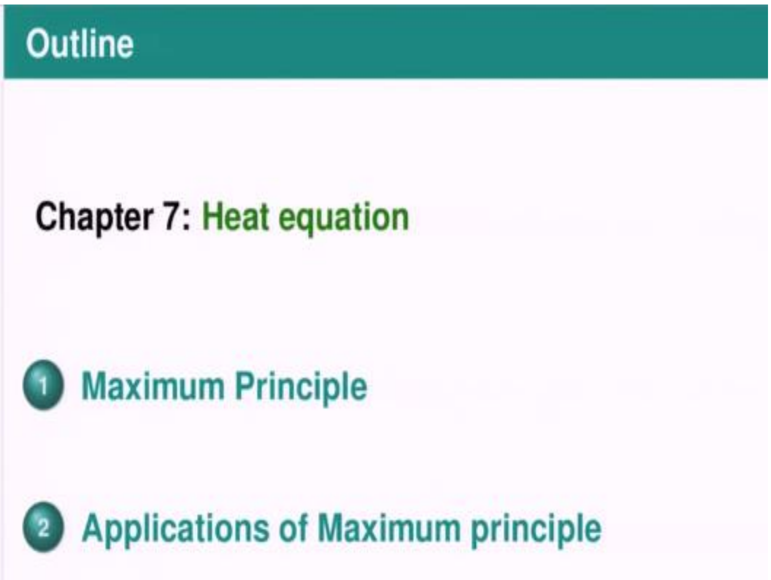
Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology – Bombay

Lecture – 58
Maximum Principle for Heat Equation

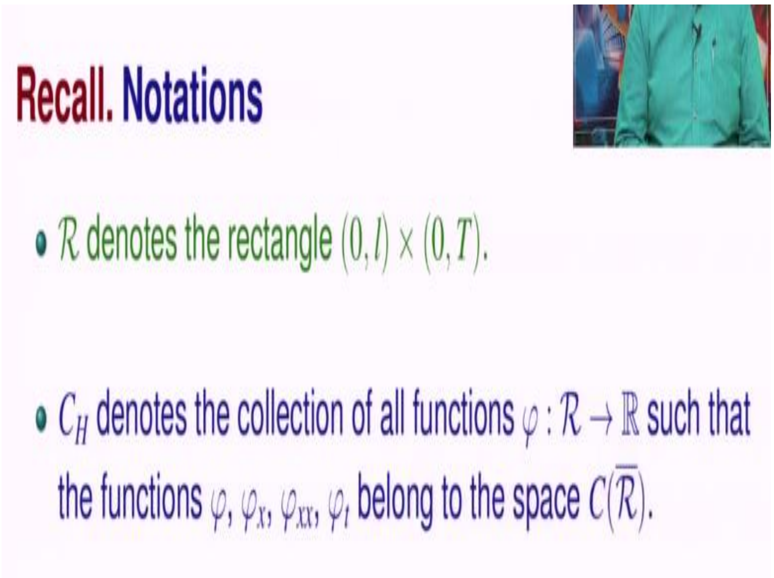
(Refer Slide Time: 00:27)



Welcome to this lecture on maximum principle for heat equation. In this lecture, we are going to prove the maximum principle stated for bounded intervals in R. Of course, there is a version of the maximum principle for X belongs to R as well, which we are not going to discuss in this course.

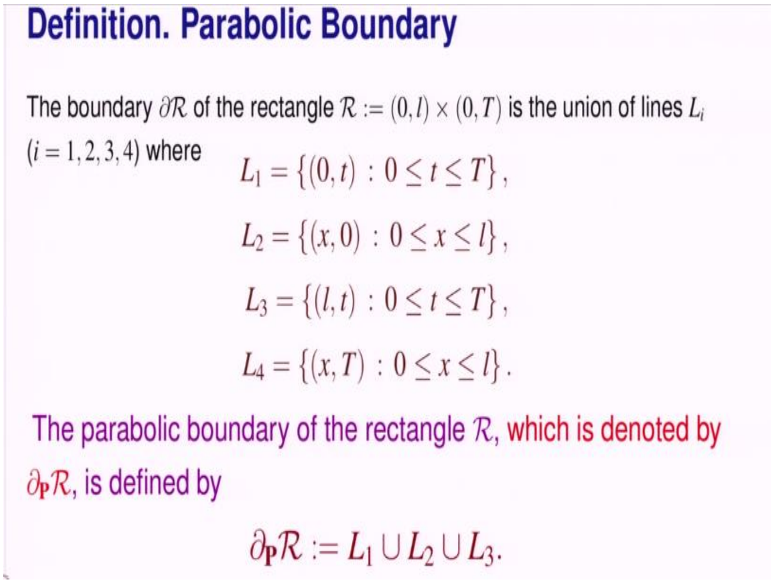
And then we give a few applications of maximum principles, one of them being justification that the formal solution obtained by the separation of variables method is indeed a solution to the initial boundary value problem, recall that we have used separation of variables method to solve an initial boundary value problem in lecture 7.3.

(Refer Slide Time: 01:00)



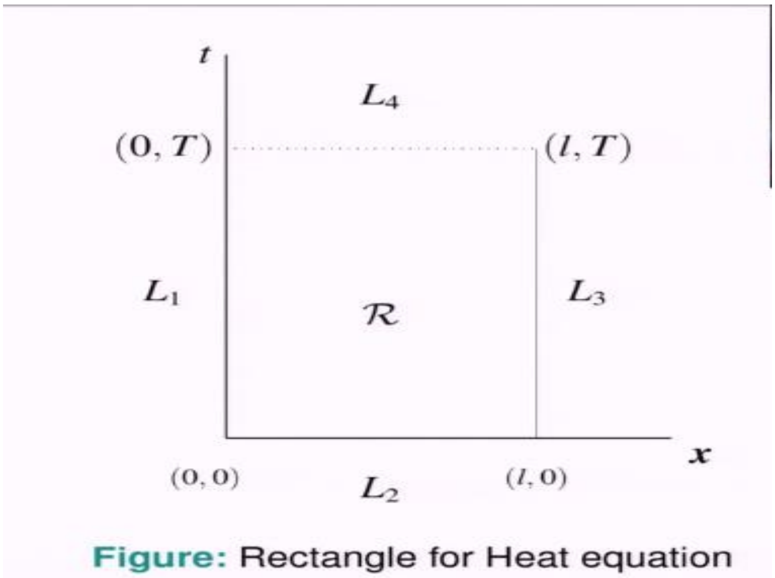
Recall notations R denote rectangle 0, l cross 0, T. This is to express X varies in 0, l and the time varies in 0 to capital T. C H which denotes the collection of all functions defined on the rectangle are taking values in the real numbers such that the function phi, the first order derivatives with respect to x and t and the second are derivative of phi with respect to x all of them belong to the space C of R closure that is their continuous functions on R closure.

(Refer Slide Time: 01:37)



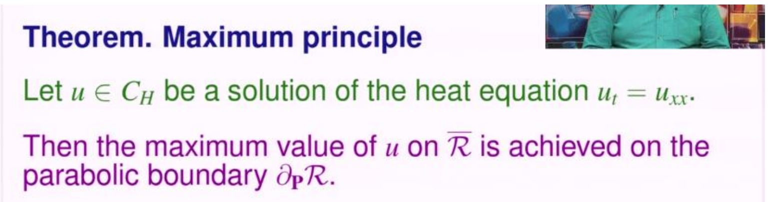
Definition of what is known as parabolic boundary. This is a subset of the boundary of the rectangle which plays a role in the maximum principle. So, the boundary of the rectangle consists of the lines L 1, L 2, L 3, L 4 and this is the rectangle R. The parabolic boundary of the rectangle R which is denoted by boundary P R, P for parabolic boundary of the rectangle, so that is parabolic boundary of the rectangle is defined as the union of L 1, L 2 and L 3. That means L 4 is not included, L 4 is part of the boundary of the rectangle but is not included in the parabolic boundary.

(Refer Slide Time: 02:45)



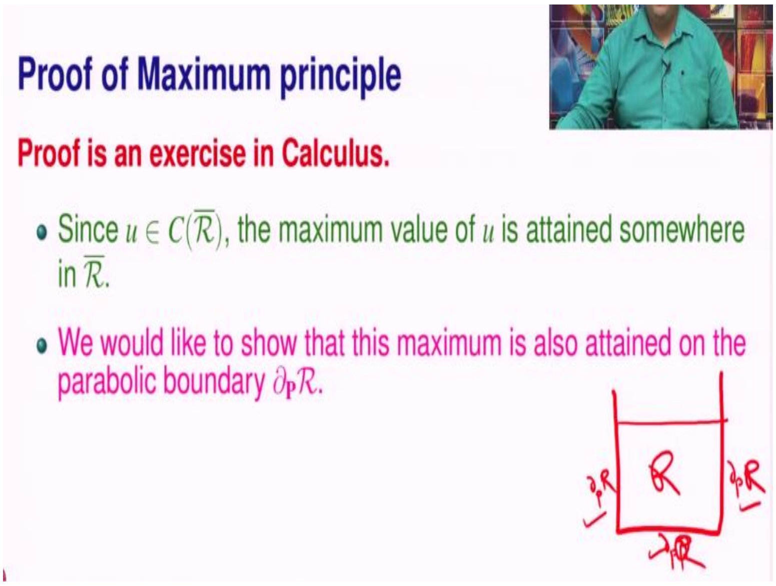
Here is a picture L 1, L 2, L 3, this is dou P R and L 4 is also a part of the boundary of rectangle R.

(Refer Slide Time: 03:02)



Let us state now the maximum principle. Let u be a solution to the heat equation u t = u xx, then the maximum value of u on R closure is achieved on the parabolic boundary dou P R.

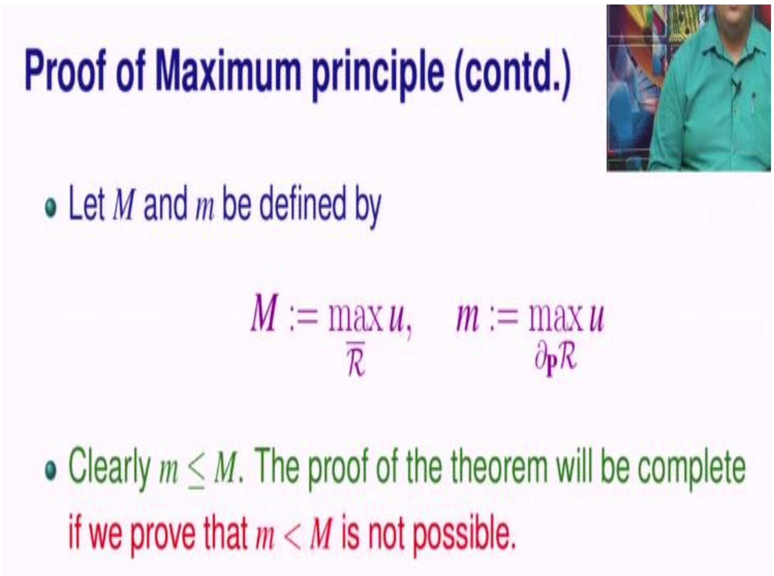
(Refer Slide Time: 03:22)



Proof of maximum principle. It is an exercise in calculus exactly like the maximum principle for the Laplace equation or for the harmonic functions which we proved is also an exercise in calculus. So, since u is a continuous function on R closure, R closure is a complex set, the maximum value of u is attained somewhere in R closure. We would like to show that this maximum is also attained on the parabolic boundary dou P R.

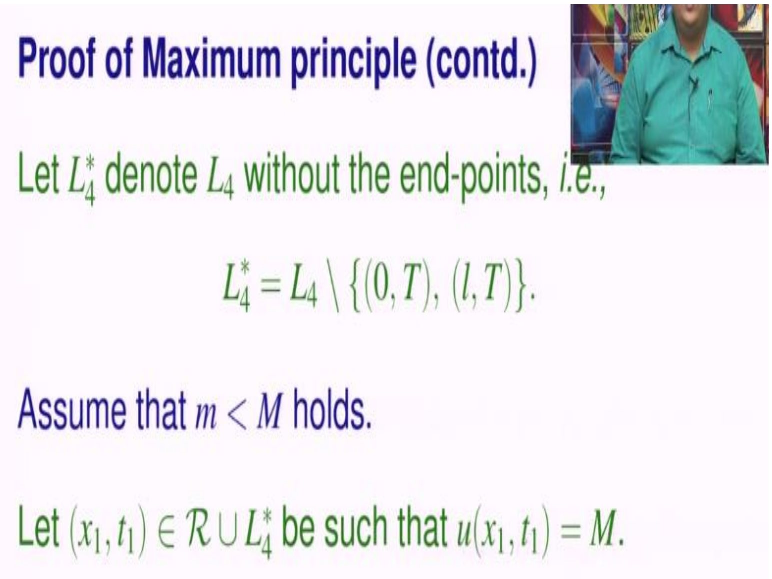
In other words, this is what we have as R and dou P R consists of these three lines. So, we would like to show that the maximum is attained on either here or here or here. It may be attained in somewhere else also, but the maximum principle does not say about that. Maximum principle says it is definitely achieved on dou P R.

(Refer Slide Time: 04:31)



So, let capital M and small m be defined by capital M is a maximum of u on R closure and small m is a maximum of u on the parabolic boundary. So, clearly m is less than or equal to M because dou P R is a subset of R closure. Therefore, m is always less than or equal to capital M. The proof of the theorem will be complete if you prove that m less than M is not possible. In that case m = M will hold and that is precisely the conclusion of the maximum principle. So, we are going to show that m less than capital M is not possible.

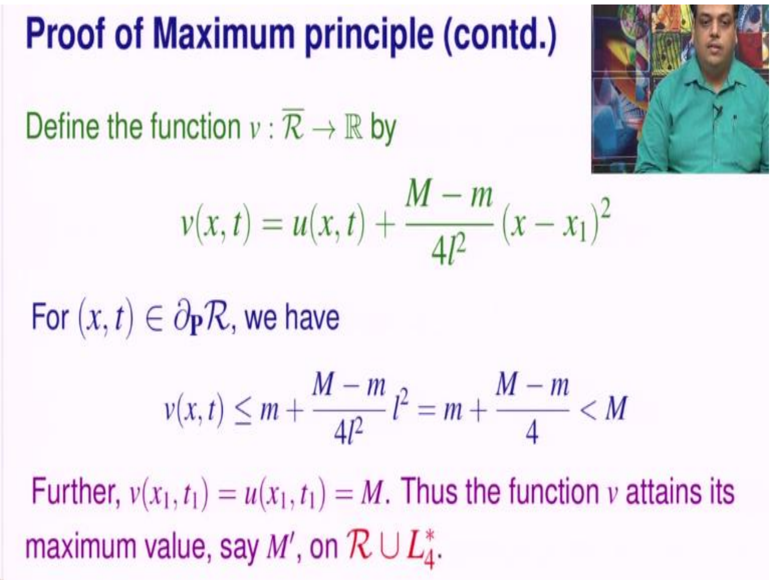
(Refer Slide Time: 05:13)



Let L 4 star denotes L 4 without the endpoints L 4 – 0, T, l, T because 0, T and l, T are already in the parabolic boundary of R. So, we want to remove that from L 4 and call it L 4 star. Assume that m is strictly less than M holds. So, let x 1, t 1 be a point in R union L 4 star be such that u of x 1, t 1 = capital M. Such x 1, one t 1 exists because capital M is strictly bigger than small m.

Capital M is a maximum of u on R closure while small m is a maximum of u on the parabolic boundary. Therefore, there will be a point x 1, t 1 which is essentially R closure minus the parabolic boundary. In other words, R union L 4 star such that u achieves the value M at the point x 1, t 1.

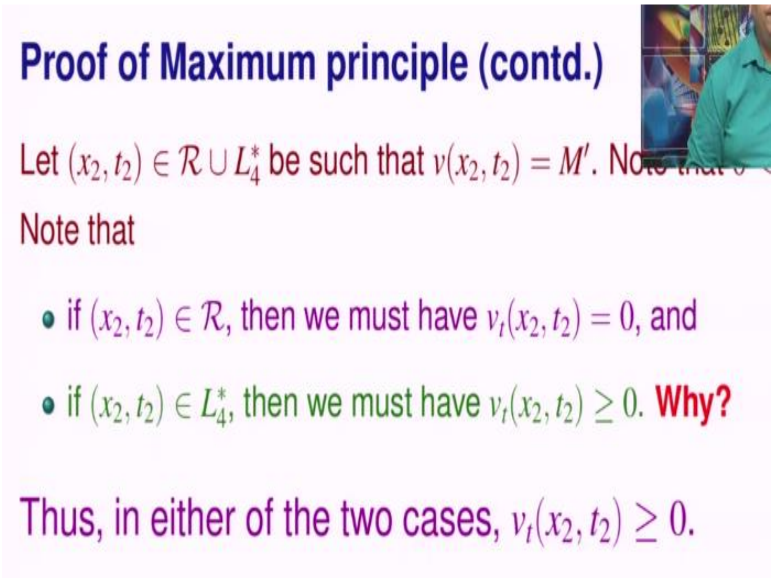
(Refer Slide Time: 06:18)



Now, we are going to define a function v by v of x, t = u of x, t + M – m by 4l square into x – x 1 whole square. So, we are adding a term to u and what we are adding is always non-negative because M is strictly bigger than small m, so it is positive, x – x 1 square is always greater than or equal to 0. So, for x, t in the parabolic boundary we have v of x, t less than or equal to small m because u of x, t is less than or equal to small m on the parabolic boundary + M – m by 4l square which is as it is from here.

Now, x – x 1 whole square is less than or equal to l square and that equal to l square, l square gets cancelled and we get M – m by 4 and that is strictly less than capital M. So, that is v of x, t is strictly less than capital M whenever x, t belongs to the parabolic boundary of R. Further v of x 1, t 1 is u of x 1, t 1 because this term is 0 when x = x 1 and u of x 1, t 1 is capital M. Thus, the function v attains its maximum value say M dash on R union L 4 star.

(Refer Slide Time: 07:52)

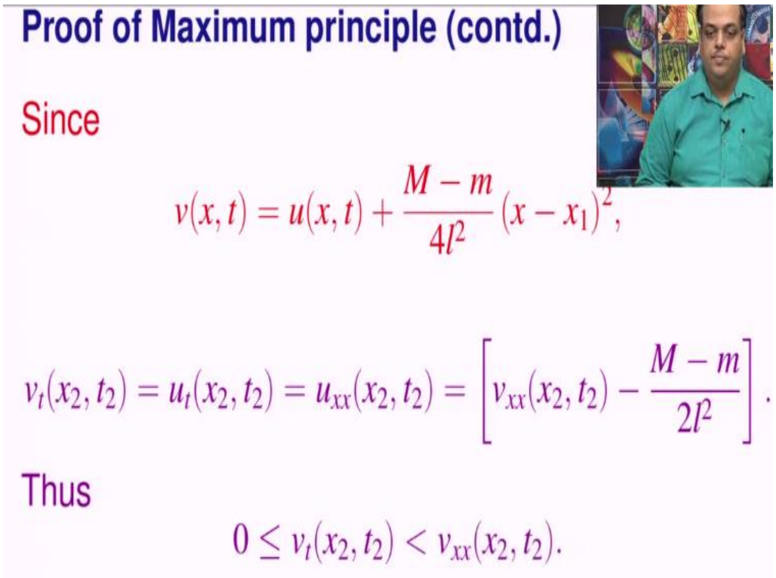


So, let x 2, t 2 belonging to R union L 4 star be such that v of x 2, t 2 is M dash. Note that x 2 lies in the open interval 0, l. Note that if the x 2, t 2 belongs to R, then we must have v t of x 2, t 2 = 0. So, what are the possibilities for x 2, t 2 if it is in R union L 4 star either it is in R or in L 4 star? If it is in R then we must have v t of x 2, t 2 = 0 and if x 2, t 2 is actually on L 4 star then v t of x 2, t 2 is greater than or equal to 0. Why? Because of this.

So, if x 2, t 2 is here, this is the situation one, x 2, t 2 belongs to R that is an interior point at which you have a maximum that is why the first order derivatives are 0. On the other hand, if it happens on L 4 star, this is L 4 star, then x 2, t 2 is here. So, t is this, right, this is a t direction. So, there is a maximum at this endpoint t 2. In other words t 2 is actually capital T. So, this point is x 2, capital T if it belongs to L 4 star.

And hence partial derivative with respect to t is greater than or equal to 0 and this follows from the different coefficients. In either of the two cases, we have v t to be greater than or equal to 0.

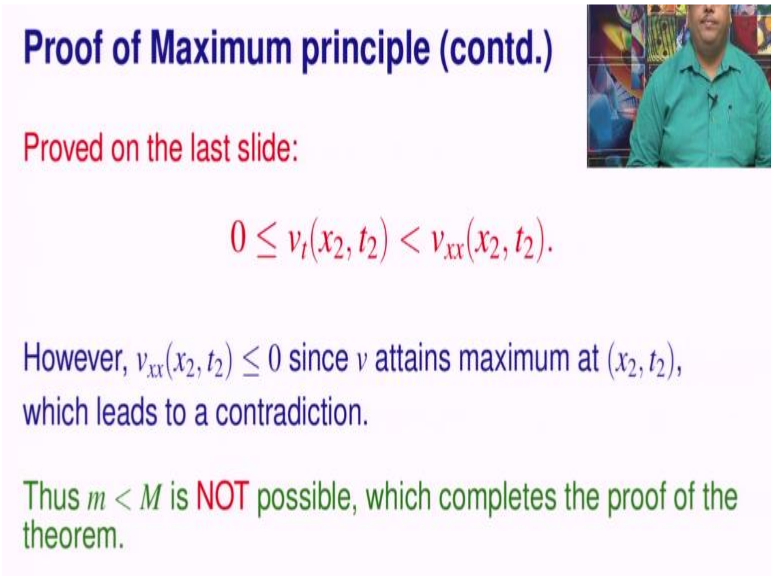
(Refer Slide Time: 10:03)



So, since v is given by this formula v t of x 2, t 2 that means we are going to differentiate with respect to t, we get to u t of x 2, t 2 and this does not depend on t. So, its t derivative is zero. Now, u is a solution to the heat equation. So, u t of x 2, t 2 is u xx of x 2, t 2. But what is u xx of x 2, t 2? We can compute from here in terms of v, u xx will be v xx – the second derivative with respect to x of this term which is here.

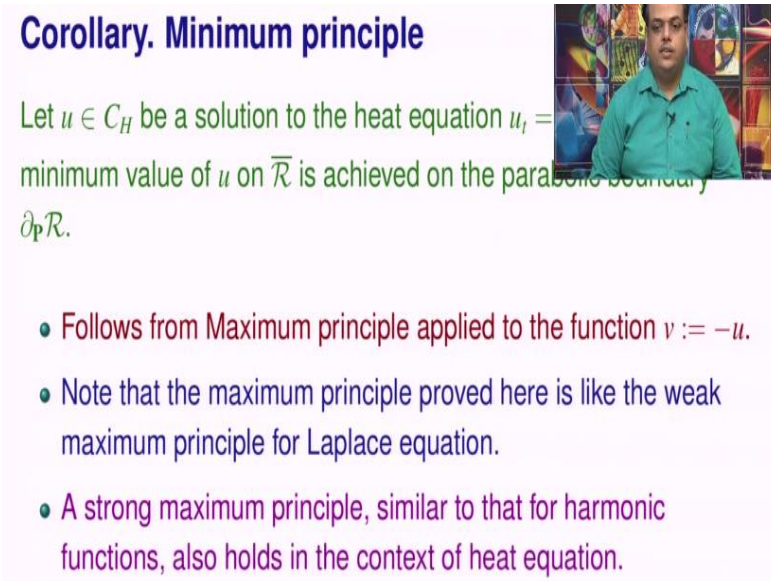
Now observe this is a nonnegative quantity, in fact a positive quantity, you are subtracting something from v xx. So, this quantity is strictly less than v xx. This we have anyway 0 less than or equal to v t of x 2, t 2 we proved on the last slide. And now that v t of x 2, t 2 is strictly less than v xx of x 2, t 2.

(Refer Slide Time: 11:13)



So, this is what we proved on the last slide. But v xx of x 2, t 2 is less than or equal to 0, why? Because v attains maximum at x 2, t 2 and this leads to contradiction because v xx is strictly positive on one hand and less than or equal to 0 on the other hand and that is a contradiction. Therefore, m less than capital M is not possible. And this completes the proof of maximum principle.

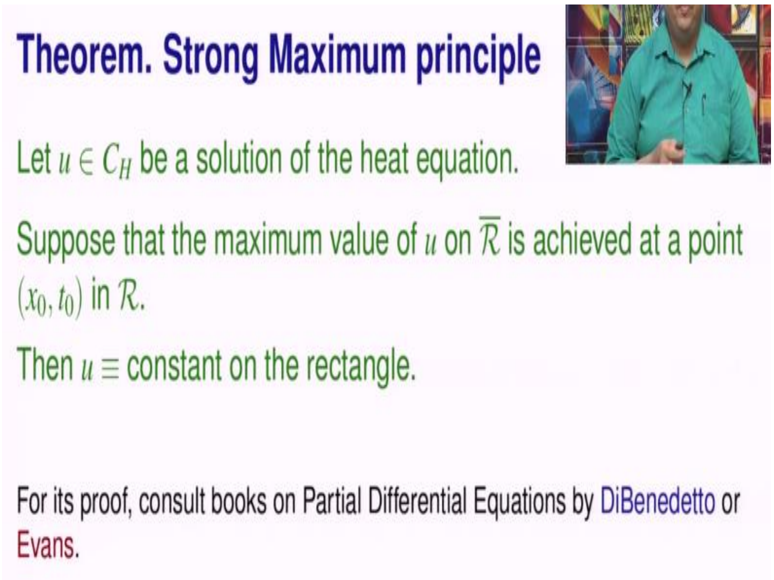
(Refer Slide Time: 11:47)



As a corollary, we can deduce minimum principle. Lt u be a solution to the heat equation. Then the minimum value of u on R closure is actually attained on the parabolic boundary. This follows from the maximum principle that we have just proved, we have to apply this to the function v = –u. If u is a solution to heat equation –u is also a solution to the heat equation. So, therefore, we can apply the maximum principle for v.

Note that the maximum principle proved here is like the weak maximum principle that we have proved for Laplace equation. A strong maximum principle similar to that for harmonic function which we have proved as a consequence of the mean value property it also holds in the context of heat equation. We are not going to discuss that; we are going to just state it.

(Refer Slide Time: 12:42)

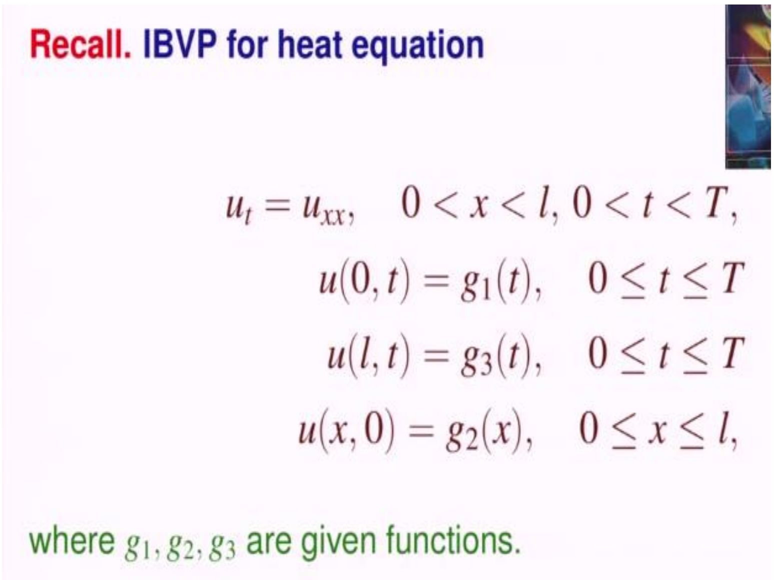


So, strong maximum principle if u is a solution of the heat equation, suppose that the maximum value of u on R closure is achieved at a point in the rectangle, then it must be constant on the rectangle. For its proof, please consult books on partial differential equations, for example by DiBenedetto or Evans. The maximum principle that I have mentioned earlier, which is stated for x belongs to R is available in this book, DiBenedetto.

(Refer Slide Time: 13:15)

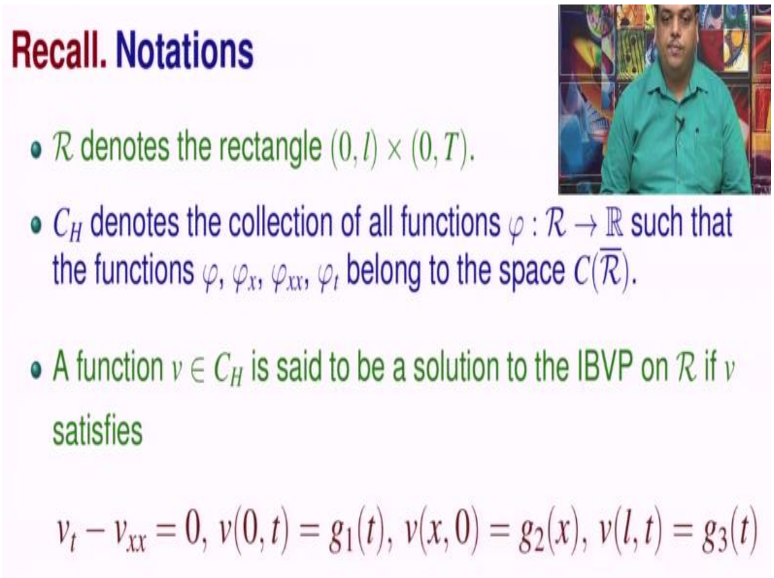
Now, as an application of maximum principle we are going to show the uniqueness of solutions to initial boundary value problems.

(Refer Slide Time: 13:24)



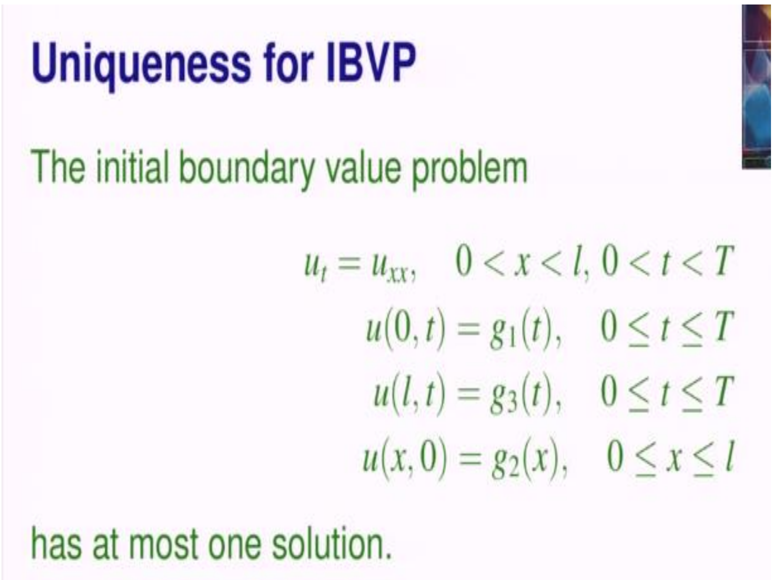
Recall the IBVP that we have considered for the heat equation u t = u xx; u 0, t g 1 of t; u l, t g 3 of t; u x, 0 g 2 of x where g 1, g 2, g 3 are given functions.

(Refer Slide Time: 13:36)



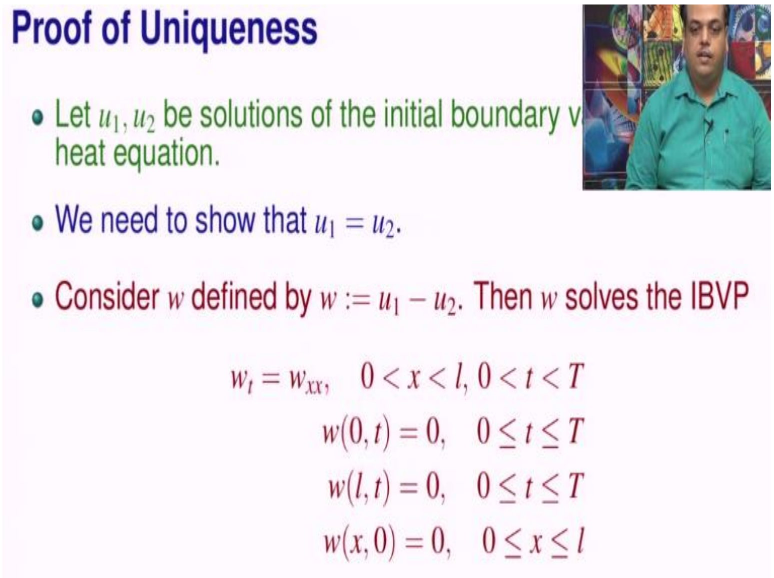
And R denotes the rectangle 0, l cross 0, T. C H denotes the collection of functions phi having this property. A function v in C H is said to be a solution to the IBVP if v satisfies the heat equation on the initial and boundary conditions.

(Refer Slide Time: 13:58)



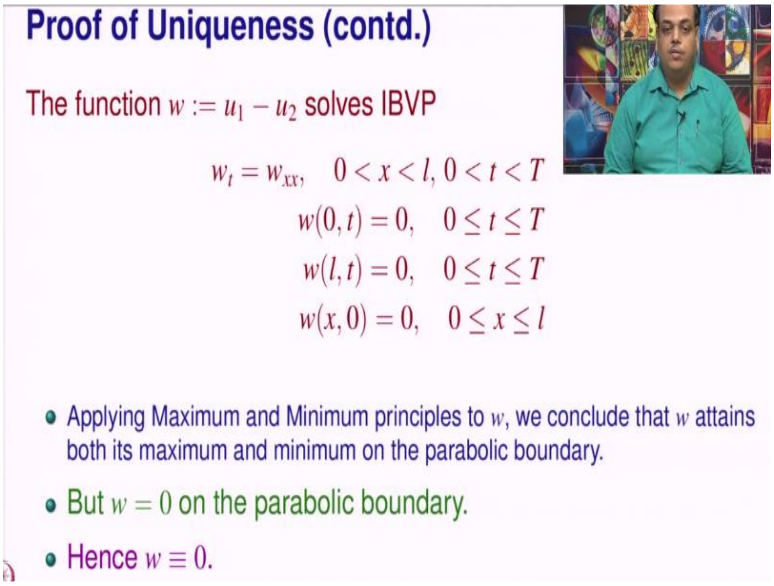
So, uniqueness for IBVP: So, the initial boundary well problem given here has at most one solution.

(Refer Slide Time: 14:07)



The proof of uniqueness, there is a standard procedure which is to assume that u 1 and u 2 are solutions of the IBVP. Consider the difference and show that the difference is 0. So if you want to show u 1 = u 2, so consider the difference u 1 – u 2, call it w. And w satisfies the IBVP for the heat equation with 0 initial boundary data.

(Refer Slide Time: 14:35)

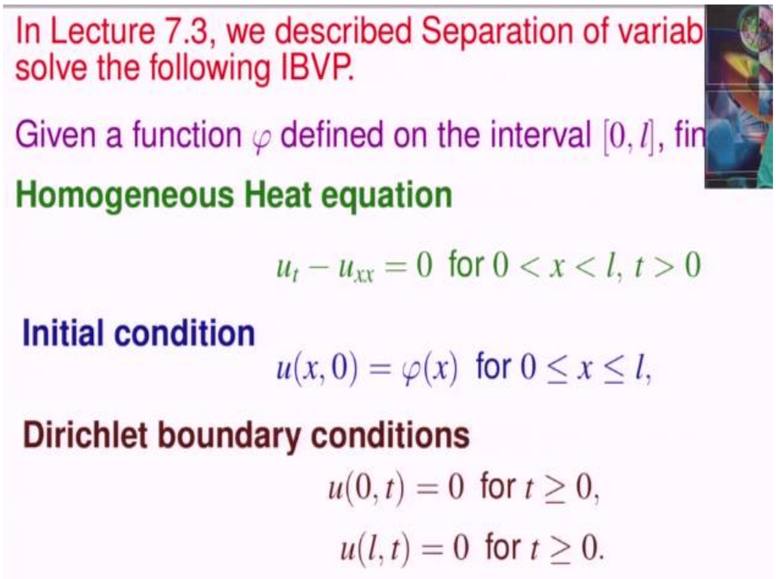


So, applying maximum and minimum principles to w, we conclude that w attains both its maximum and minimum on the parabolic boundary. But w is 0 on the parabolic boundary. This is precisely the data on the parabolic boundaries 0, therefore w is identically equal to 0 and this proves the uniqueness.

(Refer Slide Time: 15:02)

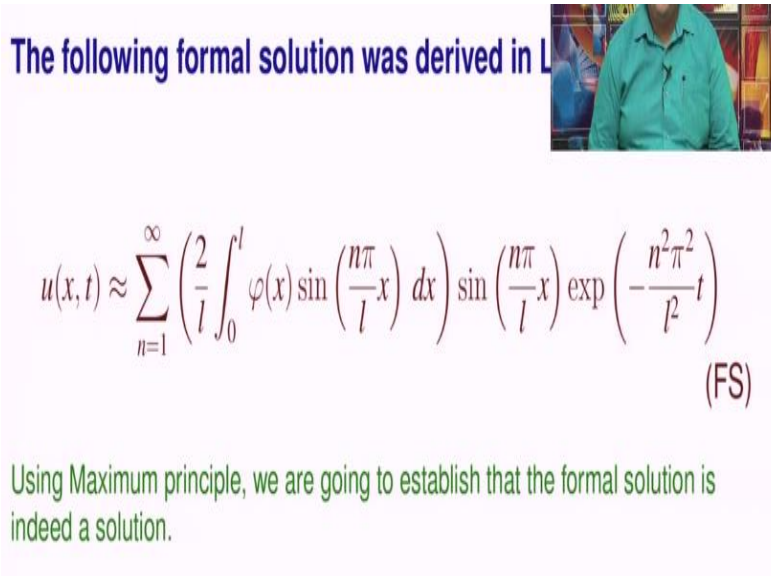
Now as another application we are going to see that the formal solution for IBVP is indeed a solution. Remember the formal solution was obtained using separation of variables method in lecture 7.3.

(Refer Slide Time: 15:16)



So, in lectures 7.3 we described separation of variables method to solve the following initial boundary value problem. Given a function phi, look at the heat equation posed on this domain and initial condition is phi of x and Dirichlet boundary condition is 0; 0 boundary conditions.

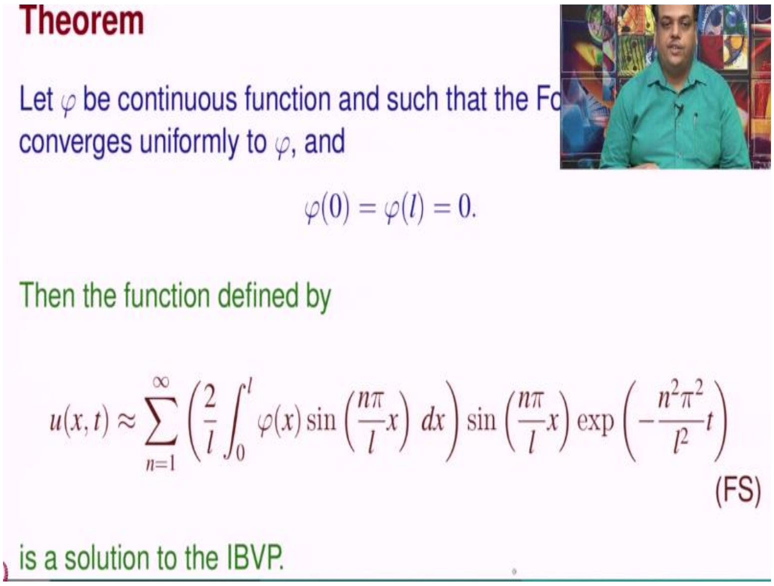
(Refer Slide Time: 15:37)



And the following formal solution was derived; u x, t is given by this infinite series. We have obtained these coefficients as coefficients in the Fourier sine series for the function phi of x. Using maximum principle, we are going to establish that the formal solution is indeed a solution. In fact, we use maximum principle to establish that the initial conditions are taken by this function.

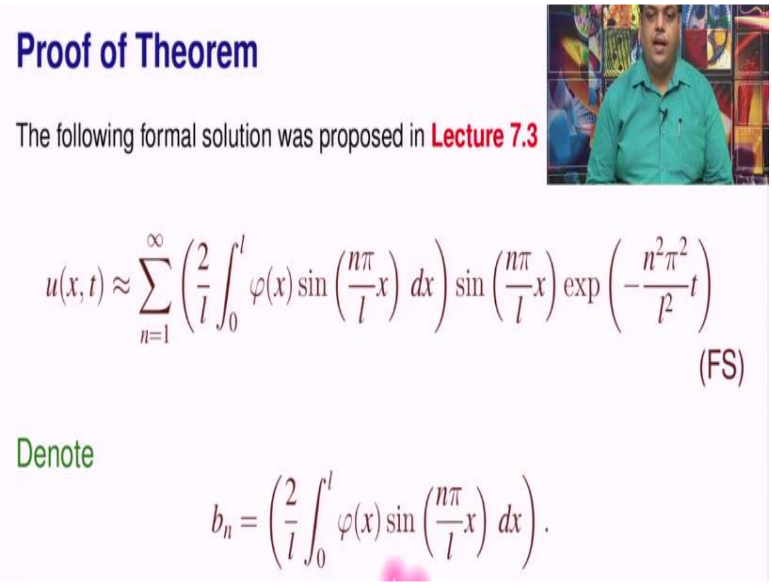
The fact that this defines a function which is twice differentiable with respect to x and once differentiable with respect to t mainly follows from this factor which is here, the exponential factor, we are going to see the proof.

(Refer Slide Time: 16:26)



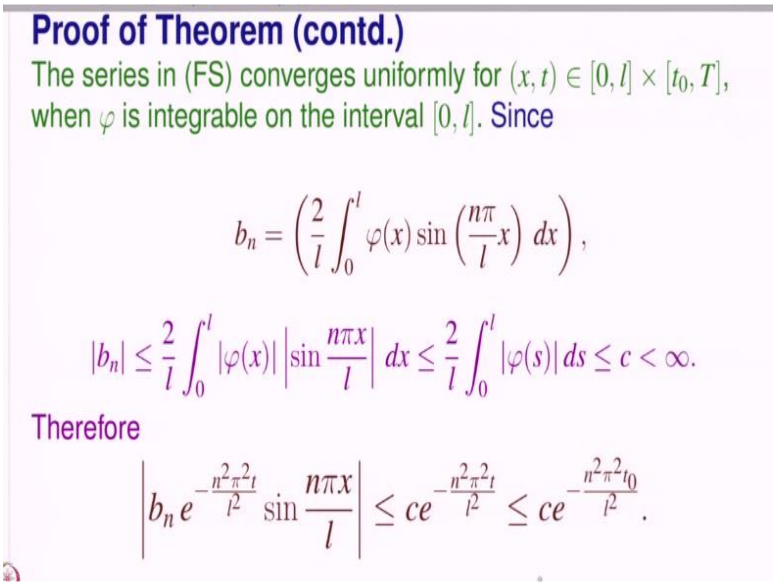
So, let us take it as a theorem. Let phi be continuous function and such that the Fourier series, in fact Fourier sine science series of phi converges uniformly to phi and phi of 0 = phi of l = 0. Then the function defined by this formal series expansion, it is indeed a solution to initial boundary value problem. So, we have to check that the series defines a function which is two times differentiable with respect to x, one time differential with respect to t and then that function actually satisfies the heat equation and the initial and boundary conditions are met.

(Refer Slide Time: 17:11)



So, this is the formal solution that was proposed as a consequence of separation of variables method. Let us denote these coefficients by b n.

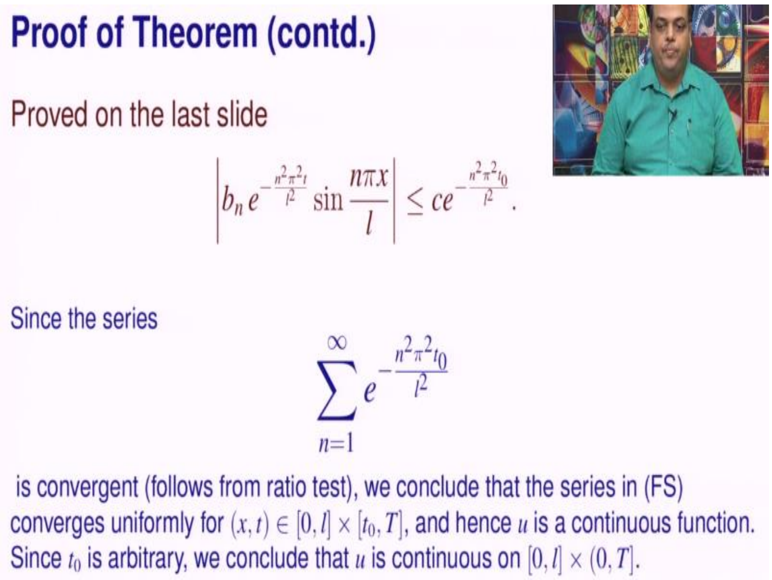
(Refer Slide Time: 17:29)



The series converges uniformly for x, t belonging to 0, l cross t 0, T and of course phi is integrable because we are assuming phi is continuous. So, b n has this expression 2 by l 0 to l phi x sine n pi by l x dx. So, mod b n is less than or equal to 2 by l modulus goes inside the integral, so mod phi into mod sine n pi x by l and that is less than or equal to 2 by l 0 l mod phis s ds which is a finite number because phi is continuous, it is bounded, it is integrable, whatever reasons you want to give.

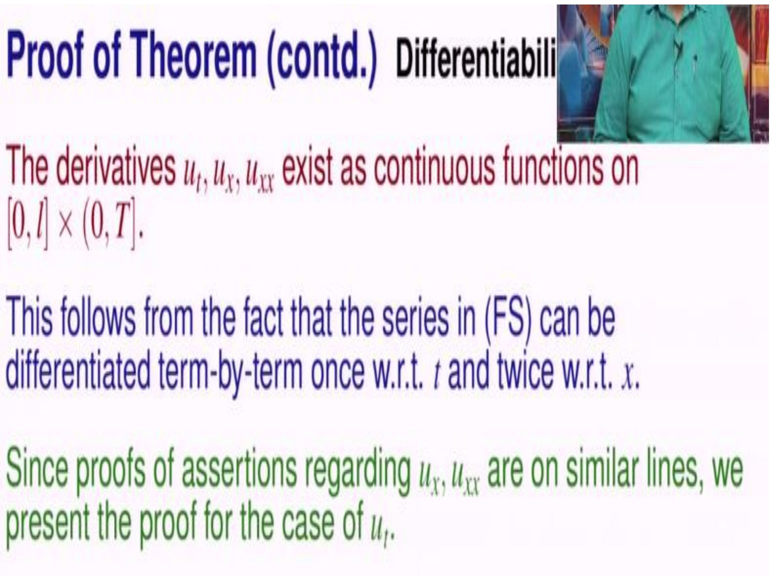
So therefore, mod b n e power –n square pi square t by l square sine n pi x by l. What is this? This is the nth term in the series that we have. This quantity now is less than or equal to the constant c for mod b n which is a non-negative quantity, so it stays as it is and of course modulus of sine is less than or equal to 1. So, we have this estimate. Not only this, because we are considering on this domain t 0, T therefore this exponential is dominated by this.

(Refer Slide Time: 18:53)



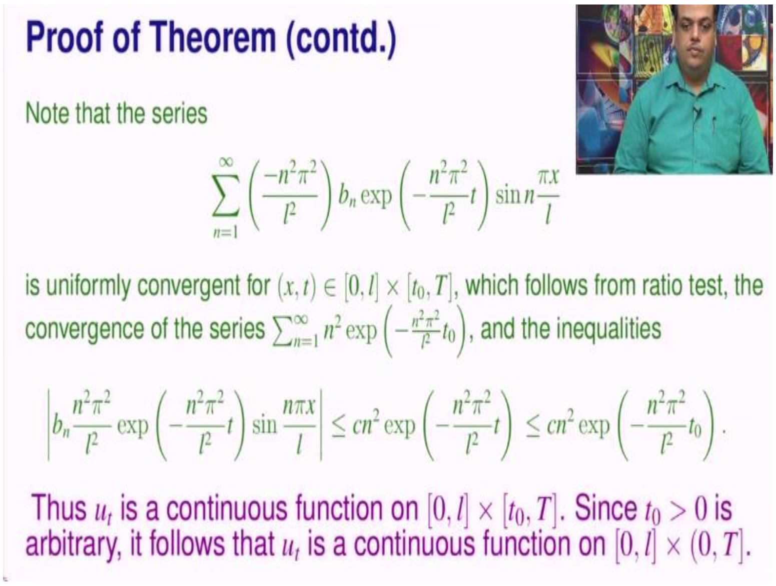
So, this is what we proved on the last slide. And this series summation nth term is this. This is convergent follows from ratio test, we conclude that the series converges uniformly for the x, t in 0, l cross t 0, T. So, whenever you have uniform convergence of the infinite series, it defines a continuous function. So, since t 0 is arbitrary we conclude that u is continuous on 0, l cross open 0 close T.

(Refer Slide Time: 19:31)



Now, differentiability. So, we have proved that series converges and defines a continuous function on the rectangle of interest. Now, we are going to show that u t, u x and u xx exist and they are continuous on this domain. This follows from the fact that the series can be differentiated term by term once with respect to t and twice with respect to x. In fact, much more that we are going to see in a remark soon after finishing this proof. So, since the proofs are similar, we are going to give a proof for u t just one derivative.

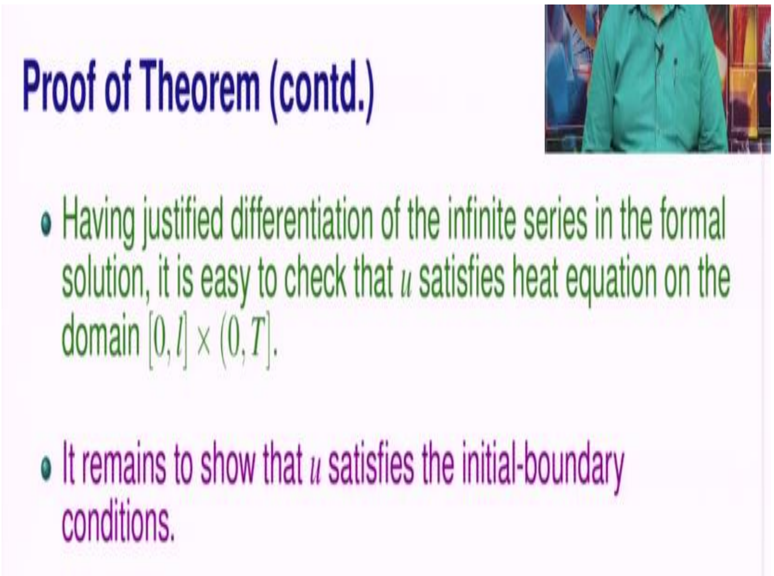
(Refer Slide Time: 20:10)



Note that the series given here this we have obtained after differentiating with respect to t once is uniformly convergent for x in 0, l and t in t 0, T. It follows once again from the ratio test and the convergence of this series exactly the same estimates. So, you have to estimate modulus of this and you will get this and a constant timestamp and this is precisely the inequality I was talking about.

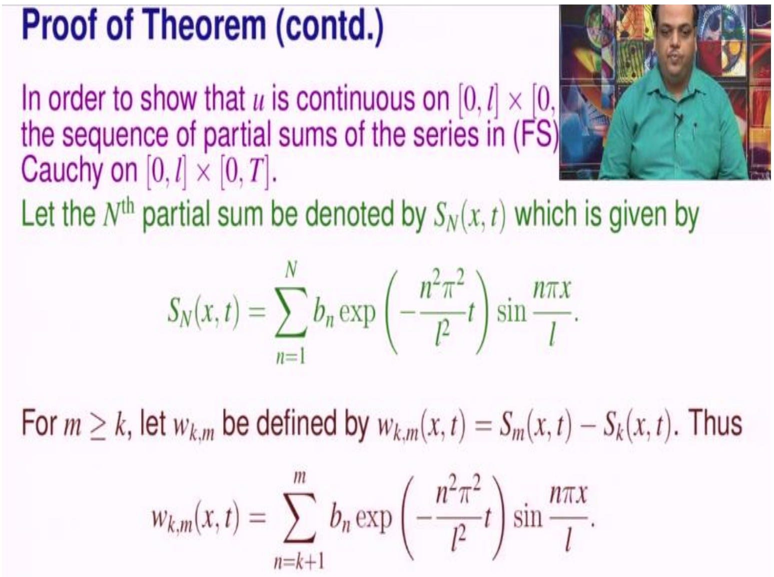
We have this and this series as stated here converges and hence this converges uniformly in t 0, T for every t 0 and hence it defines a continuous function for every t 0. Therefore, it defines a continuous function on open 0, T as well; 0, l cross open 0, T.

(Refer Slide Time: 21:14)



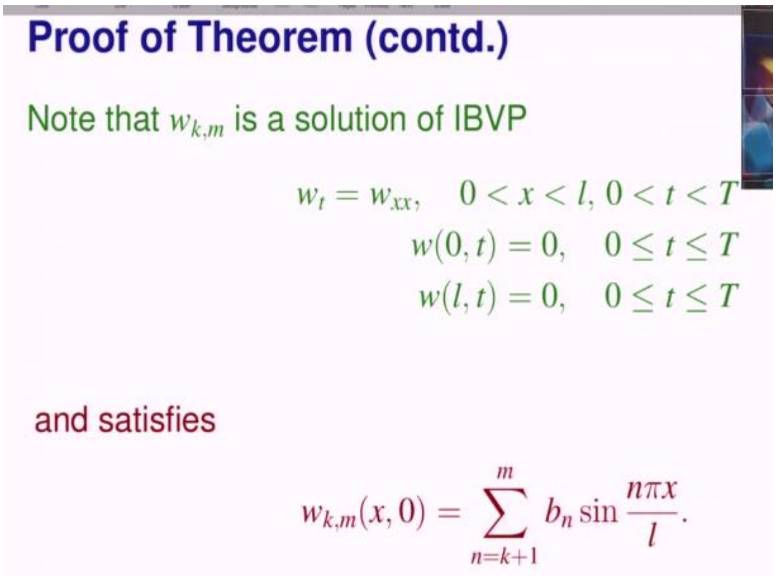
So, having justified the differentiation of the infinite series in the formal solution, it is easy to check that u satisfies the heat equation on this domain 0, l cross 0, T. It remains to show that u satisfies the initial boundary conditions.

(Refer Slide Time: 21:30)



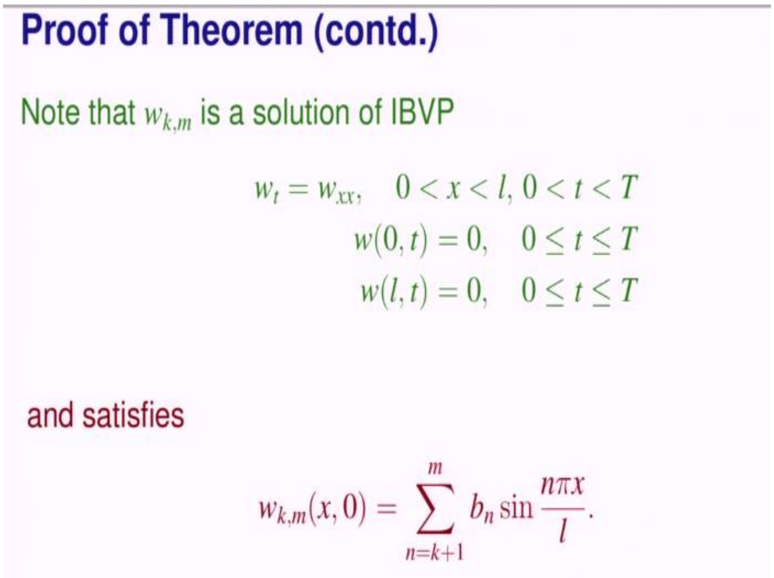
So, in order to show that u is continuous on 0, l cross 0, T we show that the sequence of partial sums is uniformly Cauchy in 0, l cross 0, T. So, let the nth partial sum be denoted by S N of x, t which is given by this. And for m greater than or equal to k, let w k, m be defined as S m – S k. Therefore, w k, m has this expression. Each term in this finite sum actually solves heat equation.

(Refer Slide Time: 22:09)



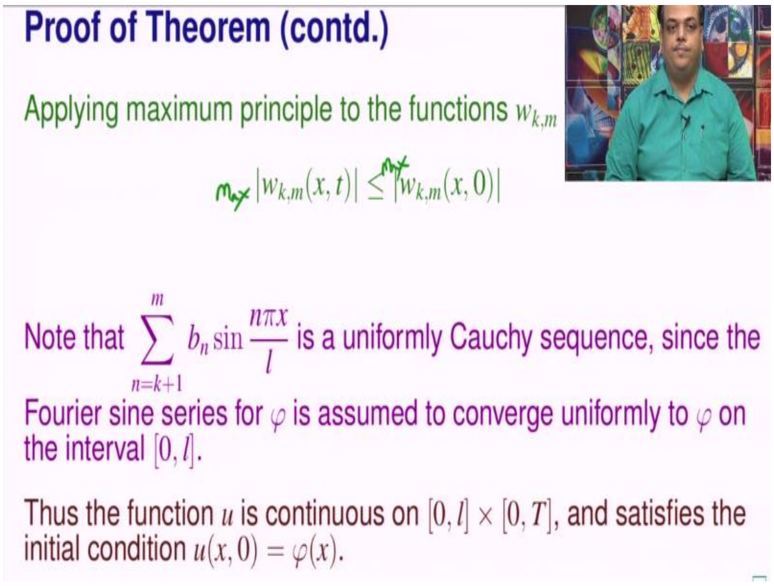
So, w k, m is a solution to the heat equation and it satisfies this boundary data when x = 0 or x = l, let us go back, yeah. When x = 0 sine 0 0, so therefore this is 0. When x = l, what we have is sine n pi that is one again 0, therefore we have 0. So, these conditions are satisfied and then w k, m of x, 0 is simply this.

(Refer Slide Time: 22:41)



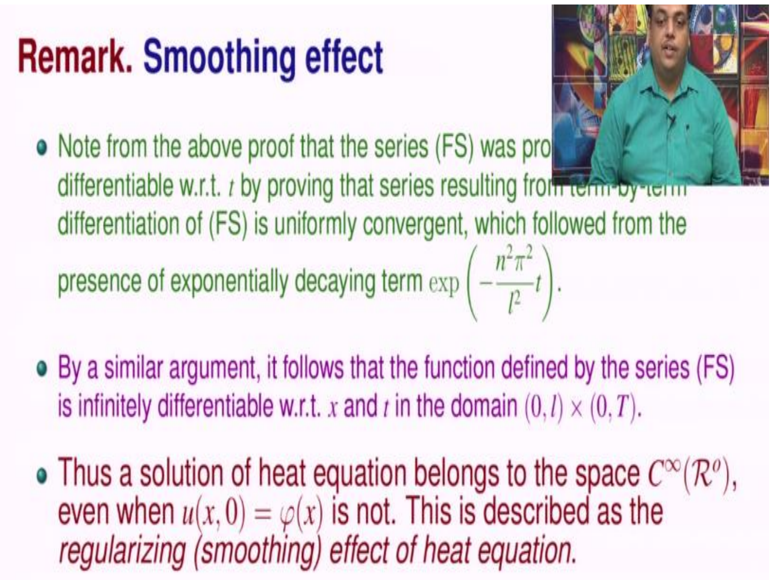
So, applying maximum principle to the function w k, m which is a solution to the heat equation and on the parabolic boundary it takes 0 on two parts of it and on the third part of it, it is this.

(Refer Slide Time: 23:00)



So, applying maximum principle to the function w k, m we get that the supremum or the maximum is less than or equal to the maximum here w k, m x, 0. So, note that this is partial sums of a uniformly convergent series. Therefore, this is uniformly Cauchy sequence. This is S m – S k that is why it is a uniformly Cauchy sequence. Since the Fourier series for phi is assumed to converge uniformly to phi on the interval 0, l. Thus does the function us is continuous on 0, l cross 0, T and satisfies the initial condition u x, 0 = phi x.

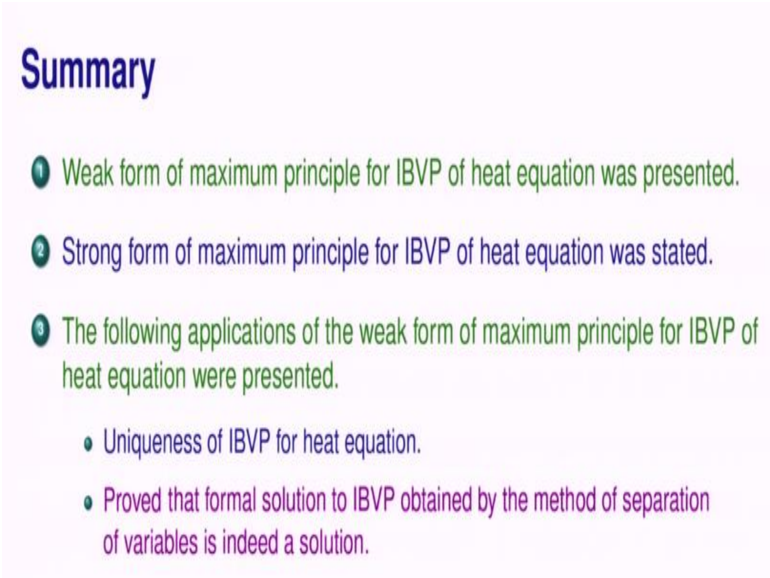
(Refer Slide Time: 23:50)



Remark, the smoothing effect: Note from the above proof that the Fourier series was proved to be differentiable with respect to t by proving that the series resulting from term-by-term differentiation is uniformly convergent, which of course followed from the presence of exponentially decaying term here. By similar argument, it follows that the function defined by the series is infinitely differentiable with respect to x and t in the domain 0, l cross 0, T.

Thus a solution of heat equation belongs to C infinity R in the interior of a rectangle even when u of x, 0 = phi x is not; phi maybe just continuous, but u is C infinity in a rectangle. This is described as the regularizing effect or smoothing effect of the heat equation.

(Refer Slide Time: 24:48)



Let us summarize what we did in this lecture. Weak form of the maximum principle for IBVP of heat equation was presented. And the strong form of the maximum principle was stated. And the following applications of the weak form of maximum principle were presented. Uniqueness for the IBVP for heat equation was proved and we have proved that the formal solution to IBVP which was obtained by method of separation of variables is indeed a solution. Thank you.