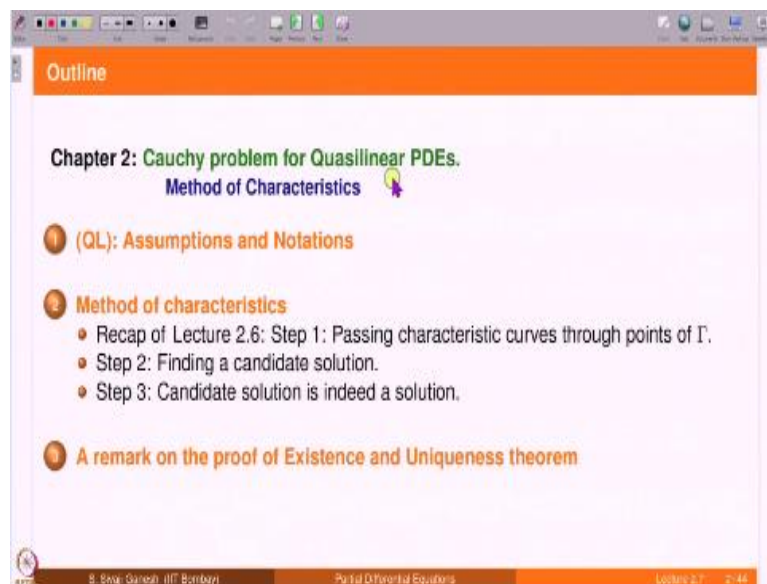


Partial Differential Equations
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Lecture – 2.7
First Order Partial Differential Equations
Method of Characteristics for Quasilinear Equations - 2

In the last lecture, we have started discussing method of characteristics for solving Cauchy problem for Quasilinear equations. In this lecture, we complete the proof of existence of solutions to Cauchy problem for Quasilinear equations using method of characteristics.

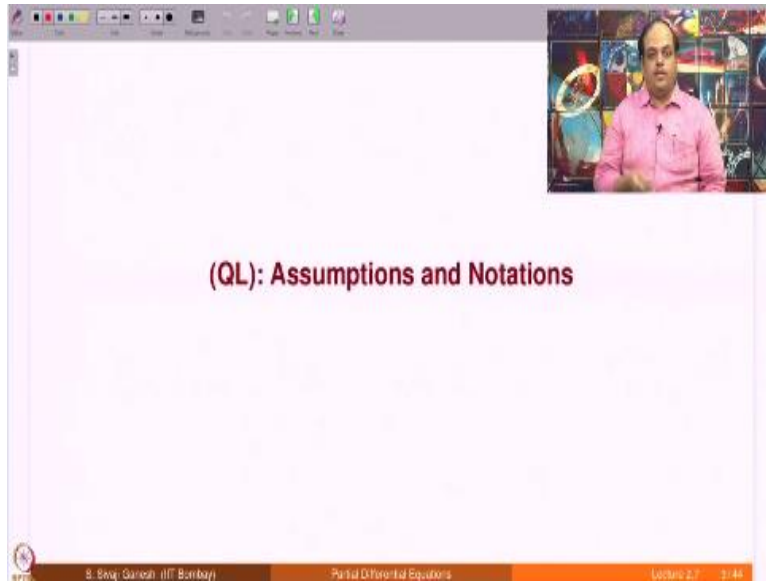
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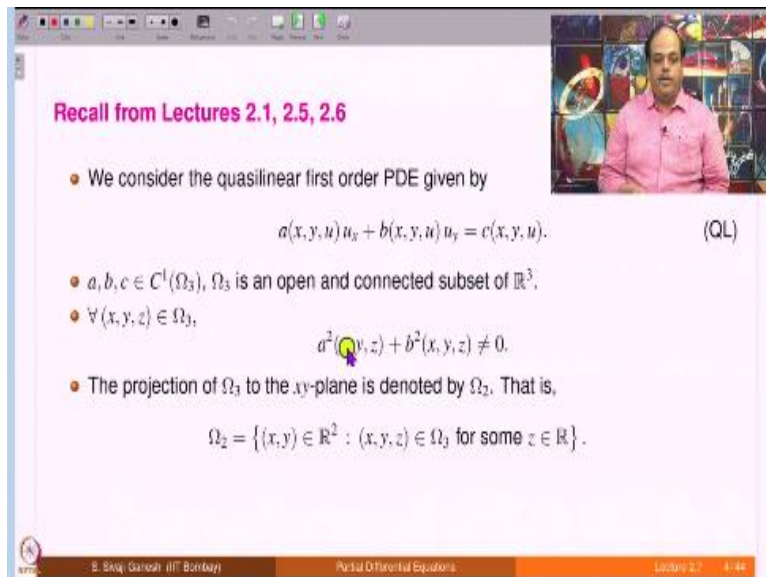
So, outline of today's lecture is, firstly recall the assumptions and notations which are made about Quasilinear equations and then we describe method of characteristics. A quick recap of lecture 2.6 where we have completed the step 1 in the method of characteristics, which is passing characteristic curves through points of Γ . Today, we start elaborating on step 2, where we say what is the finding a candidate solution and then show that that is indeed a solution.

And then a small remark on the proof of existence and uniqueness theorem that we are going to prove at the end of this step 3.

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So, assumptions and notations in the context of Quasilinear partial differential equations, first order equations, this has been visited many times 2.1- 2.5 and as last time also in lecture 2.6. So, Quasilinear equation, we denote by QL. It stands for this equation $a u_x + b u_y = c$. a, b, c are functions C^1 functions on Ω_3 . Ω_3 is an open subset of \mathbb{R}^3 such that $a^2 + b^2$ is not equal to 0 at every point of Ω_3 and the projection of Ω_3 to x, y plane is denoted by Ω_2 .

(Refer Slide Time: 02:09)

Recall from Lectures 2.1, 2.5, 2.6

Cauchy Problem (CP)

Given a space curve $\Gamma \subset \mathbb{R}^3$ described parametrically by

$$\Gamma : x = f(s), y = g(s), z = h(s), s \in I,$$

where $f, g, h \in C^1(I)$, $I \subseteq \mathbb{R}$ is an interval, and s.t. Γ_2 (projection of Γ to xy -plane) satisfies

$$(f'(s))^2 + (g'(s))^2 \neq 0$$

for all $s \in I$, find a solution u to (QL) such that

$$h(s) = u(f(s), g(s))$$

for s belonging to a **subinterval** of I .

That is, a part of the curve Γ lies on the surface $S : z = u(x, y)$.

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Cauchy problem given a space curve that is a curve in \mathbb{R}^3 , which is given parametrically in this form, where the parameter is running in the interval I and f, g, h are C^1 functions such that notation Γ_2 important, Γ_2 is a projection of Γ to xy plane that satisfies this condition. Γ_2 is actually $x = f(s); y = g(s)$ as s varies in I . We want that f' and g' do not vanish at the same time, at every point on Γ_2 .

Then we want to find a solution that means a function defined on some domain such that u on Γ_2 takes the value h which is a prescribed datum curve. Of course, we do not expect the solution domain to contain entire Γ_2 , but a small portion of Γ_2 is also allowed.

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Method of characteristics for Quasilinear equations

Recap of Lecture 2.6

S. Sajeesh (IIT Bombay) Partial Differential Equations Lecture 2.7 5/44

(Refer Slide Time: 03:22)

Inspiration for Method of Characteristics

Integral surface is a union of characteristic curves

Theorem

- Let D be an open and connected subset of Ω_2 .
- Let $u : D \rightarrow \mathbb{R}$ be a C^1 function.
- Let S denote the surface $S : z = u(x, y)$ in \mathbb{R}^3 .

Then the following two statements are equivalent.

- The surface S is an integral surface of the equation (QL).
- The surface S is a union of characteristic curves for (QL).

9. Shiv Ganeshi IIT Bombay Partial Differential Equations Lecture 2.7 7/44

Now, let us look at the method of characteristics from which was discussed in the last lecture. The inspiration for method of characteristics is the following theorem, which said give me a surface or the forms $z = u \ x \ y$, then saying that this surface is an integral surface is same as saying that this surface is union of characteristic curves corresponding to Quasilinear equations. Since, we want to construct an integral surface.

In other words, wants to find a solution or a surface corresponding to a solution. We will start with this idea of constructing some quantity, some object which is union of characteristic curve that is the idea behind this method of characteristics.

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Main steps in the Method of Characteristics

- Step 1: Passing characteristic curves through points of Γ .
- Step 2: Defining a candidate solution u using inverse function theorem.
- Step 3: Establishing that u defined in Step 2 solves the Cauchy problem.

Handwritten diagram: $z = u(x, y)$, $S \in \mathbb{R}$, $t \in \mathbb{R}$, (s, t)

9. Shiv Ganeshi IIT Bombay Partial Differential Equations Lecture 2.7 8/44

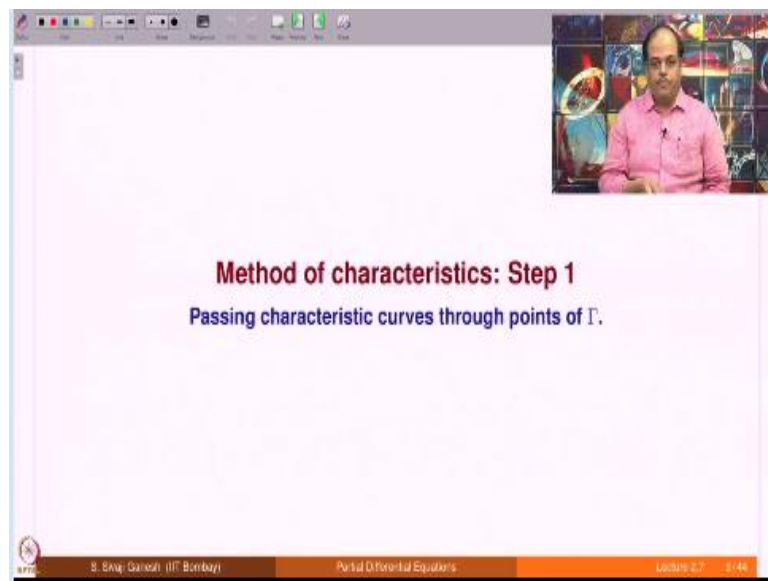
So, step 1 is through every point of gamma, you pass a characteristic curve. So, gamma is here. We want to find a surface which contains this gamma. So, what we do is, we pass

through this characteristic curves like that and you see this is going to give surface. So, characteristic is given datum curve is described by s and the characteristic curves for each fixed this point, these characteristics are passing through that is defined by t in R .

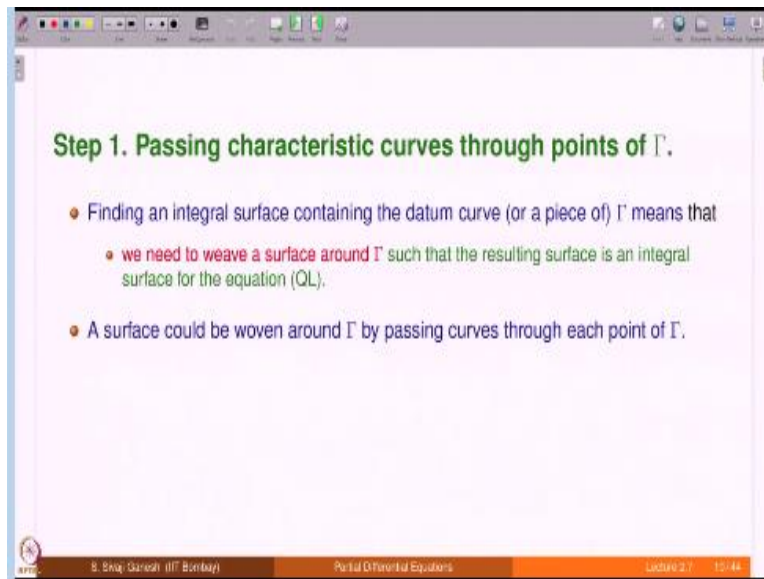
So, we have a description of some object geometrical object in R^3 which is parameterized by 2 parameters s, t . So, as s and t vary, we hope, we will get a surface that was the idea. So, that is why we start with this step 1, where we pass characteristics curve like this, then we say that the surface that we are going to generate would be of the following type where the third coordinate is going to be a function of the first 2 coordinates that is what we hope and that we will achieve in the step 2.

We will define a function u , u is a function of 2 variables x and y using inverse function theorem. And then we show that u is indeed the solution.

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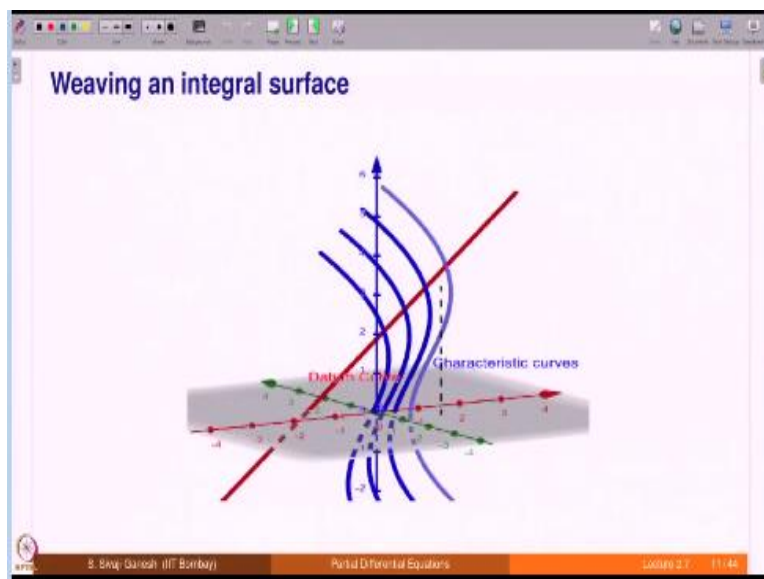


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Step 1, what we did is finding an integral surface containing the datum curve or a piece of datum curve means we need to weave a surface as I have just shown you in the picture. And if you weave that surface through passing curves through points of gamma and curves are characteristics curve, then that theorem gives us the hope that the surface we get is going to be an integral surface.

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Here, this is the datum curve. These are characteristics curve. This is the same picture which I have just drawn.

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Implementation of Step 1.

- Take a point $P \in \Gamma$. It looks like $P(f(s), g(s), h(s))$ for some $s \in I$.
- The chara. curve through P is the image/trace of a solution to (chara.ODE)

$$\frac{dx}{dt} = a(x, y, z), \quad \frac{dy}{dt} = b(x, y, z), \quad \frac{dz}{dt} = c(x, y, z) \quad (\text{chara.ODE})$$

satisfying the initial conditions

$$x(0) = f(s), \quad y(0) = g(s), \quad z(0) = h(s).$$

9. Shiv Ganesht (IIT Bombay) Partial Differential Equations Lecture 2.7 13/44

So, for this, what we do take a point and gamma, any point on gamma looks like $f(s), g(s), h(s)$ and then solve this system of characteristics ODE with initial conditions $f(s), g(s), h(s)$. It means that when you look at this gamma, you take at this point $f(s), g(s), h(s)$, the point looks like this. These are 3 tuple and then you pass a curve through that. That will be denoted by $X(t,s), Y(t,s), Z(t,s)$. This is what we get at the end of this step.

(Refer Slide Time: 07:07)

Implementation of Step 1. (contd.)

- By Cauchy-Lipschitz-Picard theorem, the above IVP has a unique solution.
- Let the solution be represented by

$$x = X(t, s), \quad y = Y(t, s), \quad z = Z(t, s), \quad (1)$$

defined for t belonging to an interval J_s in \mathbb{R} , and such that $0 \in J_s$.

- By lemma on reparametrization of Chara. curves, we may take

$$J_s = \mathbb{R}.$$

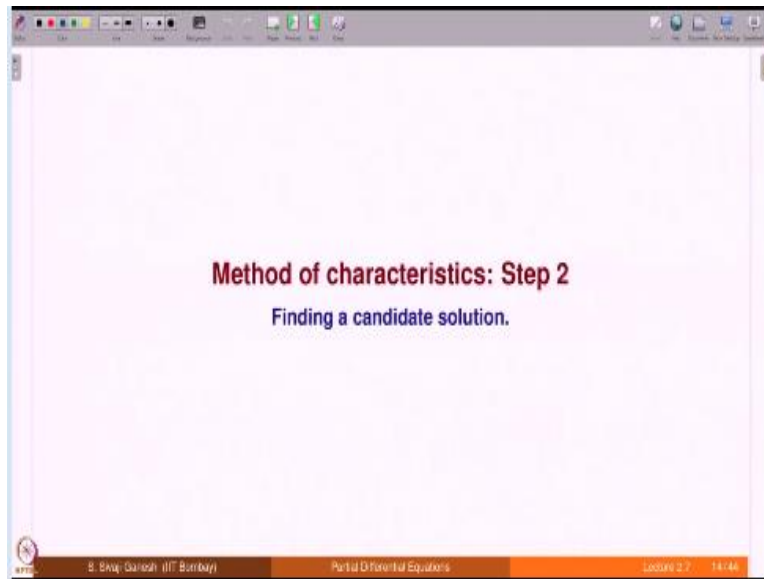
Let us proceed to the remaining Steps in the Method of characteristics.

9. Shiv Ganesht (IIT Bombay) Partial Differential Equations Lecture 2.7 13/44

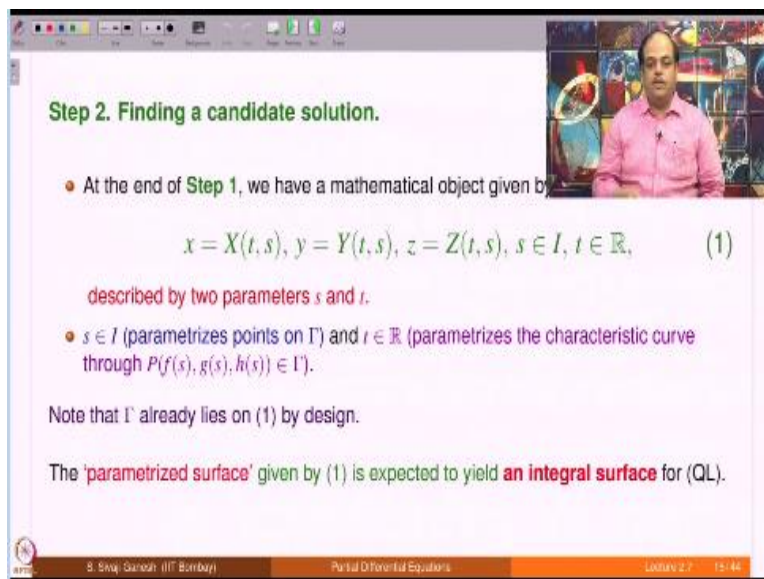
At $t = 0$, it is going through this point that is how we are solving, we are giving the initial conditions to the system of characteristic ODE. Now by Cauchy Lipschitz Picard's theorem, it has a unique solution. Let it be denoted by like this, $x = X(t,s); y = Y(t,s), z = X(t,s)$. Where is it defined? s in I and t belongs to J_s which is an interval, of course, the interval contains 0, note the dependence of s on J_s 's that is why it is written J_s . But we have seen a lemma and re-

parameterization which says, we may take $J s = R$. So, now, let us proceed to the remaining steps in this method.

(Refer Slide Time: 07:54)



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Second step is finding a candidate solution. How do we get that? Of course, we are take help of this what we have been achieved, so far. So, this mathematical object is described by 2 parameters. And this leaves in \mathbb{R}^3 . So, we hope with the surface that is the idea. As I pointed out earlier, s represents a parameter running on the datum curve γ and t in \mathbb{R} , it is parameterizing characteristic curve passing through this point and γ , which is visited when $t = 0$.

Of course, γ already lies on one by design, initial conditions. The parameterize surface given by 1, I have put it in inverted commas here, because it is not clear. it is a parameterised object is clear; surface means we expect something that is why I put here on till the confirmation comes, it is expected to be an integral surface, because we were motivated by the theorem. And therefore, we hope that we get an integral surface.

(Refer Slide Time: 09:01)

Step 2. Finding a candidate solution.(contd.)

The 'parametrized surface' given by (1) is expected to yield an **integral surface** for (QL).

Question. What does this mean?

Answer. We would like to express the parametric surface given by (1) in the form

$$S: z = u(x, y)$$

for some function u (defined on some domain in Ω_2).

In other words,

“write the third coordinate as a function of the first two coordinates” using (1).

S. Siva Ganesan (IIT Bombay) Partial Differential Equations Lecture 2.7 19/44

But this is described using parameters t and s . Now, we need to write it as $z = u$ of x y . So, what does this mean? It means that we would like to express the parametric surface given by 1 in this form $z = u$ x y for some function u . We need to find some function that means we need to find a domain on which this function is defined on this course. In other words, write the third coordinate as function of the first 2 coordinates using 1 that is the only thing that we have in our hands.

(Refer Slide Time: 09:44)

Step 2. Finding a candidate solution.(contd.)

"write the third coordinate as a function of the first two coordinates"

This suggests the following **strategy for finding u** :

- Express t and s as functions of x and y , using $x = X(t, s)$, $y = Y(t, s)$. Denote $t = T(x, y)$ and $s = S(x, y)$.
- Then substitute for t and s in $z = Z(t, s)$ and define u by

$$u(x, y) := Z(T(x, y), S(x, y)).$$

9. Shaq. Danish (IIT Bombay) Partial Differential Equations Lecture 2.7 11:44

Namely, the family of characteristic curves passing through points of comma. So, the strategy for finding u is this. Express t and s as functions of x and y , we have x and y given in terms of t and s . Now, suppose I write t and s in terms of x and y , go ahead and imagine this is the formula we write $t = T$ of x y and s as S of x y . Now, we go and substitute in these relations $z = Z$ t s and define a function this way, z of $t = T$ x y , $s = S$ x y .

So, we got some function of x y , call it to u x y . This should work because z itself was motivated like that. The equation for z was derived after using that it is going to be the value of solution along the base characteristic X t s , Y t s . Refer to lecture 2.6. So, therefore, it should work.

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Implementation of Step 2.

(1). Express t and s as functions of x and y using $x = X(t, s)$, $y = Y(t, s)$

- Recall that the equations $x = X(t, s)$, $y = Y(t, s)$ represent base characteristic curves.
- That is,

$$(x, y) = (X(t, s), Y(t, s)).$$

- This suggests applying inverse function theorem to the function

$$(t, s) \mapsto (X(t, s), Y(t, s)).$$

9. Shaq. Danish (IIT Bombay) Partial Differential Equations Lecture 2.7 11:44

But, there can be very technical difficulties, we will see what they are. So, express t and s as functions of x and y like that. Recall that x and y given by $X(t, s)$ and $Y(t, s)$ that together represent base characteristic curves that is this. Now, this suggests applying inverse function theorem to this function then you will get t and s in terms of X and Y .

(Refer Slide Time: 11:19)

Implementation of Step 2. (contd.)

To apply inverse function theorem to the function

$$(t, s) \mapsto (X(t, s), Y(t, s)),$$

we need to verify hypotheses of **inverse function theorem**.

Answers to the following questions are needed.

- 1 What is the domain of the function?
- 2 Is the function continuously differentiable?
- 3 Is the Jacobian condition satisfied?

One conveniently **ignores Questions 1 and 2**, and **discusses Question 3 at length. Why?**
Find your answer.

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So, if you want to apply inverse function theorem, you are to see what is the setup of the inverse function theorem. It involves a function. It involves C^1 function and certain Jacobian condition. So, we need to check all those hypotheses. Then only we can apply a theory. So, therefore, answers to the following questions will be needed. What is the domain of this function? Is a function continuously differentiable? Is the Jacobian condition satisfied? These 3 conditions.

Usually, one ignores questions 1 and 2 and just concentrate on third one at length. In fact, the third is not a condition. It is a Jacobian condition satisfied is never checked that is put as part of the assumption and then one proves theorem, but still 1 and 2 are still important. So, you ask this question why? You will understand later on. So, answer, I will not give. You think about this question.

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Implementation of Step 2. (contd.)

Q1. What is the domain of the function

$$(t, s) \mapsto (X(t, s), Y(t, s))?$$

- It is defined on a subset A of $\mathbb{R} \times I$ having the property that, for each fixed $s \in I$, the s -cross section of A is the interval J_s .
- It is not even clear if the set A is an open subset of \mathbb{R}^2 . May be or may not be. Definitely complicates further analysis.

However, we may assume without loss of generality that

A is equal to $J \times I$, where J is a subinterval of \mathbb{R} . In fact, we can take $J = \mathbb{R}$.

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What is the domain of this function? As we observed for each fixed s , t varies in J_s , so, it is going to be a subset of \mathbb{R} cross I for each fixed s t varies in J_s . That is all we can describe. It is not even clear what the set s is an open set etcetera. But anyway, we will do away with this kind of, answering these kinds of difficult questions. So, it complicates further analysis.

However, we may assume without loss of generality, that A is equal to J cross I where J is a subinterval of \mathbb{R} . I is where the parameters s is running in I . J is where the t is running. We take a $J = \mathbb{R}$. This is what we said thanks to our lemma tree parametrization.

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Implementation of Step 2. (contd.)

Q1. What is the domain of the function

$$(t, s) \mapsto (X(t, s), Y(t, s))?$$

We may assume without loss of generality that

A is equal to $J \times I$, where J is a subinterval of \mathbb{R} . In fact, $J = \mathbb{R}$.

This is due to

- Lemma on Reparametrization of Chara. curves.** It allows us $J = \mathbb{R}$.
- By restricting to a small piece of I' near P ,** we can always take J_s to be independent of s belonging to a subinterval of I . See ODE book by W. Wolfgang.

S. Shiv Ganesan (IIT Bombay) Partial Differential Equations Lecture 2.1 27/44

So, therefore, we may assume that without loss of generality that is A is equal to J cross I where J is a subinterval of \mathbb{R} . In fact, we can assume J is equal to \mathbb{R} . Due to this lemma that I just quoted, re-parameterization of characteristic curves are by restricting to a small piece

near gamma that is also okay. So, you have the point P at that point restrict to a small piece, then it is true that J can be chosen independent of s.

These ideas comes when one discusses differentiable dependence of or continuous dependence of solutions on the initial data. So, see the ODE by Wolfgang.

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Implementation of Step 2. (contd.)

Q2. Is the function $C^1(J \times I)$?

$$(t, s) \mapsto (X(t, s), Y(t, s))$$

Answer is Yes. Reason is

Differentiable dependence of solutions to IVPs on the parameters in the theory of ODEs
(for instance, see the ODE book by W. Wolfgang).

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For this book, it is a good book, we will be referring once more at least in this lecture, okay done. So, domain is \mathbb{R} cross I fine or \mathbb{R} cross some subinterval of I if you follow Wolfgang approach. If you follow our approach, it is \mathbb{R} cross I . Now, next question is: is it C^1 function? If you look at this X and Y or solutions of certain ODE, therefore, with respect to t , they will be C^1 .

What about with respect to s ? What about together? Answer is yes. It is C^1 function. If partial derivative with respect to t and with respect to s , both are continuous functions, then it turns out that it is differentiable with respect to the 2 variables together. As we know, existence of partial derivatives of functions of several variables does not mean that the function is differentiable.

Forget about that function need not even continuous at a point, but all directional derivatives exists. But, if partial derivatives are continuous, then the function turns out to be differentiable. And, it will be C^1 . Therefore, we are interested in the X_t , X_x whether they are continuous functions. X_t as I said it is a b, c , they are nice functions. So, there is no problem. What about with respect to s ? Various s .

s is appearing in obtaining $X(t, s)$ and $Y(t, s)$ through the initial conditions. We saw $x(0, s)$ and $y(0, s)$ as $f(s)$ and $g(s)$ and we assume f and g are C^1 functions of the variable s . That is the reason why this function is C^1 of J cross. It comes from differentiable dependence of solutions to initial value problems on the parameters in the theory of ODEs. For a instance, see the book by Wolfgang on ordinary differential equations.

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Implementation of Step 2. (contd.)

Q3. Is the Jacobian condition satisfied by the function $(t, s) \mapsto (X(t, s), Y(t, s))$?

Now we are interested in the invertibility of the function

$$\Phi: J \times I \rightarrow \Omega_2$$

$$(t, s) \mapsto (X(t, s), Y(t, s)). \quad (2a)$$

To apply inverse function theorem, the Jacobian $J(t, s)$ is required to be non-zero, where $J(t, s)$ is given by

$$J(t, s) := \frac{\partial(X, Y)}{\partial(t, s)}(t, s) = \begin{vmatrix} X_t(t, s) & X_s(t, s) \\ Y_t(t, s) & Y_s(t, s) \end{vmatrix}. \quad (3)$$

Now, let us implement step 2. Step 2 and question 3 we are considered. Is a Jacobian condition satisfied by this function? We are interested in invertibility of this function right. So, we are to compute the Jacobian. Inverse of function theorem says computed Jacobian that should be nonzero. Jacobian is this. Now, we are to compute the $X_t(t, s)$, $Y_t(t, s)$, X_s and Y_s at point t, s . Now, X_t at t, s is A , it is known.

Y_t at t, s is also known. X_s at t, s is not known. What we actually know is x_s at $0, s$. That is the reason why we stick $t = 0$ now.

(Refer Slide Time: 16:49)

Implementation of Step 2. (contd.)

- Since we do not know the function Φ given in (2) explicitly, we cannot compute $J(t, s)$ at an arbitrary point $(t, s) \in J \times I$.
- **However**, using (chara.ODE) and Initial Condns., we can compute $J(0, s)$ and is given by

$$J(0, s) = \begin{vmatrix} X_t(0, s) & X_s(0, s) \\ Y_t(0, s) & Y_s(0, s) \end{vmatrix} = \begin{vmatrix} a(f(s), g(s), h(s)) & f'(s) \\ b(f(s), g(s), h(s)) & g'(s) \end{vmatrix}.$$

If we assume that $J(0, s_0) \neq 0$ at an $s_0 \in I$, then we can apply the (local) inverse function theorem. Thereby Step 2 would be successfully implemented. This assumption on Jacobian is called **Transversality condition**.

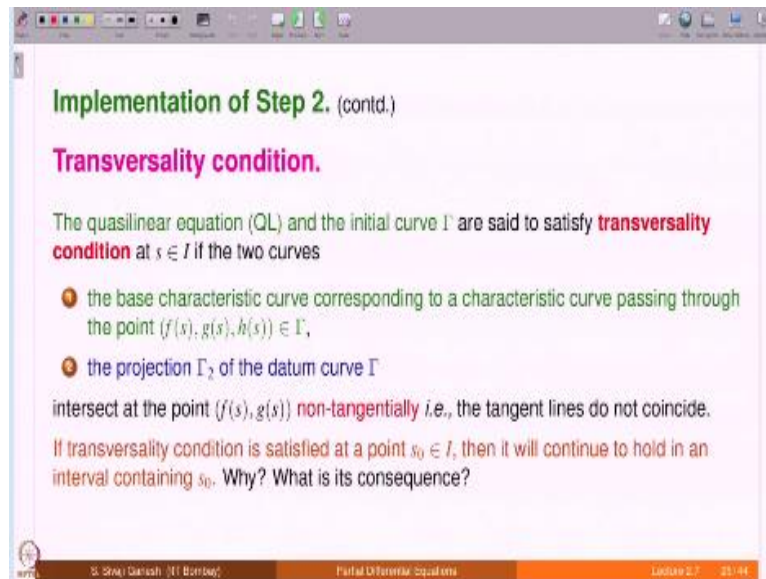
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Since, we do not know the function explicitly, we cannot compute $J(t, s)$ for an arbitrary point (t, s) in $J \times I$, which you can read it as $\mathbb{R} \times I$. However, using characteristic ODE an initial condition, because initial conditions for characteristic ODE are given at $t = 0$ therefore, they are something can be done. We can compute $J(0, s)$ and that is given by this. $X(t, s)$, $Y(t, s)$ are solving ODEs. So, from there, you get the a, b, x of $0, s$ is f, s .

Therefore, derivative that with respect to s is $f'(s)$. Similarly, y_s is $g'(s)$. So, we want this to be nonzero. So, if you assume that $J(0, s_0)$ is nonzero at a point s_0 , then we can apply the inverse function theorem. I am written in the brackets a local because there is something called something else called global inverse function theorem. But the usual inverse function theorem is a local inverse function theorem.

Thereby, step 2 will be successfully implemented. This assumption on Jacobian is called the transversality condition.

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The Quasilinear equation QL and the initial curve or the datum curve gamma are set to satisfy transversality condition. So, the transversality condition is 2 people together satisfy that. That is quadratic equation and initial curve. At a point s in I , if the 2 curves which are listed here the first curve is a base characteristic curve. So, at point s right, so, look at the point $f(s), g(s)$ that is going to be on a datum curve, through that you find the characteristic curve, projected to $\mathbb{R}^2 \times y$ plane that will be the base characteristic curve.

Thus, base characteristic curve and gamma 2 which is a projection of the datum curve gamma, these 2 curves intersect, of course, they intersect at the point $f(s), g(s)$, non-tangentially that means, the tangent lines do not coincide. If the transversality condition is satisfied at a point, then it will continue to hold in an interval containing a 0, because the transversality condition is the nonzeroness of certain determinant.

And whatever is appearing inside the determinant are continuous functions of s , therefore, at some point nonzero, it will continue to be nonzero nearby that point. That is the reason. In this case, what would happen is imagine this is our gamma 2 that base characteristic curve will come like that, it will actually cut base characteristic curve will not be like this. For example, it will not be like that, because here, they share the same tangent.

So, this is not allowed, because then transversality condition is not satisfied because if you see in the determinant, what are the 2 things which are coming the 2 columns x, y , t is the tangent of the base characteristic curve and this is a tangent of the gamma 2 and asking that

these nonzero means they are not parallel and they are passing through this point the $f(s), g(s)$; $0, s$, sorry, they are passing through the point $f(s), g(s)$. Yes.

And we are asking that the tangent line is not the same. That is why it is the transversality condition. So, here it is transversality. So, here it is not transversality. Now, such a things happens, if you look at nearby also, the curves all going to be like that. They will always be cutting and you get a surface that is a consequence.

(Refer Slide Time: 20:55)

Implementation of Step 2. (contd.)

Analytically, the **transversality condition** translates to

$$\begin{vmatrix} a(f(s), g(s), h(s)) & f'(s) \\ b(f(s), g(s), h(s)) & g'(s) \end{vmatrix} \neq 0.$$

The **transversality condition** rules out

- the possibility of base characteristic curves **touching** the projected datum curve Γ_2 ,
- or **intersecting along a piece of it.**

This justifies the use of the word '**transversal**'

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So, analytically the transversality condition translates to this being nonzero, this determinant. It rules out. The possibility of base characteristic curves touching the projected datum curve namely Γ_2 that we have illustrated in the picture. Touching means sharing the tangent up and go inside of it of course. Then also you have touching for a long time. This is, maybe at a point, but this is, if we intersect along a piece of it, it means that they share tangent up throughout the piece because curve is same.

So, such things are rolled out. Please spend some time and understanding this geometrically, because it will be useful when we are trying to give examples or solve or guess what would happen in certain situations for Quasilinear equations. Some examples, it will help. So, this justifies the use of the word transversal. It means to cut, it cuts.

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Implementation of Step 2. (contd.)

Assume $J(0, s_0) \neq 0$. Note that $s = s_0$ corresponds to the point $P_0(f(s_0), g(s_0), h(s_0)) \in \Gamma$.

By **inverse function theorem**, there exist

- an open subset E of $J \times I$ containing the point $(0, s_0)$, and an open subset F of Ω_2 containing the point $(X(0, s_0), Y(0, s_0))$ i.e., $(f(s_0), g(s_0))$,
- a continuously differentiable function $\Psi : F \rightarrow E$ such that

$$\Psi \circ \Phi = I_E, \quad \Phi \circ \Psi = I_F,$$

where I_E and I_F are the identity functions defined on E and F respectively.

So, assume that Jacobian is not 0 at the point $s = 0$. Even if Jacobian is not 0 for all s in I , J of 0 s is not 0 for all s in I , it is not a useful because the only way you apply inverse function theorem is at a point $s = s_0$. So, I am assuming at a point, it is nonzero. Of course, $s = s_0$ means the point on Γ will be $f(s_0), g(s_0), h(s_0)$. By inverse function theorem, there exists an open subset of J cross I containing this point $(0, s_0)$.

And an open set on the other side Ω_2 containing the point $(X(0, s_0), Y(0, s_0))$ because our function is that which is actually a $f(s_0), g(s_0)$ and a continuously differentiable function from F to E , because this should be the inverse of the function Φ . So, $\Psi \circ \Phi$ will be identity on E and $\Phi \circ \Psi$ will be identity on F . These are the identity functions.

(Refer Slide Time: 23:09)

Implementation of Step 2. (contd.)

Denoting

$$\Psi(x, y) = (T(x, y), S(x, y)),$$

the equations

$$\Psi \circ \Phi = I_E, \quad \Phi \circ \Psi = I_F$$

yield

$$t = T(x, y), \quad s = S(x, y).$$

$I \in E$ means I . Now, we have a picture. So, this is things. Φ , we have $E \rightarrow F$. So, originally ϕ is defined on some other side, which is $J \times I$ or $\mathbb{R} \times I$ or $J \times \text{some } I$ and then we assume that at $0, s = 0$, Jacobian condition is satisfied, nonzeroness of the Jacobian. Therefore, you can find an open set E which contains a point $0, s = 0$. And open set F , which contains the image of ϕ under that.

Whatever is a map ϕ we have, it maps t, s to $X(t, s), Y(t, s)$. So, the X of 0 is $s = 0$; Y of 0 is $s = 0$ which is $f(s = 0), g(s = 0)$ and a mapping size is that these compositions, you go from E to F , come back from F to E or go from F to E first and then come back to F , both will be identity functions. This is where we are using t, s coordinates. Here, we are using x, y coordinates, the variables names.

So, therefore, χ of x, y , you denoted as $T(x, y), S(x, y)$ and this equalities holder because they are inverses of each other that will give you $t = T(x, y), s = S(x, y)$.

(Refer Slide Time: 24:23)

Implementation of Step 2. (contd.)

From last slide,

$$t = T(x, y), s = S(x, y).$$

Recall that $Z(t, s)$ was 'designed' to be the value of a solution at the point $(X(t, s), Y(t, s))$. This motivates the definition of a candidate solution for the Cauchy problem as

$$u : F \rightarrow \mathbb{R} \text{ given by } u(x, y) = Z(T(x, y), S(x, y)).$$

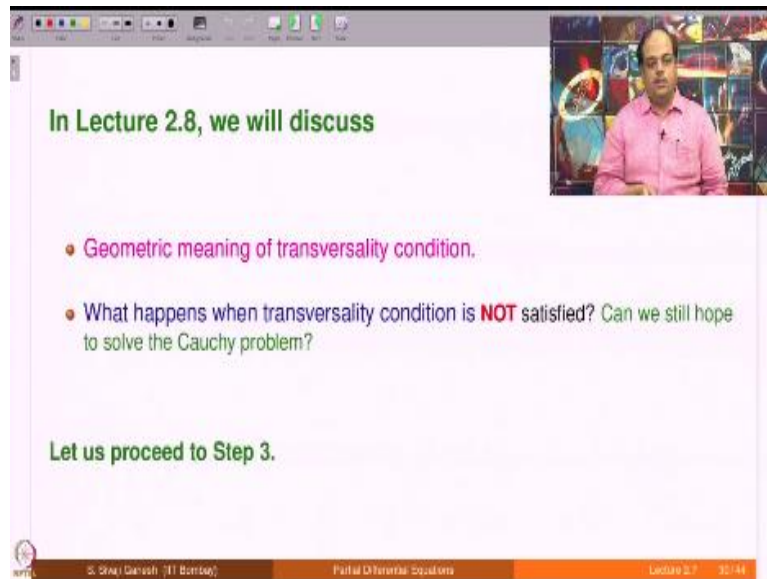
As the function u is a composition of two C^1 functions, by Chain rule $u \in C^1(F)$. **Step 2 successfully completed!**

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As I told you earlier $Z(t, s)$ was designed to be the value of the solution at this point $X(t, s), Y(t, s)$, referred to lecture 2.6. Therefore, this motivates us the definition of a candidate solution by this. u defined on $F \rightarrow \mathbb{R}$ given by $u(x, y) = Z$ of $T(x, y), S(x, y)$ as a function is a composition of 2 C^1 functions. This is C^1 inside and Z itself is C^1 function. Therefore, the composition will be C^1 function.

Therefore, u is a C^1 function. Step 2 is successfully completed. We have defined what is a candidate solution.

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In Lecture 2.8, we will discuss

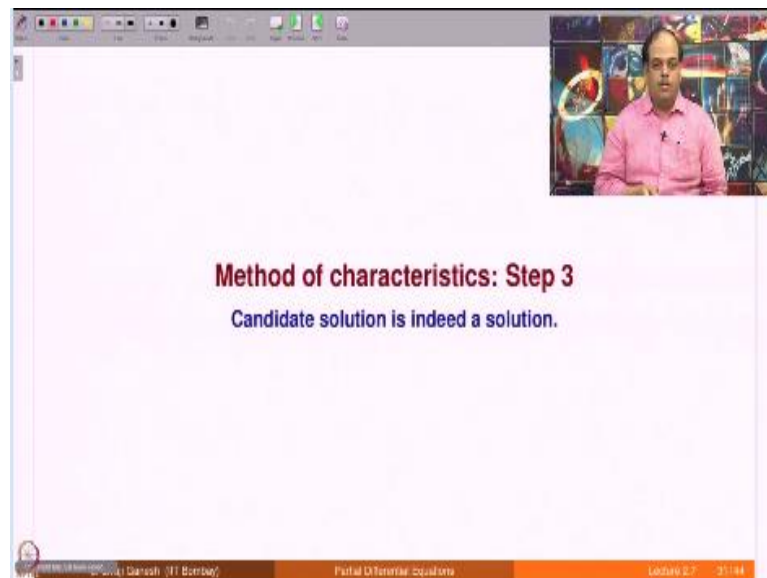
- Geometric meaning of transversality condition.
- What happens when transversality condition is **NOT** satisfied? Can we still hope to solve the Cauchy problem?

Let us proceed to Step 3.

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So, in lecture 2.8 that is a next lecture, we will discuss the geometric meaning of transversality condition and what happens when this condition is not satisfied, what will happen? When it is satisfied, we are going to prove a result today and what happens if it is not satisfied will be analysed in the next lecture. Can we still hope to solve the Cauchy problem? We hope to see. Answers will be in the next lecture. Now, let us proceed to Step 3.

(Refer Slide Time: 25:43)



Method of characteristics: Step 3

Candidate solution is indeed a solution.

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(Refer Slide Time: 25:48)

Implementation of Step 3.

Step 3. Candidate solution is indeed a solution.

By definition of u , we have

$$u(x, y) = Z(T(x, y), S(x, y)) \text{ for } (x, y) \in F,$$

$$Z(t, s) = u(X(t, s), Y(t, s)) \text{ for } (t, s) \in E.$$

Differentiating the last equation w.r.t. the variable t , we get

$$Z_t(t, s) = u_x(X(t, s), Y(t, s)) X_t(t, s) + u_y(X(t, s), Y(t, s)) Y_t(t, s).$$

Whenever we have a change of variables and PDEs, and we need to differentiate, there are always two equations. Only one of them would be useful!

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What is step 3? Candidate solution which we got in step 2 is actually a solution. By definition of u , we have this formula and $Z(t, s) = u(X(t, s), Y(t, s))$ also have this. Now, what is that we want to show? u solves PDE right. So, we have to come to a u_x , u_y etcetera. But that may not be useful, let us see. Differentiating the last equation that is this equation with respect to t , we get of course, we use chain rule and we get this.

Now, why did not we not choose the first equation? We could have chosen that differentiate with respect to x , with respect to y , go back and substitute in the equation and check whether it is satisfied or not. This is usually the situation all the time whenever we have changed variables and PDE are there, way to differentiate some equation, but there are always 2 equations for that.

Only one of them would be useful. In the sense, it may be a quicker way of getting the solution, not wrong other. One is not wrong, but you may not be able to achieve anything with that, but this one will definitely give you.

(Refer Slide Time: 27:03)

Implementation of Step 3.(contd.)

Since $X(\cdot, s), Y(\cdot, s), Z(\cdot, s)$ are solutions to (chara.ODE), equation

$$Z_t(t, s) = u_x(X(t, s), Y(t, s)) X_t(t, s) + u_y(X(t, s), Y(t, s)) Y_t(t, s)$$

becomes (use $P_{t,s}$ to denote $(X(t, s), Y(t, s), Z(t, s))$)

$$c(P_{t,s}) = u_x(X(t, s), Y(t, s)) a(P_{t,s}) + u_y(X(t, s), Y(t, s)) b(P_{t,s}).$$

In terms of $(x, y) \in F$, the last equation reads as

$$c(x, y, u(x, y)) = u_x(x, y) a(x, y, u(x, y)) + u_y(x, y) b(x, y, u(x, y)).$$

This means that $u : F \rightarrow \mathbb{R}$ solves the equation (QL).
 The function u satisfies the Cauchy data was already checked.
Step 3 successfully completed!

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You will learn this. As many times as you make mistake of differentiating this u that is true. Now, it is going to work differentiate, we got this formula. Now, $X(t, s), Y(t, s)$ and $Z(t, s)$ are solution to chara ODE, therefore, here, points will come out be too long. Just to make the notation short. I am using $X(t, s), Y(t, s)$ and $Z(t, s)$ has $P(t, s)$. Now, $Z(t, s)$ is c at $X(t, s), Y(t, s)$ and $Z(t, s)$, but now it is $P(t, s)$; u_x is as it is; $x(t, s)$ is A at $X(t, s), Y(t, s)$ and $Z(t, s)$. Now, here it is $P(t, s)$. Similarly the other term, we have this.

So, in terms of x, y in F , the last equation reads as C of $x, y, u(x, y) = u_x$ of x, y and a of $x, y, u(x, y), u_y$ of $b(x, y, u(x, y))$. Now, this means u is a solution to the Quasilinear equation. Cauchy data is already taken care. We already observed that Cauchy data will be satisfied. The datum curve will be on the integral surface. So, step 3 is also successfully completed.

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After the 3 steps

We have obtained an integral surface $S : z = u(x, y)$

- defined by the function $u : F \rightarrow \mathbb{R}$
- S contains the point $P_0(f(s_0), g(s_0), h(s_0))$
- S contains a piece of the datum curve Γ (near P_0). In fact that much part of Γ for which $(f(s), g(s)) \in F$.

If we could apply global inverse function theorem, then

- we would have got a solution defined on a subset of Ω_2 such that the corresponding integral surface would contain the entire datum curve Γ .
- the solution may still be not defined on whole of Ω_2 , some reasons for the same will be explored later.

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After the 3 steps, we have obtained an integral surface defined by function u from F to \mathbb{R} . Of course, here we have used the inverse function theorem that means, existential to start with and s contains this point P_0 ; s contains a piece of the datum curve near P_0 . In fact, that much part of γ for which $f(s), g(s)$ belongs to this open set F . If you could apply, if there is a global inverse theorem, if you could apply that, then we would have got a solution defined still on a subset of Ω_2 .

But, that would have contained the entire datum curve γ . That integral surface would contain an entire datum curve γ . The solution may still not be defined on whole of Ω_2 . This local, global, we have to be very, very careful, what we have been dealing with is we are looking at a point on the datum curve and showing solution a surface exists nearby that which is integral surface that means, we are discussing something like local with respect to the datum curve.

And if you say globally inverse function theorem, it must be giving global with respect to the datum curve but not with respect to the domain. More on this, we will see in a future lecture. Local, there are 2 different concepts; local with respect to datum curve, local with respect to the domain Ω_2 .

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Thus we have proved the existence part of the following
Local existence and uniqueness theorem

Hypotheses

- Standard Hypotheses on (QL) and the Cauchy data

$$\Gamma: x = f(s), y = g(s), z = h(s), \quad s \in I$$
- Assume that transversality condition holds at $s_0 \in I$. Denote

$$P_0(x_0, y_0, z_0) := (f(s_0), g(s_0), h(s_0))$$

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Theorem is here hypothesis. Standard hypothesis on QL, I am not going to recall here on Cauchy data. Assume that the transversality condition holds at a point s_0 and denote $P_0 = (f(s_0), g(s_0), h(s_0))$; same thing, let us call x_0, y_0, z_0 for convenience.

(Refer Slide Time: 30:14)

Local existence and uniqueness theorem

Conclusions

- 1 The Cauchy problem has a solution defined on a neighbourhood of the point $(x_0, y_0) \in \Omega_2$.
- 2
 - Let D_1 and D_2 be open neighbourhoods of the point (x_0, y_0) .
 - Let $u_1 : D_1 \rightarrow \mathbb{R}$ and $u_2 : D_2 \rightarrow \mathbb{R}$ be solutions to the Cauchy problem.

Then $u_1 \equiv u_2$ on some subset of $D_1 \cap D_2$ containing the point (x_0, y_0) .

That is, solution to the Cauchy problem is locally unique, near the datum curve Γ .

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Conclusions: Cauchy problem has a solution defined and a neighbourhood of a point x_0, y_0 and let D_1, D_2 be open neighbourhood of the point x_0, y_0 . Let u_1, u_2 to be defined on D_1, D_2 respectively be solutions to Cauchy problem and then u_1 will be identically equal to u_2 on some subset of $D_1 \cap D_2$. Of course, it contains the point x_0, y_0 that is solution to Cauchy problem is locally unique near the datum curve Γ .

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Proof of Conclusion (1)

Steps 1 through 3 constitute a proof of Conclusion (1).

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So, let us prove conclusion 1 that is precisely steps 1, 2, 3 that gives the proof of conclusion 1 that is about existence.

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Proof of Conclusion (2)

- Let $S_1 : z = u_1(x, y)$ and $S_2 : z = u_2(x, y)$ be integral surfaces containing the point $P_0(x_0, y_0, z_0) \in \Gamma$.
- Thus $P_0 \in \Gamma \cap S_1 \cap S_2$, and as a consequence a part of Γ , say Γ' , lies on $S_1 \cap S_2$.
- By a Corollary from Lecture 2.6, some part of each of the characteristic curves of (QL) passing through points of Γ' will lie on S_1 as well as on S_2 .

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Now, let us do the uniqueness. Let us S_1 and S_2 to be as given. One is defined on D_1 ; other one is defined on D_2 and take this point P_0 which is in Γ . P_0 is there on Γ , it is there in S_1 as well as S_2 and as a consequence P_0 is there on S_1 ; P_0 is there on S_2 , therefore, a part of, what is S_1 ? It is an integral surface. Therefore, a part of Γ will be there on $S_1 \cap S_2$. It will be there on S_1 and other parts will be in S_2 .

There will be a common portion which will be there on both S_1 and S_2 that is there. Then we have seen a corollary in lecture 2.6 in the last lecture, it said some part of the each of characteristic curves of QL passing through points of Γ will also be there on S_1 and also on S_2 .

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Proof of Conclusion (2) (contd.)

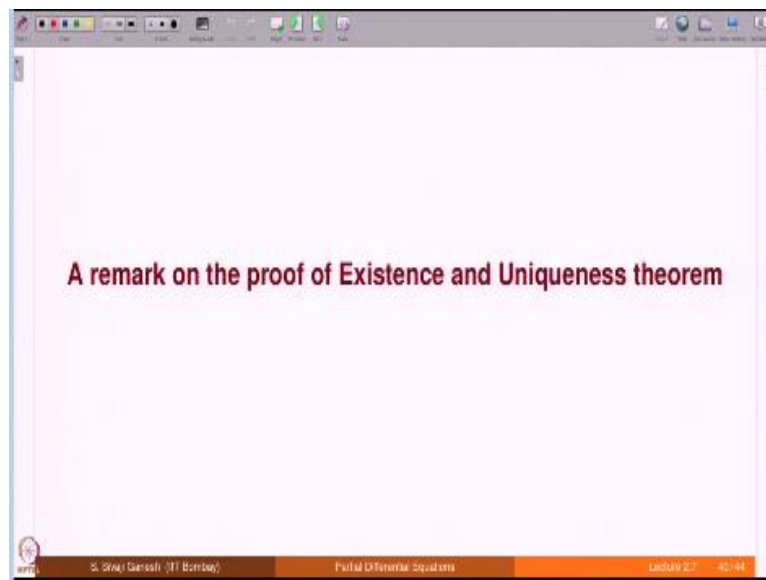
- Due to the transversality condition, as seen in Step 2, any point on a characteristic curve has the form $(x, y, u_1(x, y))$ as well as $(x, y, u_2(x, y))$ for $(x, y) \in F \cap D_1 \cap D_2$.
- This proves that $S_1 \equiv S_2$ in a neighbourhood of P_0 .
- Hence $u_1 \equiv u_2$ on some neighbourhood of (x_0, y_0) . □

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Of course, the corollary was with respect to one surface, but here with respect 2, a 2 surfaces, yes. You can see the corollary. This characteristic curves will be there on both S_1 and S_2 . Due to transversality condition that is why step 2 is very, very important. Step 2, the t s and x y the maps ϕ and χ are inverses of each other that is very, very important.

Due to the transversality condition, any point on a characteristic curves has the form x y , u_1 of x y and also x y , u_2 of x y whenever you are in this common area, F intersection D_1 intersection D_2 . F , you remember, is the domain of definition for χ , this proves that S_1 coincides with S_2 in a neighbourhood of P_0 . Hence, u_1 is identically equal to u_2 on some neighbourhood of x_0 y_0 .

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This proves uniqueness. Now, quick remark on the proof of existence and uniqueness theorem.

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Remark on proof of Theorem

Recall that in Step 3 of the proof existence theorem, we proved that the function

$$u(x, y) := Z(T(x, y), S(x, y))$$

is a solution to (QL).

We give a **geometric proof** of the same.

- 1 The surface S was described using parametric equations

$$x = X(t, s), y = Y(t, s), z = Z(t, s), s \in I, t \in \mathbb{R}.$$
- 2 The curves $s = \text{constant}$ are the characteristic curves of (QL), and at any point on these curves the tangential direction is given by (X_t, Y_t, Z_t) .

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In fact, it is going to be another proof of step 3. Step 3 was very simple. We had to simply compute the derivative, substitute, we see that the candidate solution is indeed the solution. We can give a geometric proof of that. Surface s well described using these parametric equations right $X t s, Y t s, Z t s$ fine. The curves s equal to constant. What are they? They are characteristic curves. This is Gamma, fix s on this; this is $X t s$.

So, if S is fixed, it is a characteristic curve. Characteristic curve is; what is the tangential direction? First of all, it is a curve therefore, $X t, Y t, Z t$ is the tangential direction. So, $X t, Y t, Z t$, this is tangent, but it is characteristic curves. Therefore, the direction is actually a b c remember this.

(Refer Slide Time: 34:08)

Remark on proof of Theorem (contd.)

- 3 Since characteristic curves lie on S , (X_t, Y_t, Z_t) is a tangential direction to S as well.
- 4 Hence the normal to S is perpendicular to the direction (X_t, Y_t, Z_t) , which is nothing but the characteristic direction.
- 5 This implies S is an integral surface.
- 6 Any proof of the uniqueness assertion in Theorem is incomplete, if the transversality assumption is not used.
 - For, the uniqueness fails with the failure of transversality condition It will be discussed later.

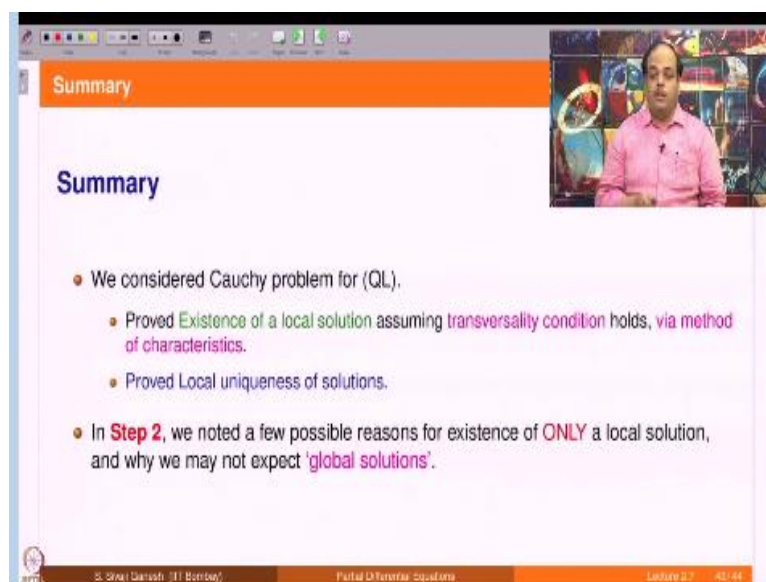
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And characteristic curves lie on the surface S . For s , we know something which is normal. Characteristic curves lie on S ; X_t, Y_t is also a tangential direction to S and X_t, Y_t , it is nothing but characteristic redirection, we have observed. What is the normal? Normal is a, b, c . Now, $u_x, u_y, -1$ yeah, $u_x, u_y, -1$ that is a normal. $z = u(x, y)$ if it is your surface normal is along this direction u_x, u_y and -1 normal.

And what is the tangential direction? One of them is a, b, c . That has to be 0 and this is nothing but the equation. Therefore, the surface S is an integral surface. So, any proof of the uniqueness assumption in the theorem is incomplete if the transversality assumption is not used, because uniqueness fails with the failure of transversality condition. When transversality condition fails, there are examples, where uniqueness fails.

There are examples where existence fails, anything can happen. So, we will discuss that later. So, if you are proving uniqueness, you must be using transversality condition is satisfied. So, people may not write this much explicitly in books, but they will say that step 2 which is what we have said right, they make use of step 2 without explicitly mentioning to you. So, you should not get confused.

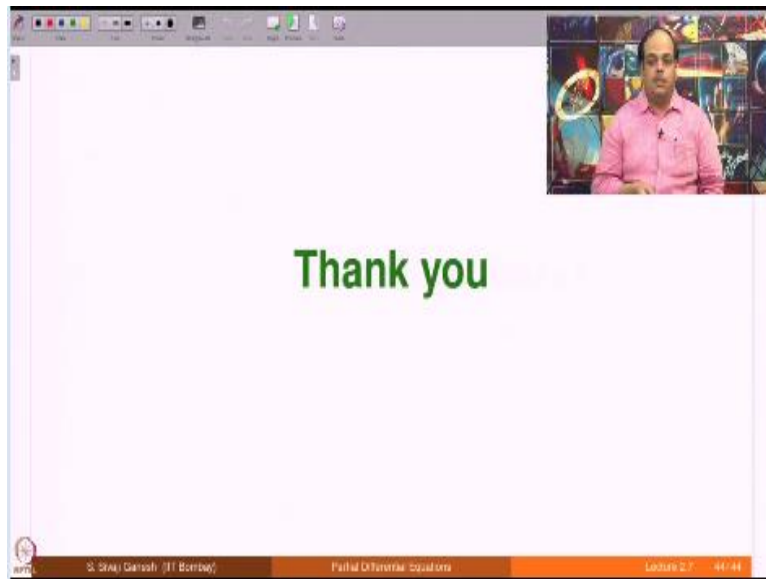
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The summary is that we consider the Cauchy problem for QL. we prove the existence of a local solution. Assuming transversality condition holds nothing, no more assumptions via method of characteristics. We prove local uniqueness of solutions. And in step 2, we noted a few possible reasons for existence of only a local solution, which will be seen in next lectures

and why you may not expect global. Global, I already mentioned; there are at least 2 notions of local solutions and hence global solutions.

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So, in the next lecture, we will take up as promised, what is transversality condition geometrically and what happens when it fails? What are the possibilities? Thank you.