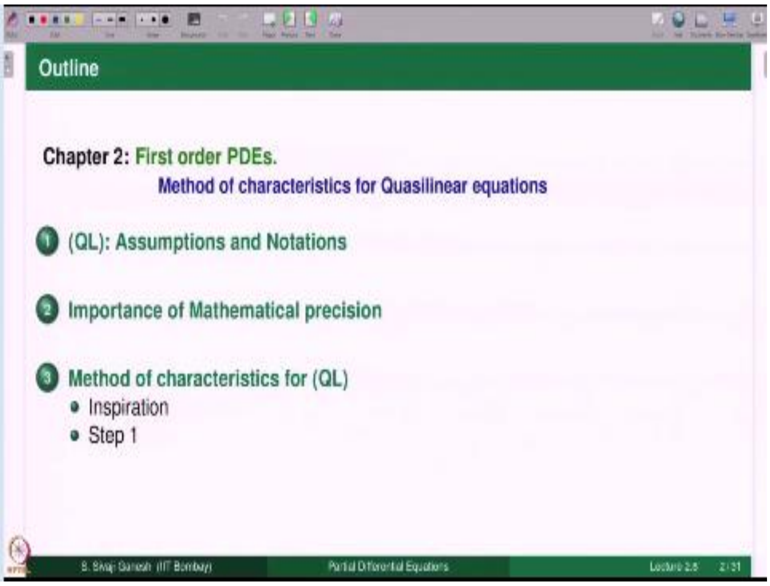
Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology – Bombay

Lecture – 2.6
First Order Partial Differential Equations
Method of characteristics for Quasilinear Equations - 1

We start the discussion of method of characteristics for Quasilinear equations from this lecture onwards.

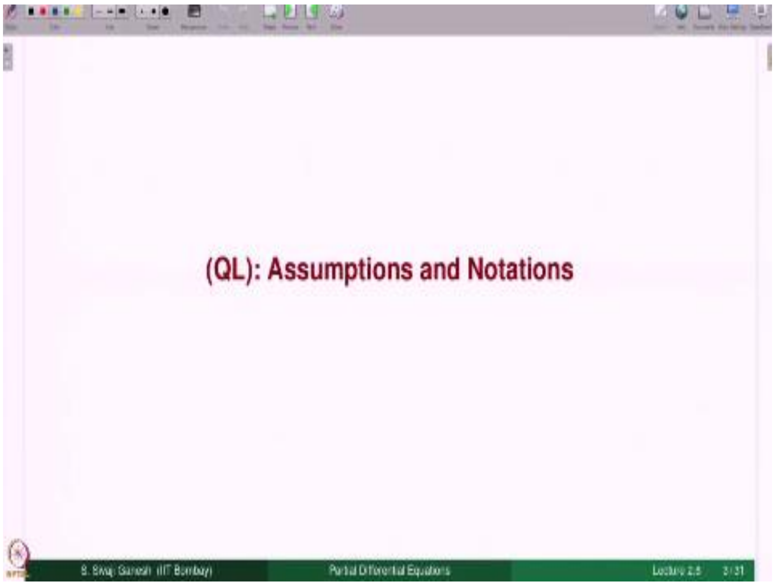
(Refer Slide Time: 00:27)



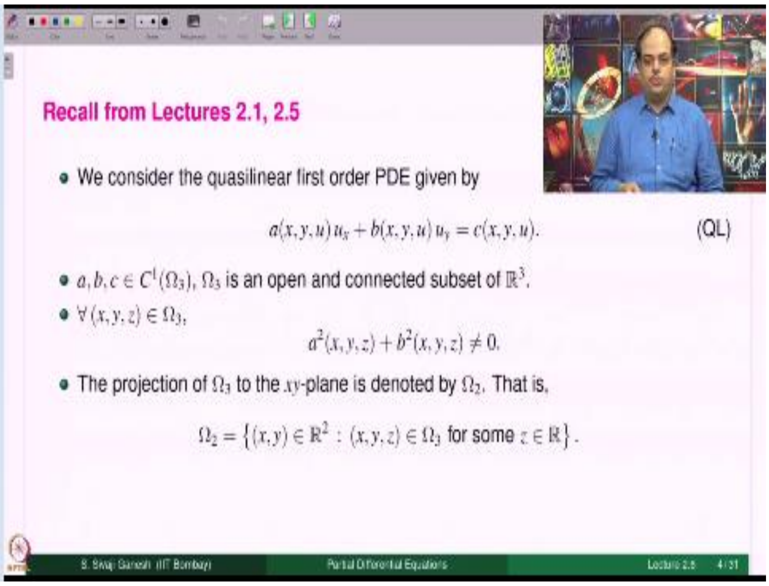
The outline for today's lecture is, first, we recall the assumptions and notations which are used in the context of Quasilinear equations. And we emphasise the importance of mathematical precision, we see couple of statements which could wrongly imply wrong things, which are written in language and that can be read in many ways. So, therefore, this highlights the importance of writing or importance of understanding mathematically what we write in language.

And then we take first steps into method of characteristics for Quasilinear equations. First, we present the inspiration behind this method and we carry out the step 1 out of the 3 steps. So, in the next lecture, step 2 and step 3 will be discussed.

(Refer Slide Time: 01:17)



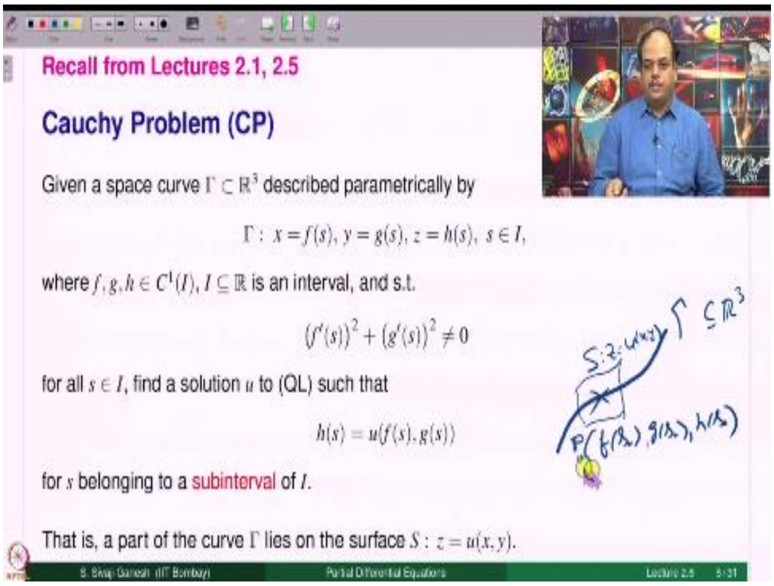
(Refer Slide Time: 01:22)



So, assumptions and notations used in the context of Quasilinear equations, recall from lectures 2.1 and 2.5, Quasilinear equation we denote by QL and that is an equation a x y u u x + b x y u u y = c x y u, where a b c are assumed to be C 1 functions defined on omega 3. Needless to say, omega 3 is an open subset of R 3 and connected, we already discussed whether we should have the connectedness or not. This is the most important thing.

We do not want the coefficients of the partial derivatives u x and u y, namely a and b vanish simultaneously at the same point. So, therefore, we require that at least one of the a and b must be nonzero at each and every point in omega 3 and the projection of omega 3 to x y plane is denoted by omega 2. Omega 2 is those elements of R 2 such that x y z belongs to omega 3 for some set in R.

(Refer Slide Time: 02:25)



Now, Cauchy problem: given a space curve gamma described parametrically by gamma x = f s, y = g s and z = h s, s belongs to I where I is an interval and f g h are C 1 functions defined on the interval I and such that f dash s square plus g dash s square is not equal to 0 for all s in I. This, we discussed that this corresponds to the projection gamma 2, x = f s and y = g s, these curve in a plane and this curve is what is called regular.

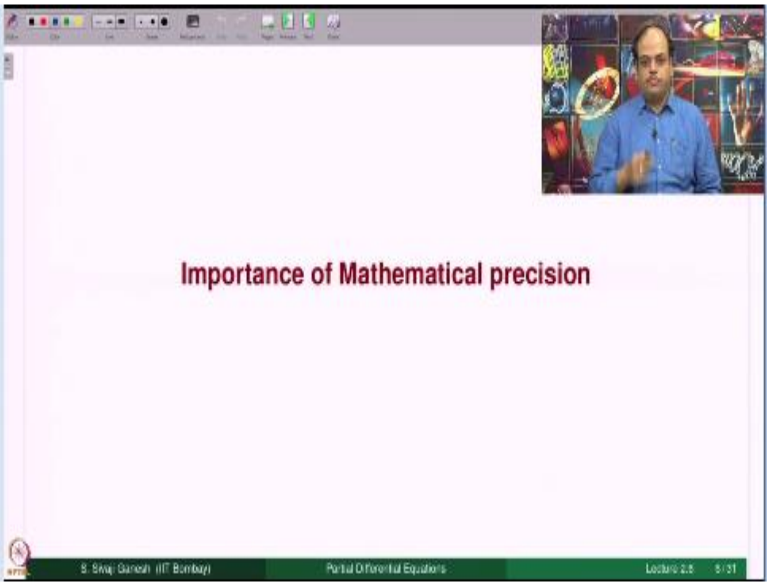
This assumption is something to do with the smoothness of the curve gamma 2. This means that tangent is well defined and each and every point of gamma 2. For example, this is gamma 2 or you take a point. The point looks like f s, g s and at this point, the tangent line, this is not a good picture; will illustrate a good place this point, this has the direction of s prime is g prime s. Find a solution to the Quasilinear equation such that u of f s, g s = h s that means, on the curve gamma 2 where f s, g s describes a typical point on gamma 2.

When you take u of that, then you will get into 3 dimensions z equal to that. z is going to be h f s that means, it is lying on the surface z = u of x y and this, we require for s belonging to a subinterval of I. So, it means that a part of the curve gamma lies on the surface. In other words, we are looking to construct a surface, construct an integral surface which contains a part of the curve gamma.

So, what we are going to see in the future lectures is that if you are given this curve gamma; this is in R 3 and given any point on that, that looks like f s 0, g s 0, h s 0 under some conditions. We are going to show that let us call this point P 0. Some conditions, we are going to show that there is a surface integral surface, s, z = u x y such that it contains this point P 0 on some curve nearby that, some gamma the initial data.

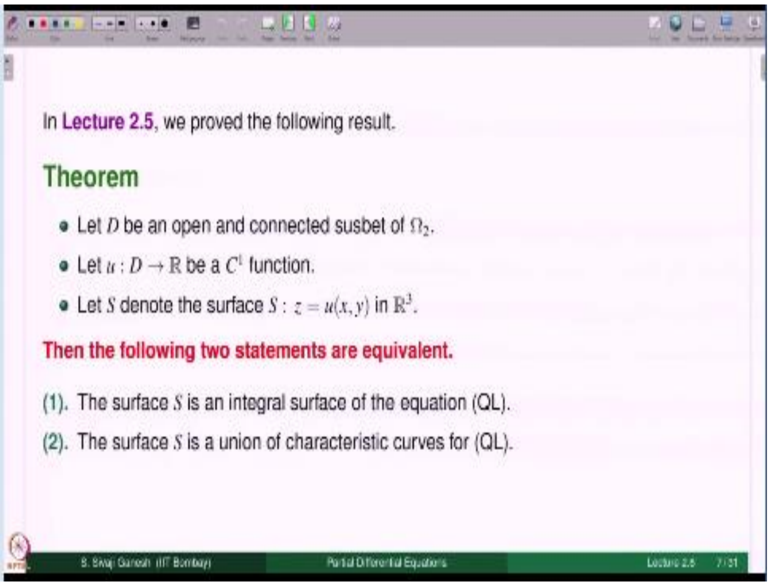
In a datum, curve gamma is contained on the surface. Of course, we need some assumptions to assert that there is such a function u. We have already seen that this question arose; already in the case of linear and semilinear equations, when we try to solve Cauchy problems. Some assumptions need to be made and then we will show such a function exists. This will be the final result at the end of implementing step 3 of the method of characteristics.

(Refer Slide Time: 05:44)



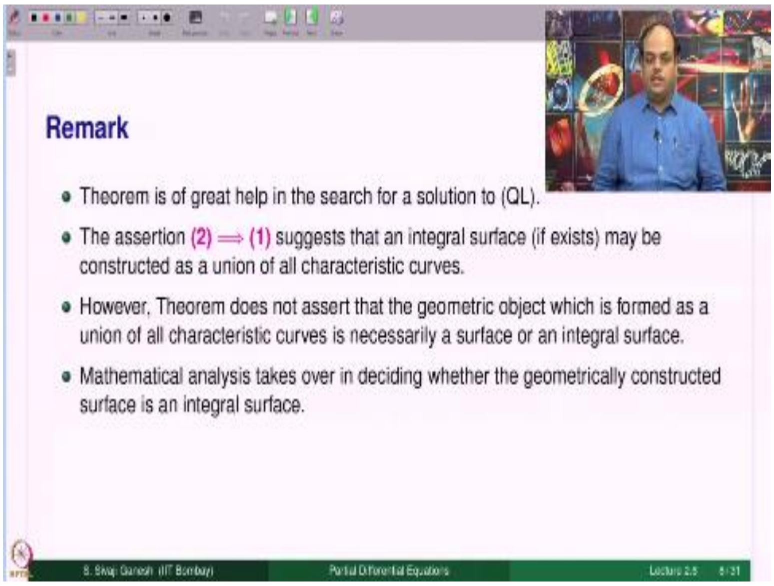
Now, few points about the importance of mathematically precision.

(Refer Slide Time: 05:50)



Before that let us state this theorem, this theorem we proved in the last lecture, lecture 2.5. So, this said that given a function u defined on a domain D in omega 2 which is a C 1 function and s denotes the surface z = u x y that is the graph of this function in R 3, then saying that this s is an integral surface is same as saying that s is a union of characteristic curve for Quasilinear equations is what we discussed in lecture 2.5. We proved this theorem.

(Refer Slide Time: 06:26)



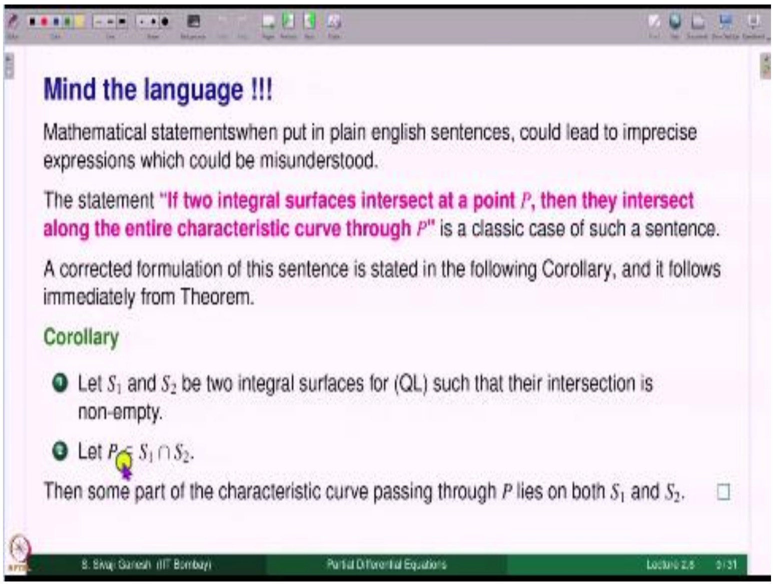
Now, a remark about this theorem. Theorem is of great help in the search for a solution to Quasilinear equations, namely the Cauchy problem for Quasilinear equation CP for Quasilinear equations. The assertion 2 implies 1. What is 2? 2 says, the surface s is a unique characteristic curve that implies that surface s is an integral surface. Of course, here u is in the background which is fixed in the background.

So, the assertion 2 implies 1 suggests that an integral surface if exists; in other words, a surface z = u x y where u is a solution if exists may be constructed as a union of all characteristic curves. However, theorem does not assert that the geometric object is formed as a union of all characteristic curves is necessarily a surface. In other words, if you take the union of characteristic curves, it is not necessary that the third component is expressible as a function of first 2 coordinates. It does not say that.

The theorem says, you give me a function u and look at the graph of u. Now, it is a comment about the graph of u. The graph of u, u is surface; it is s; it is denoted by s; s is an integral surface if and only if it is union of characteristic curves that u has to be brought beginning. So, therefore, 2 implies 1 suggests that an integral surface may be constructed. It does not assert the geometric object which is found as a union of characteristic curves is necessarily surfaced that means that u hidden behind that.

Such that the third component is a u of the first 2 components x y. And moreover, even it is surface; it is not clear whether it is going to be an integral surface. So, here surface mean z = u x y for some function u, that is what we mean in both of this. Now, mathematically analysis takes over in deciding whether the geometrically constructed surface is an integral surface or not. So, we will come across this analysis in a future lectures, maybe in the next one.

(Refer Slide Time: 08:49)



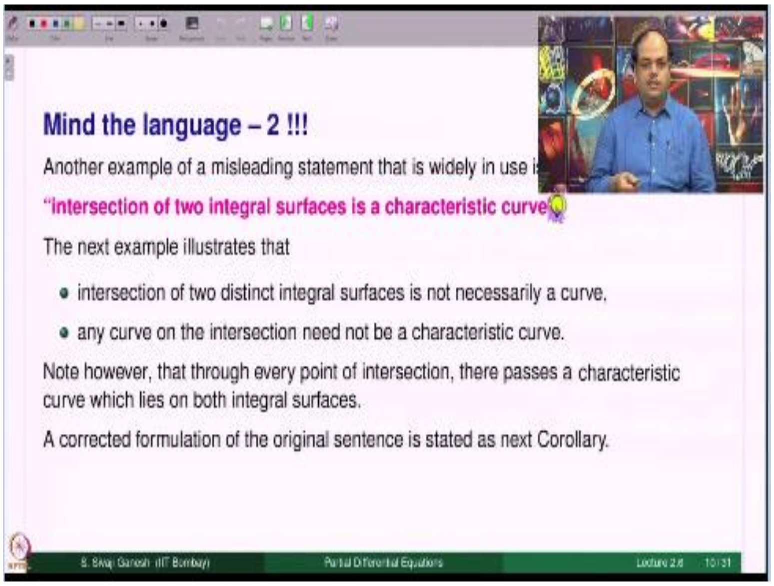
Important thing is this precision, mathematical precision is very important. So, I call this mind the language. So mathematical statements, when put in plain English sentences or any other language sentences could lead to imprecise expressions, which could be misunderstood. The statement: if 2 integral surfaces intersect at a point, then they intersect along the entire characteristic curve through P. This is a statement.

Ii is a classic case of such as a sentence. What does the sentence say? Suppose you have 2 integral surfaces, they intersect at a point P, then they intersect along the entire characteristic curve through P. What do you mean the entire characteristic curve through P? The longest possible characteristic curve passing through the point P, what is it? So, a corrected formulation of the sentence is stating in the following corollary and it follows immediately from theorem.

What is the corollary? Let S 1 and S 2 be 2 integral surfaces for Quasilinear equation QL such that their intersection is non empty. Of course, that is not a big deal because here we are saying it intersects at a point P. Therefore, we have not made much difference. This sentence is as it is, non empty. Let P be a point in the intersection. So, that takes care of the first condition.

The conditional statement: if 2 integrals surface intersect at a point P that is captured in 1 and 2. Now, we have to see how we are capturing the conclusion, they intersect along the entire characteristic curve. Then some part of the characteristic curve passing through P lies on both S 1 and S 2. So, we can only say some part near P. Alright, there is a big characteristic curve passing through P, but the entire curve why will it be there on both of them? Some parties there that is reasonable to believe and that is what is true.

(Refer Slide Time: 10:51)

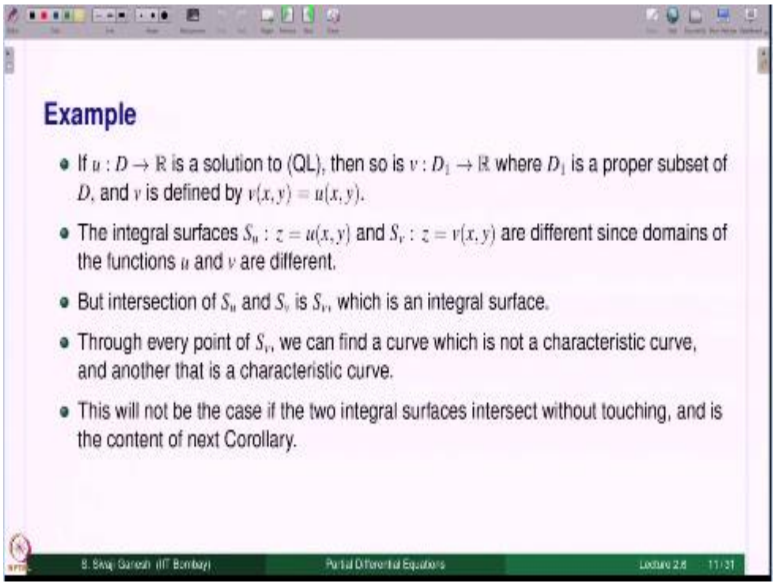


Now, second point. Another example of a misleading statement that is widely in use is intersection of 2 integral surfaces is a characteristic because we believe that if 2 surfaces intersect, imagine 2 planes intersect, it is a line; is that true always? If the intersection of 2 planes is not aligned always, it could be a plane, it could be the same plane, right? It need not be line all the time.

So, that is in fact happens, which is a counter example this statement. So, we have to be careful. The next example illustrates that intersection of 2 distinct integral surfaces is not necessarily a curve. Forget about characteristic curve. Any curve on the intersection need not be a characteristic curve. Even that is true. That simply because the example we are going to see, the 2 integral surfaces intersect and give a surface.

Note however, that through every point of intersection, there passes a characteristic curve which lies on both integral surfaces. So, a corrected formulation of the original sentence is stated as next corollary. So, intersection of 2 integral surfaces the characteristic curve, this is not precise, it is not correct as we usually understand this. The standard meaning of this turns out to be that the statement is incorrect.

(Refer Slide Time: 12:18)



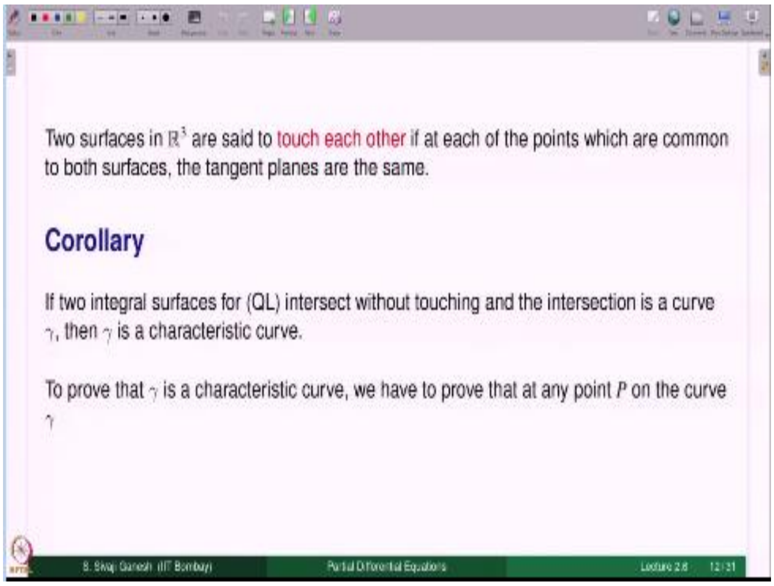
So, corrected formulation we are going to give. Before that, let us do this example. Suppose, u is a solution to QL defined on D. Then so is another function v defined on D 1. Now, how do I define this v? It is going to be using u. Therefore, D 1, I will take it to be a subset of D. It is a proper subset of D. Of course, v remain solution to the QL. I look at the integral surfaces.

S u to denote that the surface is defined using the function u. This is used, this S v, this surface z = v x y is defined using the function v. They are different because domains of the functions u and v are different. It is true that S v is a subset of S u. But they are different. Two functions are different, the moment their domains are different. So, but intersection of S u and S v is S v, which is an integral surface. So, it is not a curve. It is an integral surface.

Through every point of S v, we can find a curve which is not a characteristic curve. And another, that is a characteristic curve. This will not be the case with the 2 integral surfaces intersect without touching and that is what is the content of the next corollary. So, when is the intersection of 2 planes is a straight line? When they do not match right? Some the 2 planes, it is not a line if and only if they are the same planes. For a surface, the plane R approximation will be the tangent plane.

So, if we say that the tangent planes are not same, then it will be a curve that is what is the next corollary.

(Refer Slide Time: 14:00)

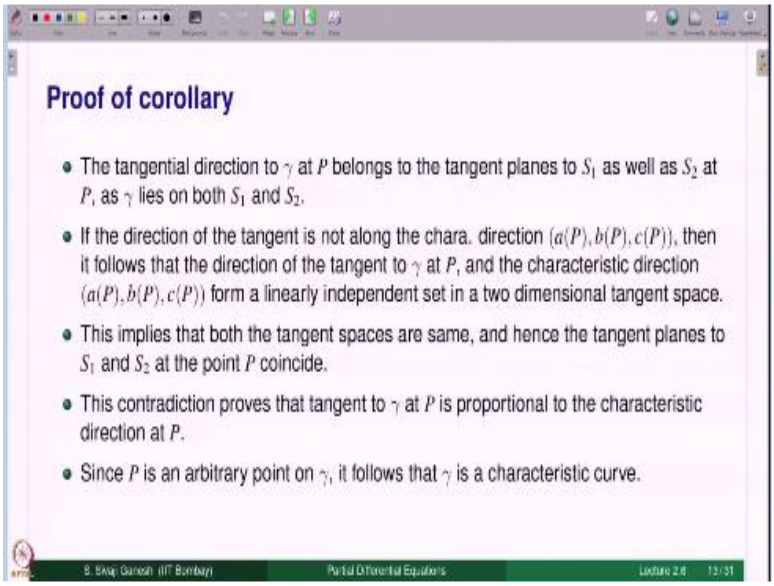


So, 2 surfaces in R 3 are set to touch each other. If at each of the points which are common that means wherever intersection, whichever point is the intersection of the 2 surfaces at those points, the tangent planes are the same. So, our theorem is going to be for surfaces which do not touch each other. If 2 integral surfaces intersect without touching each other and the intersection is a curve, then it is a characteristic curve. This is a correct corollary.

If touch each other, it is like, a tangent planes are one in the same whenever planes are same, we got the intersection to be plane right for planes. So here, we do not expect a straight line there for planes. Therefore, same thing here. Here 2 surfaces if they intersect by touching each other, then we are not making any statement, but if they do not touch each other and intersect that means points are in common, then the intersection if it is a curve, so, which means it may not be curve as well it can be a point, is a curve, then gamma is a characteristic.

So, to prove that gamma is a characteristic, what we need to do? We have to prove that the tangential direction at any point on the curve is the characteristic direction.

(Refer Slide Time: 15:29)

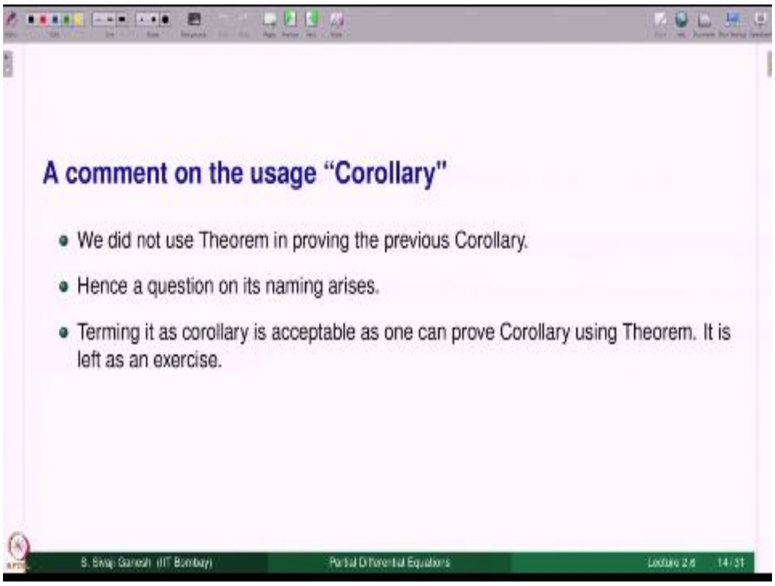


Let us prove that. So, the tangential direction to gamma at P belongs to the tangent plane to both surfaces S 1 and S 2 at that point P as gamma is laying on both of them. If the direction of the tangent is not along the characteristic direction, then it follows that the direction or the tangent to gamma at P and the characteristic direction a P, b P, c P, they form a linearly independent set in a 2 dimensional tangent space.

This implies that both the tangent spaces are the same. All the directions in the tangent planes for both S 1 and S 2 at the point P are the same. And hence, tangent planes coincide which means, they touch each other. Therefore, we have assumed, they do not touch each other. Therefore, this contradiction proves that the tangent to gamma at P is not independent of characteristic duration. It is proportional.

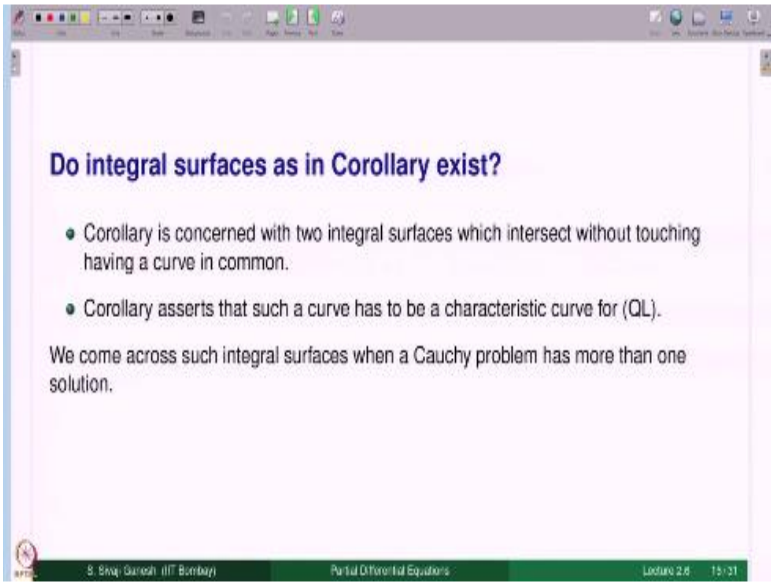
It is linearly dependent, proportional to the characteristic direction at P which means it has a curve gamma has characteristic division at P and P is arbitrary point that means, a curve is a characteristic curve.

(Refer Slide Time: 16:56)



Now, in the proof, we are not use a theorem. Therefore, why do we use our corollary? Therefore, this question arises terming it as a corollary is acceptable as one can prove corollary using theorem. You can use a theorem and prove the corollary that is left as an exercise to you.

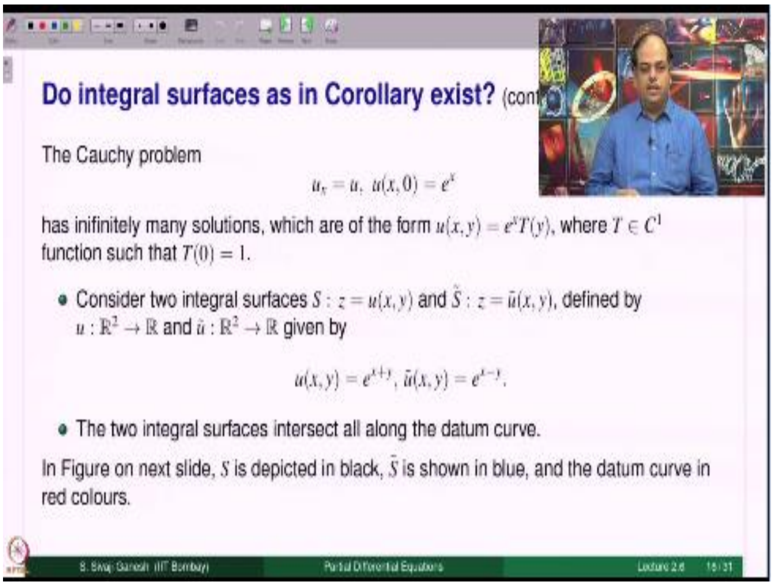
(Refer Slide Time: 17:20)



We have seen one proof. Other proof uses a theorem directly the statement of theorem and that is left as an exercise. Now, the question is: do integral surfaces as in the corollary exists? That is 2 integral surfaces which intersect but do not touch each other whether such services exist. Corollary is concerned with the 2 integral surfaces which intersect without touching having a curve in common.

And corollary assert that such a curve is necessarily a characteristic curve for Quasilinear equation. We come across such integral surfaces when a Cauchy problem has more than one solution.

(Refer Slide Time: 18:00)

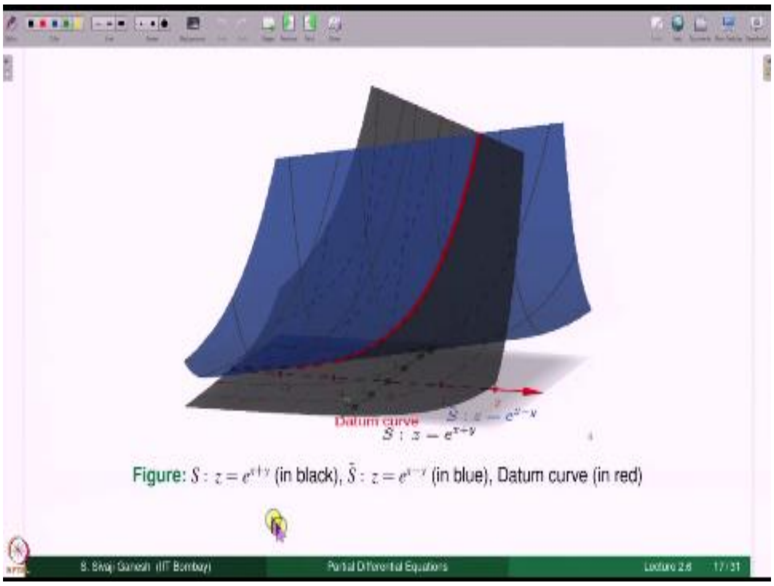


Look at this example, u x = u. This equation, we are always considering partial differential equations in 2 independent variables unless otherwise stated. So, this is a function of 2 variables x and y and the equation is u x = u. So, this is like a ODE in the x variable and we are given initial condition u 0 = e power x. If you want Cauchy data, what is it? x = s, y = 0, z t = e power s, s belongs to R. This problem has infinitely many solutions.

We saw this already. They are the form u of x y = e power x into T y where T is C 1 function. T of 0 should be 1. Therefore, as many C 1 functions as you have with the property T = 1. We have so many solutions, clearly infinitely many. Now, consider 2 integral surfaces, z = u x y and z = u tilde of x y. Two integral surfaces. Defined by these formulae, one is e power x + y; other one is e power x – y. Both are solutions to this Cauchy problem.

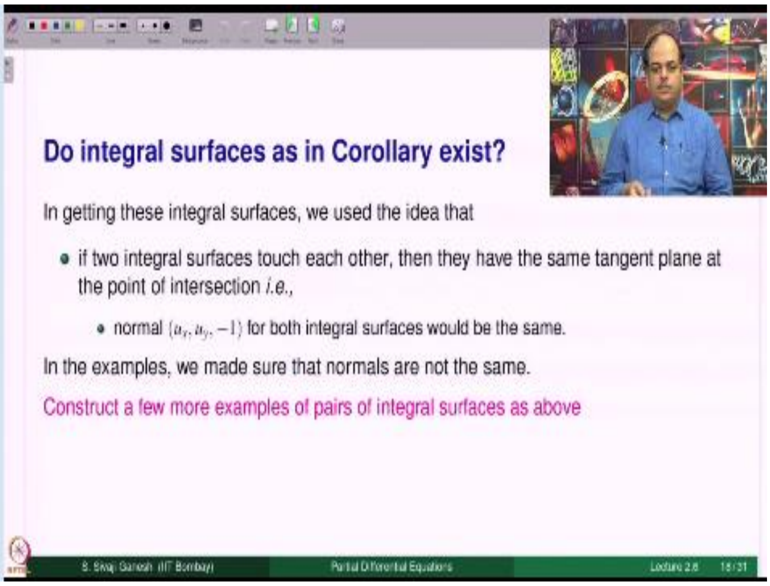
The two integral surfaces intersect all along the datum curve that is S 0 e power s that is intersection. In figure on the next slide, the surface S corresponding to u e power x plus y is depicted in black, the one for S tilde is depicted in blue and datum curve is in red colour.

(Refer Slide Time: 19:30)



Here. So, 3D picture. So, intersection is shown here, that is the datum curve. Blue is one integral surface. Other one is other integral surface.

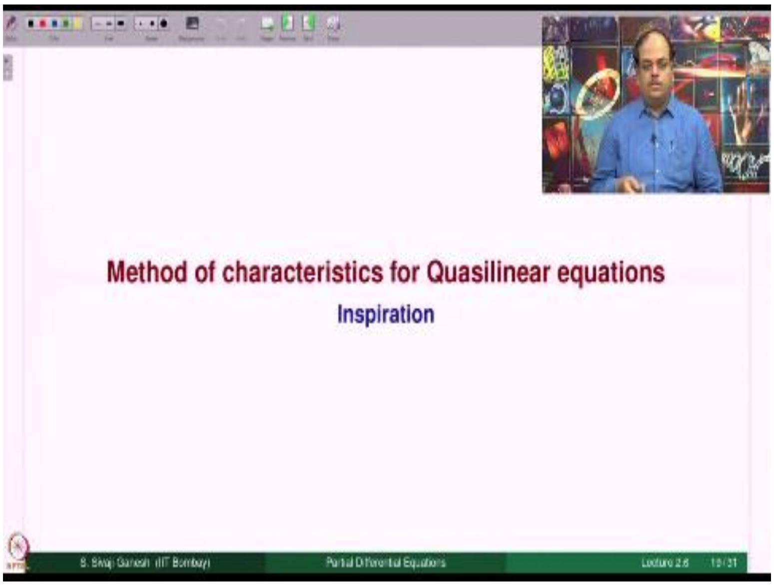
(Refer Slide Time: 19:43)



I am getting these integral surfaces. How did I get this example? We use the idea that if 2 integrals surfaces touch each other, what happens? The tangent planes are same. The directions in the tangent planes are same. And u x u y – 1 for both integral surfaces would be the same because a tangent plane is the same; normal has to be same plus or minus. We are not insisting that normal has unit length etcetera.

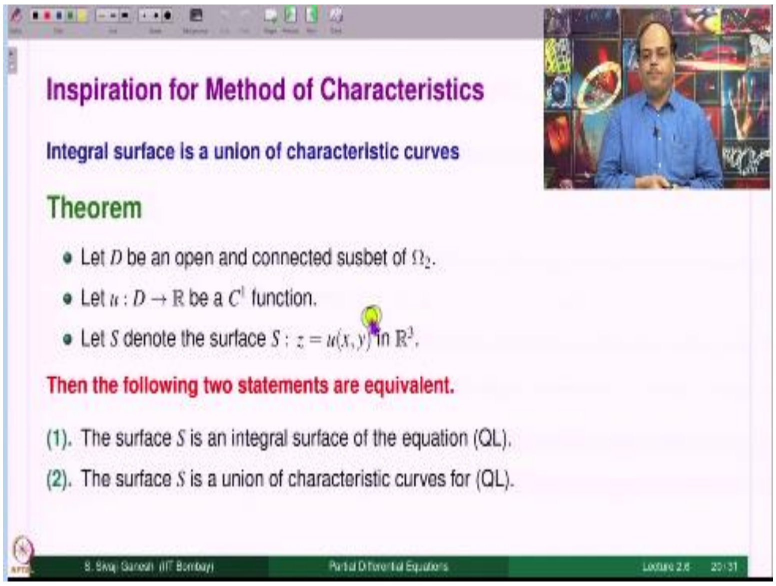
So, therefore, direction is direction; any other normally be proportional to this; they will be same. So, in these examples, we made sure that normals are not the same. Therefore, tangent plane will not be the same. That was what was done. And please do a few more examples of pairs of integral surface as above.

(Refer Slide Time: 20:35)



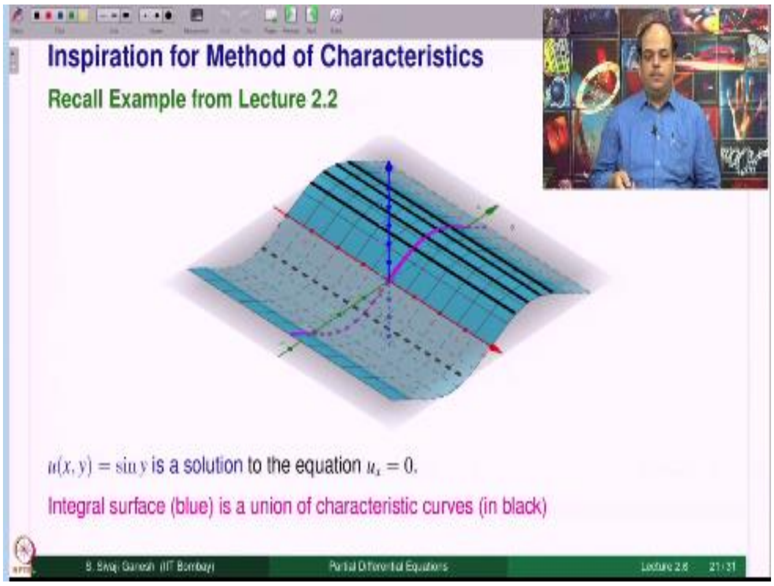
Now, let us discuss method of characteristics for Quasilinear equations. First, we start with an inspiration for this method. What inspired this method? Of course, it is no secret, because we just saw one theorem at the beginning of today's lecture and even in lecture 2.5 that is the inspiration.

(Refer Slide Time: 20:55)



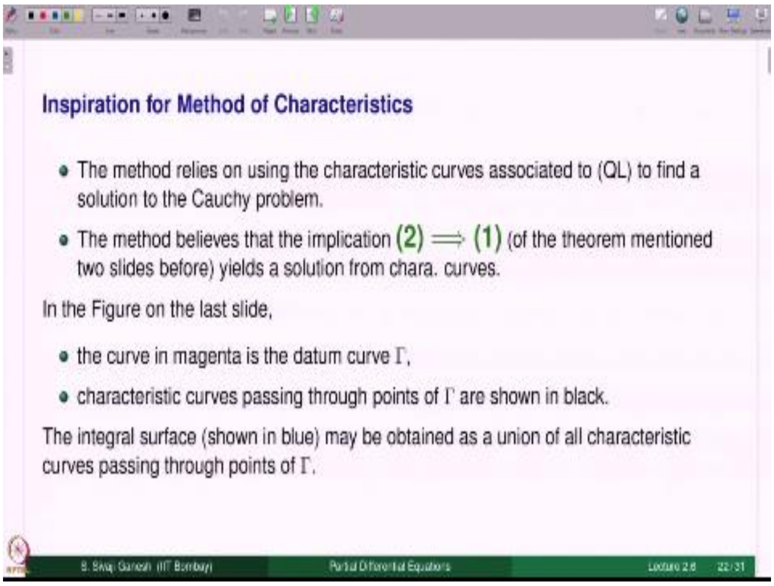
Integrals surface is a union of characteristic curves that is the idea. So, this is the theorem. This is the inspiration. It is not saying that you construct a union of characteristic curves that is automatically an integral surface. It does not say that. It still requires you that the surface is given by z = u x y. We still have to do some work, but inspiration it works.

(Refer Slide Time: 21:19)



So, in this example, u x = 0; u x y = sin y, an integral surfaces blue and that can be obtained as union of these black lines and this magenta is the datum curve that is 0 y sin y, 0 s sin s. This is that one.

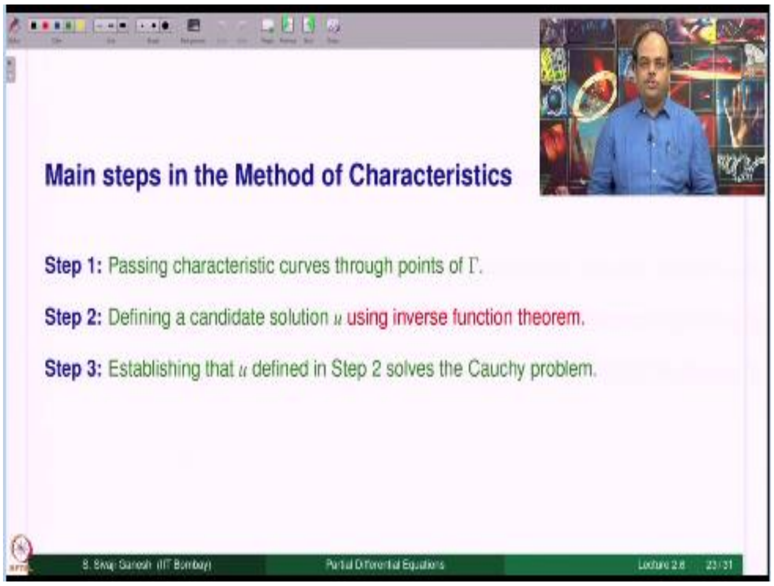
(Refer Slide Time: 21:48)



So, the method relies on using characteristic curves associated to Quasilinear equations to find a solution to Cauchy problem. It believes that this implication 2 implies 1 yields a solution from characteristic curves; it believes that. Let us take the union of characteristic curves, somehow we can get that function u and it will be alright that is what the method believes.

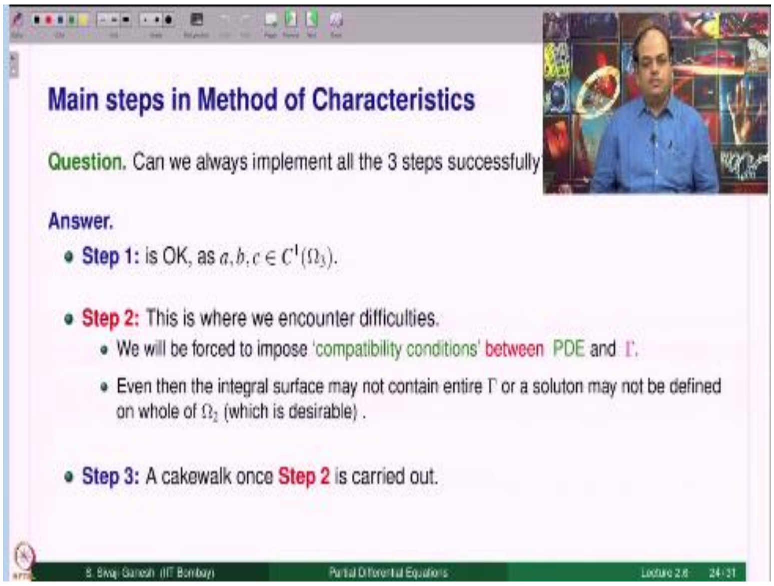
In the figure on the last slide, the curve in magenta is a datum curve; characteristic curves passing through points of gamma are in black. And we saw that the blue thing is a union of black lines. The integral surface may be obtained as a union of all characteristic curves passing through points of gamma. So, it is working in that example.

(Refer Slide Time: 22:40)



Of course, another point that we knew the answer beforehand the function, but inspiration is fine. So, what are the main steps? First step passing characteristic curves through points of gamma, these are strategy we tried for linear and semilinear equations in the last lecture. Defining a candidate solution u using inverse function theorem. This is a key step. Step 3 is establishing that the function u define in step 2 is actually a solution to the Cauchy problem.

(Refer Slide Time: 23:17)

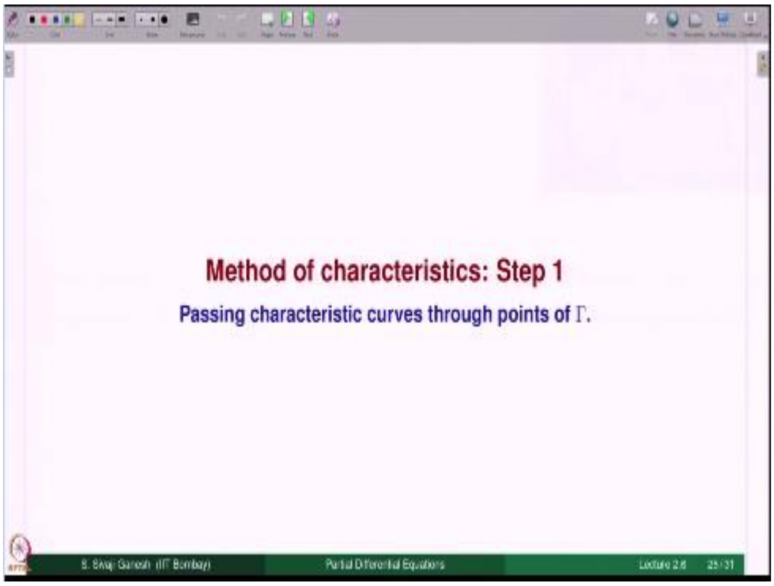


Question: you have given 3 steps. Can we always implement those 3 steps successfully? That is the question. Step 1 is OK. What is the step 1? it is to pass characteristic curves through points of gamma. That is the step 1 passing characteristic curves through points of gamma. That is okay. Because a b c are C 1 functions in omega 3, because theorem will give you that characteristics pass through every point of gamma.

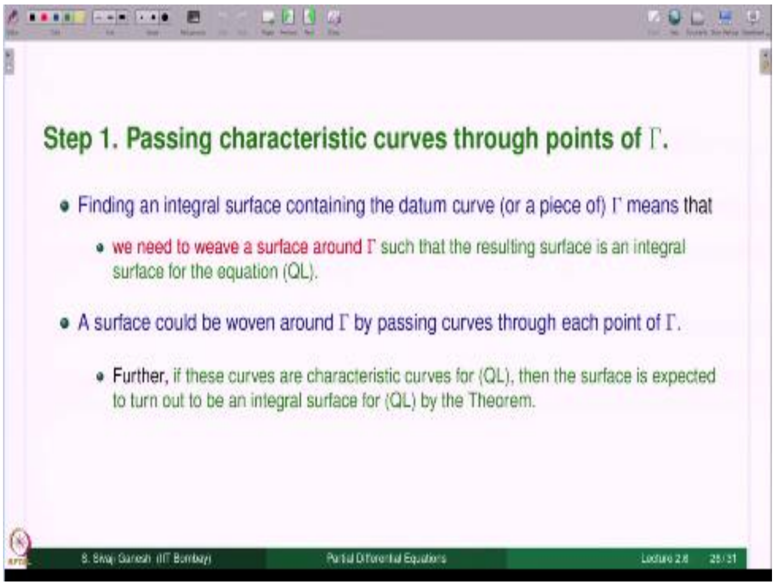
Second step, this is where we encounter difficulties, is where the problem lies. Because we need to apply inverse function theorem. Inverse function theorem requires some conditions to be met. So, we will be forced to impose compatibility conditions between the PDE and gamma. Even then, the integral surface may not contain entire gamma. Or, a solution may not be defined on whole of omega 2. These are the 2 things we somehow want.

I want a Cauchy problem. So, that the integral surface contains the entire datum curve which is given to me and it is defined on a whole of omega 2, omega 2 being the projection of omega 3. So, this is desirable but neither of these 2 may happen. We will see using examples. Step 3 is a cakewalk once step 2 is carried out.

(Refer Slide Time: 24:59)



(Refer Slide Time: 25:06)

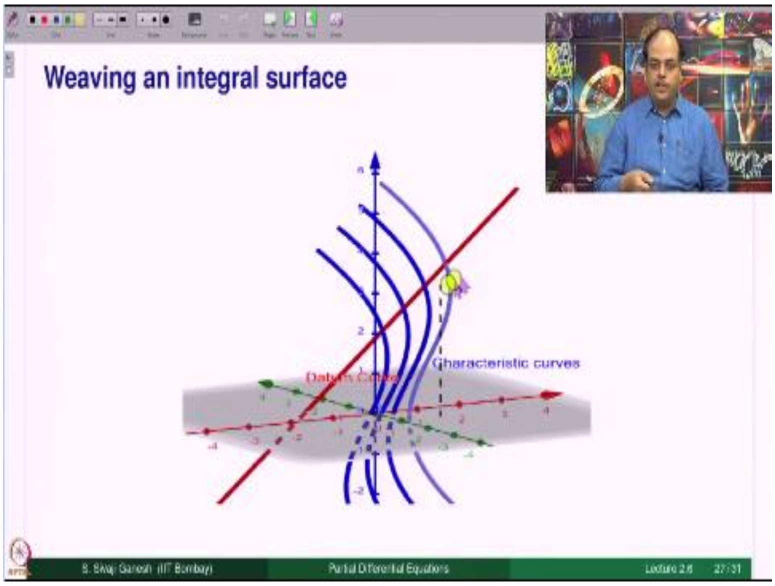


Let us discuss step 1 passing characteristic curves through points of gamma. Finding an integral surface containing the datum curve. Now, we have decided a piece of datum curve means what does that mean? We need to view a surface around gamma. Gamma is given to us and we need to find a surface something like that a surface s such that the resulting surface is an integral surface for the equation.

A surface could be wove around gamma by actually take a point of gamma, take a characteristic curve through that, through that, through that, like that, repeat this at every point of gamma by passing curve through each point of gamma. Further, if these curves are characteristic curves for QL, then the surface is expected to turn out to be an integral surface. You have to be very careful, I use the word expected.

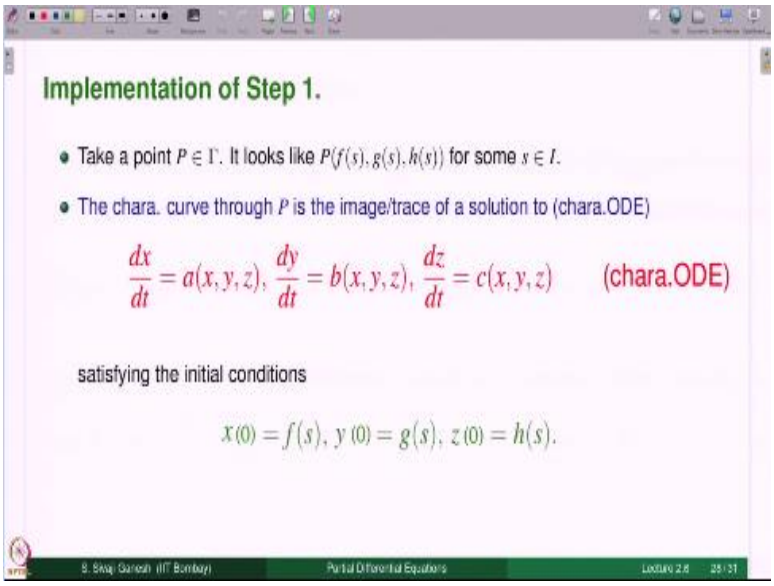
Usually people mean expected means it will happen. Now, this may not happen. Hoped, maybe that is the correct word. Then the surface is hoped to be turned out to be an integral surface by the theorem.

(Refer Slide Time: 26:37)



So, weaving an integral surface, this is a computer generated picture of what I have just written. So, this is the datum curve and you pass characteristic curve through the each point and then hopefully, you will get a surface and that surface is expressed like z = u x y and u is a solution to the PDE.

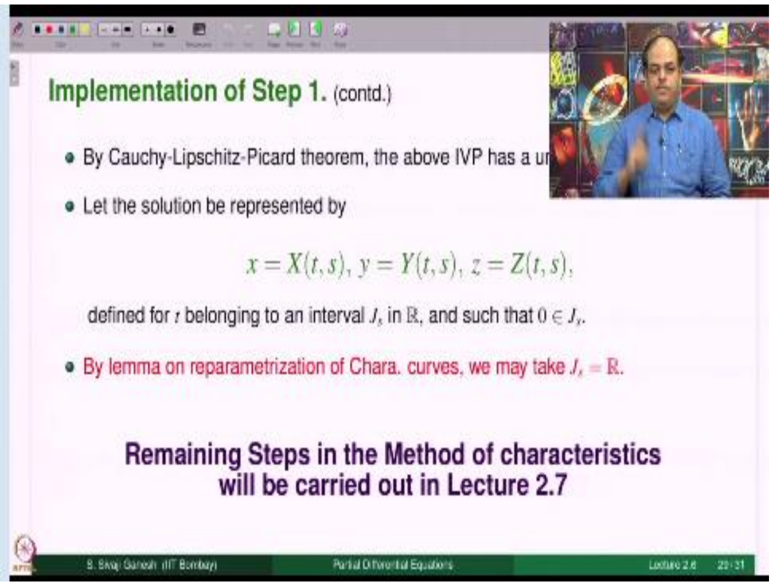
(Refer Slide Time: 27:00)



It will be a solution to the Cauchy problem in the sense Cauchy data will be satisfied that how you are getting the solution. How do we implement the step 1? Take a point P on the datum curve, it looks like f s, g s, h s for some s in I. The characteristic curve through P is the image or trace of solutions to characteristic ODE. This is the system of characteristic ODE, is dx by dt = a dy by dt = b dz by dt = c.

What are a b c? a b c are in QL. What is QL? u x + b u y = c satisfying the initial conditions. I need the solutions of this ODEs to pass through this point. So, at t = 0, x of 0 is f s, y of 0 is g s; instead of 0 equal to h s.

(Refer Slide Time: 27:57)

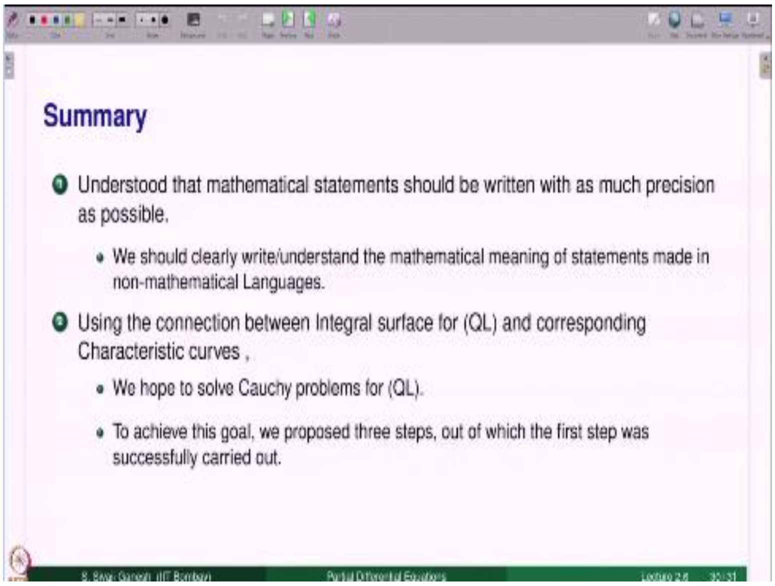


By Cauchy Lipschitz Picard's theorem, we are assuming a b c are C 1 function, they are local Lipschitz. IVP will have a unique solution. Let the solution be represented by this notation x = X of t s; y = Y of t s; z = Z of t s. Of course, ODEs were in the variable t. Then why are we writing s here? It is because the characteristic ODE, we have solved using initial conditions which depend on s, the initial conditions depend on s to remember that we write except s, x of t s, y of t s, z of t s.

So, it is defined for t belonging to some interval. Now, you see J s, the interval may change from s to s. For some s, it may be one interval; for another s, it may be a different interval. The only thing that we can assure is that 0 belongs to J s. Now, we recall a lemma that we did on re-parameterization of characteristic curves. We may take J s = R according to that lemma. The reason being let me recall the reason, the system of characteristic ODE is a system of autonomous equations.

And for that, if you are looking only at the trajectories, you can always change the independent variable that is namely t there to make that the interval J s is actually equal to R. Trace will be the same. Now remaining steps in the method of characteristics will be carried out in lecture 2.7 the next lecture. Let us summarise what we did.

(Refer Slide Time: 29:38)



We understood that mathematical statement should be written with as much precision as possible, full precision. Mathematical statement, it has to be fully precise and everyone should understand the same meaning of the sentence. We should clearly write or understand the mathematical meaning of statements made in non mathematical languages. Using the connection between integral surface and characteristic curves, we hope to solve Cauchy problems for Quasilinear equations.

To achieve this goal, we propose the 3 steps out of which the first step was carried out successfully. Thank you.

(Refer Slide Time: 30:14)