Partial Differential Equations
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Lecture – 60
Heat Equation
Infinite Speed of Propagation, Energy, Backward Problem

Welcome. In this lecture, we are going to discuss certain concepts associated to solutions of heat equation and heat equation itself. One of them is infinite speed of propagation and then we look at the energy associated to the heat equation and using energy we will show that initial boundary value problem that we have considered in earlier lectures has a unique solution.

Then we go on to state a couple of examples which illustrate that heat equation is a kind of irreversible, that means we cannot consider T to be negative unlike the wave equation. So, this backward problem for the heat equation is not well posed.

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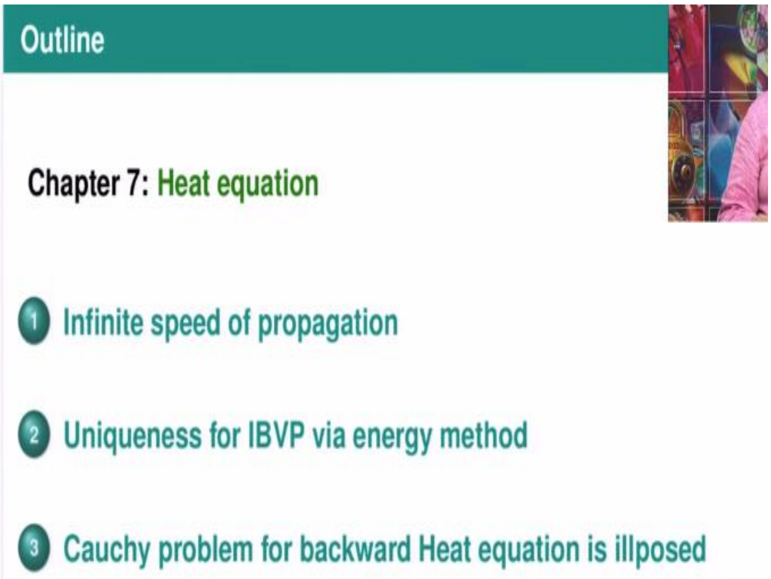

So, the outline is we start with infinite speed of propagation, we will show that the solutions to heat equation have this property. Then we show uniqueness for IBVP via energy method and then we finally conclude by showing two examples which illustrate that backward heat equation Cauchy problem for it is not well posed.

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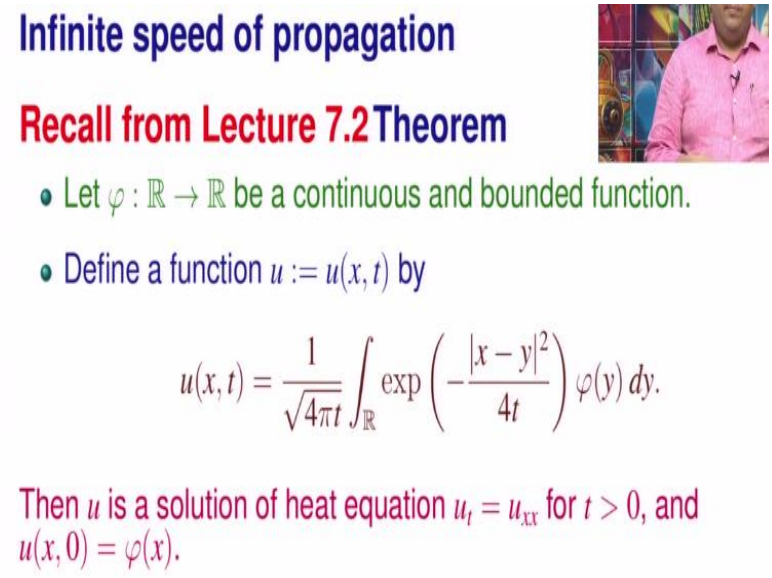

Recall from lecture 7.2 where we have established this theorem. Whenever phi is a continuous and bounded function, if you define u of x, t by this formula, then u is a solution to the heat equation u t = u xx for t positive and satisfies the initial condition u of x, 0 = phi x. In other words, this formula represents a solution to the Cauchy problem for homogeneous heat equation.

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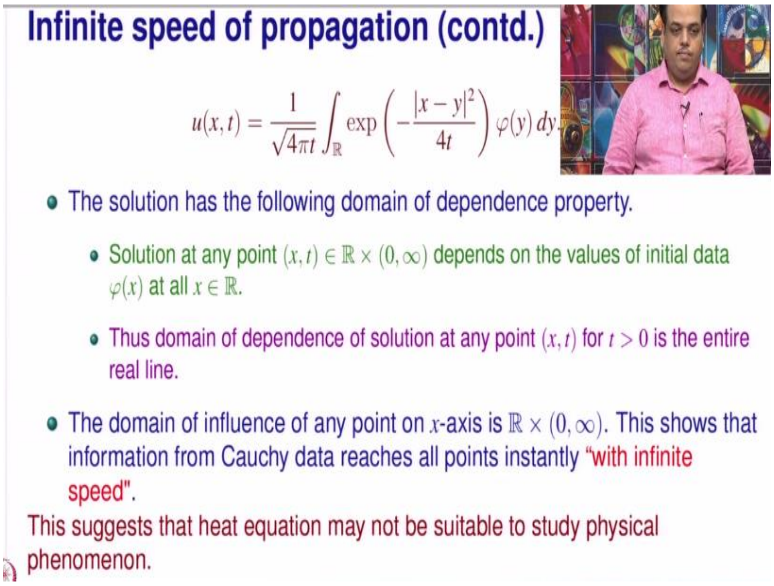

So, the solution has the following domain of dependence property. Like we have discussed domain of dependence and domain of influence for wave equation, we are going to discuss the same for heat equation. So solution at any point x, t depends on the values of the initial data phi x at all x in R that is visible from this integral. All the values of phi are used to find u of x, t at any point x, t.

Thus domain of dependence of solution at any point x, t is the entire real line because all the values of phi matter, phi of y as y belongs to R they matter in determining the solution at any point x and at any time t positive. The domain of influence of any point on x-axis is R cross 0, infinity for the same reason. This shows that the information from Cauchy data reaches all points instantly because Cauchy data is given a t = 0.

Whereas u of x, t for whatever may be the x all the values of phi matter in this. They influence the solution at every point. So, with infinite speed this is unlike the wave equation, where an initial disturbance takes some time to reach other points. But here, it reaches instantly with infinite speed. This suggests that heat equation may not be suitable to study physical phenomena.

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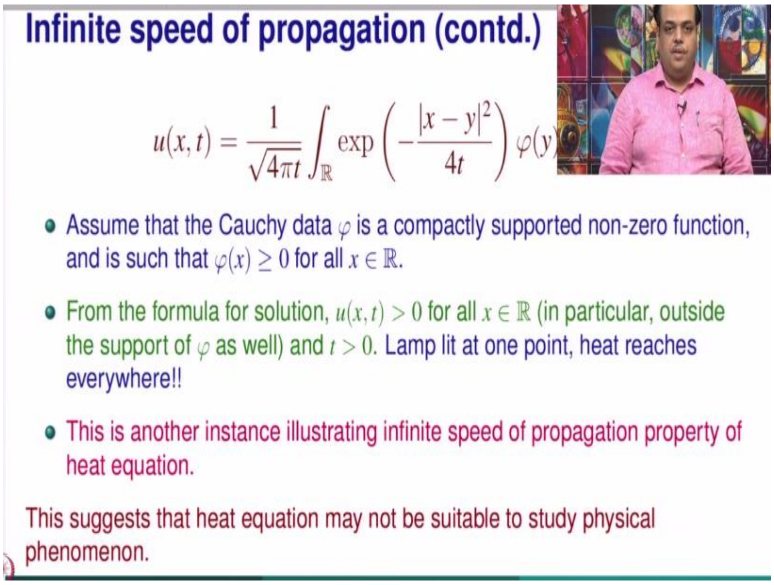

Let us look at the formula again. Assume that the Cauchy data phi is also compactly supported function now and it is a nonzero function and such that phi is greater than or equal to 0 for every x in R. That is a nonnegative, compactly supported, continuous function. From the formula for solution, u of x, t is strictly positive because integrands are greater than or equal to 0, but it is strictly greater than 0 somewhere because the phi is nonzero function.

And hence this integration, after the integration we get strictly positive quantity. So u of x, t is strictly greater than 0 for every x in R, in particular outside the support of phi also and t positive. So, it is like lamp lit at one point, heat reaches everywhere. There is a story in the literature where a person is trying to cross a river where the water is very chill and suddenly he notices a lamp is lit at other bank and then he starts to feel the warmth.

This happens in literature and perhaps in mind, not the physical world. So, this is another instance illustrating infinite speed of propagation property of the heat equation. This suggests again that heat equation may not be suitable to study physical phenomena.

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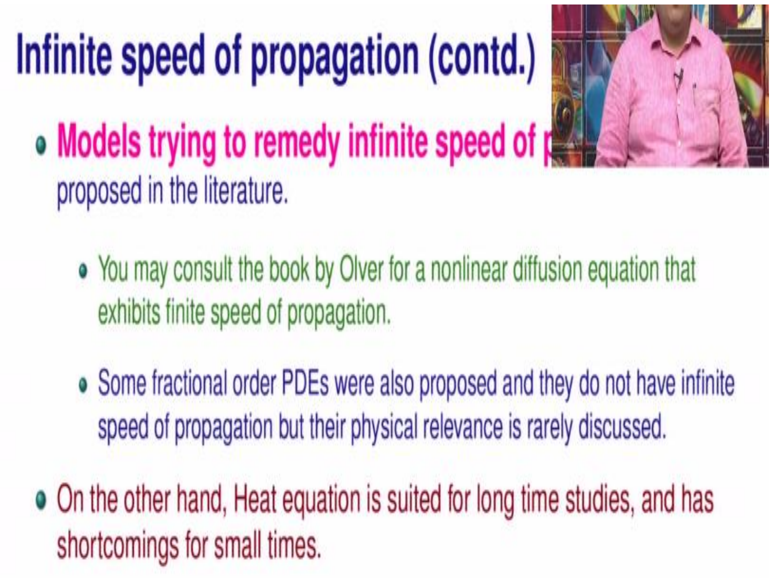

So, models trying to remedy infinite speed of propagation. So, people have created models, which do not have this infinite speed of propagation, they were proposed in literature. You may consult the book by Olver, the book of Olver partial differential equations for a nonlinear diffusion equation that exhibits finite speed of propagation.

There are some fractional order partial differential equations, they are also proposed and they do not have infinite speed of propagation, maybe you are happy because it has finite speed of propagation, but the physical relevance is rarely discussed. On the other hand, heat equation is suited for long time studies as t large and has shortcomings for small times.

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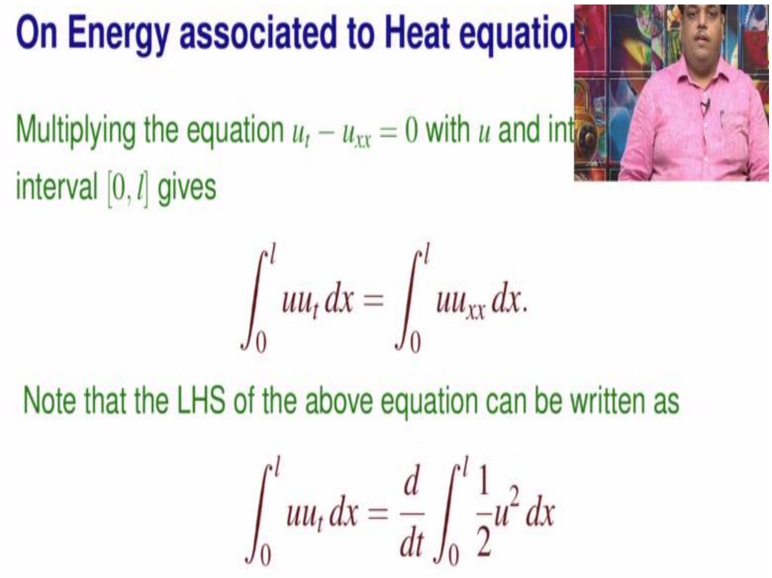

So, let us now discuss uniqueness for IBVP via energy method. So, an energy associated to heat equation. So, we have to get what we want to declare as energy. So, multiply the heat equation u t – u xx = 0 with u and integrate over the interval 0, l that gives us 0 to l u u t d x = 0 to l u u xx dx. Note that the LHS of the above equation namely this can be written as this.

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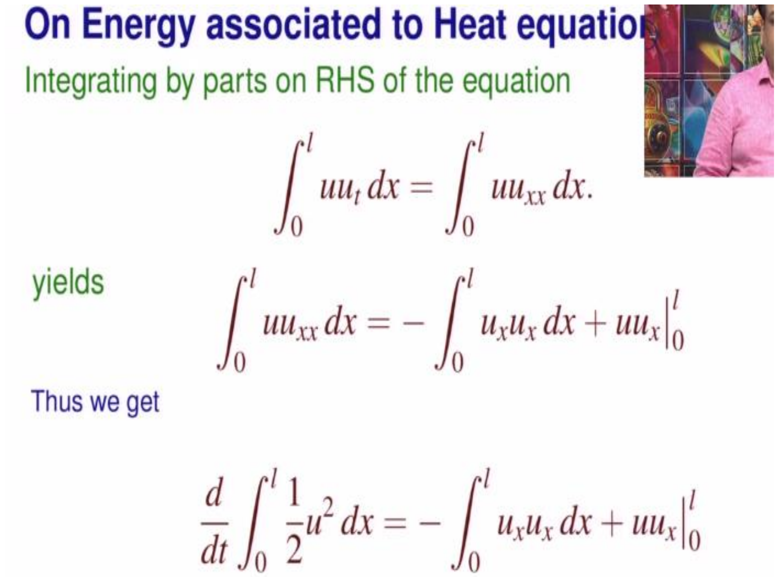

So, integrating by parts on the RHS of this equation, we get these terms. So, therefore what we have is d by dt of 0 to l half u square dx = – 0 to l ux ux dx + u ux at the point 0 and l.

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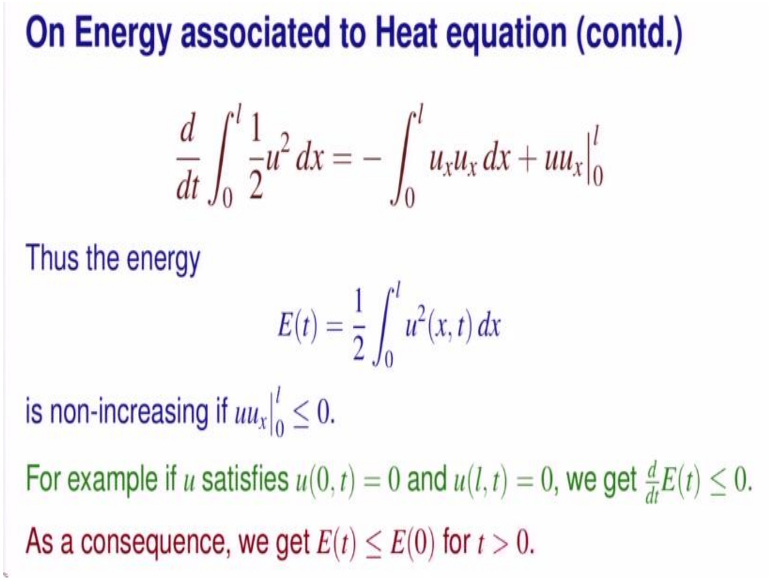

So, define energy to be half 0 to l u square of x, t dx. This this is non-increasing if u u x 0 to l is less than or equal to 0 because its derivative which is here this is non-positive and this is also non-positive, then d by dt of the energy is non-positive that means energy is non-increasing.

For example, if u satisfies u of 0, t = 0 and u of l, t = 0 these are the kind of conditions that we have in initial boundary value problem that actually is 0. So, what we have is d by dt of E t is less than or equal to 0. As a consequence, E of t is less than or equal to E of 0 for t positive.

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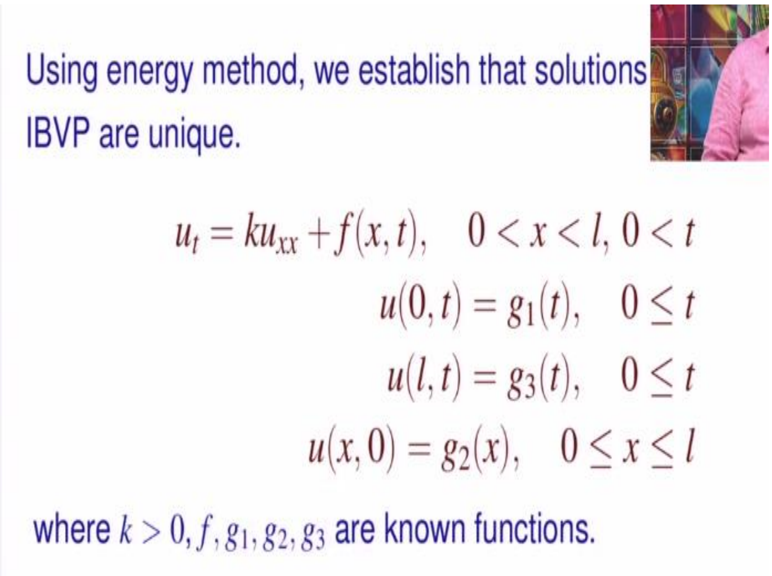

So, using energy method, we established that solutions to the following IBVP are unique. The IBVP is u t = u xx + f of x, t, x in 0 to l and t positive; u of 0, t is g 1 t; u of l, t is g 3 t; u of x, 0 is g 2 x. So, these two are boundary condition, this is the initial condition. So, this is the initial boundary value problem for nonhomogeneous heat equation; f, g 1, g 2, g 3 are known functions. But I am taking k = 1 here.

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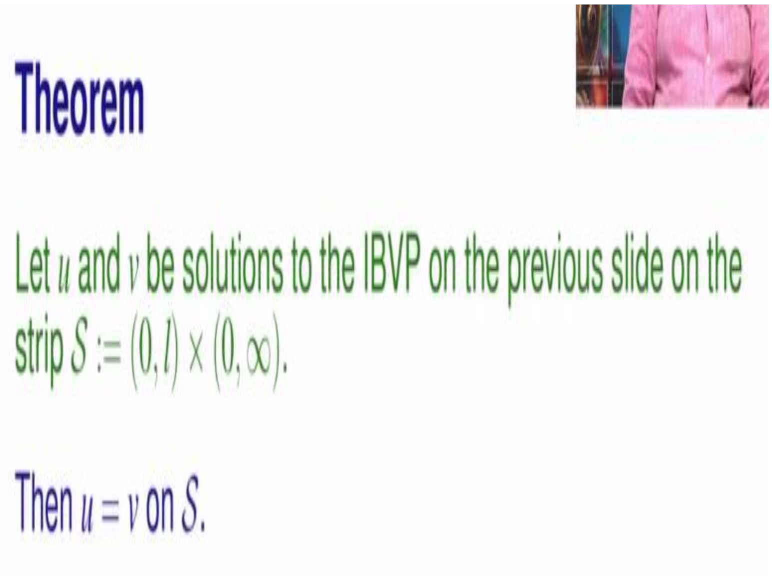

So, let u and v be solutions to the IBVP on the previous slide on the strip 0, l cross 0, infinity then u = v on S.

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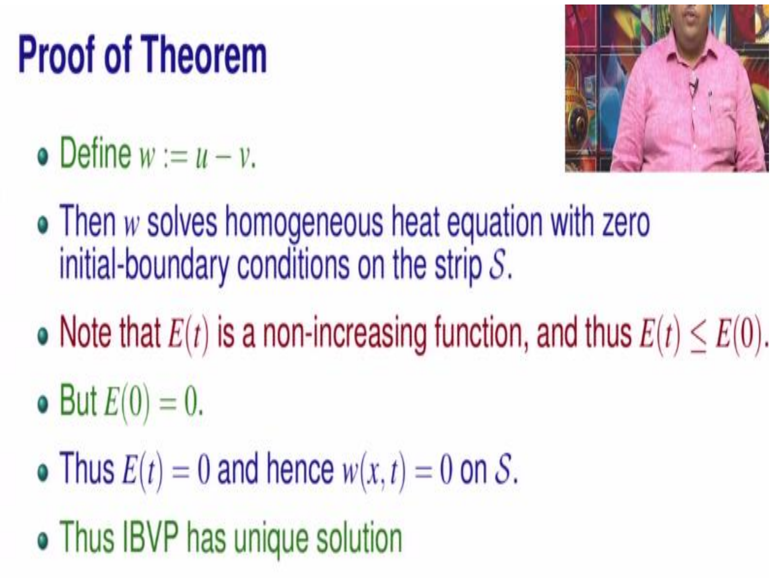

Let us see the proof of the theorem. How do we show uniqueness? We consider the difference u – v and show that that is 0. So, that leads us to defining w = u – v. Then we look at the problem solved by w, w solves homogeneous heat equation because u and v solve nonhomogeneous heat equation, therefore w solves homogeneous heat equation and the 0 initial boundary conditions on the strip.

Note that E of t is a non-increasing function that is E of t is less than or equal to E of 0. If you remember this, we have derived on the assumption that u of 0, t and u of l, t are 0. And in this case w of 0, t, w of l, t are 0; therefore this is valid. So, the energy associated to w has this property. But at time t = 0 the energy is 0, half u square. So, if it is w, half integral w square and w is 0. Therefore, E of 0 is 0. That means E of t is 0 and hence w of x, t is 0 on the strip. Thus IBVP has unique solution.

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Cauchy problem for backward Heat equation

is illposed

Now let us discuss backward heat equation and show through two examples that the Cauchy problem for it is illposed.

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Example 1. Non-existence of solutions

Cauchy problem for Backward Heat equation Let $T > 0$.

$$u_t - u_{xx} = 0,\ x \in \mathbb{R}, t \in (-T, 0),$$

$$u(x,0) = \varphi(x), \quad \text{for } x \in \mathbb{R}.$$

Assume that φ is continuous and bounded function.

The above problem is NOT solvable in the class of bounded functions.

The example 1, non-existence of solutions. Cauchy problem for backward heat equation let T be positive. Now it is exactly the heat equation here u t – u xx = 0 x in R but now t belongs to –T, 0. That means I am considering this equation for negative t and u of x, 0 = phi x for x in R. So, that is why this is called backward because you have prescribed t = 0 and you are interested in solving the heat equation for t less than 0 that is why it is called backward heat equation.

Assume that phi is a continuous and bounded function. The above problem is not solvable in the class of bounded functions. So, that means there is no solution. If a problem is not well posed it can happen because of three reasons, either there are no solutions or uniqueness is violated or there is no stability of solutions. So here we are showing about non-existence of solution that is the reason why this problem is illposed.

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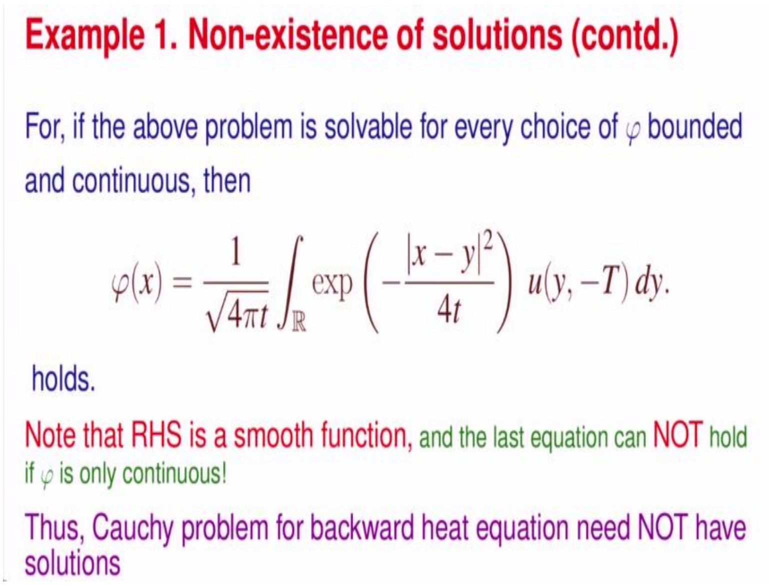

So, what is the reason? Suppose the above problem is solvable for every choice of phi which is bonded and continuous, then phi of x is given by this formula. We have derived this formula, we have shown that in the class of bounded and continuous functions the solutions are unique. So this, I am looking at u which the solution to the backward problem and so that is why this is the condition like the initial condition there, then phi of is at a later time, right. This is at t = –T, this is at time t = 0. So it is given in terms of this integral, so this holds.

So once this holds, we can see that the RHS is a smooth function. There is s no doubt about that it is a smooth function. And the last equation cannot hold if phi is only continuous because if this is smooth and phi is not even differentiable, then this cannot hold, right.

Therefore, if phi is only continuous this equation cannot hold and hence our supposition that there is a solution for every choice of phi which is bounded and continuous is not true. Therefore, the Cauchy problem for backward heat equation need not have solutions.

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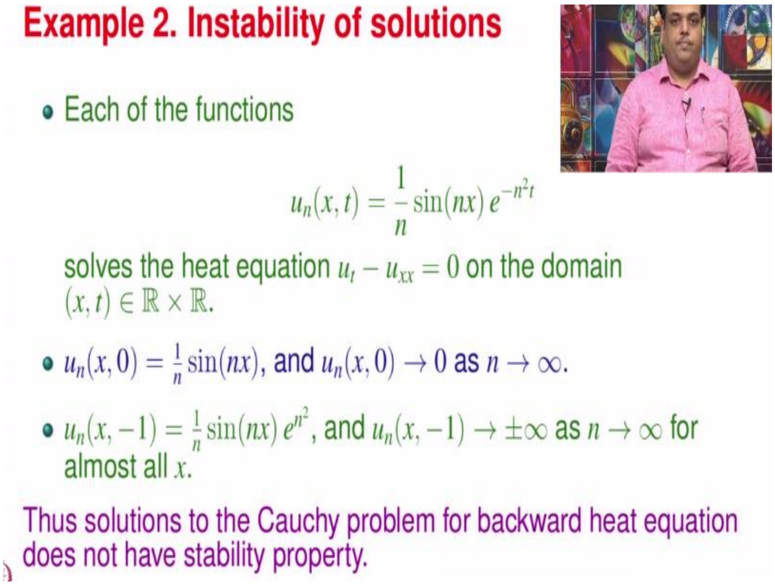

So, second example is about instability of solutions. Look at this function, sequence of functions, u n of x, t = 1 by n sine nx into e power –n square t. Each of them solves the homogeneous heat equation for every x and t, x in R, t in R. Look at u n of x, 0. If we put t = 0, this is 1, therefore you have 1 by n sine nx. So, u n of x, 0 is 1 by n sine nx and that goes to 0 as n goes to infinity, in fact uniformly.

Let us look at u n of x, –1; –1 is just one instance for one negative time t okay, instead of writing let t 0 be negative, less than 0, we have chosen t 0 = –1. Then this u n of x, –1 what we get is 1 by n sine nx e power n square. Now, as n goes to infinity, this term 1 n of course goes to 0, but e power n square goes to infinity. So, this is going to infinity and sine nx will be oscillating between –1 and 1.

So, therefore u n of x, –1 keeps oscillating towards plus infinity and minus infinity. In any case, it is not going to 0. Except for x being integral multiples of pi, this will never go to 0. So, initial conditions are going to 0, but at a time t = –1 it does not go anywhere. The solutions to the Cauchy problem for backward heat equation does not have stability property. Thank you.