Partial Differential Equations
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Lecture-54
Tutorial 2 on Laplace Equation

Welcome to tutorial 2 on Laplace equation. Problem 1.

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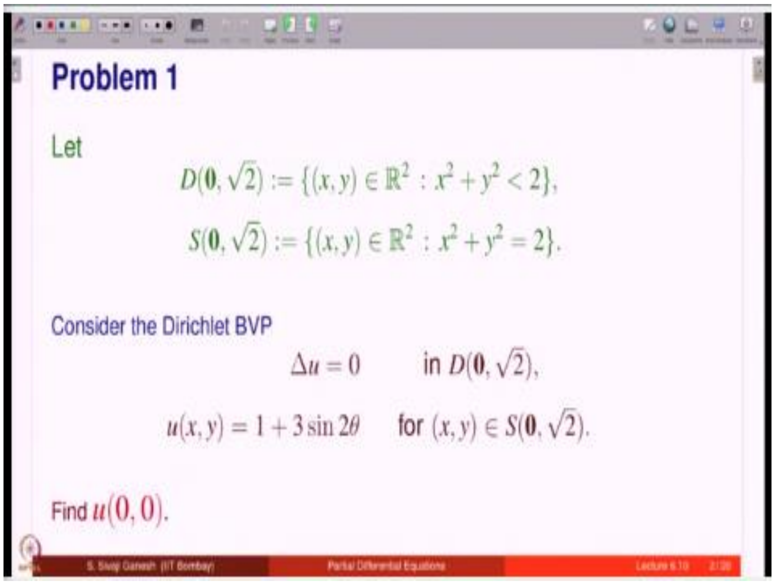

Let D of 0, root 2 denote the open disk of radius root 2 and S of 0 root 2 denotes the circle of radius root 2. Consider the Dirichlet boundary value problem Laplacian u = 0 in the disk D 0 root 2 and u of x, y = 1 + 3 sine 2 theta for x, y belongs to S of 0, root 2, find u 0, 0. If you look at this problem it looks somewhat strange because you are prescribing u of x, y and you are not giving a function of x, y here, u are giving a function of theta.

So, usually this is how people may pose sometimes but then you have to realize that because this is the disk theta is that angle which is coming into the picture and in the background there is r cos theta r sine theta kind of description for the circle x square + y square = r square. So, now how do you find this u of 0, 0? One way of course is to find out the solution of this problem and find the value at 0, 0 but do we have to do that much?

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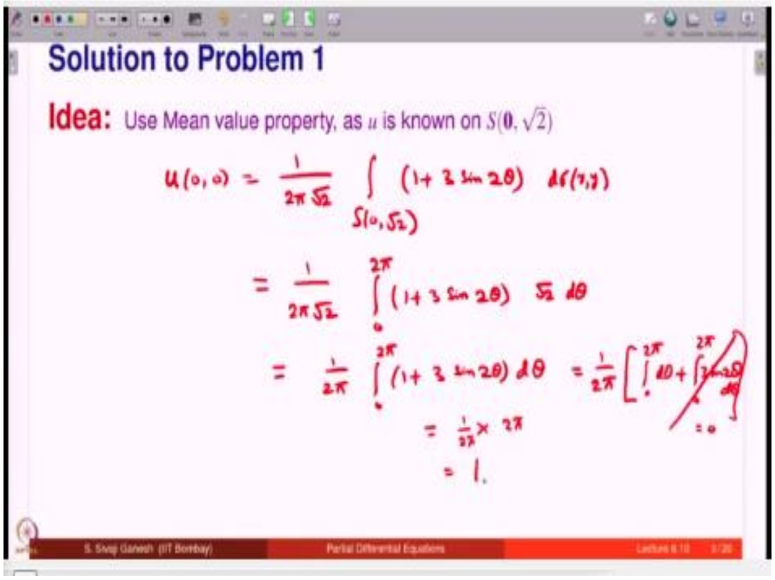

Idea is use mean value property, u is known on the circle you are asking what is the value of u at the centre and precidely the mean value property is what connects both the things. So, u of 0, 0 = 1 by 2 pi r, here r is root 2, S of 0 root 2 1 + 3 sine 2 theta into d sigma of x, y, we will expand this integral. So, that is equal to 1 by 2 pi root 2 integral from 0 to 2 pi because the angle theta runs from 0 to 2 pi and root 2 d theta.

Therefore there is nothing but 1 by 2 pi 0 to 2 pi 1 + 3 sine 2 theta d theta, so that is equal to 1 by 2 pi the integral can be split into 2 integrals, namely 0 to 2 pi d theta + 0 to 2 pi of 3 sine 2 theta d theta, this is 0 therefore we get 1 by 2 pi into 2 pi that is equal to 1.

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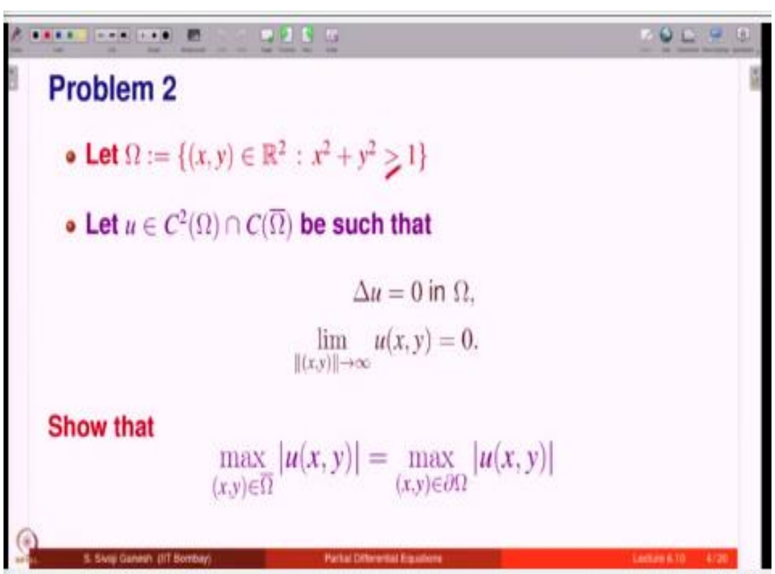

Let us move on to problem 2. Let omega be the complement of the closed unit disk centered at origin that is set of all x, y and R 2, such that x square + y square is greater than or equal to 1. Let u be C 2 omega intersection C of omega bar function such that Laplacian u is 0, that is u is harmonic in omega and such that limit of u as norm x, y goes to infinity is 0. Show that the maximum of modulus u x, y as x, y varies in omega bar is same as maximum of mod u x, y as x, y varies in boundary of omega.

In other words we are saying the maximum of mod u or omega bar is attained on the boundary rather is definitely attained on the boundary of omega. The requirement here suggests application of maximum principle, namely the weak maximum principle but we do not have that on unbounded domains and our omega is unbounded domain, let us see how to do that? Let us see how to cleverly apply the weak maximum principle and then show what is required in this problem?

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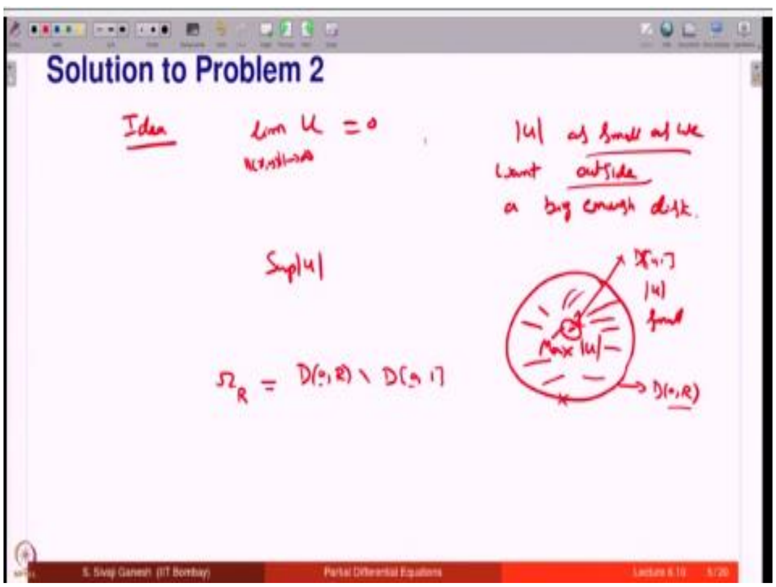

So, what is the idea to solve this problem? We are given that limit of u is 0 as x, y goes to infinity, this is 0. So, that means that one can make mod u as small as we want outside a big enough disks. That is for example this and you can make mod u is small here. So, that will tell us that the supremum of u is meaningful because u is now bounded and then that actually becomes maximum of u inside this region.

And that is when we plan to apply maximum principles and get the answer. Of course, recall this picture is not exactly correct because our domain is complement of the unit disk, now it is fine. So, the natural choice would then to consider this annular region and apply some kind of maximum principles. So, let us denote this circle by D 0, R this disk and this disk is anyway is D 0, 1.

So, look at this annual region D 0, R - D 0, 1 if you call this omega R, so we can apply your maximum principles and conclude that the maximum of mod u, it can be minimum of u or maximum of u that is attained on the boundary. And we conclude that it cannot be this boundary because we have made it very small. The R is chosen such that it is very small here therefore it has to be on this and that is what exactly what we want to show, we will go into the details now.

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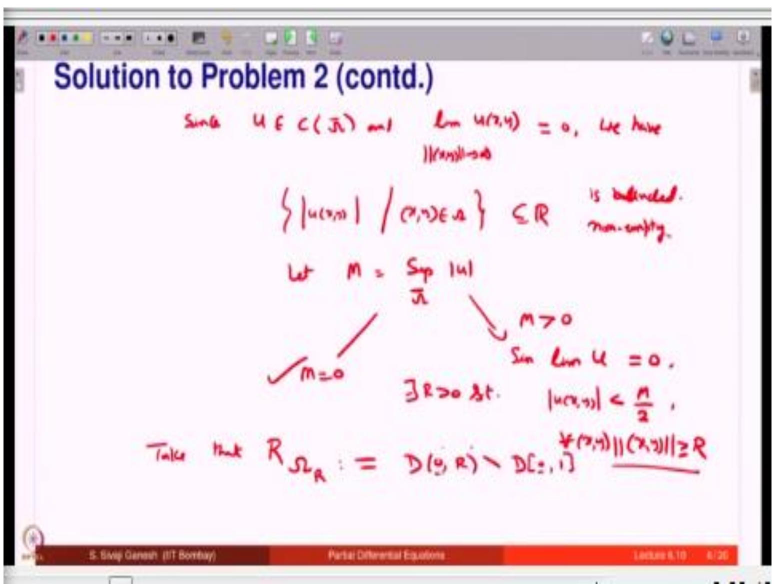

So, since u belongs to c of omega bar and the limit is given to be 0. We have looked at modulus of u of x, y as x, y vary in omega, of course this is a subset of R. This set is bounded set, of course non empty, therefore it has a supremum, so let that M denote the supremum. Now suppose M is 0 then there is nothing to prove, if M is 0 it means u itself is 0, therefore what we are supposed to prove holds automatically? There is nothing to prove.

So, we would as well assume that M is a positive quantity and show what is required? Namely, that maximum of mod u on omega closure is same as maximum of mod u on boundary of omega.

Boundary of omega, recall, is the unit circle. Now in this case since the limit of u is 0 there exists R positive such that modulus of u of x, y, we said it can be made as small as I please so I will make it less than M by 2.

And this will hold for every x, y with what property x, y is outside the disk of radius R, that is for every x, y such that norm x, y is bigger than or equal to R. So, now we are going to take that R which we have just chosen here by this constraint. And then we propose the annular region omega R let us denote this by disk of radius R - the disk with centre 0, radius 1.

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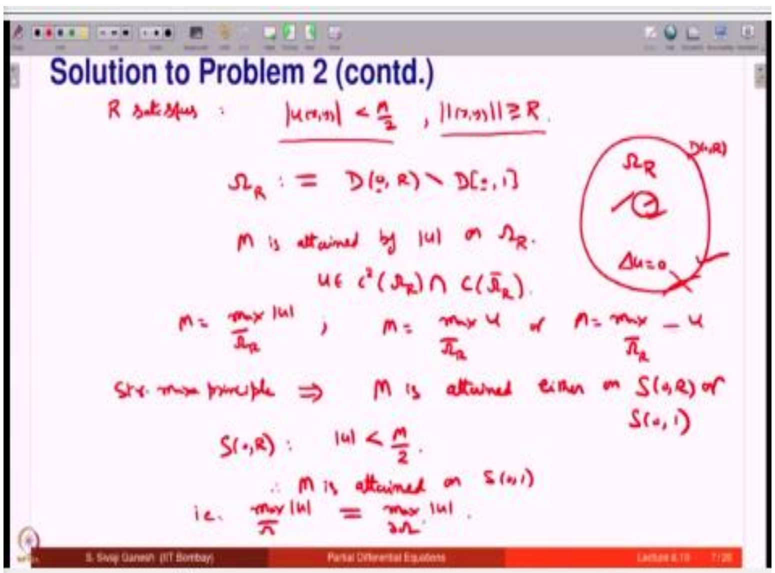

R satisfies, R has the following property that whenever norm x, y is bigger than equal to R mod u x, y is less than M by 2, this is the way we have chosen R. Now let me recall the notation again omega R let us denote this by disk of radius R - the disk with centre 0, radius 1. So, this is an annular region, so this is disk of radius 1, this is disk of radius R and this is my domain omega R. So, what are the properties? Now I have on this domain omega R, the value M is attained by mod u on omega R.

And Laplacian u = 0 in this angular region, because it is given to be 0 on omega, this is a subset of omega, therefore Laplacian is 0 .And u is c 2 of omega R intersection c of omega R closure, this is also true. Now what is this M? M is maximum of mod u over omega R bar also; therefore

M is either maximum of u on omega R bar or maximum of -u on omega R bar. Now by strong maximum principle, of course we have state only for strong maximum principle.

We can also state a strong minimum principle as a consequence we can deduce that from strong maximum principle. So, that tells us that whether it is minimum or maximum it is attained inside the domain it has to be a constant function. The function u that we are dealing with is not a constant function, we know that because the behavior of u at infinity. We assume that limit of u is 0 at infinity, so if u had been a constant function then the function should have been the 0 function, that is the situation when M = 0.

Now we are in the case where M is positive, therefore u is not a constant function, therefore the maximum of u or minimum of u is attained only on the boundary of omega R by strong maximum principle or and strong minimum principle. So, M is attained either on the circle of radius R or on the circle of radius 1, that is this or this. But what happens on the circle of radius R? Circle of radius R mod u is strictly less than M by 2. Therefore M cannot be attained on the circle of radius R therefore M is attained only on the circle of radius 1. That is maximum of mod u or omega bar = maximum of mod u on boundary of omega.

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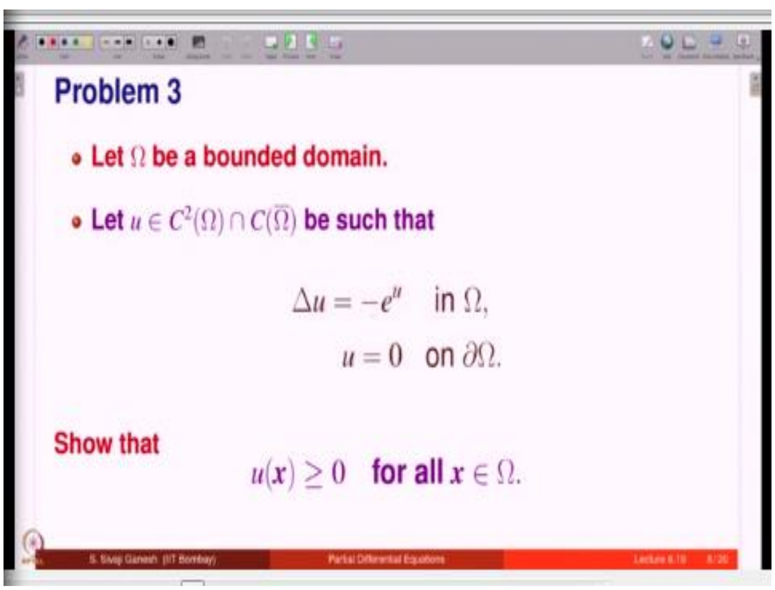

Problem 3 let omega be a bounded domain. Let u belongs to C 2 omega intersection C of omega bar be such that Laplacian u = -e power u in omega and u = 0 on boundary of omega. Show that u of x is greater than or equal to 0 for all x in omega. So, how to solve this problem? We know nothing about this kind of equations, this is actually non-linear equation or if you want to be milder semi-linear equation but then we have no idea how to handle this.

So, we start thinking the other way suppose this is not true u of x is greater than or equal to 0 is not true. It means at some point in omega u is less than 0, in particular it means the minimum is going to be a negative number. But here if you see u is 0 on the boundary therefore a minimum of u on omega closure is attained at a point in omega. At points of minimum what is the Laplacian? Laplacian u has to be greater than or equal to 0, but by this equation Laplacian u is always less than 0.

Therefore it cannot happen that u is less than 0 at some point, in other words u is always greater than or equal to 0, we are going to write down the details. As such this problem is not something to do with Laplace equation but the fact that Laplacian u has a certain sign at points of minima or maxima.

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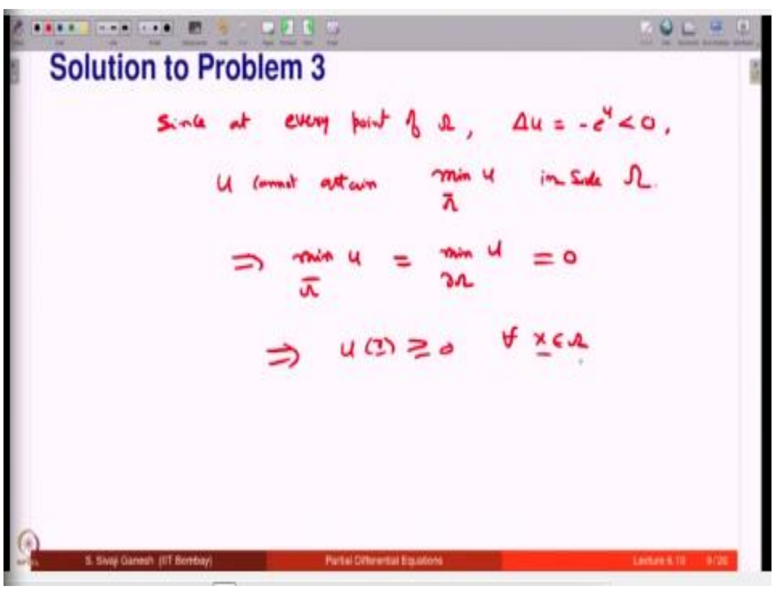

Since at every point of omega we have Laplace and u which is equal to -e power u and that is strictly less than 0, u cannot attain minimum of u on omega bar inside omega. The minimum cannot be achieved inside omega that means minimum achieved only on the boundary of omega. But on the boundary of omega u is 0, so minimum of u on omega bar is same as minimum of u on boundary of omega but that is 0. But saying that minimum of u on omega bar is 0 is same as saying that u is greater than or equal to 0 all x in omega. So, this is a problem featuring Laplacian but the solution needs knowledge of only maxima, minima in multivariable calculus.

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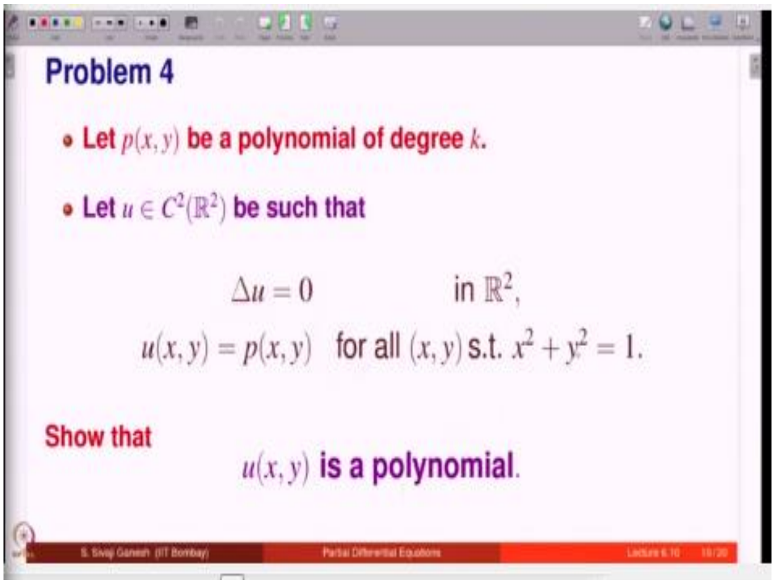

So, problem 4 let p of x, y be a polynomial of degree k, let u belongs to C 2 of R 2 be such that Laplacian u is 0 that is u is harmonic in R 2. And such that u of x, y = p of x, y for all x, y on the unit circle. That means we have a harmonic function in R 2 which equals a prescribed polynomial p on the unit circle. Then show that u itself is a polynomial and polynomial solutions of Laplacian equal to 0 are called harmonic polynomials.

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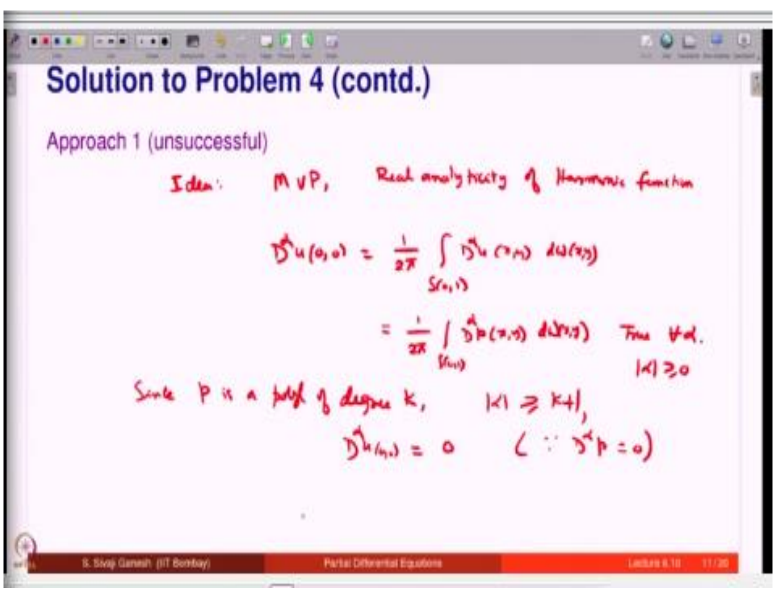

We are going to discuss 2 approaches to solve this problem. The first approach is not successful but nevertheless we mention the ideas. We think of using mean value property and also the fact that harmonic functions are known to be real analytic functions. How do we plan to use this idea or why do we think of this idea? We are given that u = p on the unit circle, p is a polynomial if p is a polynomial of degree k you know k + 1th order derivatives of p are 0.

So, therefore **we** if you think of D alpha u at the origin 0, 0 for mod alpha bigger than or equal to k + 1 by mean value property they will be 0. Because mean value property is written in terms of an integral on a circle but on the circle you have D alpha p. Whenever alpha is such that mod alpha is bigger than or equal to k + 1 d alpha p will be 0 because p is a polynomial of degree k. Therefore we get the D alpha u at 0, 0 will be 0 whenever mod alpha is bigger than or equal to k + 1.

And since u is supposed to be real analytic function I can write the Taylor series expansion at the point 0, 0, of course this will be a finite expansion. In other words it is a polynomial, it is not a series, it is going to be just polynomial. And then I hope to show that is a polynomial which we are looking for. Let us see what are the problems in this approach? So, let me just write what is idea?

Mean value property and real analyticity of harmonic functions. So, mean value property gives us that D alpha u and 0, 0 = 1 by 2 pi integral on the unit circle D alpha u at x, y d omega of x, y. That is nothing but 1 by 2 pi S of 0, 1 D alpha but on the unit circle u = p therefore it is D alpha p of x, y d omega of x, y, this is true for all alpha, that is mod alpha greater than or equal to 0. But since p is a polynomial of degree k whenever mod alpha is bigger than or equal to k + 1, we get D alpha u at 0, 0 is 0 because D alpha p is 0.

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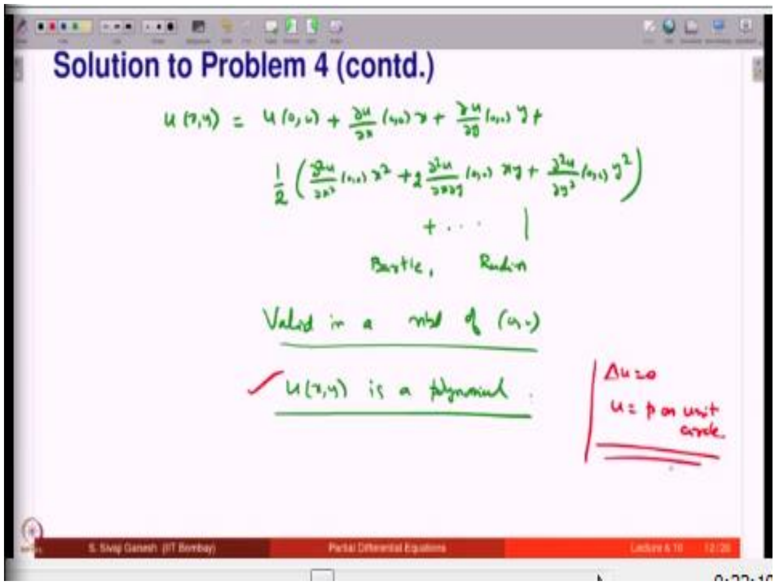

So, writing down the Taylor series, what we get is u of 0, 0 I will write only a few terms dou u by dou x at the point 0, 0 into x dou u by dou y at the point 0, 0 into y + second order derivatives dou x dou y this will come with true at 0, 0 xy + dou 2 u by dou y square into 1 by 2 and so on. So, if you want to understand the Taylor series, you can consult any book on multivariable calculus, in particular the book of Bartle or Rudin, Rudin's book on mathematical analysis and Bartle's book on elements of real analysis, we will find that. Of course this will be valid in a neighborhood of 0, 0 or in a disk containing 0, 0. But as we observed this is going to end after some time.

It is not a series, it is going to be polynomial, so there is no problem about validity, it will be valid everywhere. So, we caught hold of a polynomial expression for u of x, y. We have to show that this u is indeed Laplacian u = 0 and u = p on the unit circle. We are not applying any uniqueness theorems, if we had uniqueness to this problem to the given problem we can start to say that the u we got this answer. So, the trouble lies there, I am going to explain this later.

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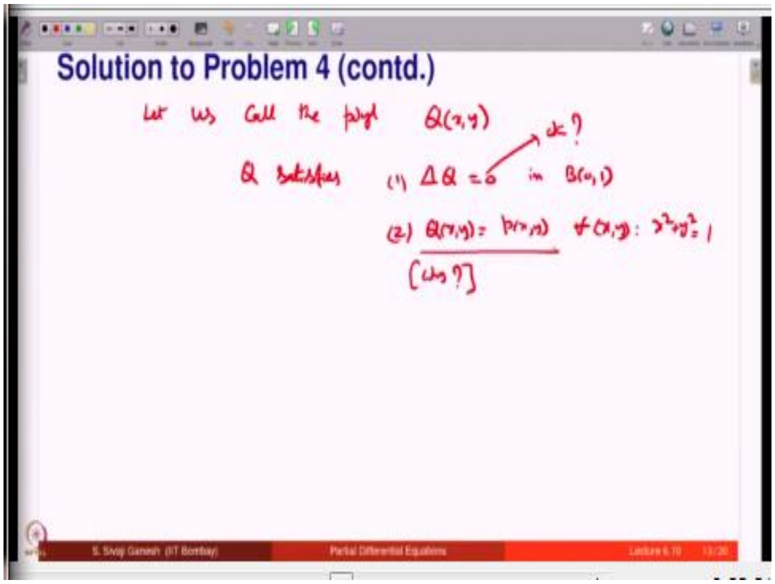

So, let us call the polynomial that we got in the previous slide as Q of x, y using the real analyticity of the function u. So, now Q satisfies Laplacian Q = 0 in the unit (()) (22:40) and Q of x, y = p x, y for all x, y such that x square + y square = 1. The question is why? This is what we would like to say but is this true? One seems to be ok, because 1 means this, **ok** this seems to be ok because Q is actually a representation of the u and we know Laplace u = 0 therefore Laplacian Q is 0. But why is this true, Q x, y = p x, y? No idea, it looks like the proof is going to break down here, so we go for another approach.

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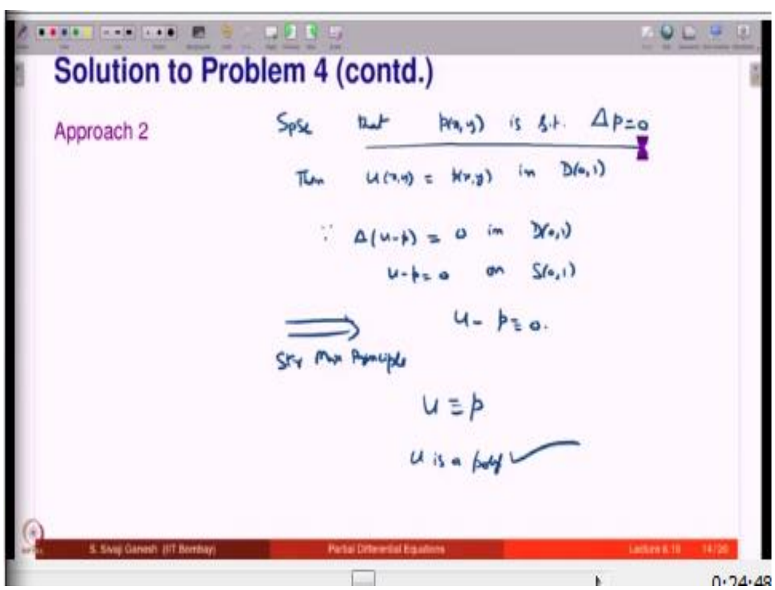

Approach 2, let us look at a simple case suppose that the given p of x, y is such that Laplacian p = 0, that means p itself is a harmonic polynomial, then what happens? u of x, y = p of x, y in the unit disk. Because Laplacian of u - p = 0 in the unit disk and u - p = 0 on the circle, therefore by strong maximum principle we get u - p identically equal to 0. So, in other words u is identically equal to p in particular u is a polynomial, done. So, trouble is that this p which is given may not be harmonic polynomial, then what you do in that case?

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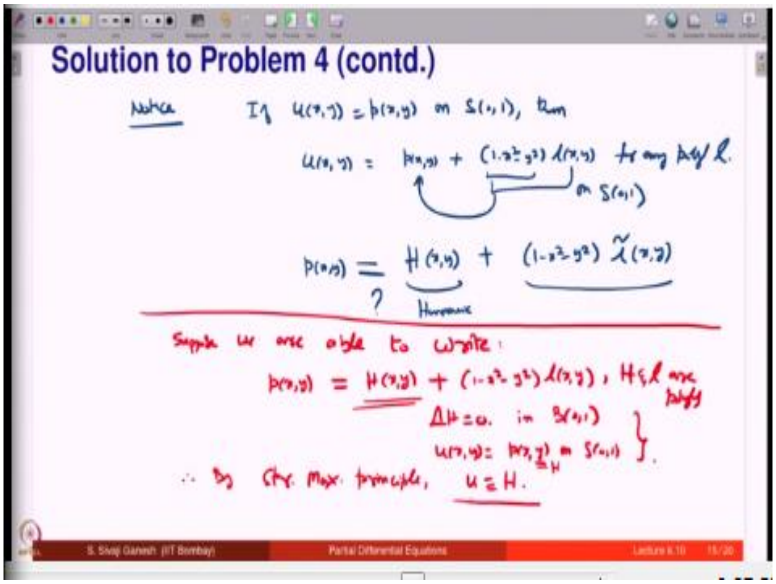

Just notice some interesting thing here which is if u x, y = p x, y on the unit circle then u x, y = p x, y + 1 - x square - y square into l of x, y for any polynomial l, that is because 1 - x square - y square is 0 on S 0, 1, so this will happen on S 0, 1. Therefore we ask, can we supply something from here into this? So, that we get a harmonic polynomial here and something here, let me write l tilde form for the moment.

So, in other words we want to write whether p of x, y can be expressed like this, where H is harmonic and you have addition of this which is vanishing on S 0, 1. So, suppose we are able to write p of x, y = H of x, y + 1 - x square - y square into l of x, y where H and l are polynomials of course polynomials and Laplacian H is 0. That is H is a harmonic polynomial in B 0, 1, let us put, for a polynomial it does not matter once Laplacian H is 0, Laplacian H is 0, everywhere.

u of x, y = p of x, y on S 0, 1 therefore by strong maximum principle we get u ideally equal to H. So, this follows from the observation that we made on the previous slide, when the given p were harmonic polynomial then we concluded that u is identically equal to p. Now here I will treat this as the given polynomial H and u = H this actually equal to H, u = H on the unit circle then H is the solution by strong maximum principle. So, u in particular is a polynomial in p 0, 1.

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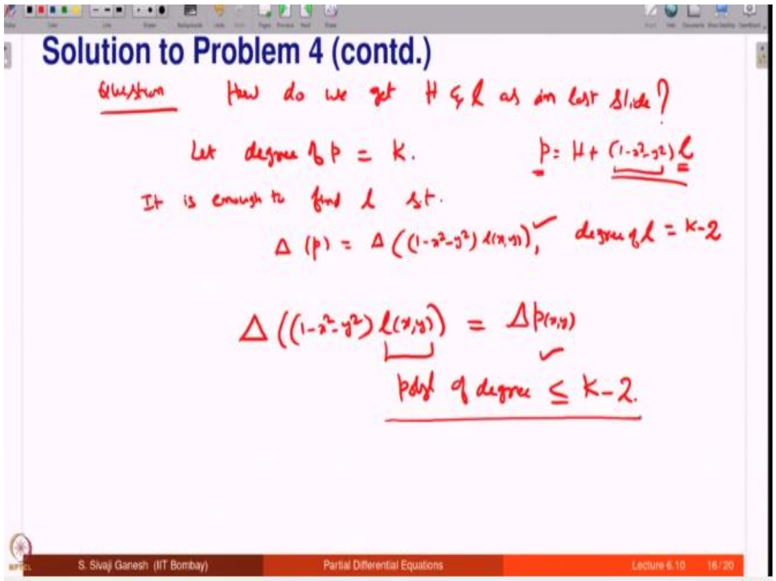

So, the question now is how do I catch hold of H and l as on the last slide? That is p = H + 1 - x square - y square into l, that H and l how do I get? Of course if you get l, you have the H. So, let degree of p in fact it is given to us equal to k. So, it is enough to find l such that Laplacian of p = Laplacian of 1 - x square - y square into l of x, y and degree of l is k - 2. Because I want to write p = H + 1 - x square - y square into l.

So, this is already a degree 2 polynomial, so l is degree k – 2, because p is given to be degree k. So, this is a natural condition we get because I want Laplacian H = 0 that is if and only if Laplacian of p = Laplacian of this quantity, that is what I have written here. Thus we are interested in solving this equation. Here notice the right hand side is known, p is given polynomial, so this is known, so we want to solve this.

Now it looks like equally difficult problem, luckily it is not that difficult. Because we are only looking for l which is polynomial of degree less than or equal to k - 2, that really helps us. Because if you are looking at only polynomials degree less than or equal to k – 2, it forms a vector space of finite dimension. That makes our job easy as we are going to see on the next slide.

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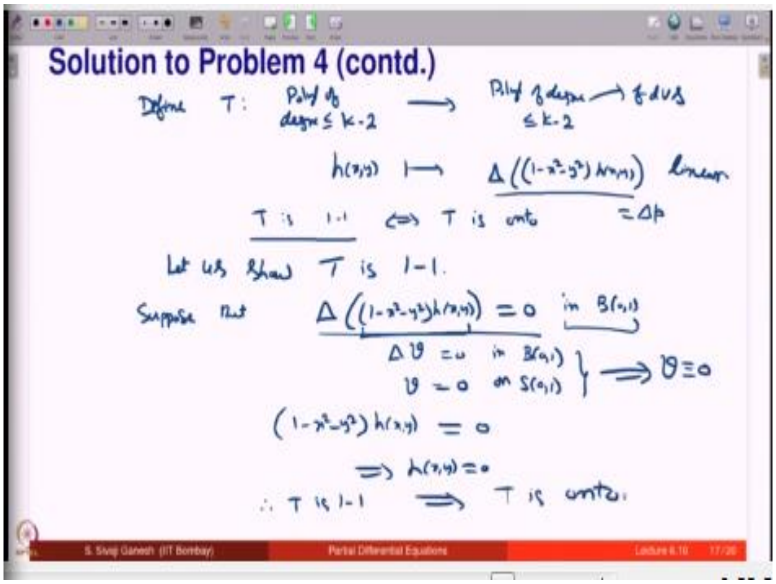

So, define T, it is a mapping from polynomials of degree less than or equal to k - 2 to same space polynomials of degree less than or equal to k - 2, what is the operator? It takes a polynomial h of x, y to this multiply with this 1 - x square - y square then you get a polynomial of degree less than or equal to k and then take Laplacian of that, so that will be a polynomial. So, this operator is well defined, it is a linear operator.

And this is a finite dimensional vector space; therefore what we know is T is 1 to 1 if and only if T is onto. What we are interested in? Is the T is onto? Because if t is onto I can find an h such that this quantity is equal to Laplacian p because p is at degree k Laplacian p will be of degree less than or equal to k - 2. That is how I get this H, which I will call it as l, once I have l, I have my H.

Therefore we will show that T is 1 to 1, it is easier to show T is 1 to 1. So, suppose that Laplacian of 1 - x square - y square into h of x, y is 0 in B 0, 1. In other words what I am going to show that the kernel of T consists of only the 0 element. So, I assume that H belongs to the kernel of T, that means this equation is satisfied everywhere. But I have written in B 0, 1 because I have plans to apply some maximum principles.

H has this property that means this is the function I am looking at, let me call it as v. So, v is a harmonic function in B 0, 1 and what is it is value on S 0, 1? It is 0 on S 0, 1. Because on S 0, 1 x square + y square = 1, therefore v is 0. Now the maximum principle tells me that v has to be identically equal to 0. That means 1 - x square - y square into h x, y is the 0 polynomial or 0 function or the 0 polynomial, that implies that h of x, y is 0 polynomial. So, we have shown that T is 1 to 1, therefore T is 1, 1 and that implies T is onto.

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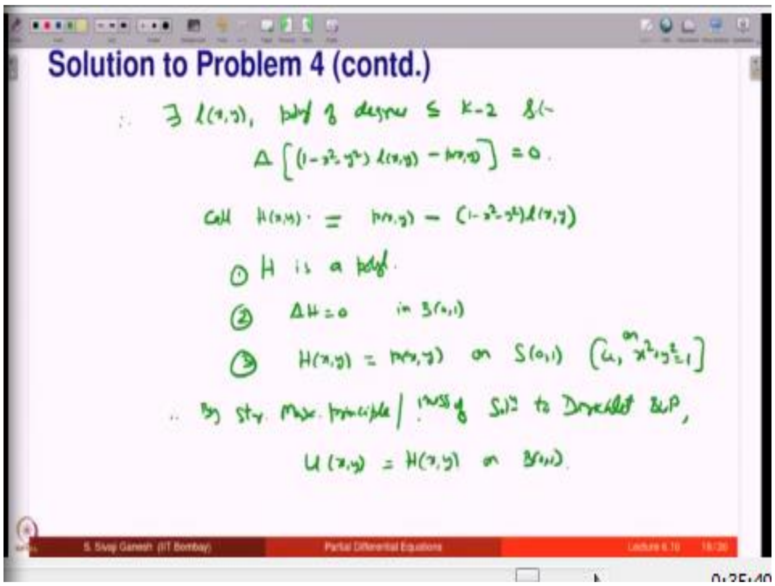

Therefore there exists l of x, y polynomial of degree less than or equal to k - 2 such that Laplacian of 1 - x square - y square into l of x, y - p of x, y is 0. So, define or call H of x, y equal to this p of x, y - 1 - x square - y square into l x, y. Now what can we say about H? H is a polynomial, Laplacian H is 0 of course in particular in B 0, 1 and H of x, y = p of x, y on S of 0, 1, that is for x y such that x square + y square = 1. Therefore by strong maximum principle or the uniqueness or uniqueness of solutions to Dirichlet boundary value problem, we get that u of x, y = H of x, y on B 0, 1.

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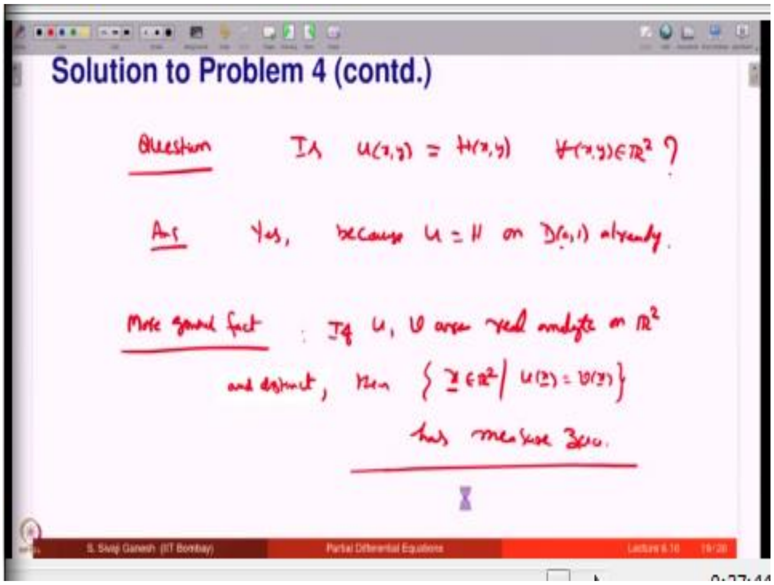

A natural question because we are dealing with polynomials. Is u of x, y = H of x, y for all x, y in R 2? We have only shown the equality and B 0, 1 because we apply strong maximum principle on the domain D 0, 1 on the disk of radius 1. Answer is yes, because u = H on the disk of radius 1 centre origin already this we already showed, we will not prove this. A more general fact is that if u and v are real analytic on R 2 and distinct, then look at the set x in R 2 such that u of x = v of x. This set has measure 0, it is more general. So, we already have seen that this set in our example or in our problem this set contains the disk of radius 1 already which is non zero measure. Therefore we can apply this and also say that u = H in R 2 itself, thank you.