Partial Differential Equations
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Lecture – 2.7
First Order Partial Differential Equations
Method of Characteristics for Quasilinear Equations - 2

In the last lecture, we have started discussing method of characteristics for solving Cauchy problem for Quasilinear equations. In this lecture, we complete the proof of existence of solutions to Cauchy problem for Quasilinear equations using method of characteristics.

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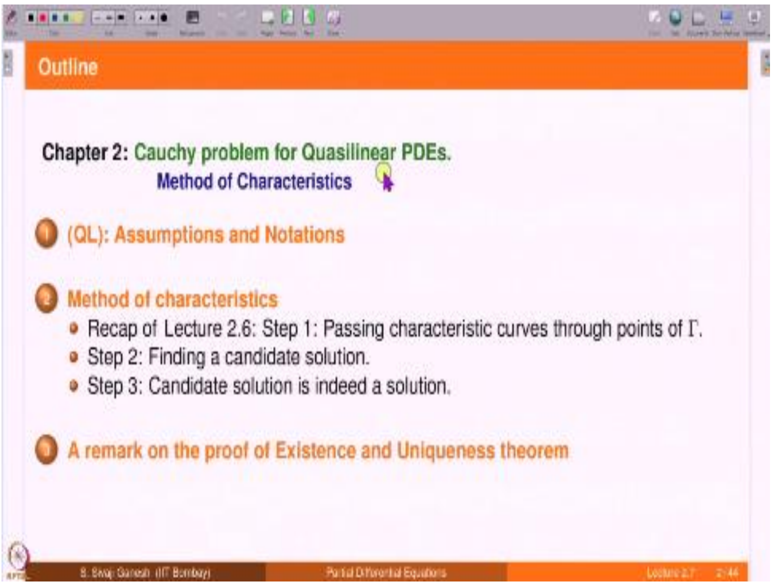

So, outline of today's lecture is, firstly recall the assumptions and notations which are made about Quasilinear equations and then we describe method of characteristics. A quick recap of lecture 2.6 where we have completed the step 1 in the method of characteristics, which is passing characteristic curves through points of gamma. Today, we start elaborating on step 2, where we say what is the finding a candidate solution and then show that that is indeed a solution.

And then a small remark on the proof of existence and uniqueness theorem that we are going to prove at the end of this step 3.

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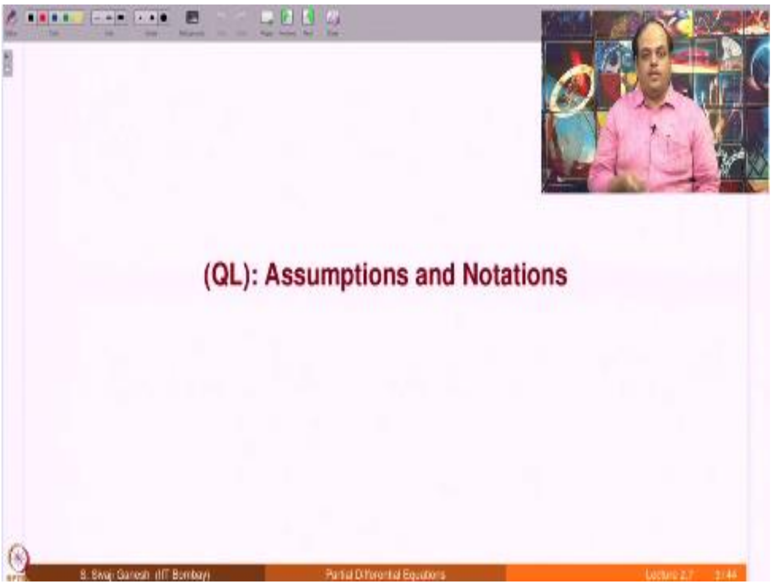

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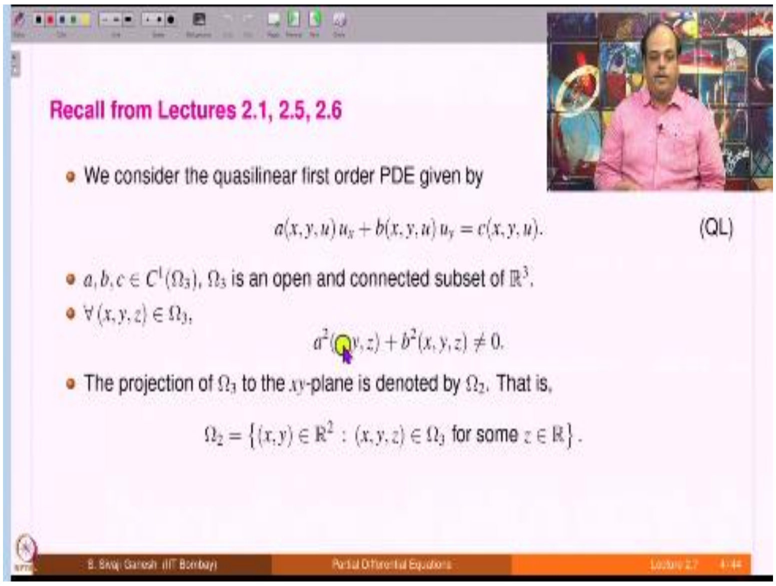

So, assumptions and notations in the context of Quasilinear partial differential equations, first order equations, this has been visited many times 2.1- 2.5 and as last time also in lecture 2.6. So, Quasilinear equation, we denote by QL. It stands for this equation a u x + b u y = c. a b c are functions C 1 functions on omega 3. Omega 3 is an open subset of R 3 such that a square + b square is not equal to 0 at every point of omega 3 and the projection of omega 3 to x y plane is denoted by omega 2.

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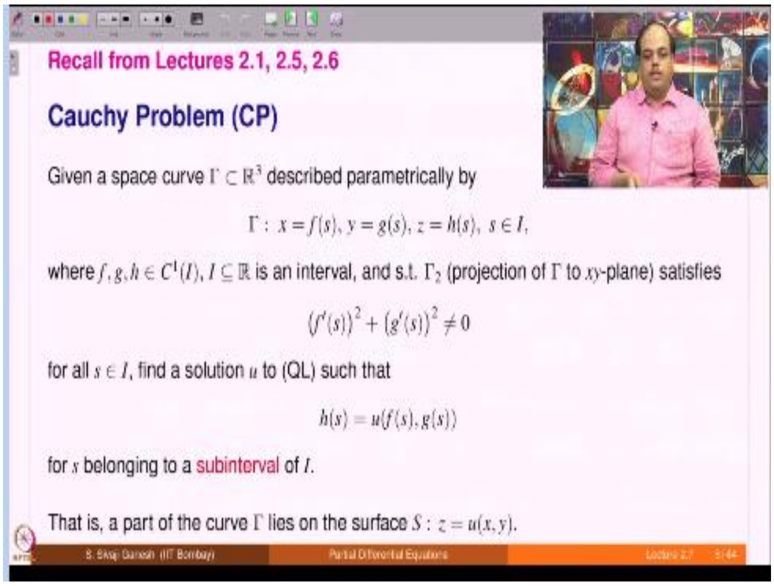

Cauchy problem given a space curve that is a curve in R 3, which is given parametrically in this form, where the parameter is running in the interval I and f g h are C 1 functions such that notation gamma 2 important, gamma 2 is a projection of gamma 2 x y plane that satisfies this condition. Gamma 2 is actually x = f s; y = g s as s varies in I. We want that f prime and g prime do not vanish at the same time, at every point on gamma 2.

Then we want to find a solution that means a function defined on some domain such that u on gamma 2 takes the value h which is a prescribed datum curve. Of course, we do not expect the solution domain to contain entire gamma 2, but a small portion of gamma 2 is also allowed.

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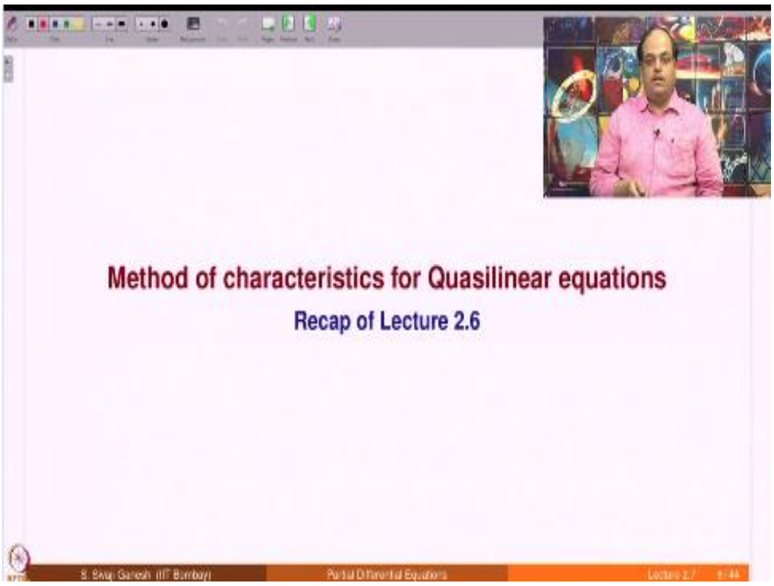

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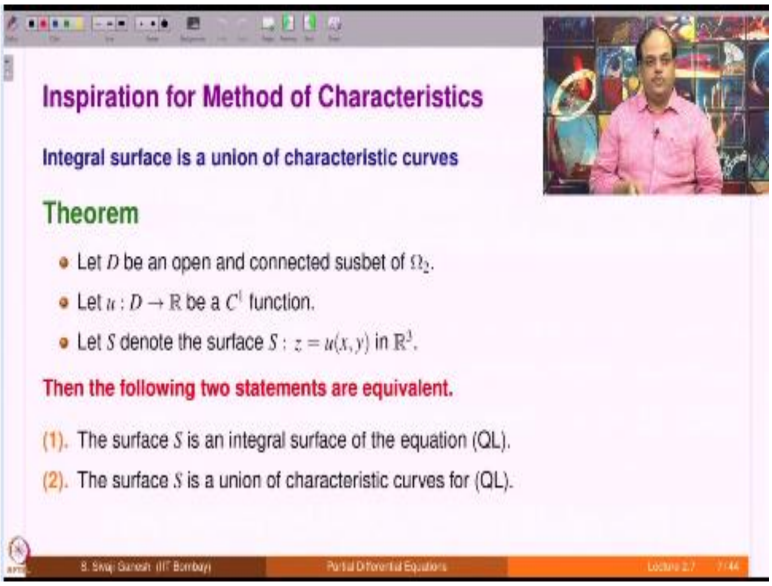

Now, let us look at the method of characteristics from which was discussed in the last lecture. The inspiration for method of characteristics is the following theorem, which said give me a surface or the forms z = u x y, then saying that this surface is an integral surface is same as saying that this surface is union of characteristic curves corresponding to Quasilinear equations. Since, we want to construct an integral surface.

In other words, wants to find a solution or a surface corresponding to a solution. We will start with this idea of constructing some quantity, some object which is union of characteristic curve that is the idea behind this method of characteristics.

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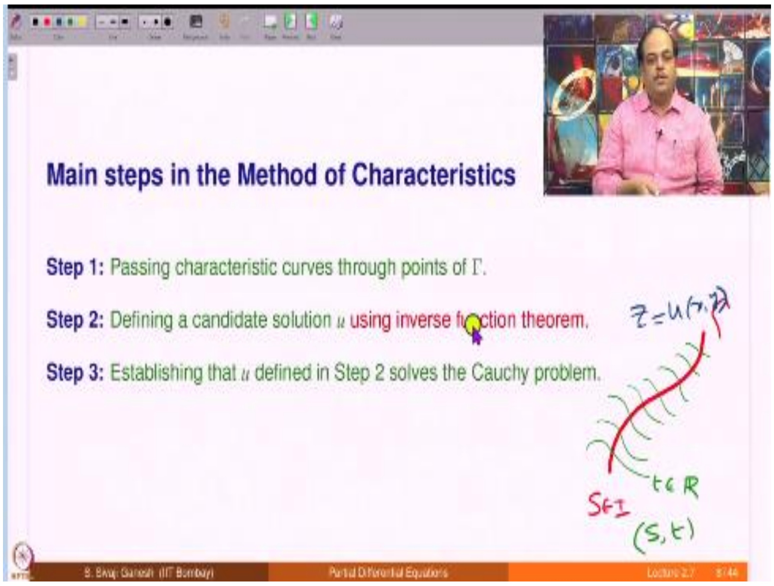

So, step 1 is through every point of gamma, you pass a characteristic curve. So, gamma is here. We want to find a surface which contains this gamma. So, what we do is, we pass through this characteristic curves like that and you see this is going to give surface. So, characteristic is given datum curve is described by s and the characteristic curves for each fixed this point, these characteristics are passing through that is defined by t in R.

So, we have a description of some object geometrical object in R 3 which is parameterized by 2 parameters s, t. So, as s and t vary, we hope, we will get a surface that was the idea. So, that is why we start with this step 1, where we pass characteristics curve like this, then we say that the surface that we are going to generate would be of the following type where the third coordinate is going to be a function of the first 2 coordinates that is what we hope and that we will achieve in the step 2.

We will define a function u, u is a function of 2 variables x and y using inverse function theorem. And then we show that u is indeed the solution.

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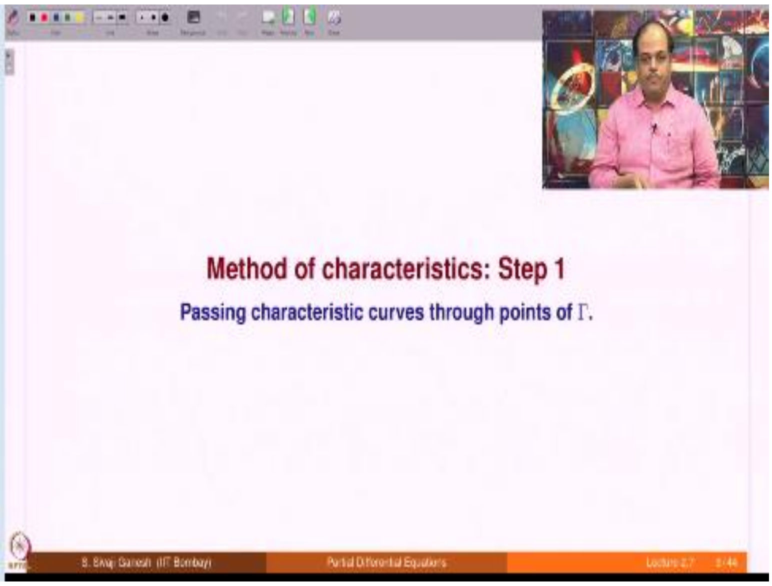

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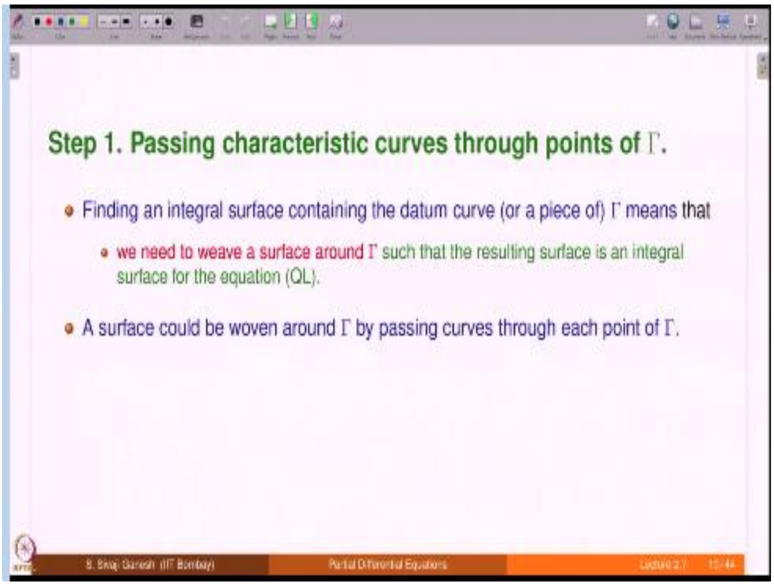

Step 1, what we did is finding an integral surface containing the datum curve or a piece of datum curve means we need to leave a surface as I have just shown you in the picture. And if you leave that surface through passing curves through points of gamma and curves are characteristics curve, then that theorem gives as the hope that the surface we get is going to be an integral surface.

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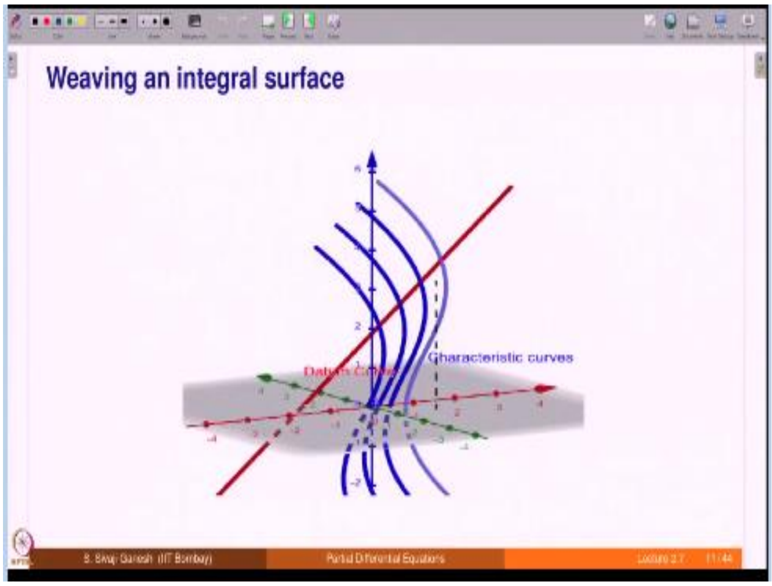

Here, this is the datum curve. These are characteristics curve. This is the same picture which I have just drawn.

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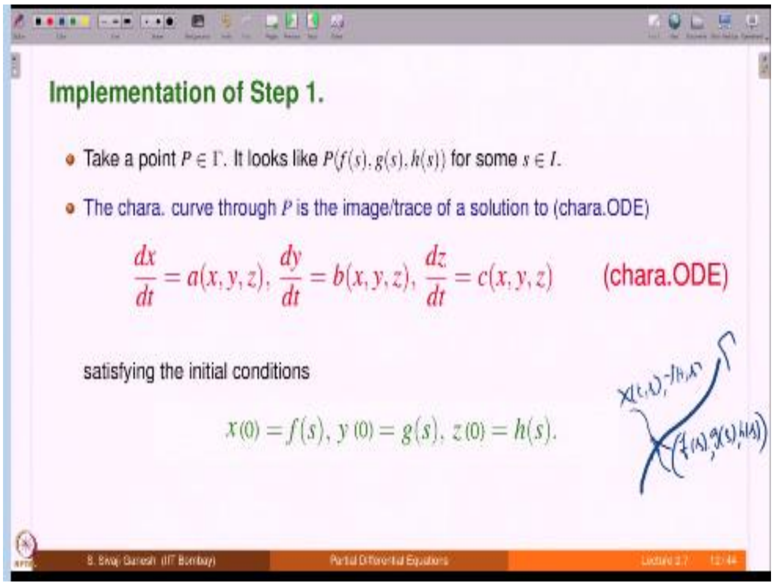

So, for this, what we do take a point and gamma, any point on gamma looks like f s, g s, h s and then solve this system of characteristics ODE with initial conditions f s, g s, and h s. It means that when you look at this gamma, you take at this point f s, g s, h s, the point looks like this. These are 3 tuple and then you pass a curve through that. That will be denoted by X t s, Y t s, Z t s. This is what we get at the end of this step.

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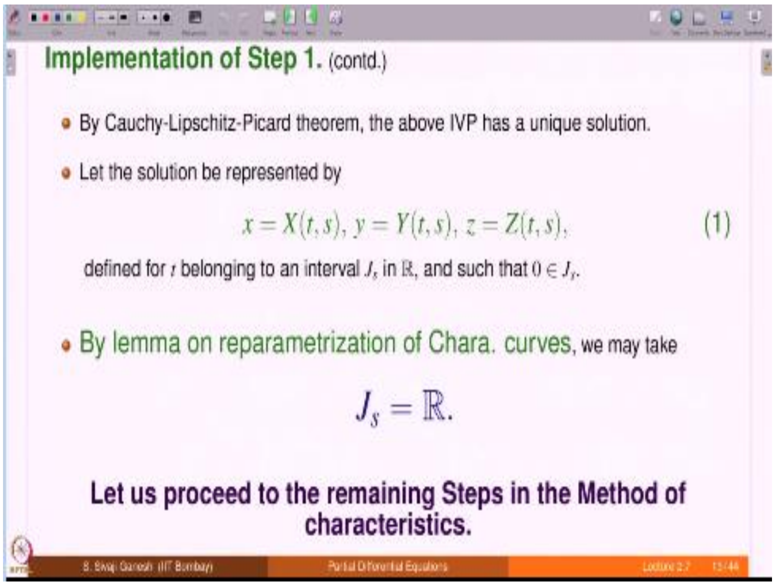

At = 0, it is going through this point that is how we are solving, we are giving the initial conditions to the system of characteristic ODE. Now by Cauchy Lipschitz Picard's theorem, it has a unique solution. Let it be denoted by like this, x = X t s; y = Y t s, z = X t s. Where is it defined? s in I and t belongs to J s which is an interval, of course, the interval contains 0, note the dependence of s on J's that is why it is written J s. But we have seen a lemma and re-parameterization which says, we may take J s = R. So, now, let us proceed to the remaining steps in this method.

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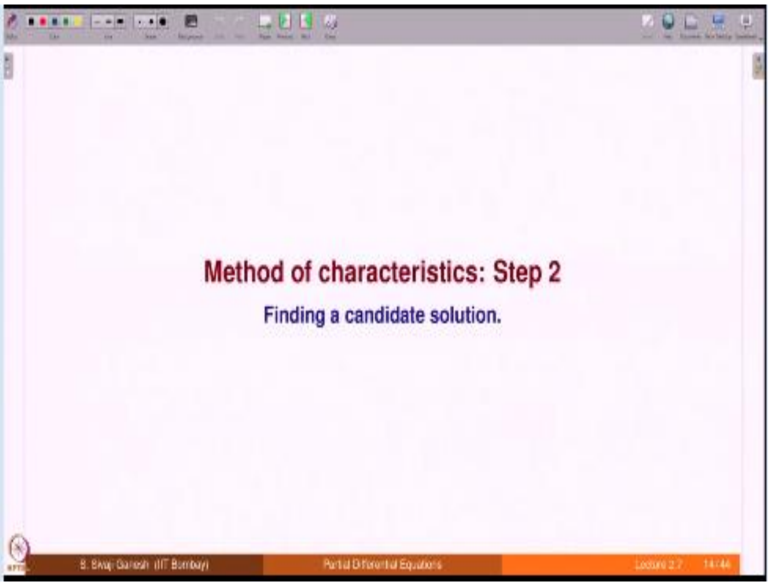

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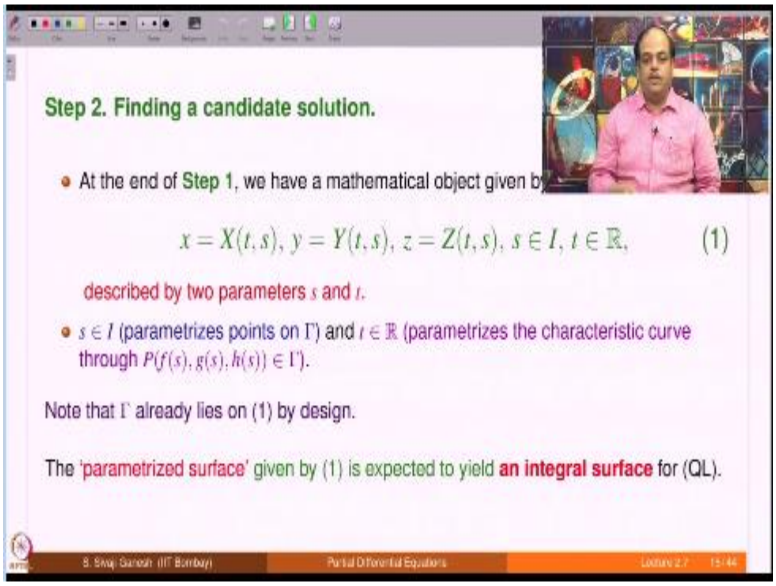

Second step is finding a candidate solution. How do we get that? Of course, we are take help of this what we have been achieved, so far. So, this mathematical object is described by 2 parameters. And this leaves in R 3. So, we hope with the surface that is the idea. As I pointed out earlier, s represents a parameter running on the datum curve gamma and t in R, it is parameterizing characteristic curve passing through this point and gamma, which is visited when t = 0.

Of course, gamma already lies on one by design, initial conditions. The parameterize surface given by 1, I have put it in inverted commas here, because it is not clear. it is a parameterised object is clear; surface means we expect something that is why I put here on till the confirmation comes, it is expected to be an integral surface, because we were motivated by the theorem. And therefore, we hope that we get an integral surface.

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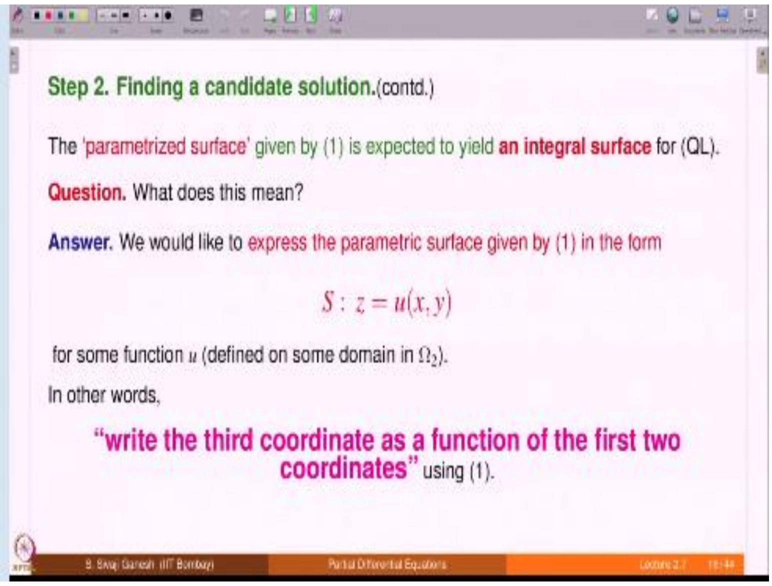

But this is described using parameters t and s. Now, we need to write it as z = u of x y. So, what does this mean? It means that we would like to express the parametric surface given by 1 in this form z = u x y for some function u. We need to find some function that means we need to find a domain on which this function is defined on this course. In other words, write the third coordinate as function of the first 2 coordinates using 1 that is the only thing that we have in our hands.

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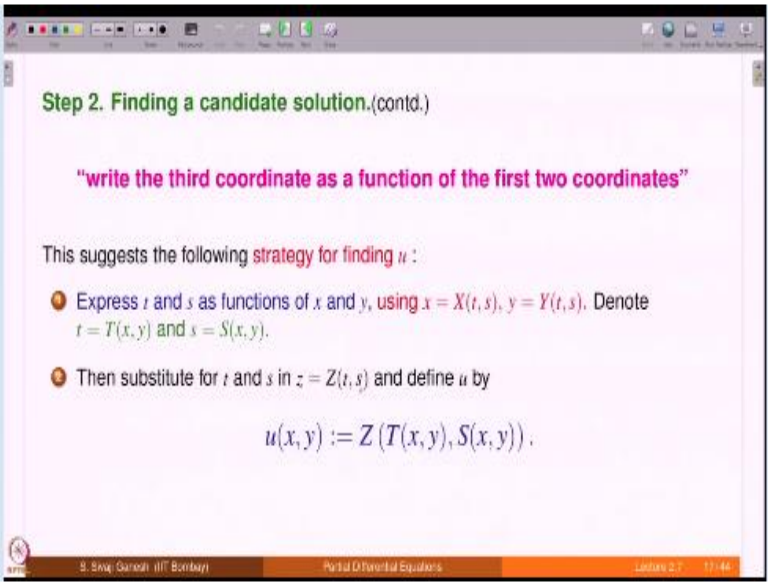

Namely, the family of characteristic curves passing through points of comma. So, the strategy for finding u is this. Express t and s as functions of x and y, we have x and y given in terms of t and s. Now, suppose I write t and s in terms of x and y, go ahead and imagine this is the formula we write t = T of x y and s as S of x y. Now, we go and substitute in these relations z = Z t s and define a function this way, z of t = T x y, s = S x y.

So, we got some function of x y, call it to u x y. This should work because z itself was motivated like that. The equation for z was derived after using that it is going to be the value of solution along the base characteristic X t s, Y t s. Refer to lecture 2.6. So, therefore, it should work.

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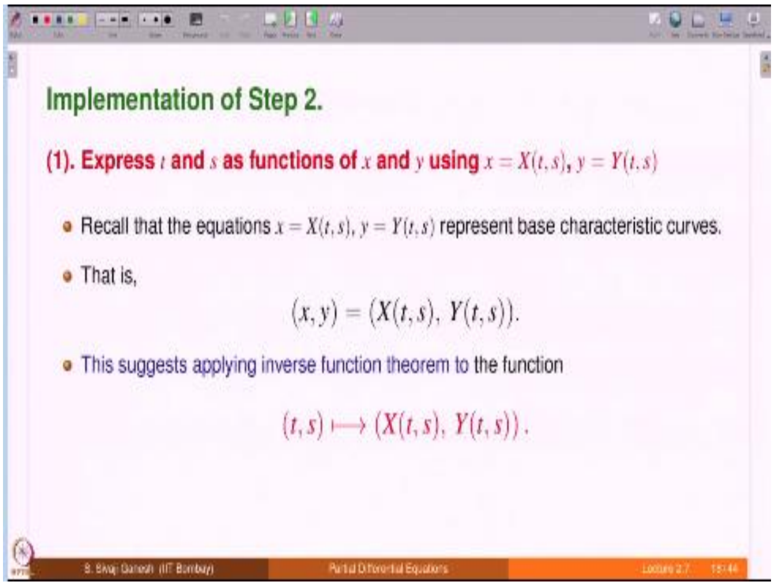

But, there can be very technical difficulties, we will see what they are. So, express t and s as functions of x and y like that. Recall that x and y given by X t s and Y t s that together represent base characteristic curves that is this. Now, this suggests applying inverse function theorem to this function then you will get t and s in terms of X and Y.

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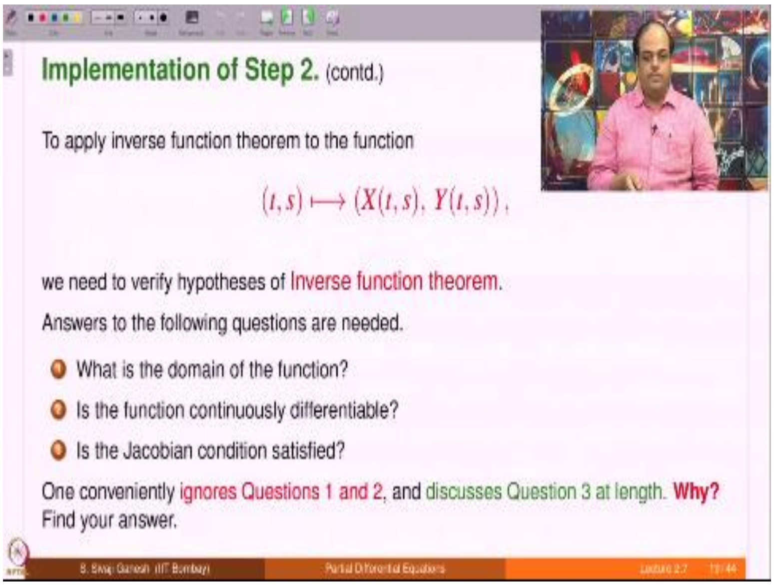

So, if you want to apply inverse function theorem, you are to see what is the setup of the inverse function theorem. It involves a function. It involves C 1 function and certain Jacobian condition. So, we need to check all those hypotheses. Then only we can apply a theory. So, therefore, answers to the following questions will be needed. What is the domain of this function? Is a function continuously differentiable? Is the Jacobian condition satisfied? These 3 conditions.

Usually, one ignores questions 1 and 2 and just concentrate on third one at length. In fact, the third is not a condition. It is a Jacobian condition satisfied is never checked that is put as part of the assumption and then one proves theorem, but still 1 and 2 are still important. So, you ask this question why? You will understand later on. So, answer, I will not give. You think about this question.

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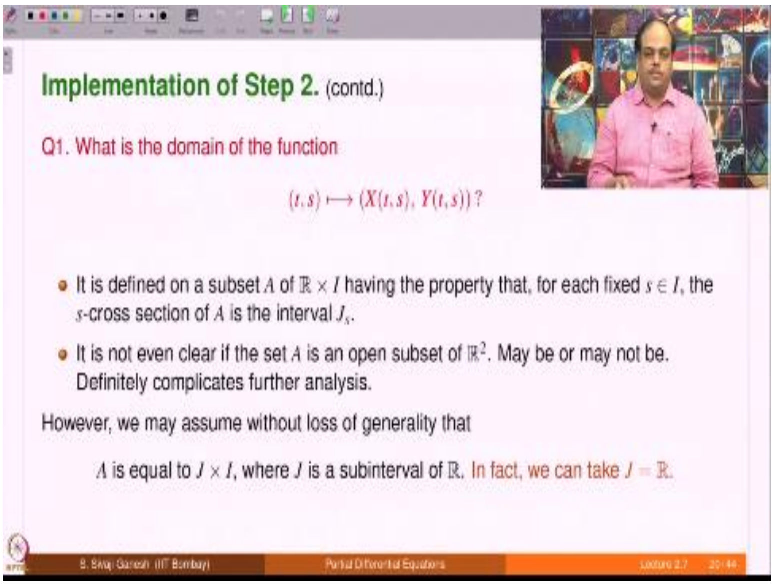

What is the domain of this function? As we observed for each fixed s, t varies in J s, so, it is going to be a subset of R cross I for each fixed s t varies in J s. That is all we can describe. It is not even clear what the set s is an open set etcetera. But anyway, we will do away with this kind of, answering these kinds of difficult questions. So, it complicates further analysis.

However, we may assume without loss of generality, that A is equal to J cross I where J is a subinterval of R. I is where the parameters s is running in I. J is where the t is running. We take a J = R. This is what we said thanks to our lemma tree parametrization.

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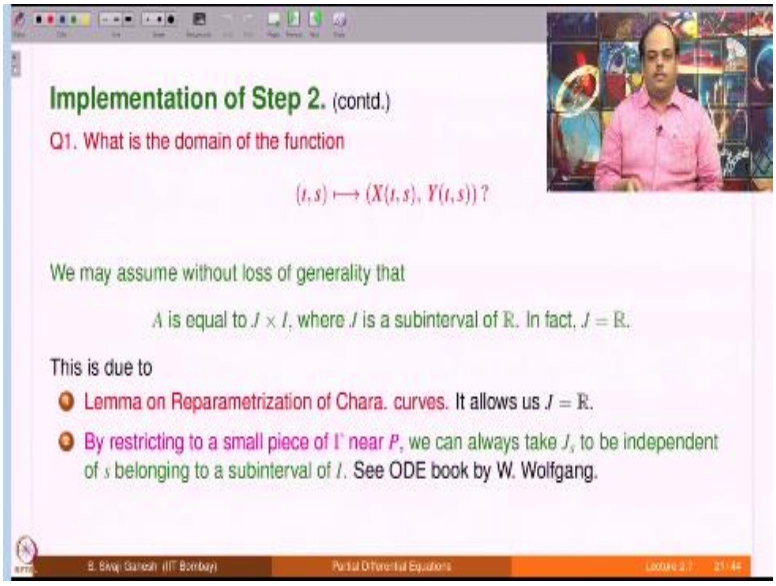

So, therefore, we may assume that without loss of generality that is A isequal to J cross I where J is a subinterval of R. In fact, we can assume J is equal to R. Due to this lemma that I just quoted, re-parameterization of characteristic curves are by restricting to a small piece near gamma that is also okay. So, you have the point P at that point restrict to a small piece, then it is true that J can be chosen independent of s.

These ideas comes when one discusses differentiable dependence of or continuous dependence of solutions on the initial data. So, see the ODE by Wolfgang.

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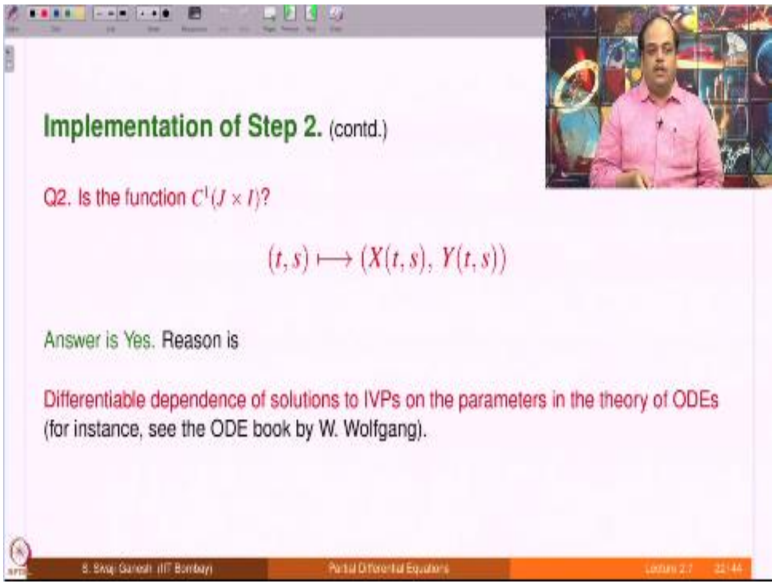

For this book, it is a good book, we will be referring once more at least in this lecture, okay done. So, domain is R cross I fine or R cross some subinterval of I if you follow Wolfgang approach. If you follow our approach, it is R cross I. Now, next question is: is it C 1 function? If you look at this X and Y or solutions of certain ODE, therefore, with respect to t, they will be C 1.

What about with respect to s? What about together? Answer is yes. It is C 1 function. If partial derivative with respect to t and with respect to s, both are continuous functions, then it turns out that it is differentiable with respect to the 2 variables together. As we know, existence of partial derivatives of functions of several variables does not mean that the function is differentiable.

Forget about that function need not even continuous at a point, but all directional derivatives exists. But, if partial derivatives are continuous, then the function turns out to be differentiable. And, it will be C. Therefore, we are interested in the X t, X x whether they are continuous functions. X t as I said it is a b c, they are nice functions. So, there is no problem. What about with respect to s? Various s.

s is appearing in obtaining X t s and Y t s through the initial conditions. We saw x of 0 s and y of 0 as f s and g s and we assume f and g are C 1 functions of the variable s. That is the reason why this function is C 1 of J cross. It comes from differentiable dependence of solutions to initial value problems on the parameters in the theory of ODEs. For a instance, see the book by Wolfgang on ordinary differential equations.

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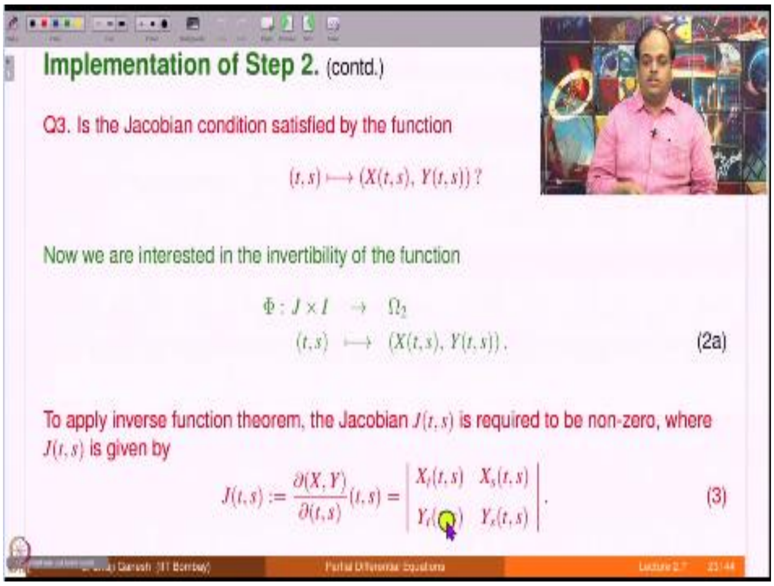

Now, let us implement step 2. Step 2 and question 3 we are considered. Is a Jacobian condition satisfied by this function? We are interested in invertibility of this function right. So, we are to compute the Jacobian. Inverse of function theorem says computed Jacobian that should be nonzero. Jacobian is this. Now, we are to compute the X t t s, Y t t s, X s and Y s at point t s. Now, X t at t s is A, it is known.

Y t at t s is also known. X s at t s is not known. What we actually know is x s at 0, s. That is the reason why we stick t = 0 now.

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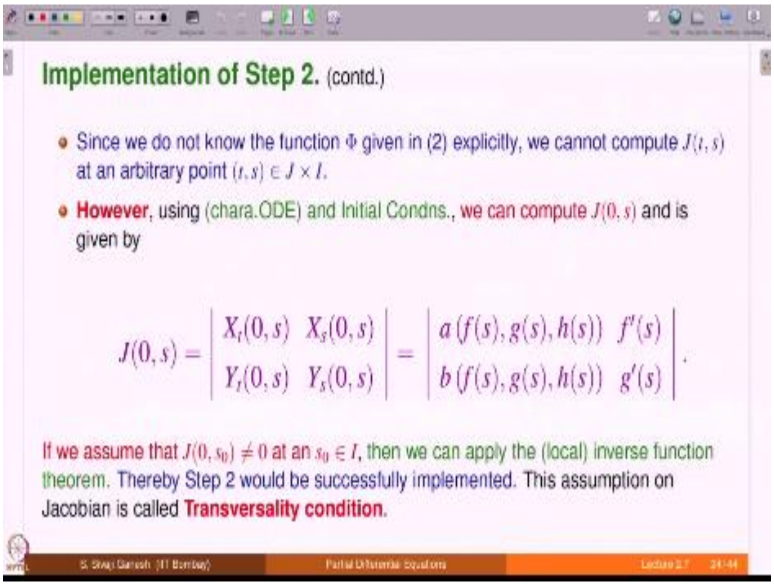

Since, we do not know the function explicitly, we cannot compute J t s for an arbitrary point t s in J cross I, which you can read it as R cross I. However, using characteristic ODE an initial condition, because initial conditions for characteristic ODE are given a t = 0 therefore, they are something can be done. We can compute J 0, s and that is given by this. X t, Y t, are solving ODEs. So, from there, you get the a b. x of 0 0, s is f s.

Therefore, derivative that with respect to s is f dash s. Similarly, y s is g dash s. So, we want this to be nonzero. So, if you assume that J of 0, s 0 is nonzero at a point s 0, then we can apply the inverse function theorem. I am written in the brackets a local because there is something called something else called global inverse function theorem. But the usual inverse function theorem is a local inverse function theorem.

Thereby, step 2 will be successfully implemented. This assumption on Jacobian is called the transversality condition.

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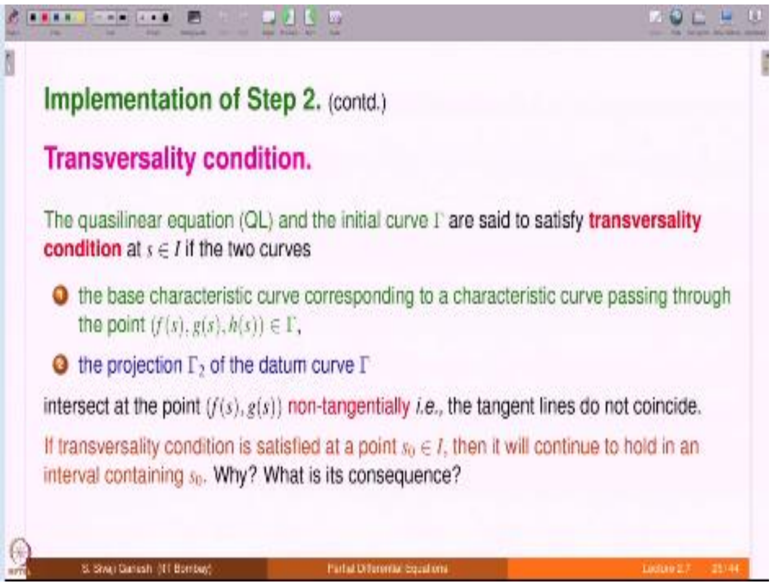

The Quasilinear equation QL and the initial curve or the datum curve gamma are set to satisfy transversality condition. So, the transversality condition is 2 people together satisfy that. That is quadratic equation and initial curve. At a point s in I, if the 2 curves which are listed here the first curve is a base characteristic curve. So, at point s right, so, look at the point f s, g s that is going to be on a datum curve, through that you find the characteristic curve, projected to R 2 x y plane that will be the base characteristic curve.

Thus, base characteristic curve and gamma 2 which is a projection of the datum curve gamma, these 2 curves intersect, of course, they intersect at the point f s, g s, non-tangentially that means, the tangent lines do not coincide. If the transversality condition is satisfied at a point, then it will continue to hold in an interval containing a 0, because the transversality condition is the nonzeroness of certain determinant.

And whatever is appearing inside the determinant are continuous functions of s, therefore, at some point nonzero, it will continue to be nonzero nearby that point. That is the reason. In this case, what would happen is imagine this is our gamma 2 that base characteristic curve will come like that, it will actually cut base characteristic curve will not be like this. For example, it will not be like that, because here, they share the same tangent.

So, this is not allowed, because then transversality condition is not satisfied because if you see in the determinant, what are the 2 things which are coming the 2 columns x t, y, t is the tangent of the base characteristic curve and this is a tangent of the gamma 2 and asking that these nonzero means they are not parallel and they are passing through this point the f s, g s; 0, s, sorry, they are passing through the point f s, g s. Yes.

And we are asking that the tangent line is not the same. That is why it is the transversality condition. So, here it is transversality. So, here it is not transversality. Now, such a things happens, if you look at nearby also, the curves all going to be like that. They will always be cutting and you get a surface that is a consequence.

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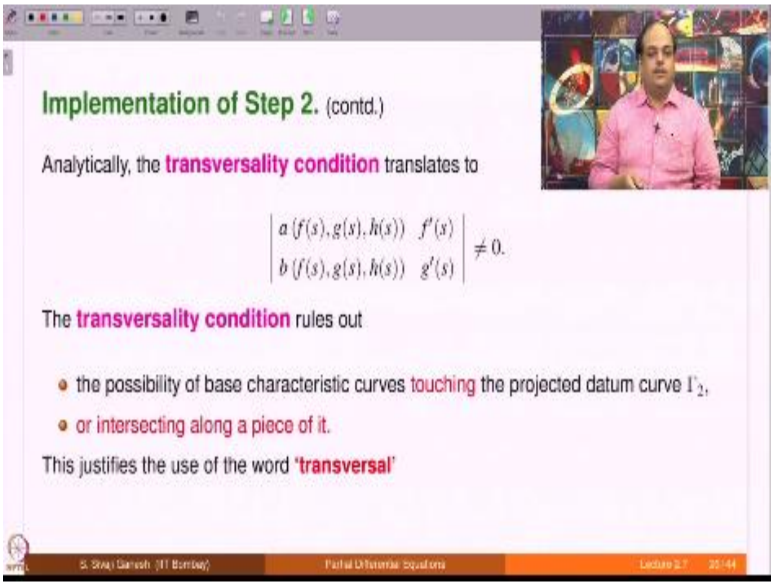

So, analytically the transversality condition translates to this being nonzero, this determinant. It rules out. The possibility of base characteristic curves touching the projected datum curve namely gamma 2 that we have illustrated in the picture. Touching means sharing the tangent up and go inside of it of course. Then also you have touching for a long time. This is, maybe at a point, but this is, if we intersect along a piece of it, it means that they share tangent up throughout the piece because curve is same.

So, such things are rolled out. Please spend some time and understanding this geometrically, because it will be useful when we are trying to give examples or solve or guess what would happen in certain situations for Quasilinear equations. Some examples, it will help. So, this justifies the use of the word transversal. It means to cut, it cuts.

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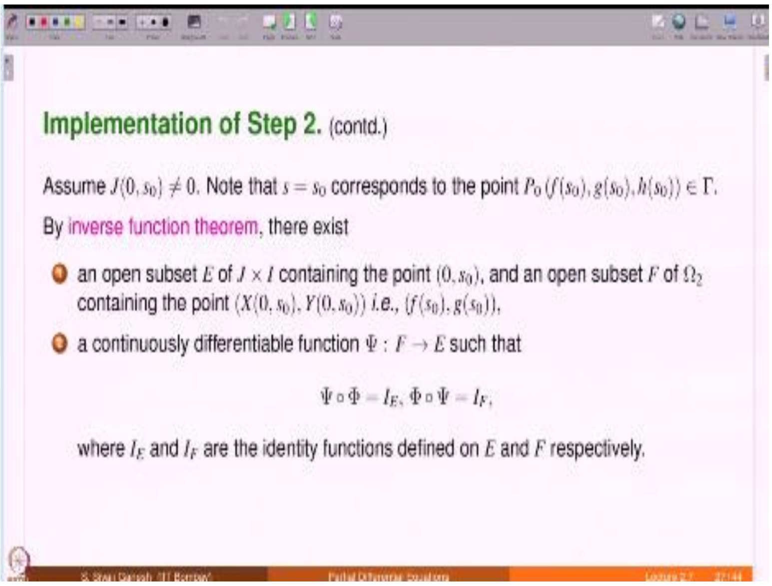

So, assume that Jacobian is not 0 at the point s 0. Even if Jacobian is not 0 for all s in I, J of 0 s is not 0 for all s in I, it is not a useful because the only way you apply inverse function theorem is at a point s = s 0. So, I am assuming at a point, it is nonzero. Of course, s = s 0 means the point on gamma will be f s 0, g s 0, h s 0. By inverse function theorem, there exists an open subset of J cross I containing this point 0, s 0.

And an open set on the other side omega 2 containing the point X of 0 s 0, Y of 0 s 0 because our function is that which is actually a f s 0, g s 0 and a continuously differentiable function from F to E, because this should be the inverse of the function phi. So, chi circle phi will be identity on E and phi circle chi will be identity on F. These are the identity functions.

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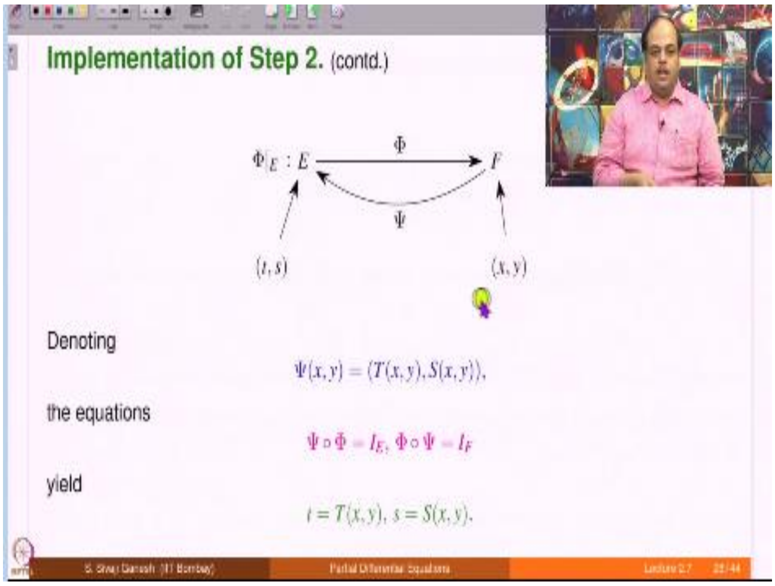

I E means I. Now, we have a picture. So, this is things. Phi, we have E to F. So, originally phi is defined on some other side, which is J cross I or R cross I or J cross some I dash and then we assume that at 0, s 0, Jacobian condition is satisfied, nonzeroness of the Jacobian. Therefore, you can find an open set E which contains a point 0, s 0. And open set F, which contains the image of phi under that.

Whatever is a map phi we have, it maps t s to X t s, Y t s. So, the X of 0 is s 0; y of 0 is s 0 which is f s 0, g s 0 and a mapping size is that these compositions, you go from E to F, come back from F to E or go from F to E first and then come back to F, both will be identity functions. This is where we are using t s coordinates. Here, we are using x y coordinates, the variables names.

So, therefore, chi of x y, you denoted as T x y, S x y and this equalities holder because they are inverses of each other that will give you t = T x y, s = S x y.

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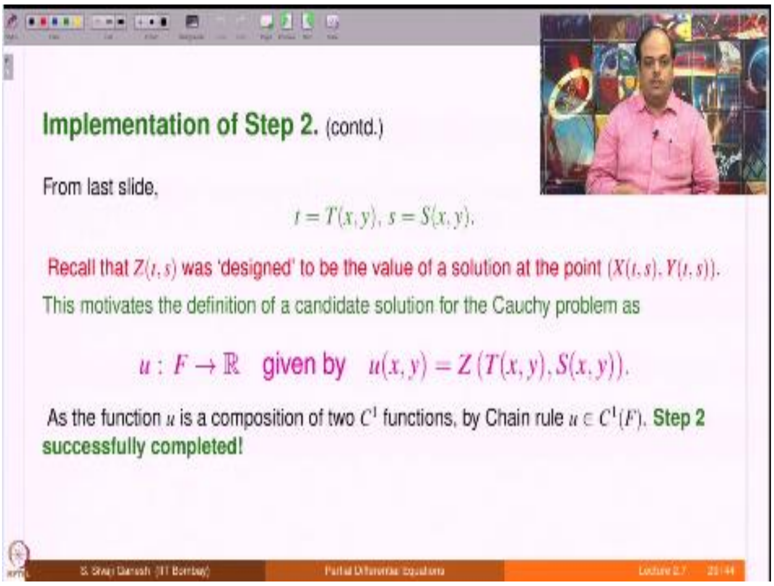

As I told you earlier Z t s was designed to be the value of the solution at this point X t s, Y t s, referred to lecture 2.6. Therefore, this motivates us the definition of a candidate solution by this. u defined on F to R given by u x y = Z of T x y, S x y as a function is a composition of 2 C 1 functions. This is C 1 inside and Z itself is C 1 function. Therefore, the composition will be C 1function.

Therefore, u is a C 1 function. Step 2 is successfully completed. We have defined what is a candidate solution.

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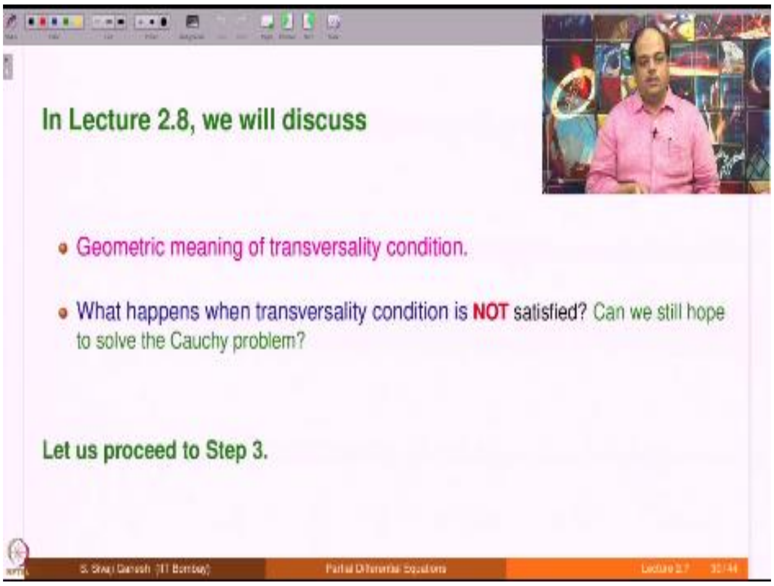

So, in lecture 2.8 that is a next lecture, we will discuss the geometric meaning of transversality condition and what happens when this condition is not satisfied, what will happen? When it is satisfied, we are going to prove a result today and what happens if it is not satisfied will be analysed in the next lecture. Can we still hope to solve the Cauchy problem? We hope to see. Answers will be in the next lecture. Now, let us proceed to Step 3.

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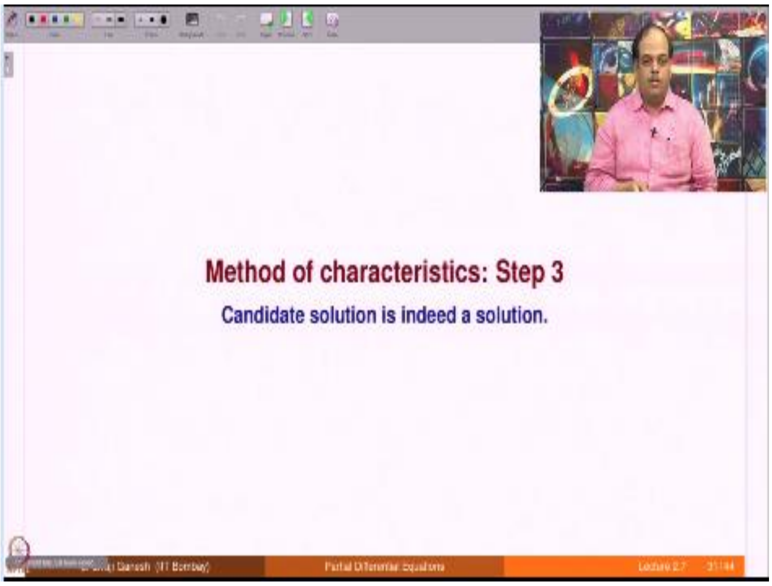

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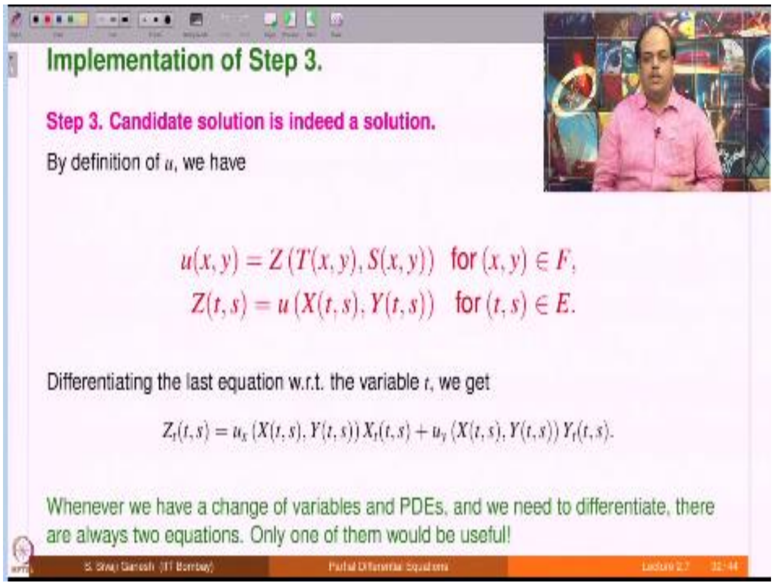

What is step 3? Candidate solution which we got in step 2 is actually a solution. By definition of u, we have this formula and Z t s = u X t s, Y t s also have this. Now, what is that we want to show? u solves PDE right. So, we have to come to a u x, u y etcetera. But that may not be useful, let us see. Differentiating the last equation that is this equation with respect to t, we get of course, we use chain rule and we get this.

Now, why did not we not choose the first equation? We could have chosen that differentiate with respect to x, with respect to y, go back and substitute in the equation and check whether it is satisfied or not. This is usually the situation all the time whenever we have changed variables and PDE are there, way to differentiate some equation, but there are always 2 equations for that.

Only one of them would be useful. In the sense, it may be a quicker way of getting the solution, not wrong other. One is not wrong, but you may not be able to achieve anything with that, but this one will definitely give you.

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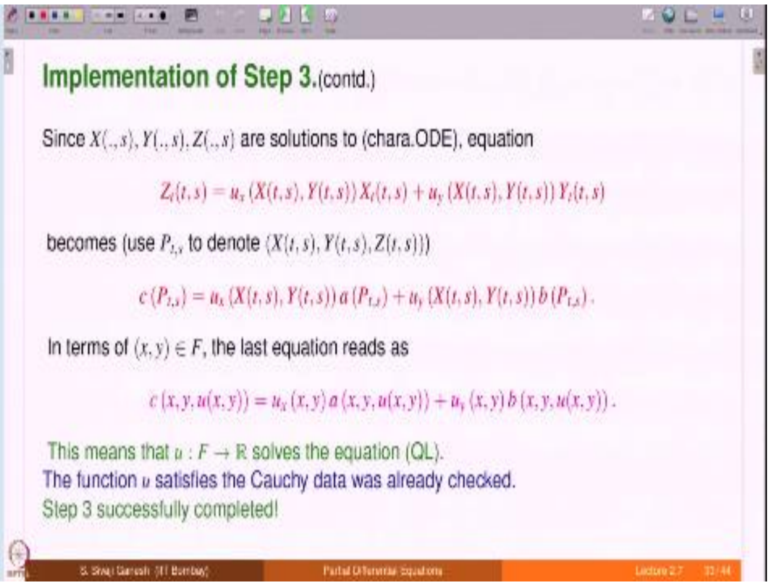

You will learn this. As many times as you make mistake of differentiating this u that is true. Now, it is going to work differentiate, we got this formula. Now, X t s, Y t s and Z t s are solution to chara ODE, therefore, here, points will come out be too long. Just to make the notation short. I am using X t s, Y t s and Z t s has P t s. Now, Z t is c at X t s, Y t s and Z t s, but now it is P t s; u X is as it is; x t is A at X t s, Y t s and Z t s. Now, here it is P t s. Similarly the other term, we have this.

So, in terms of x y in F, the last equation reads as C of x y u x y = u x of x y and a of x y u x y, u y of b x y u y. Now, this means u is a solution to the Quasilinear equation. Cauchy data is already taken care. We already observed that Cauchy data will be satisfied. The datum curve will be on the integral surface. So, step 3 is also successfully completed.

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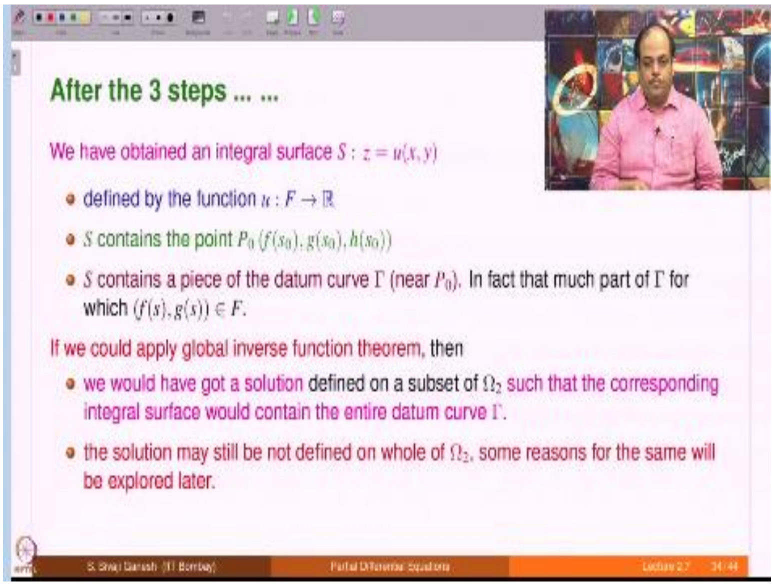

After the 3 steps, we have obtained an integral surface defined by function u from F to R. Of course, here we have used the inverse function theorem that means, existential to start with and s contains this point P 0; s contains a piece of the datum curve near P 0. In fact, that much part of gamma for which f s, g s belongs to this open set F. If you could apply, if there is a global inverse theorem, if you could apply that, then we would have got a solution defined still on a subset of omega 2.

But, that would have contained the entire datum curve gamma. That integral surface would contain an entire datum curve gamma. The solution may still not be defined on whole of omega 2. This local, global, we have to be very, very careful, what we have been dealing with is we are looking at a point on the datum curve and showing solution a surface exists nearby that which is integral surface that means, we are discussing something like local with respect to the datum curve.

And if you say globally inverse function theorem, it must be giving global with respect to the datum curve but not with respect to the domain. More on this, we will see in a future lecture. Local, there are 2 different concepts; local with respect to datum curve, local with respect to the domain omega 2.

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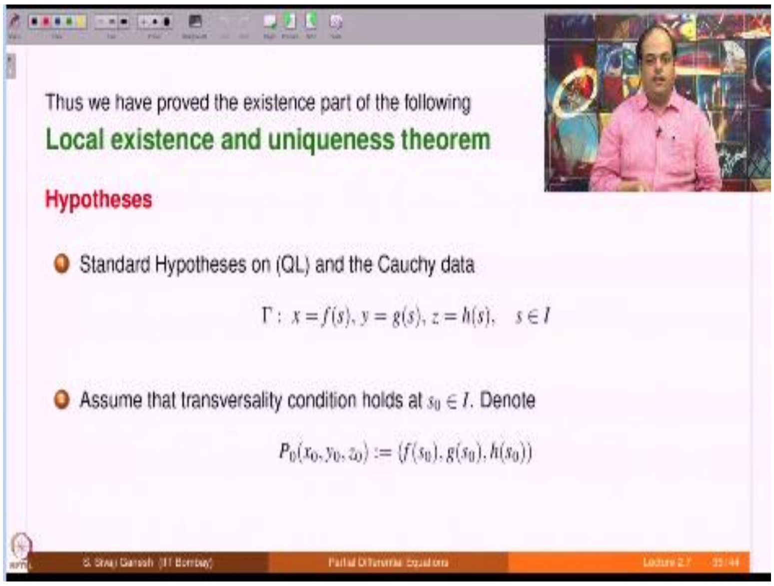

Theorem is here hypothesis. Standard hypothesis on QL, I am not going to recall here on Cauchy data. Assume that the transversality condition holds at a point s 0 and denote P 0 f s 0, g s 0, h s 0; same thing, let us call x 0, y 0, z 0 for convenience.

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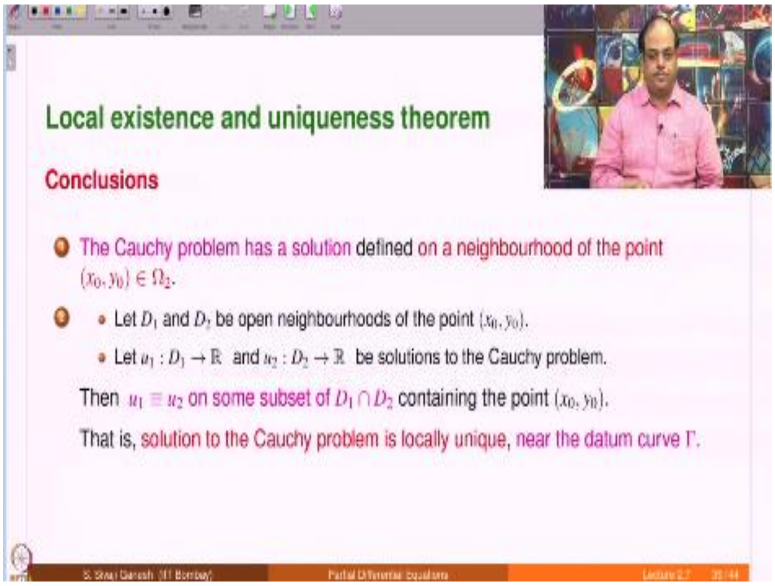

Conclusions: Cauchy problem has a solution defined and a neighbourhood of a point x 0, y 0 and let D 1, D 2 be open neighbourhood of the point x 0, y 0. Let u 1, u 2 to be defined on D 1, D 2 respectively be solutions to Cauchy problem and then u 1 will be identically equal to u 2 on some subset of D 1 intersection D 2. Of course, it contains the point x 0, y 0 that is solution to Cauchy problem is locally unique near the datum curve gamma.

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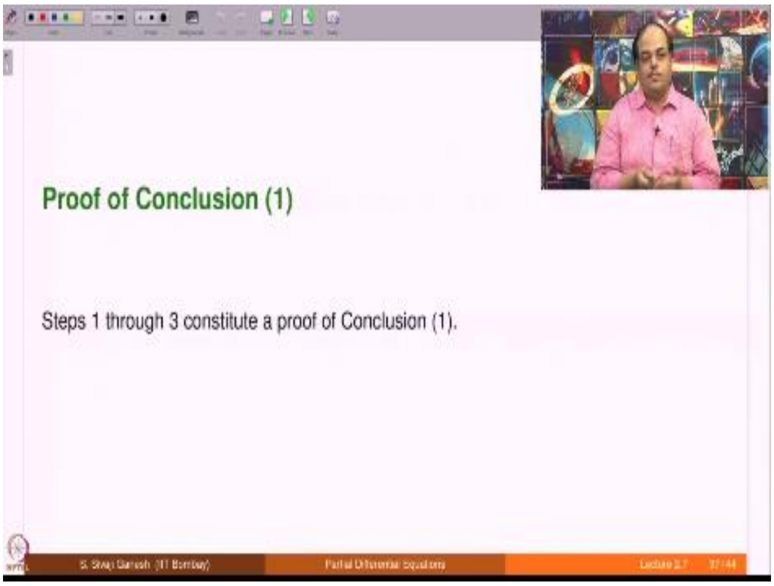

So, let us prove conclusion 1 that is precisely steps 1, 2, 3 that gives the proof of conclusion 1 that is about existence.

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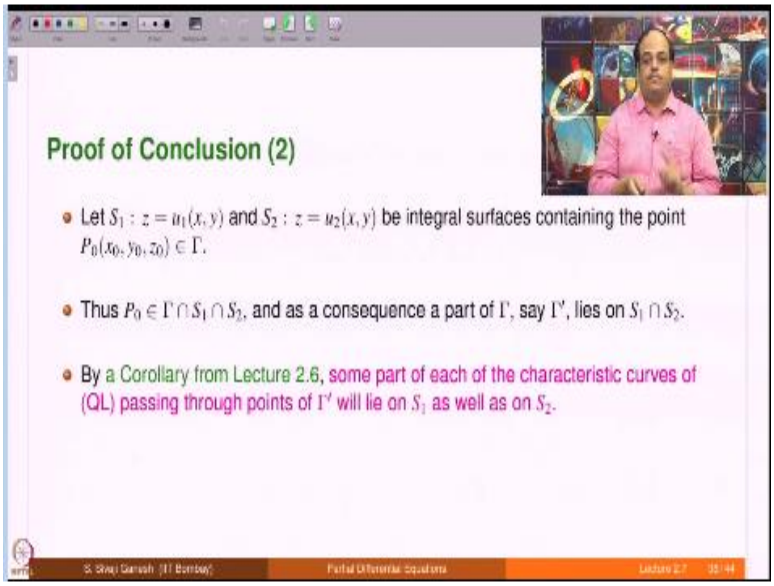

Now, let us do the uniqueness. Let us S 1 and S 2 to be a as given. One is defined on D 1; other one is defined on D 2 and take this point P 0 which is in gamma. P 0 is there on gamma, it is there in S 1 as well as S 2 and as a consequence P 0 is there on S 1; P 0is there on S 1, therefore, a part of, what is S 1? It is an integral surface. Therefore, a part of gamma will be there on S 1 intersection S 2. It will be there on S 1 and other parts will be in S 2.

There will be a common portion which will be there on both S1 and S 2 that is there. Then we have seen a corollary in lecture 2.6 in the last lecture, it said some part of the each of characteristic curves of QL passing through points of gamma dash will also be there on S 1 and also on S 2.

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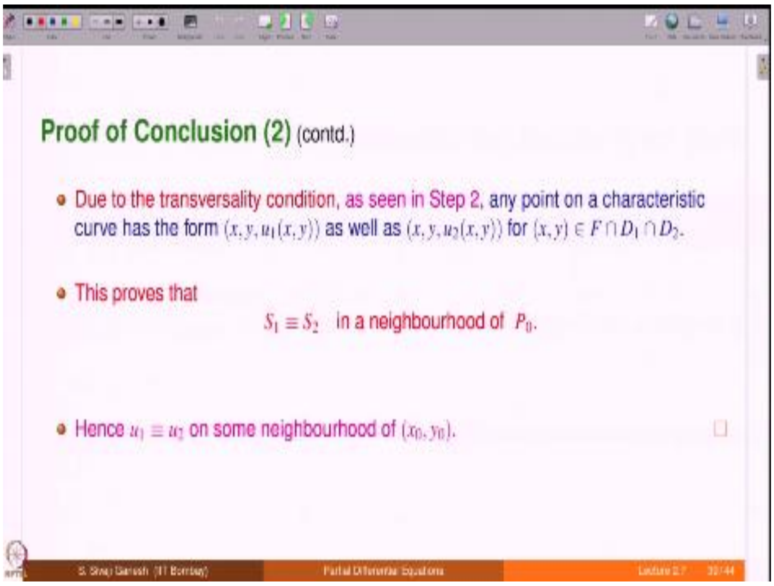

Of course, the corollary was with respect to one surface, but here with respect 2, a 2 surfaces, yes. You can see the corollary. This characteristic curves will be there on both S 1 and S 2. Due to transversality condition that is why step 2 is very, very important. Step 2, the t s and x y the maps phi and chi are inverses of each other that is very, very important.

Due to the transversality condition, any point on a characteristic curves has the form x y, u 1 of x y and also x y, u 2 of x y whenever you are in this common area, F intersection D 1 intersection D 2. F, you remember, is the domain of definition for chi, this proves that S 1 coincides with S 2 in a neighbourhood of P 0. Hence, u 1 is identically equal to u 2 on some neighbourhood of x 0 y 0.

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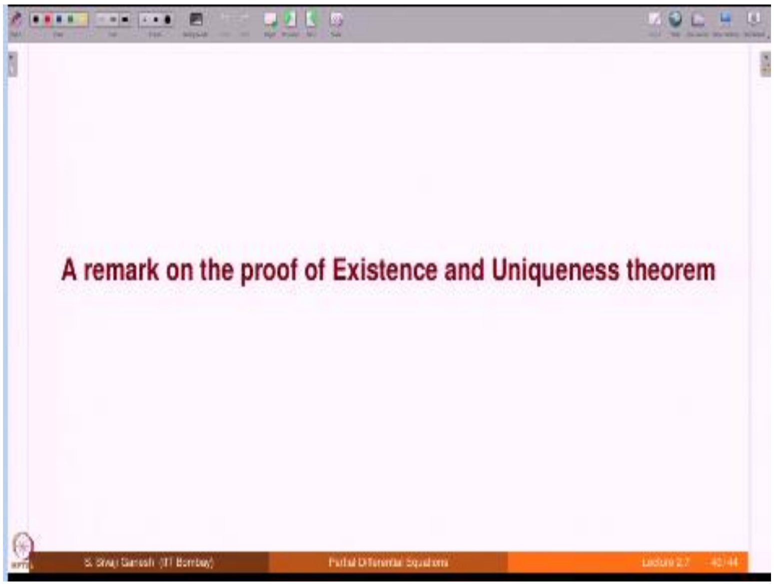

This proves uniqueness. Now, quick remark on the proof of existence and uniqueness theorem.

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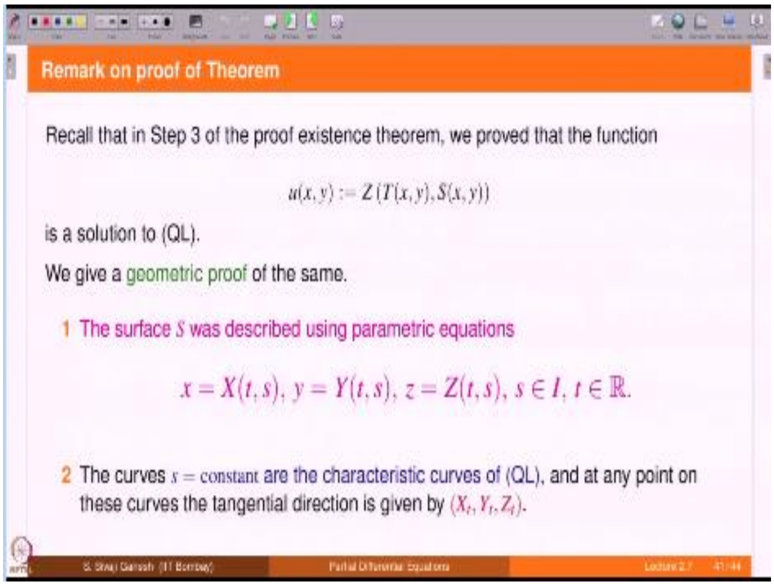

In fact, it is going to be another proof of step 3. Step 3 was very simple. We had to simply compute the derivative, substitute, we see that the candidate solution is indeed the solution. We can give a geometric proof of that. Surface s well described using these parametric equations right X t s, Y t s, Z t s fine. The curves s equal to constant. What are they? They are characteristic curves. This is Gamma, fix s on this; this is X t s.

So, if S is fixed, it is a characteristic curve. Characteristic curve is; what is the tangential direction? First of all, it is a curve therefore, X t, Y t, Z t is the tangential direction. So, X t, Y t, Z t, this is tangent, but it is characteristic curves. Therefore, the direction is actually a b c remember this.

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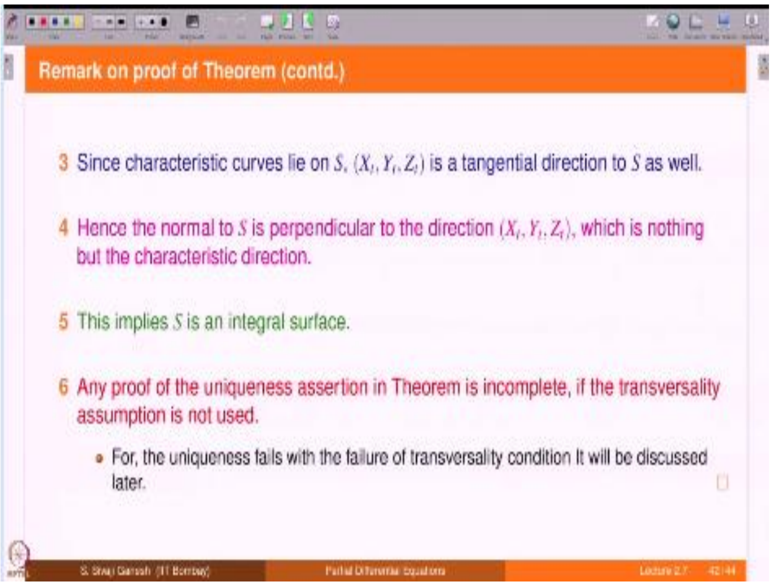

And characteristic curves lies on the surface S. For s, we know something which is normal. Characteristic curves lie on S; X t, Y t is also a tangential direction to S and X t, Y t, it is nothing but characteristic redirection, we have observed. What is the normal? Normal is a b c. Now, u x, u y, – 1 yeah, u x, u y, – 1 that is a normal. z = u x y if it is your surface normal is along this direction u x, u y and – 1 normal.

And what is the tangential direction? One of them is a b c. That has to be 0 and this is nothing but the equation. Therefore, the surfaces S is an integral surface. So, any proof of the uniqueness assumption in the theorem is incomplete if the transversality assumption is not used, because uniqueness fails with the failure of transversality condition. When transversality condition fails, there are examples, where uniqueness fails.

There are examples where existence fails, anything can happen. So, we will discuss that later. So, if you are proving uniqueness, you must be using transversality condition is satisfied. So, people may not write this much explicitly in books, but they will say that step 2 which is what we have said right, they make use of step 2 without explicitly mentioning to you. So, you should not get confused.

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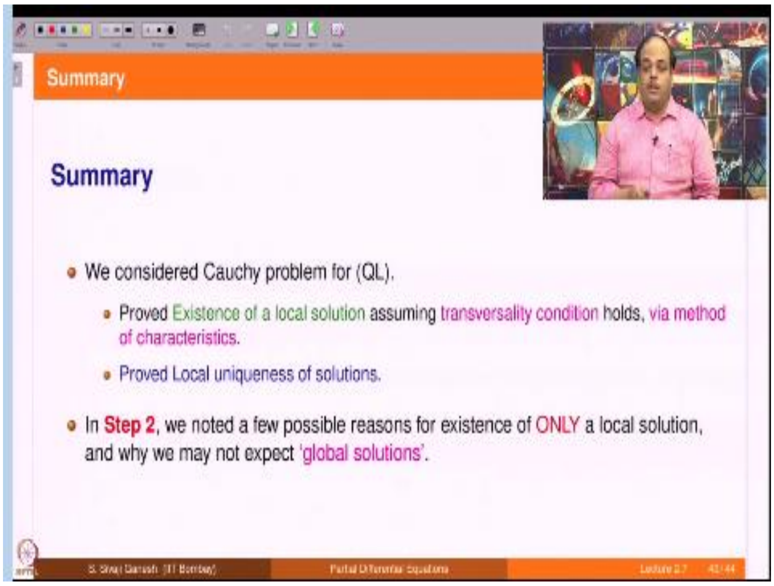

The summary is that we consider the Cauchy problem for QL. we prove the existence of a local solution. Assuming transversality condition holds nothing, no more assumptions via method of characteristics. We prove local uniqueness of solutions. And in step 2, we noted a few possible reasons for existence of only a local solution, which will be seen in next lectures and why you may not expect global. Global, I already mentioned; there are at least 2 notions of local solutions and hence global solutions.

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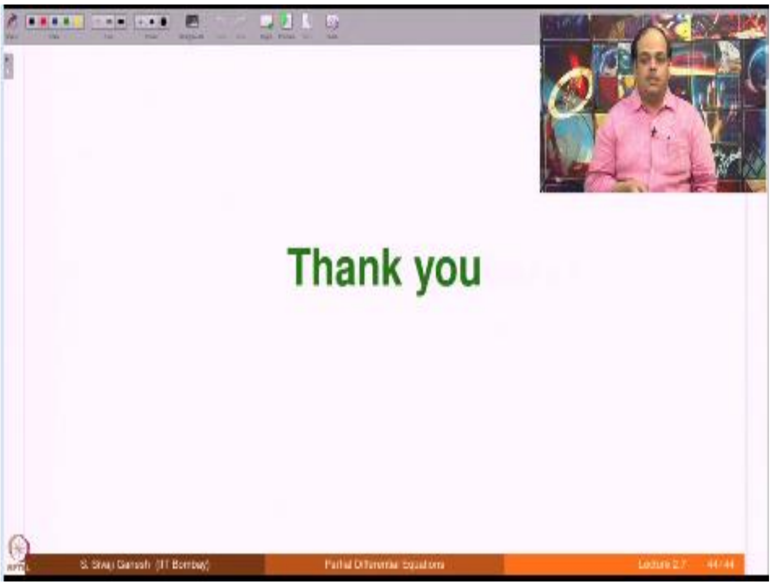

So, in the next lecture, we will take up as promised, what is transversality condition geometrically and what happens when it fails? What are the possibilities? Thank you.