Partial Differential Equations
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Lecture – 56
Cauchy Problem for Heat Equation - 2

Welcome. We resumed our study of Cauchy problem for heat equation that we have started in lecture 7.1. We ended lecture 7.1 by defining what is known as heat kernel or the fundamental solution associated to heat equation. Using that, we are going to express a solution of the Cauchy problem both for homogeneous heat equation and for nonhomogeneous heat equation. So, the outline of this lecture is as follows.

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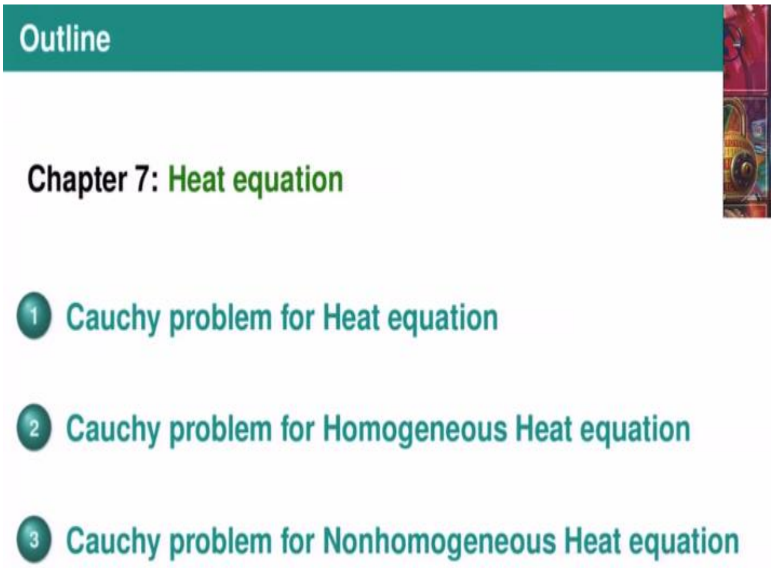

We briefly recall the Cauchy problem for heat equation that we have discussed in the last lecture that is lecture 7.1 and then we solve the Cauchy problem for homogeneous heat equation and then go on to solve for the nonhomogeneous heat equation.

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Given functions f, φ, Cauchy problem

$$u_t - u_{xx} = f(x,t),\ x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \varphi(x), \quad \text{for } x \in \mathbb{R}.$$

So, Cauchy problem consists of solving the heat equation u t – u xx equals to f with the right hand side the nonhomogeneous heat equation with prescribed initial conditions u of x, 0 = phi of x. The functions f and phi are given and we have to find a function u, which satisfies these two conditions.

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Notation: $C^{2,1}(\mathbb{R} \times (0,\infty))$

The function space $C^{2,1}(\mathbb{R} \times (0,\infty))$ consists of ALL functions

$$u : \mathbb{R} \times (0,\infty) \to \mathbb{R}$$

such that

- u is continuous on $\mathbb{R} \times (0,\infty)$
- u_t, u_x are continuous on $\mathbb{R} \times (0,\infty)$ *i.e.*, $u \in C^1(\mathbb{R} \times (0,\infty))$.
- u_{xx} is continuous on $\mathbb{R} \times (0,\infty)$

In $C^{2,1}$, 2 and 1 stand for the number of derivatives w.r.t. x and t variables respectively.

We have introduced this notation of C 2,1 R cross 0, infinity. C 2,1 stands for two derivatives with respect to x and one derivative with respect to t and it consists of all the functions defined on the domain R cross 0, infinity such that u, u t, u x, u xx all continuous on R cross 0, infinity.

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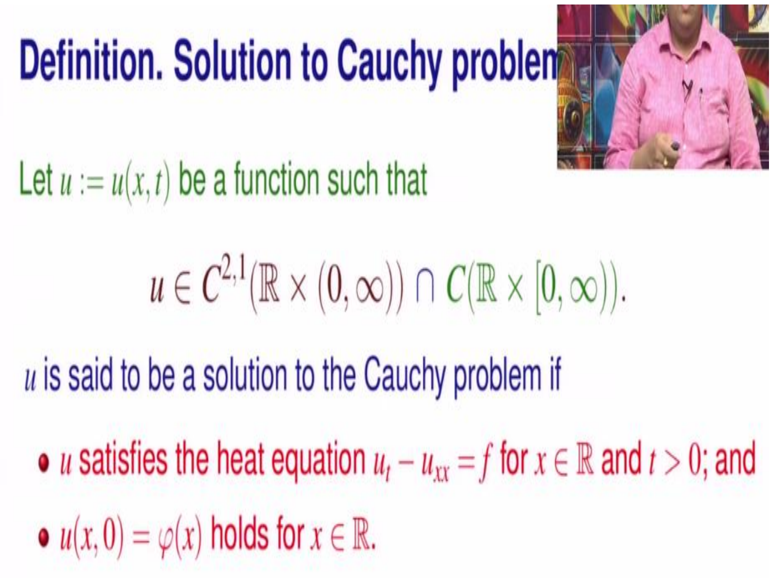

So, solution to the Cauchy problem was defined as follows. A function u which is C 2,1 of R cross 0, infinity intersection continuous function on R cross closed 0, infinity is a solution to the Cauchy problem if it satisfies the heat equation with the right hand side f and u x, 0 = phi of x.

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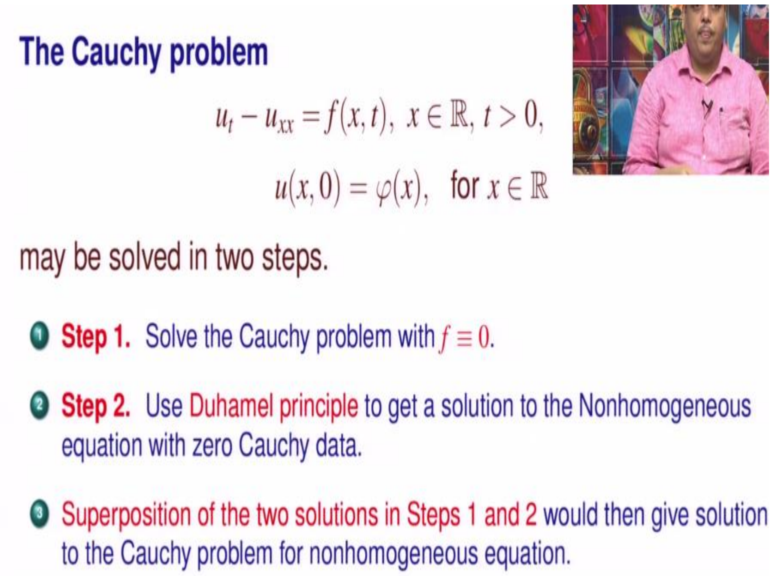

And the Cauchy problem for the nonhomogeneous equation can be solved in two steps. First, you solve the Cauchy problem for the homogeneous heat equation that is your set f = 0 So, you solve u t – u xx = 0; u x, 0 = phi x. In the second step we use Duhamel principle to get a solution to the nonhomogeneous equation namely u t – u xx = f x, t with 0 Cauchy data that is u of x, 0 = 0. And superposition of the two solutions that we obtained in step 1 and step 2 will give you a solution to the problem that we want to solve.

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Cauchy problem for Homogeneous Heat equation

$$u_t - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \varphi(x),\ \text{ for } x \in \mathbb{R}.$$

So, first we start looking at the homogeneous heat equation and the Cauchy problem for that.

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Solving Heat equation using a similarity transformation

Key observation: For every $a > 0$, Heat equation is invariant under the change of coordinates

$$z := ax,\ \tau := a^2 t.$$

- $$\frac{z^2}{\tau} = \frac{x^2}{t}.$$
- We look for solutions to Heat equation respecting the above symmetries.

$$v := v(z),\ u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$$

We observed that under this change of variables z = ax, tau = a square t, the heat equation remains invariant and therefore we thought of this variable x by root t here. So, we look for the solution u of x, t as v of x by root t where v is a function of one variable.

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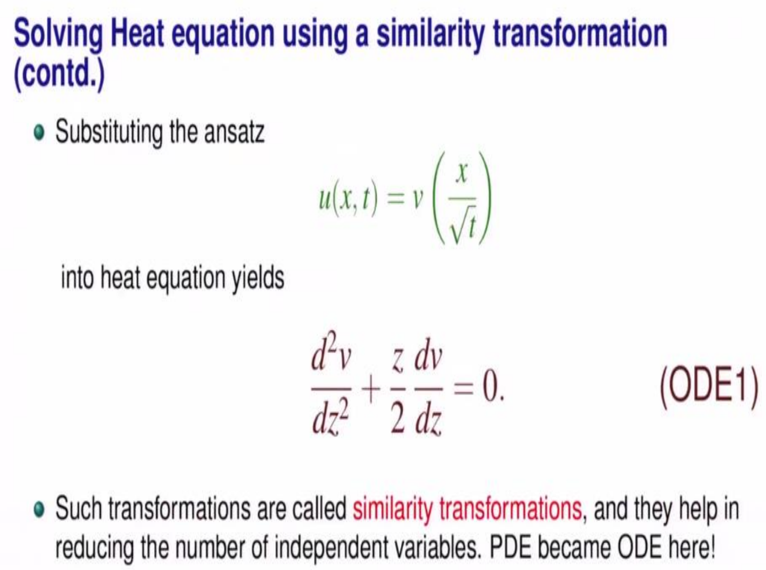

So, when we substitute this ansatz into the heat equation, we end up getting a second order ODE.

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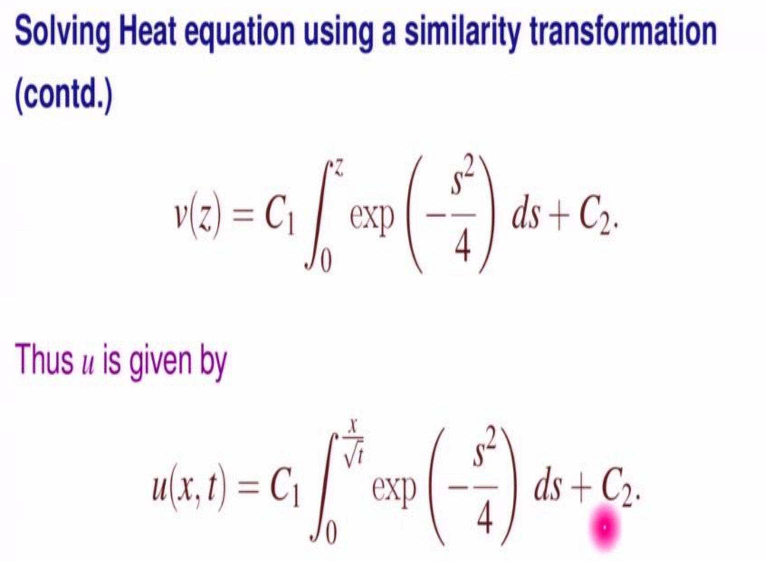

Which can be totally solved and this is a general solution of the ODE and therefore y of x, t is expressed as this, C 1 and C 2 are constants.

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Solving Heat equation using a similarity transformation (contd.)

$$u(x,t) = C_1 \int_0^{\frac{x}{\sqrt{t}}} \exp\left(-\frac{s^2}{4}\right) ds + C_2.$$

Note that

$$u(x,0) := \lim_{t \to 0+} u(x,t) = \begin{cases} -C_1\sqrt{\pi} + C_2 & \text{if } x < 0, \\ C_1\sqrt{\pi} + C_2 & \text{if } x > 0. \end{cases}$$

$u := u(x,t)$ is smooth for $t > 0$, however, $u(x,0)$ has a jump discontinuity. No way to obtain a solution to Cauchy problem!

Then we thought we can get the solution to the Cauchy problem by specializing the C 1, C 2 we will get the solution, but that was not to be the case. And we have found out that u of x, 0 is given by –C 1 root pi + C 2 for x negative and C 1 root pi + C 2 for x positive. So, there is no way that we can solve the given Cauchy problem using this analysis that we have done so far. And then we observed that this u has a jump assuming that C 1 is nonzero.

If C 1 is 0, then u is constant, right, u of x, 0 C 2 and u of x, t is C 2, so it is constant. So constant solutions are known to be solutions, in any case they do not solve the Cauchy problem. So, if C 1 is nonzero, there is a jump that we observed. The moment we observe a jump in the function, the derivative will pick up a Dirac delta that observation was made.

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In Lecture 7.1, We observed that

$$u_x(x,t)$$

would be useful to solve Cauchy problem.

$$u_x(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right). \qquad \text{(FSol1)}$$

We briefly recall reasons for the same.

So in lecture 7.1, we observed that u x of x, t would be useful to solve the Cauchy problem. What is u x of x, t? It is 1 by root 4 pi t exponential – x square by 4 t is what we have obtained. We briefly recalled the reasons why u x of x, t is expected to be useful in solving the Cauchy problem.

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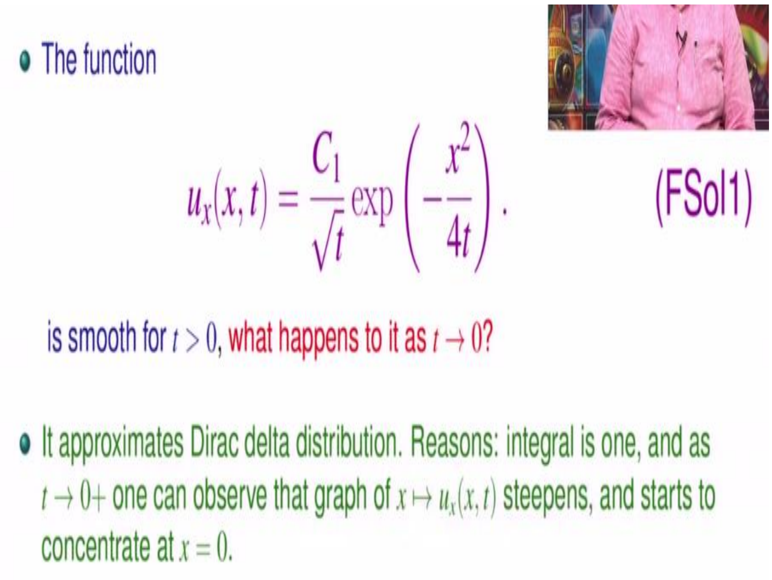

So, u x of x, t has this expression, where C 1 = 1 by root 4 pi which I have not written here. It is smooth function for t positive, but what happens to t goes to 0 that approximates Dirac delta function. The reasons were explained that the integral is 1 and as t goes to 0, we can observe that the graph of u x steepens and starts to concentrate at x = 0.

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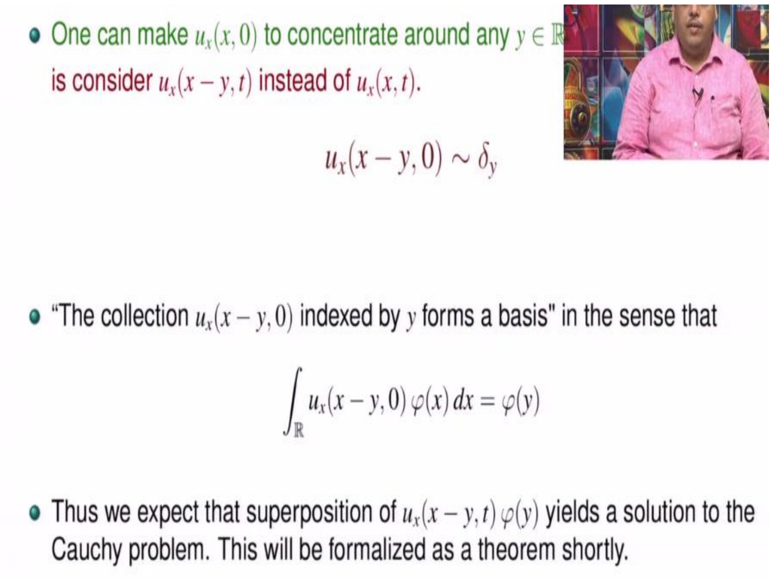

Then we said that the concentration can be made to happen at any point y in R instead of 0 by translation that is if you look at u x of x – y, t instead of u x of x, t, then u x of x – y, 0 is like delta y. So, the collection u x of x – y, 0 indexed by y forms a basis in the sense of this equation. Therefore, we expected that the superposition of u x of x – y, t into phi y yields a solution to the Cauchy problem. This will be formulated as a theorem very shortly.

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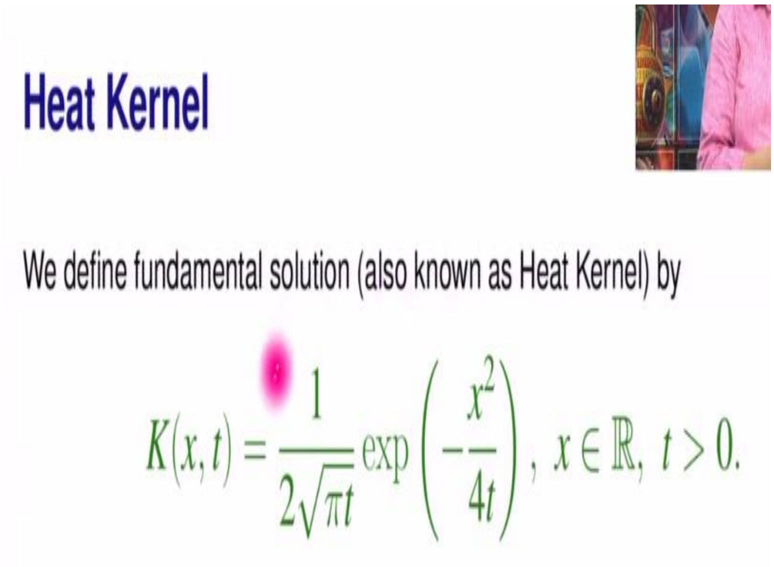

So, we defined what is called heat kernel and the fundamental solution by this formula K x, t = 1 by 2 root pi t exponential of – x square by 4 t. This is in fact u x of x, t that we have obtained.

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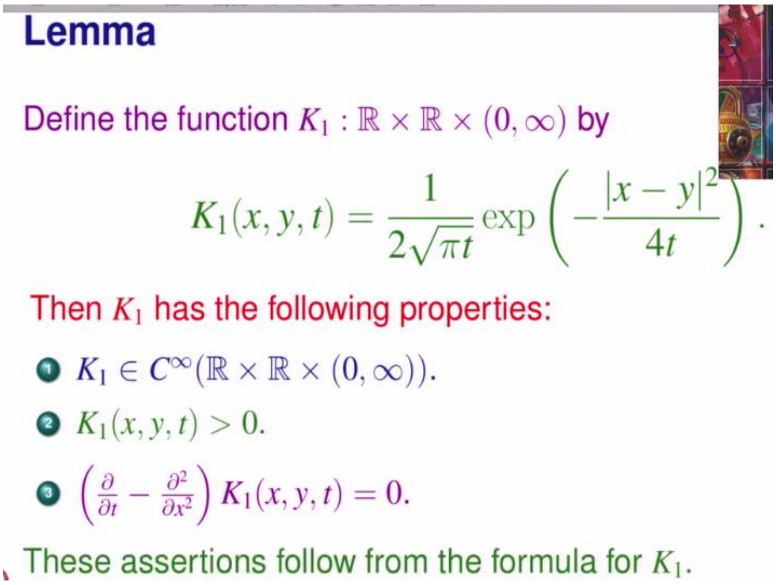

So, Lemma: Define the function K 1 from R cross R cross 0, infinity to R by K 1 of x, y, t = 1 by 2 root pi t exponential of minus mod x – y square by 4 t. This is nothing but K of x – y, t. So, then K 1 has the following properties, what are they? K 1 is a C infinity function in this domain, very obvious because this is defined to exponential function and inside you have – x

– y whole square by 4 t. So, this is a polynomial divided by polynomial and t is not 0 in the domain.

Therefore, this is always a smooth function, so C infinity. And it is positive function because defined through exponential, so this is always strictly greater than 0. And more importantly dou by dou t – dou square by dou x square of K 1 x, y, t is 0. Why is this true? What is K 1 of x, y, t? It is K of x – y, t. Yes, but what is K of x – y, t? u x of x – minus y, t. What is u? u is solving heat equation that is how we have found you through v.

So, u x being a derivative of the solution it is still a solution that is why you get this one that dou by dou t – dou 2 by dou x square K 1 of x, y, t = 0 and this is of course only a translation by y. So, we are using here some of the exercises that we have stated in lecture 7.1. Whenever u is a solution, u of x – y, t is a solution for every y. Whenever u is a solution u x of x, t is also a solution. So, these are assertions follow from the formula for K 1. I have already explained it.

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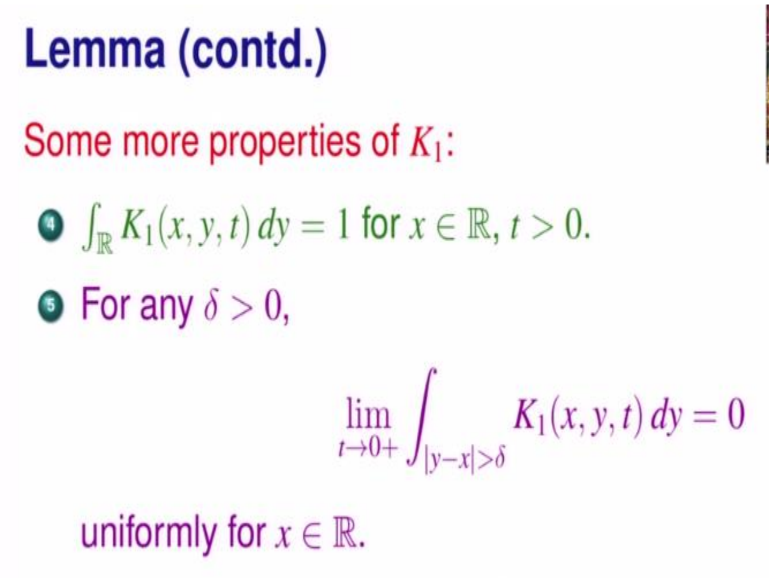

Some more properties of K 1: Integral of K 1 is 1. For any delta positive limit t goes to 0+, integral over mod y – x strictly greater than delta of K 1 x, y, t dy is 0 uniformly for x in R.

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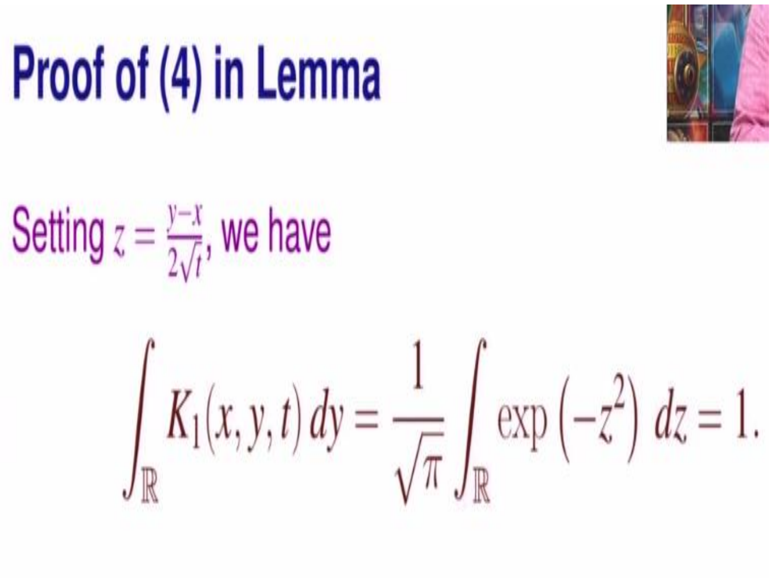

So, proof of 4 in the lemma. So, setting z = y – x by 2 root t. We have integral of K 1 equal to this quantity and that is equal to 1 because this integral is a standard integral, this integral is actually root pi.

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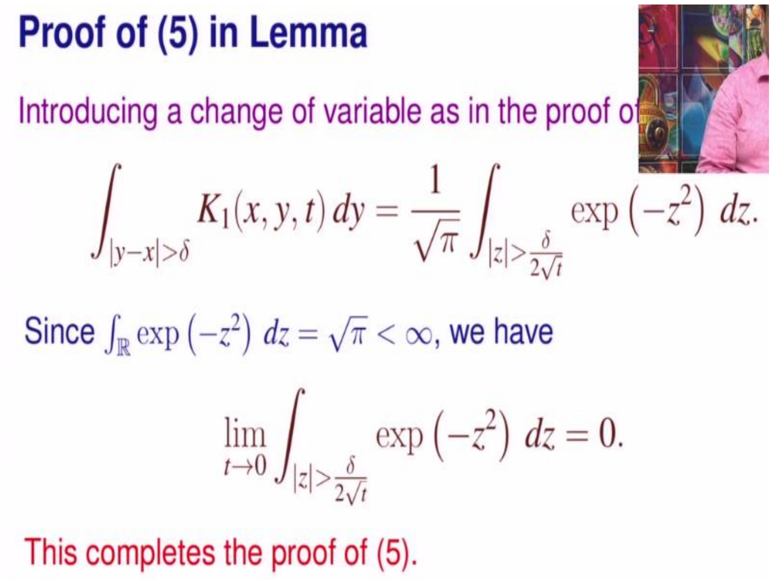

So, proof of 5. Introducing a change of variable exactly as in the proof of 4, we get integral of K 1 on this domain to be this integral. Now, because this integral is root pi and it is finite integral as t goes to 0, limit of this is 0. So, this completes the proof of 5.

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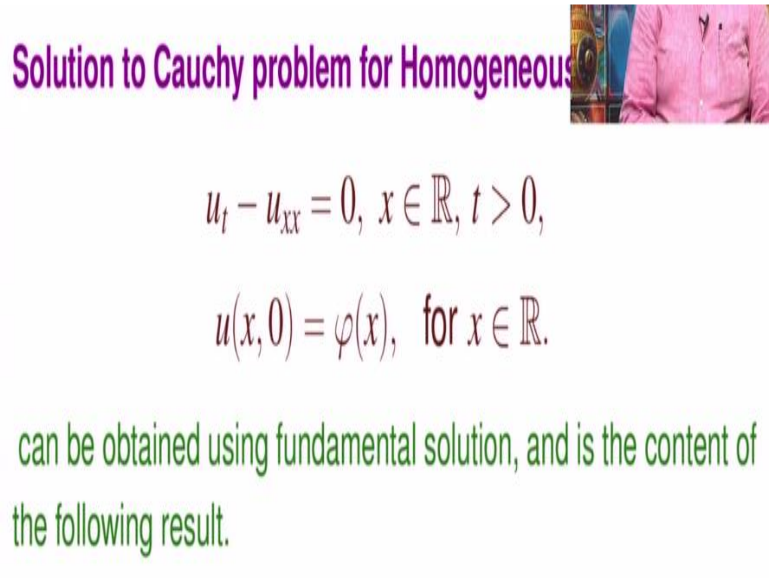

So solution to Cauchy problem for homogeneous heat equation. This is a Cauchy problem for the homogeneous heat equation, can be obtained using the fundamental solution or the heat kernel of K 1. K 1 of x, y, t is nothing but K of x – y, t and that is the content of the following result.

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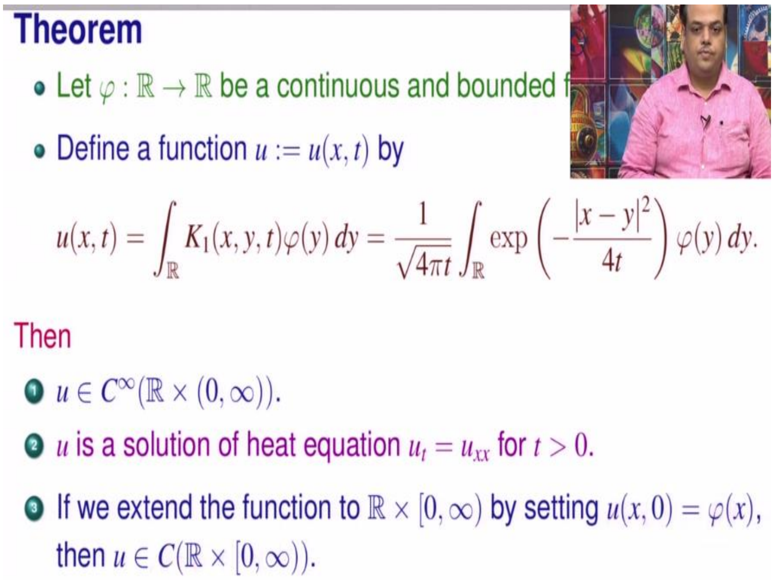

Theorem: Let phi be a continuous and bounded function. Define function u by this integral substituting the expression for K 1, it is this. So u of x, t is proposed to be this integral. Then u is a C infinity function on R cross 0, infinity; u is a solution to the heat equation u t – u xx = 0. If we extend the function by setting u of x, 0 = phi x, then it is a continuous function on R cross closed 0, infinity.

This says that the initial condition is realized by this function which is only defined for t positive, it can be extended continuously up to t = 0 that is why the closed 0 here and in a continuous manner that is why C here and it assumes value phi x.

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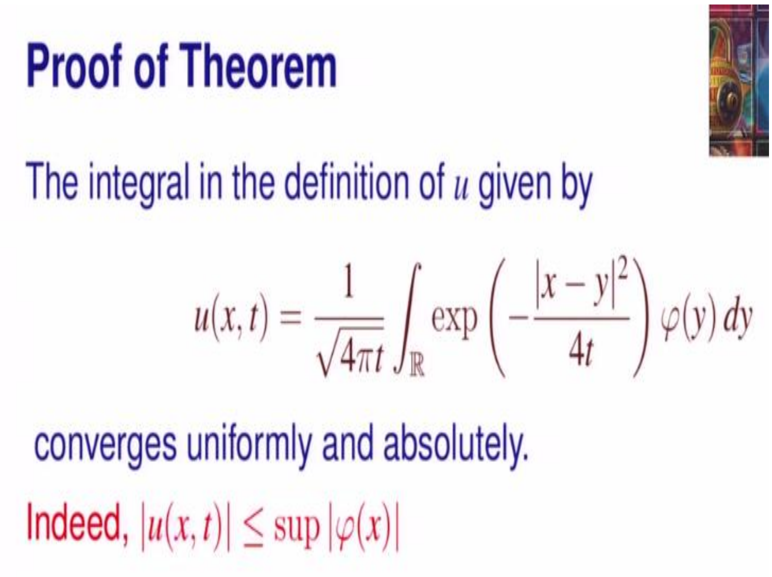

So, the integral in the definition of u which is here converges uniformly and absolutely. Indeed modulus of u of x, t; if you look modulus of this integral R modulus of this, it is a positive function. So, mod phi y and phi is bounded, therefore that comes out as the supremum and what remains is 1 by root 4 pi t integral exponential of this quantity which is 1. So, phi is bounded that is why supremum is finite, then u is also bounded.

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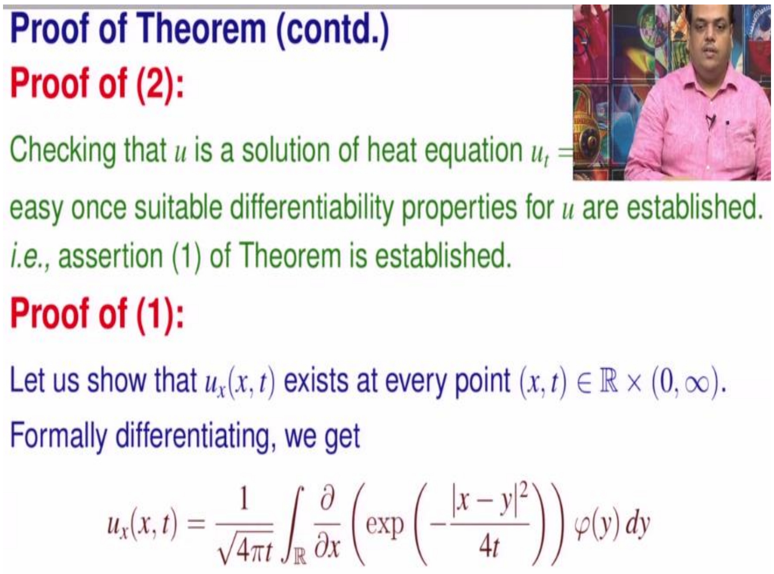

So, checking that u is a solution to heat equation is easy once suitable differentiability properties for you are established that is assertion 1 is established. Of course, assertion 1 is C infinity. What we need here is only that u is in C 2,1. So, proof of 1: Let us show that u x exists. Remaining derivatives u x, u t, u xx, u xt, u tt, and all hierarch derivatives showing that the exist is similar. So, we will show that u x of x, t exists at every point in R cross 0, infinity.

So, formally differentiating we get this. If it is differentiable, u of x, t is differentiable with respect to x, then u x of x, t must be this that is idea. So, what we will now show is this integral converges absolutely and uniformly and hence differentiation and the integral sign is valid and then we have this expression that is idea.

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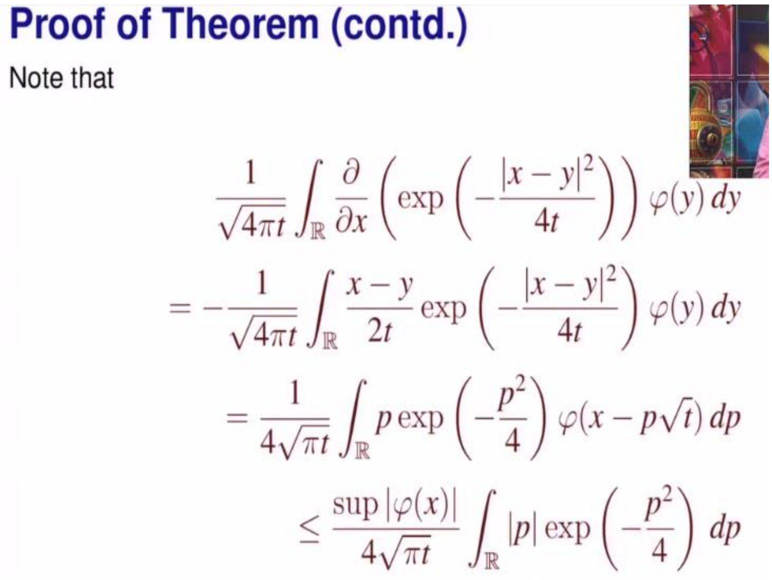

So, note that this expression I am going to differentiate with respect to x and write down, so is this. Now, we do a change a variable x – y by root t = p, then this integral will turn out to be this integral. Now, this is less than or equal to, this phi comes out as supremum, rest stay as it is and you have a mod p instead of p and this integral is finite.

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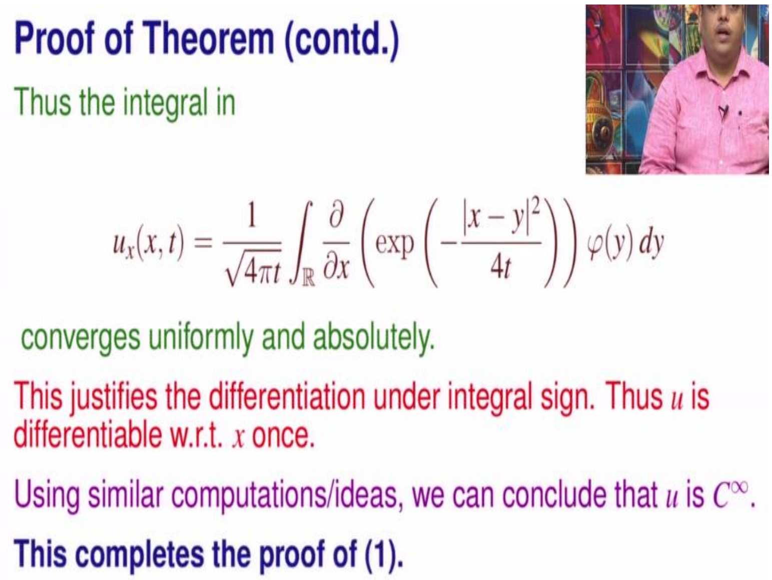

So, therefore, the integral in this converges uniformly and absolutely and this justifies the differentiation under integral sign. Thus, u is differentiable with respect to x once. Using similar computations and ideas, we can conclude that u is C infinity. This completes the proof of 1.

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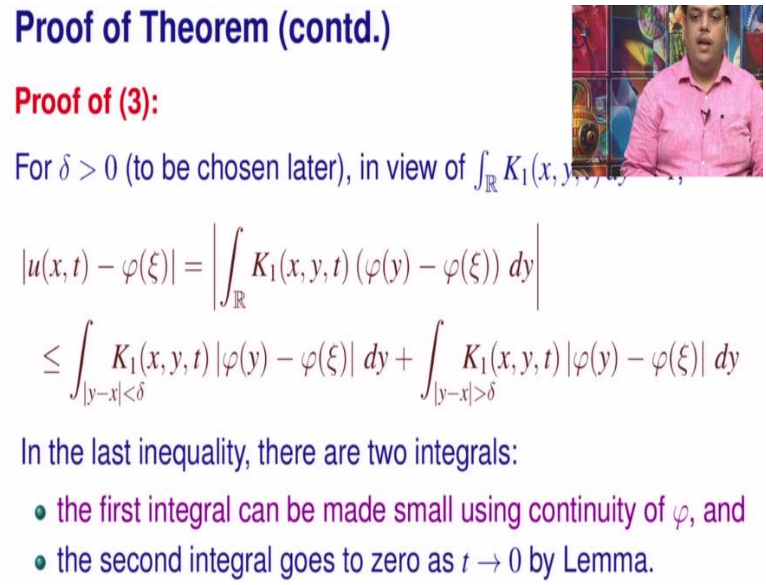

Now, what remains is the proof of 3 that is initial condition is achieved that is u of x, 0 makes sense and that is phi of x. So, this delta we will choose later. In view of integral being 1 of K 1 we can write this u x, t is the first term here, integral K 1 phi y dy, phi of psi, psi some point okay, phi of psi into integral K 1, integral K 1 is 1, you multiply that with phi psi. So, this equal to this because of this.

Now, that is less than or equal to mod y – x less than delta, in fact this integral we write it on mod of y – x less than delta and mode y – x greater than delta and then use triangle inequality. So, in this inequality there are two integrals on the RHS; one is this, one is this. So, the first integral can be made small using continuity of phi at the point psi and the second integral goes to 0 as t goes to 0 by Lemma.

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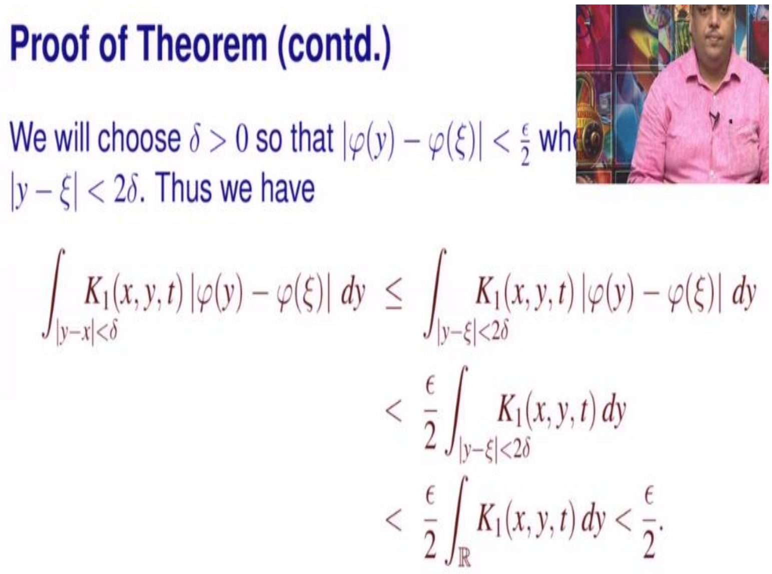

So, we will choose delta so that mod phi y – phi psi is less than epsilon by 2 whenever y – psi is less than 2 delta. This is precisely consequence of the continuity of the function phi at the point psi. Thus, we have this integral less than or equal to this integral because this is a bigger, this is contained in this, mod y – x less than delta is contained in mod y – psi less than 2 delta. And now by contiguity where this is valid, this quantity less than epsilon by 2 is valid, therefore we get this.

And this integral on sum set is always less than or equal to the integral in the bigger set which we know to be 1. So, this is actually equal to epsilon by 2, I have written less than but it should be equal. So, what we have what is this less than epsilon by 2.

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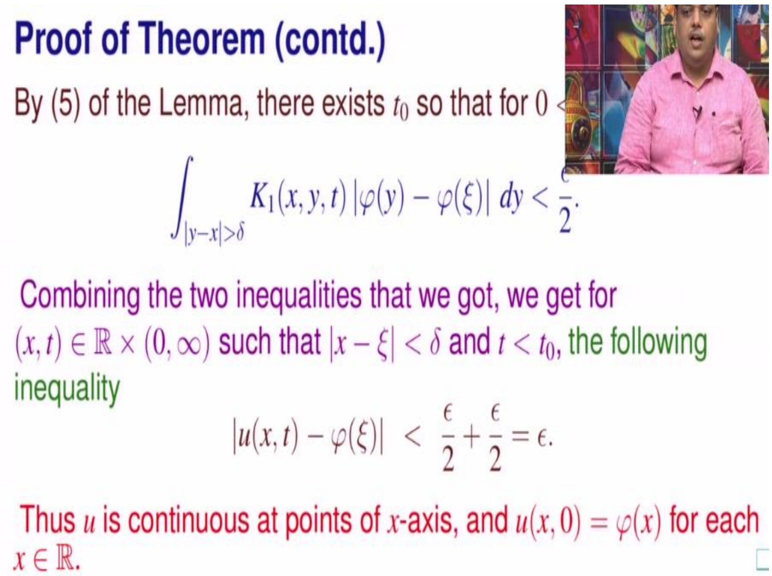

Now, by 5 of the Lemma there exists t 0 so that for t less than t 0 we have this quantity less than epsilon by 2. Combining the two inequalities that we got, one is here and one is on the last slide, what we get is that whenever mod x – psi is less than delta and t is less than t 0, modulus of u x, t – phi psi is less than or equal to epsilon. So, thus u is continuous at points of x-axis and u x, 0 = phi x for each x in R.

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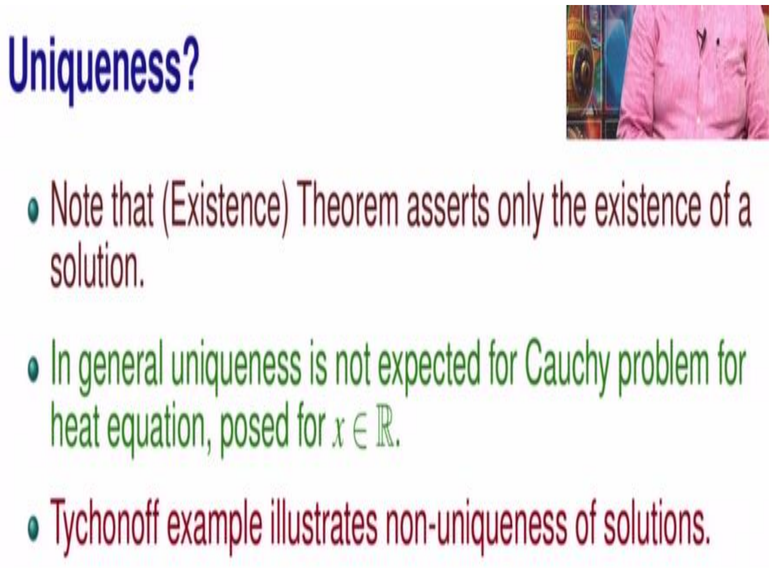

What about uniqueness? Note that the theorem that we have stated is existence theorem, we are proposing a formula and then we are saying that formula gives rise to a solution. So, essentially it is an existence by exhibition of the solution. So, in general uniqueness is not expected for Cauchy problem posed in x in R. So, Tychonoff example illustrates non-uniqueness of solutions.

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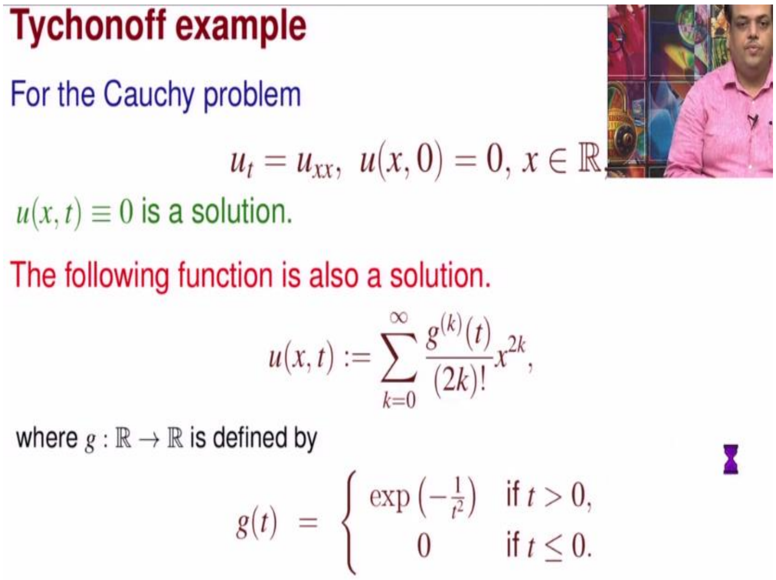

So, what is Tychonoff example. It is concerning the Cauchy problem u t = u xx that is the homogeneous heat equation and 0 initial data u x, 0 = 0. Of course, u x, t identically equal to 0 is obviously a solution to this, but unfortunately there is one more solution. The following function which is given as a series is also solution. What is g? g is defined by this formula. Now, the analysis to show that this solution is not easy, but it can be understood, it is elementary but not easy.

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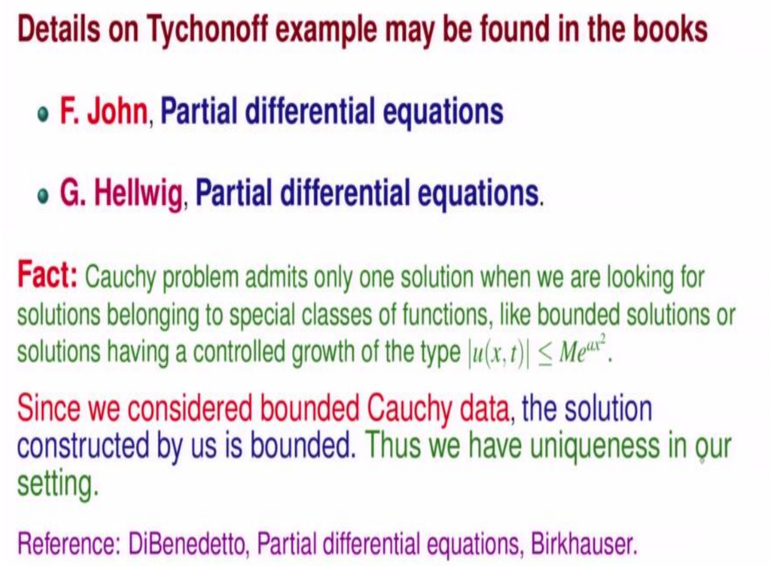

So, details on Tychonoff example, may be found in this book by Fritz John on partial differential equations or Hellwig partial differential equations and many more books, which has this. Now, we state a fact with our proof. So, Cauchy problem admits only one solution when we are looking for solutions belonging to special classes of function that is uniqueness can be regained, but we need to put conditions on the kind of function that we are looking at or admitting as solutions, so like bounded solutions.

So, if you are looking only for bounded solutions, then the solution is unique or solutions having a controlled growth of this type then also solution is unique okay. In our case, we are considering the bounded Cauchy data and therefore the solution is bounded, which we have constructed that solution is bounded. Therefore, solution is unique.

Reference, you can see this book DiBenedetto, PDE or you can look at again Fritz John and Hellwig or Evans. So, in the case of Tychonoff example, bounded solution is 0 solution, so that is the only bounded solution. The series solution is not bounded solution.

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Cauchy problem

$$u_t - u_{xx} = f(x,t),\ x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \varphi(x),\ \text{ for } x \in \mathbb{R}.$$

So, now let us discuss how to solve Cauchy problem for nonhomogeneous heat equation using Duhamel principle. We have already seen Duhamel principle when we were discussing wave equation and how we obtained solution to the nonhomogeneous wave equation and the corresponding initial value problem our initial boundary value problems. So, it is a very general principle.

We will apply that and obtain a solution to the nonhomogeneous heat equation and corresponding Cauchy problem. The Cauchy problem for the nonhomogeneous heat equation is u t – u xx = f of x, t and u of x, 0 = phi of x. We are going to apply Duhamel principle to obtain a solution to the Cauchy problem. Duhamel principal expresses the solution of the nonhomogeneous equation in terms of solutions of the Cauchy problem for homogeneous heat equation and that passes through what is known as source operator.

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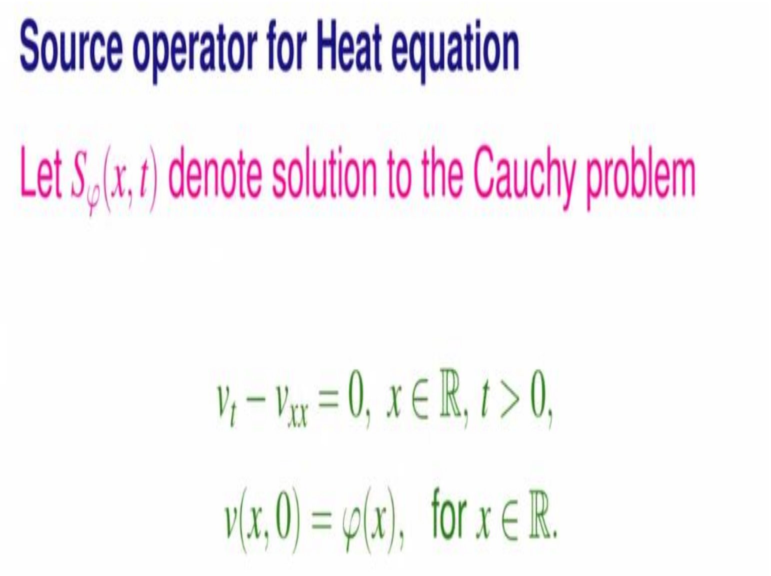

Source operator for heat equation. Let S phi of x, t denotes the solution to the Cauchy problem for the homogeneous heat equation with initial data given us phi x. Now, the question is, is it well defined? Answer is yes because given a function phi, which is bounded and continuous, there is exactly one bounded solution to this Cauchy problem. So, S phi denotes that solution.

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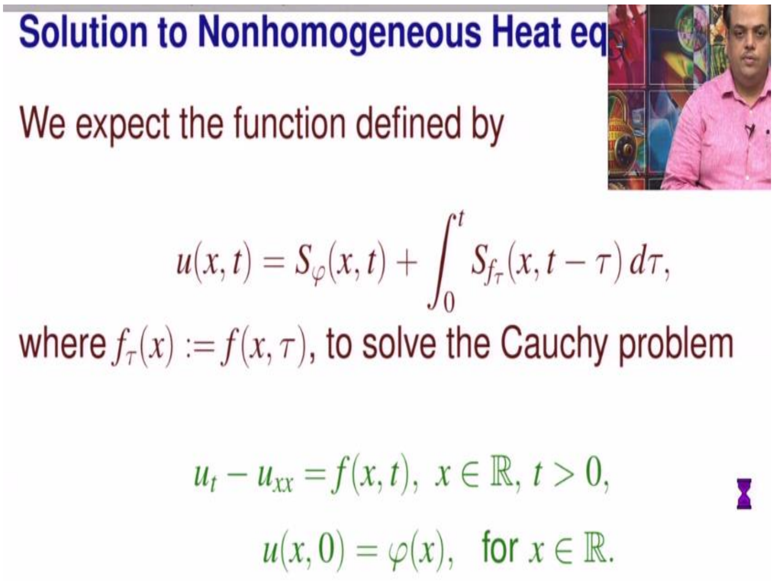

So, we expect the function defined by this where f tau of x is f of x, tau to solve the Cauchy problem for the nonhomogeneous heat equation.

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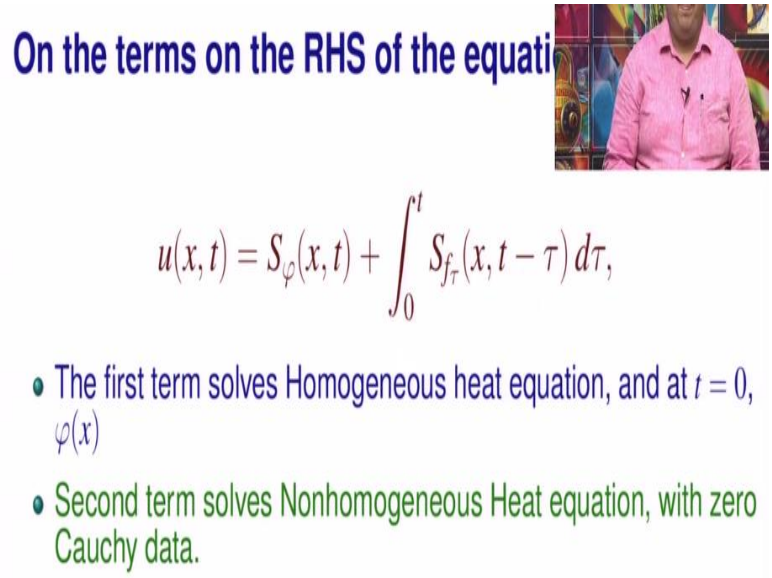

A comment on the terms that we see on the RHS of this equation, the proposed solution. The first term solves homogeneous heat equation by definition and a t = 0 it takes the value phi of x by the definition of the source operator. The second term solves the nonhomogeneous heat equation with the zero Cauchy data.

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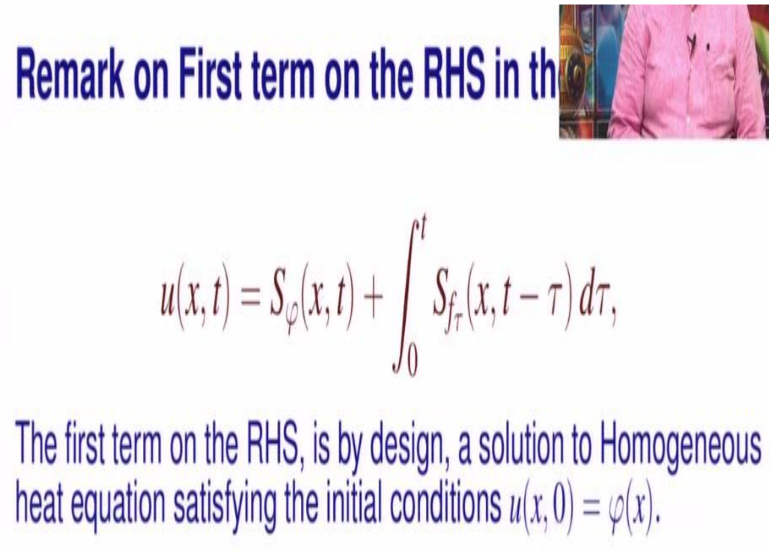

So, remark on the first term once again. The first term on the RHS is by design or with the construction of a source operator is a solution to the homogeneous heat equation and satisfies the initial conditions u of x, 0 = phi x for x in R.

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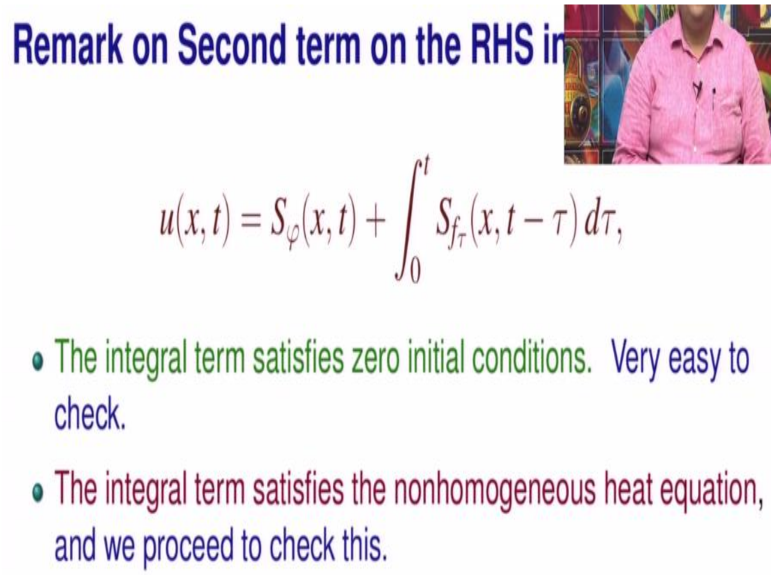

Remark on the second term on RHS in this formula, the integral term satisfies 0 initial conditions t = 0. Very easy to check. The integral term satisfies the nonhomogeneous heat equation and we proceed to check this.

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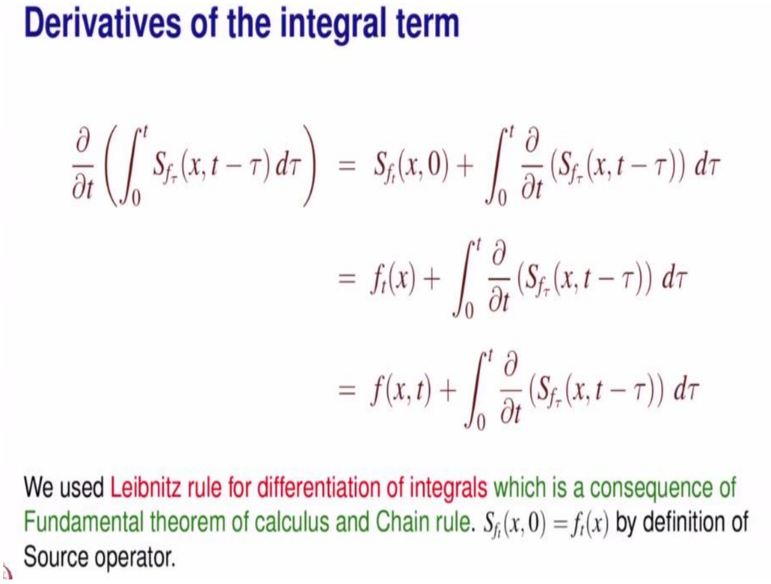

So, we have to compute the derivative of the integral term. So, what is dou by dou t of that, that is S f t of x, 0 + 0 to t, derivative goes inside the integral sign. S f t of x, 0 is f t of x by definition of the source operator plus we have this. So, we have f of x, t plus this time. So, we use Leibnitz rule for differentiation of integrals which is a consequence of Fundamental theorem of calculus and Chain rule.

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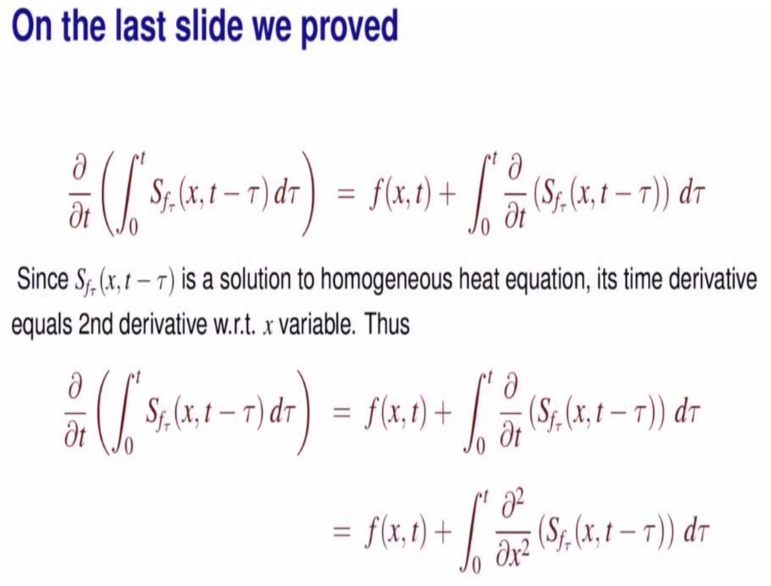

So, this is what we proved on the last slide for the first derivative of the integral. Now, since S f tau of x, t – tau is a solution to the homogeneous heat equation, its time derivative equals second derivative with respect to x variable. Therefore, we have this equal to this dou by dou t is dou 2 by dou x square.

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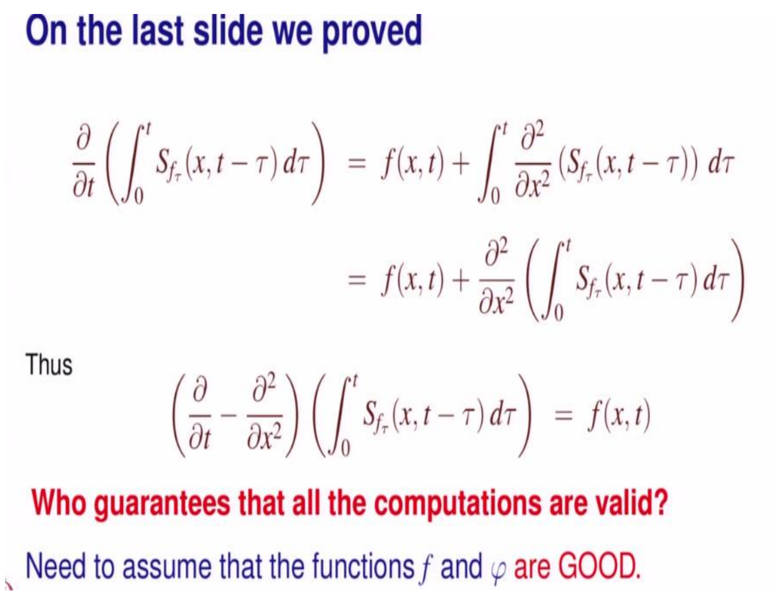

So, bring dou 2 by dou x square outside this integral. That means we are assuming that the differentiation and integral commute with each other and we get this. So, that means the integral term satisfies the nonhomogeneous heat equation. Who guarantees that all the computations that we have done are valid? We need to assume that the functions f and phi are good.

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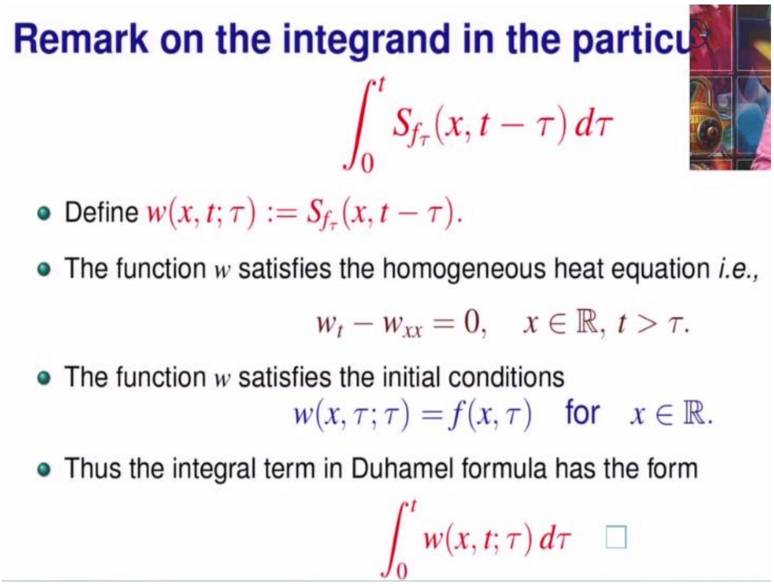

So, remark on the integrand in the particular solution, this one 0 to t S f tau x, t – tau say, so define w of x, t, tau to be the integrand of this. Then w satisfies homogeneous heat equation in this domain x belongs to R, t bigger than tau and it satisfies the initial conditions w of x, tau, tau is f of x, tau for x in R. Thus the integral term in the Duhamel formula has this expression 0 to t w, w of solution to the homogeneous heat equation with this as the initial data. Initial data is coming from the source term that is a general idea in Duhamel principle.

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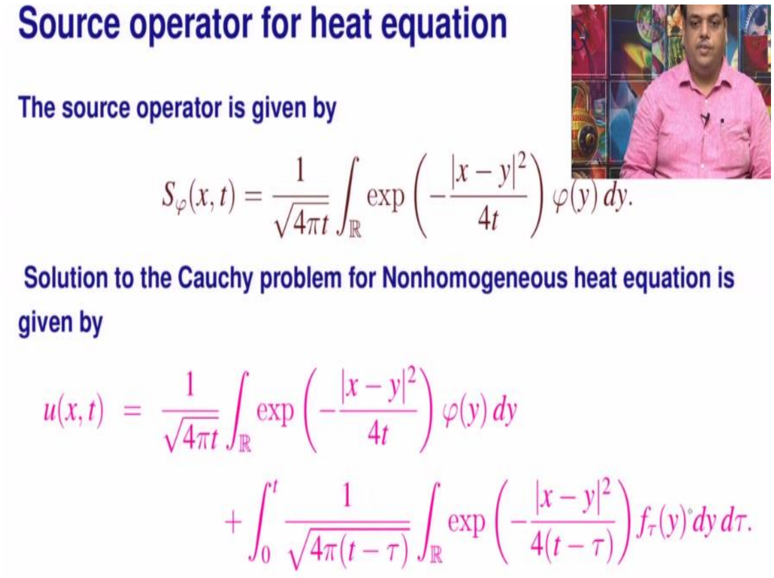

So, the source operator is given by this because this is expression for the solution of the homogeneous heat equation with Cauchy data phi. Therefore, u of x, t is now given by this which is a solution to the nonhomogeneous heat equation.

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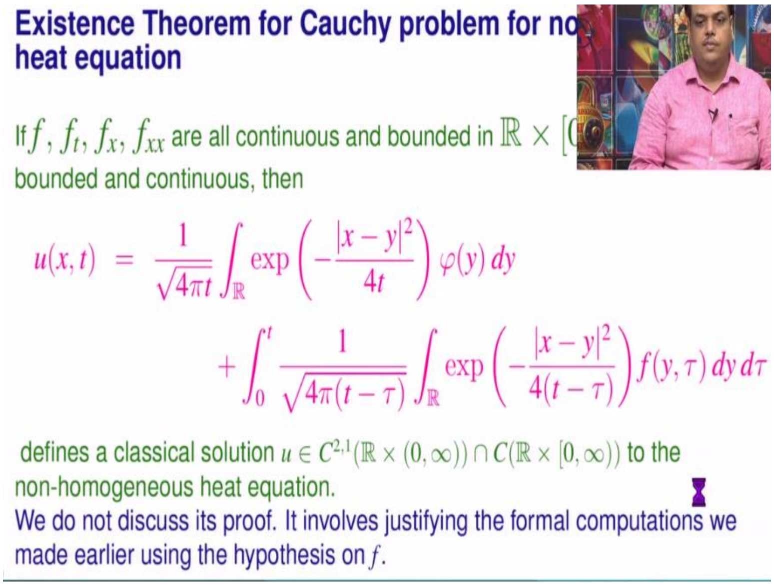

We can state a theorem with our proof. If f, f t, f x, f xx are all continuous and bounded in this domain R cross 0 to t for every t positive and of course phi is bounded and continuous, then u that we obtained using the Duhamel principle is indeed a solution. Defines a classical solution C 2,1 R cross 0, infinity and it is continuous up to t = 0 and it replaces the initial condition that part we have already checked, so we do not discuss it proof.

It involves justifying the formal computations we made earlier, namely which involved interchanging the differentiation and the integration that justification using the hypothesis on f. Thank you.