**Partial Differential Equations**
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**Lecture-51**
**Laplace Equation**

## Mean Value Property

Welcome, in this lecture, we are going to discuss about mean value property and some of it is consequences.

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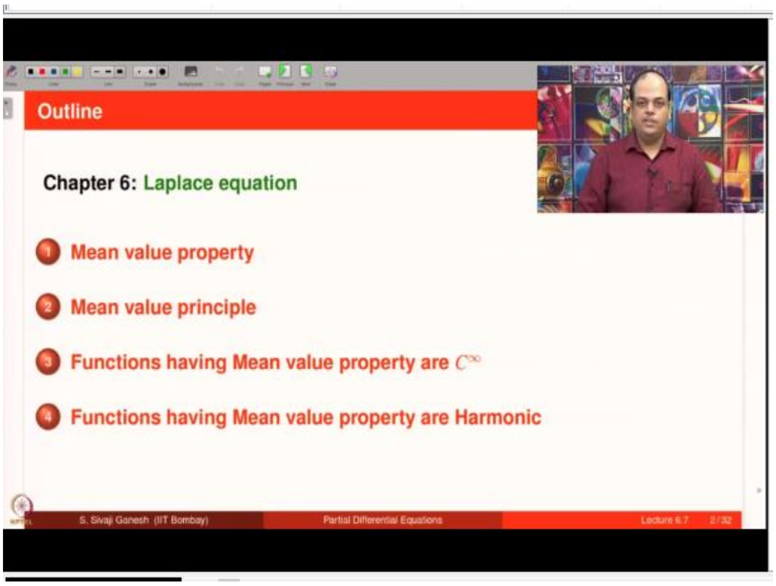


The outline of this lecture is as follows. First we introduce what is called mean value property. And then we go on to show mean value principle namely mean value property holds for harmonic functions, that is what is called mean value principle. Then we show that functions having mean value property are C infinity functions and functions having mean value property are harmonic functions.

In 3 and 4 points here, we are going to assume only continuity of the function. A continuous function has mean well property automatically it follows C infinity; if a continuous function has mean well property then it is automatically a harmonic function.

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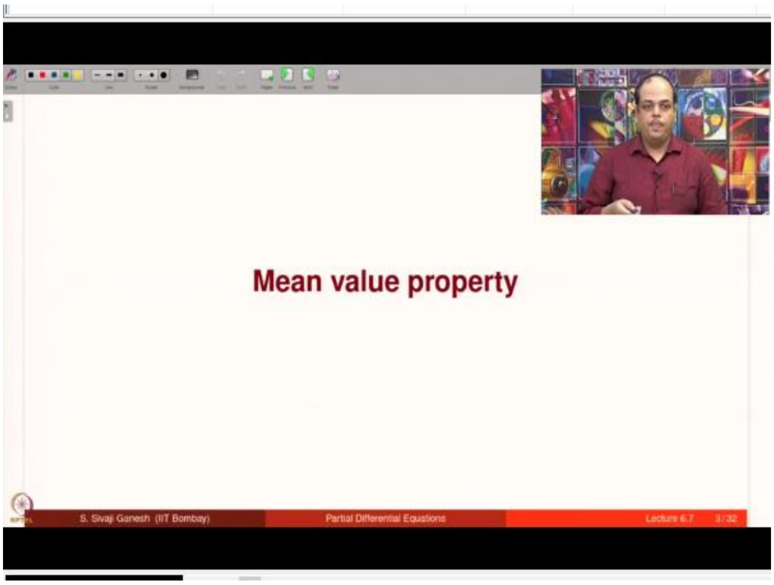


So, what is mean value property?

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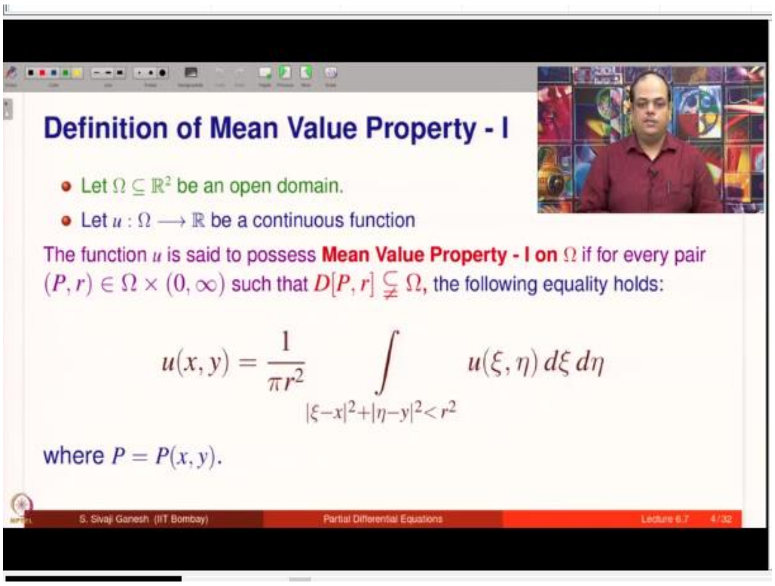


So, we are going to define 2 notions of mean value property that they are called mean value the property I and mean value property II. First let us define what is mean value property I? Let omega instead R to be an open domain, let u from omega to R be a continuous function. The function u is said to possess mean value property I on omega if for every pair P, r such that the close disk of radius r with center at D is properly contained in omega. What does this mean? Take any point in omega and take any r positive such that this close disk is contained in omega.

Then the following equality holds, u of x, y, x, y is P, P of x, y, P is P of x, y. So, u of x, y equal to integral of u over the disk D P, r and this is pi r square is precisely the area of the disk. That means, I am integrating the function on this domain D P, r and dividing it with the area of D P, r. Therefore, this is called mean value. So, u of x, y is nothing but you take a disk around x, y which is this D P, r integrate and divide with 1 by pi r square. In other words, the value of u at the center of a disk is given by its average on the disk.

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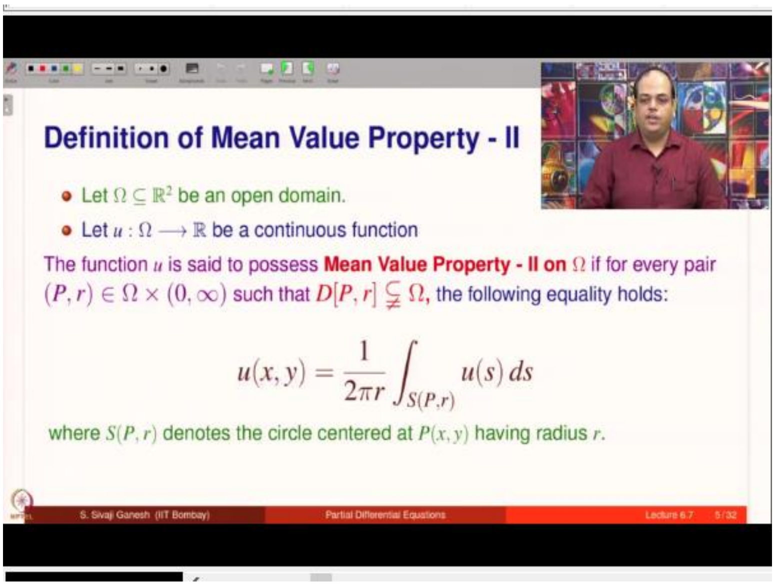


Definition of mean value property II: Let omega be once again an open domain in R 2 and let u be a continuous function on omega. Then the function is said to possess mean value property II if for every point P in omega and for every r positive such that this closed disk is contained in omega something should happen. What would happen is u of x, y is nothing but integrate u over the circle S P, r denotes a circle centered at P having radius r and divided with perimeter of a circle. In other words, this is the mean value of u on the circle is equal to the value of u at the center of the circle. So, this is called mean value property II.

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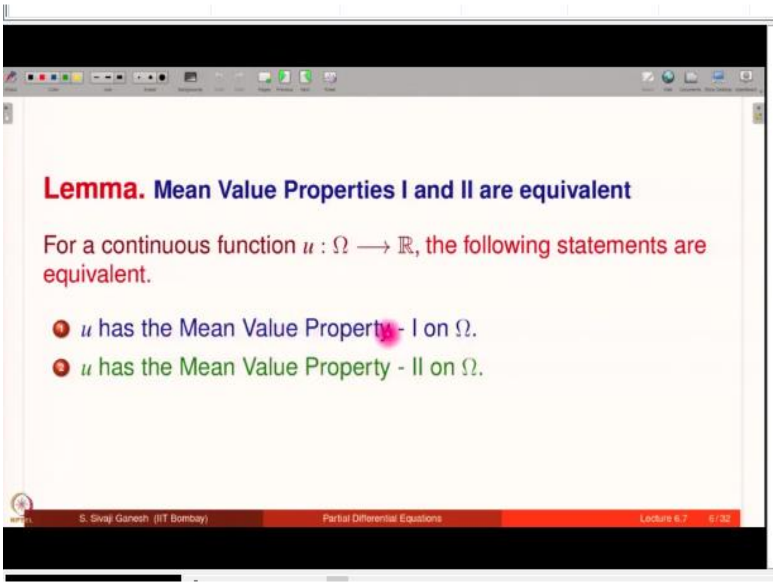


So, we have a result which says that mean value properties I and II are equivalent, what does that mean? If you have a continuous function u on omega, then the following statements are equivalent, u has a mean value property I on omega is same as u has a mean value property II on omega. So, this means that if you have a continuous function and it has the mean value property I then automatically it has mean value property II. Similarly, if the function u has a mean value property II on omega then definitely will also have the mean value property I on omega.

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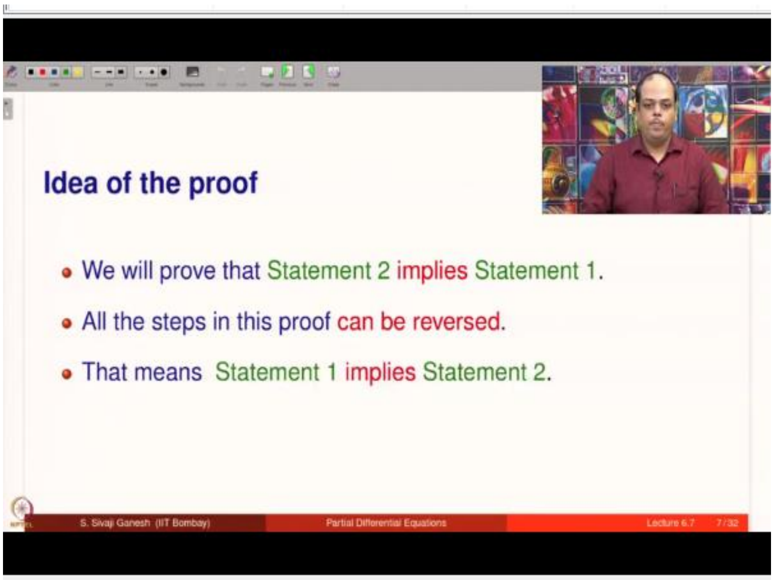


So, what is the idea of the proof? We are going to prove the statement 2 implies statement 1. All the steps in this proof can be reversed. That means statement 1 implies statement 2.

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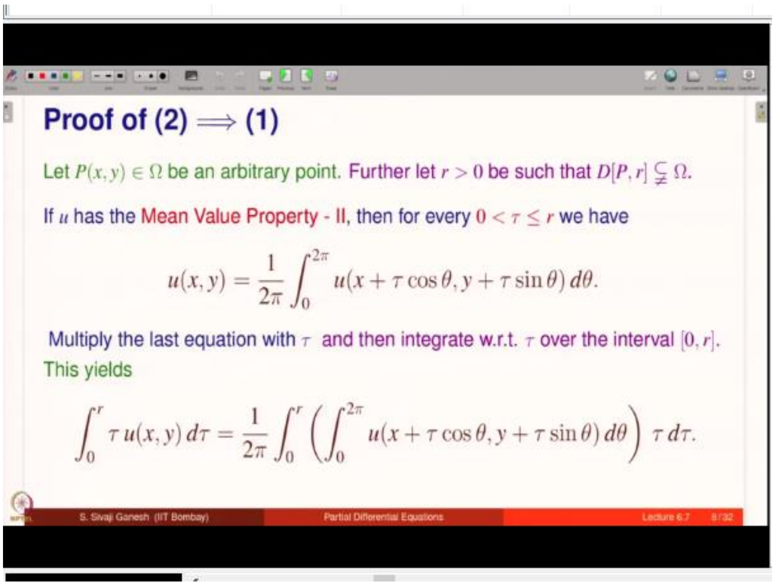


So, let us prove 2 implies 1. So, let P be an arbitrary point in omega. Further let r positive be such that the closed disk with center P radius r is inside omega. If you have the mean value property II then for every tau less than or equal to r we have this equality. This is a mean on the circle equal to u at the center. Now, of course, by the definition it holds for r, why does it hold for every tau less than or equal to r?

It is because if D of P, r is in omega then D of P tau will also be a subset of omega. For every towel less than or equal to r therefore, this holds for every towel less than or equal to r. Now, multiply the last equation with tau and then integrate with respect to tau over the interval 0 to r. So, I have simply multiply this equation with tau and wrote this integral 0 to r on both sides.

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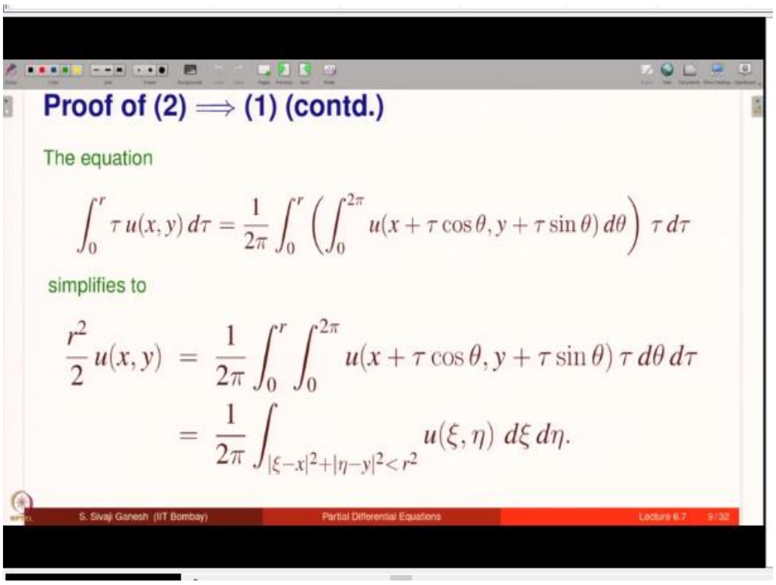

Now, we can integrate the left hand side because u of x, y does not depend on tau. So, it is a constant. So, integral 0 to r of tau d tau that will give us tau square by 2 with limits 0 to r. So, that gives us r square by 2 into u of x, y. Right hand side I have written as it is. Now, what is right hand side? It is nothing but the integral on the disk expressed in the polar coordinates. In polar coordinates tau d theta d tau is the area element. And therefore, this integral is equal to this integral. Now, from here, you get what is u of x, y because you bring r square by to this side you get 2 by r square then you get 1 by pi r square into integral u on the disk.

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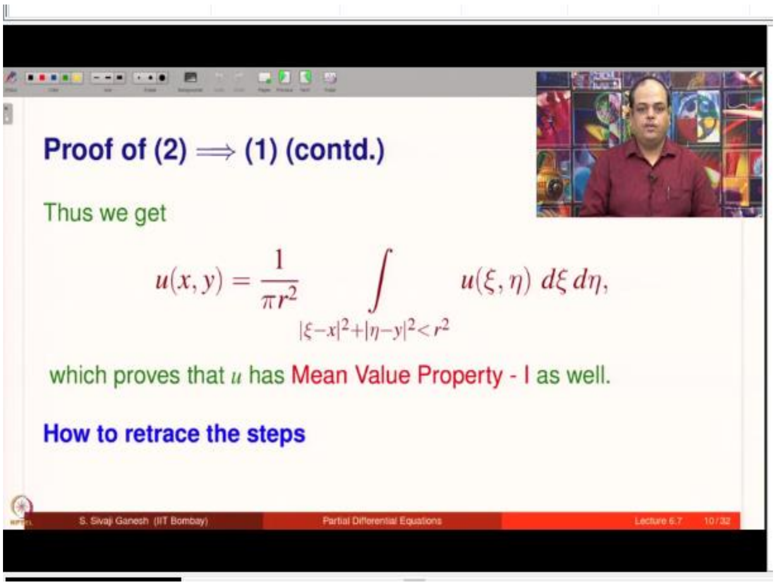


So, that finishes the proof of mean value property I. Now, how to retrace the steps and prove 1 implies 2. Let us do that. This is what we have; this is what we assume. If you have mean value property I we have this equation that is on multiplying with r square by 2. We have r square by 2 x, y equal to this. Now, we can express this as this integral in the polar coordinates. Now, I want to pass from this equation to this equation; what do I do?

Right hand side is the same, left hand side is precisely this is nothing but this integral. So, we have got this. That means, we have this last equation on the slide. Now, from here to I want to pass to this; how do I do that? Apply fundamental theorem of calculus then 0 to r will go away and what you will have is r times u of x, y and here 1 by 2 pi 0 to 2 pi and tau replaced by r then we will have this then cancel r u of this. So, therefore, we can retrace the steps and through that 1 implies 2.

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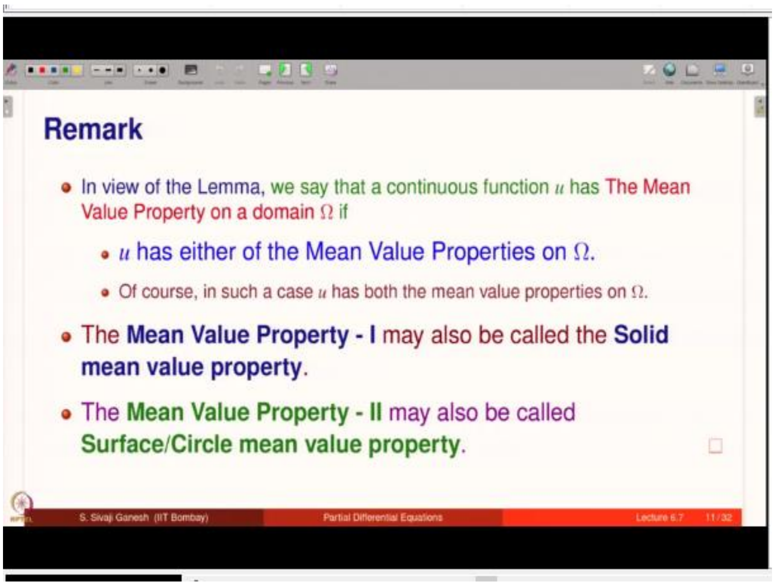


Remark; in view of this lemma that we have just proved we say that a continuous function has the mean value property on a domain omega, we do not mention which one. If u have either of the mean value properties I am going to say that the function u has the mean value property, we need not mention which one of them because they are equivalent. Of course, in such a case you have both the mean value properties.

If it has one mean value property, it has both the mean value properties. That is the reason we would not mention which one. So, the mean value property I may also be called the solid mean value property. This because in mean value the property I features mean on the disk that means I am integrating the entire disk and taking the average. That is why it is called solid mean value property.

And the mean value property II may be called as a surface or circle mean value property. This terminology is not very standard I am using this terminology because we can easily understand by names. Names represent a quantity much more than a very generic name like mean value property I, II, III etcetera. Solid mean value property means I understand it, how to write it. Similarly surface mean value property I know what does it mean I can easily recall.

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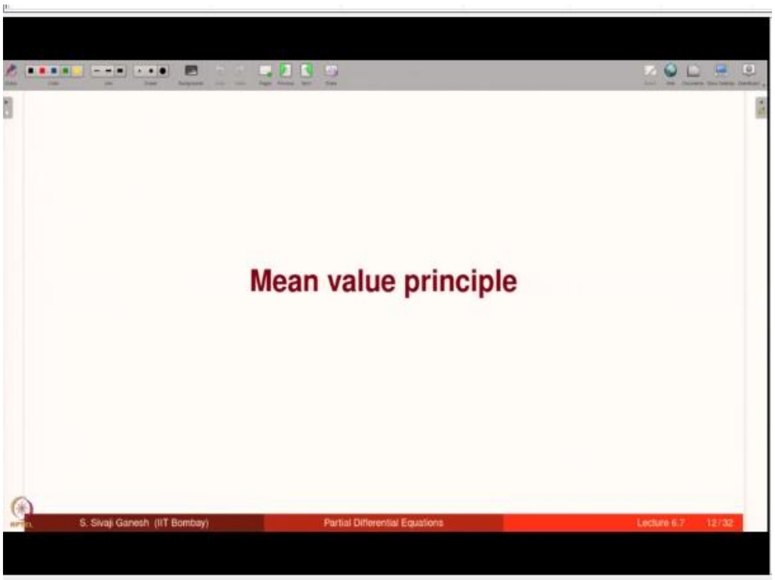


Let us prove now mean value principle.

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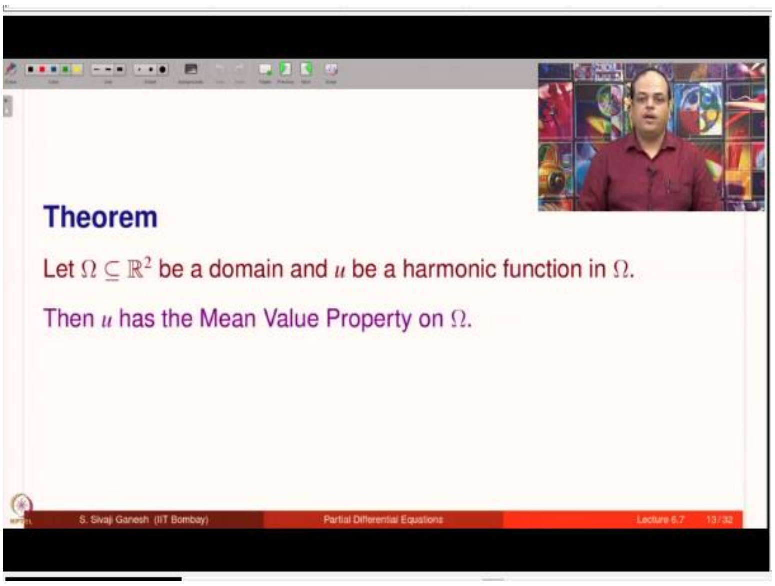


Select omega be a domain in R 2 and u be a harmony function on omega. Then u has the mean value property on omega.

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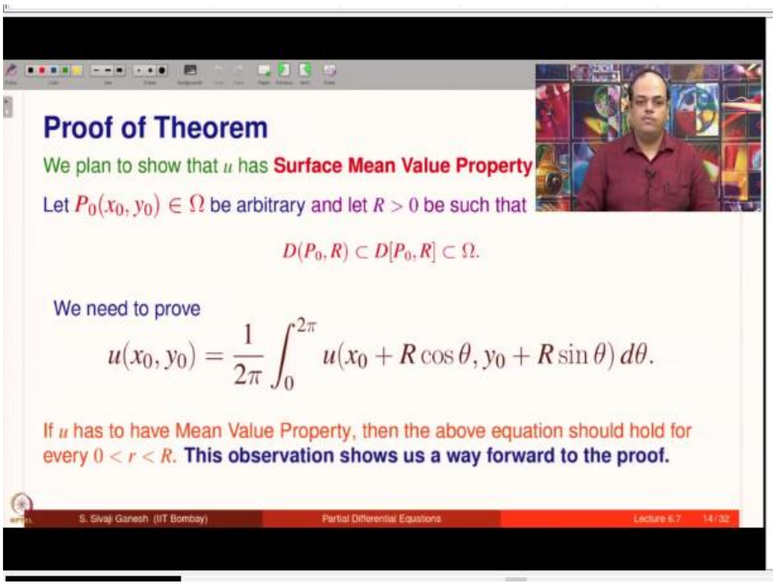


So, proof of the theorem. We plan to show that u has a surface mean value property. So, if you want to show that u has mean value property I can choose to prove any one of them which is convenient. I am going to show that u has surface mean value property. Select P 0 be a point x 0, y 0 in omega be arbitrary, unlike capital R be such that this open disk is contained the closed that is contained in omega.

And we need to prove that u are at the center of the disk is given by the average of u on the circle. If u has to have mean value property then the above equation namely this should hold when I replaced this capital R with any r which is less than r. This observation shows us a way how to proceed forward.

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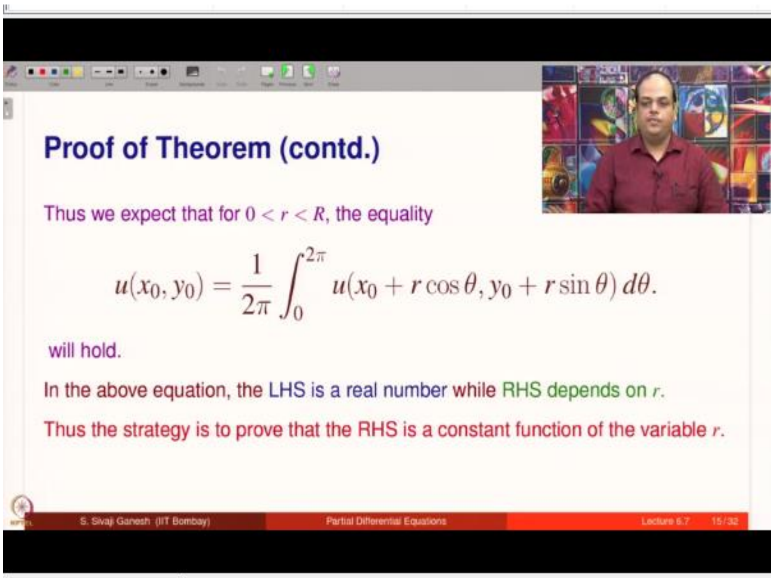

So, that we expect that for r between 0 and R this equality will hold. In the above equation, the LHS is a real number u at the point x 0, y 0 is a number. Whereas, the right hand side is a function of r, it depends on r. Therefore, the strategy is show that the right hand side is a constant function of the variable r.

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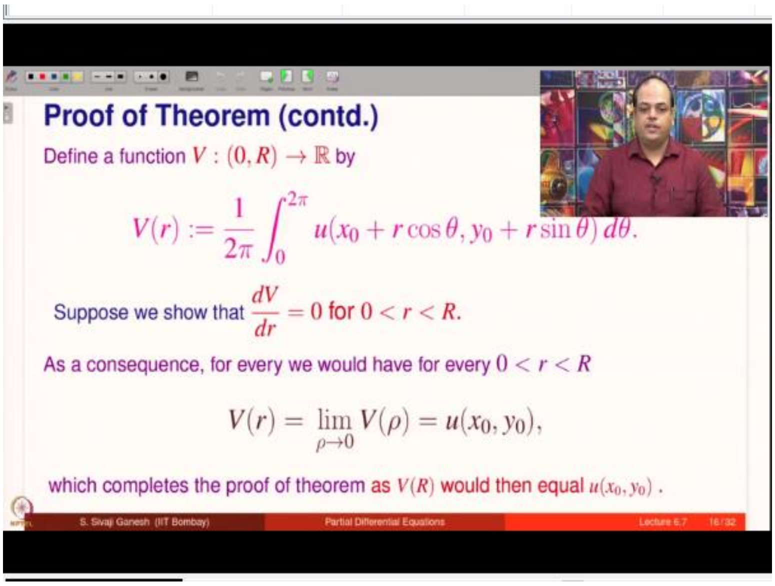


So, therefore, define a function V on this interval 0, R by this formula, which is precisely the right hand side on the last equation in the previous slide. Suppose, we show that dV by dr = 0 what does it mean? V is the constant function on this interval 0, r. So, as a consequence for every r in interval 0 R we get that V of r is constant, but what is that? That constant can be evaluated by taking limit rho goes to 0 of V of rho.

And that will be u of x 0, y 0. On the other side is a constant V is a continuous function of r therefore, V of r is also equal to limit rho goes to R of V of rho and that will give you V of R which is precisely the right hand side in the circuit mean value property. So, this completes the proof. As V of r = u of x 0, y 0.

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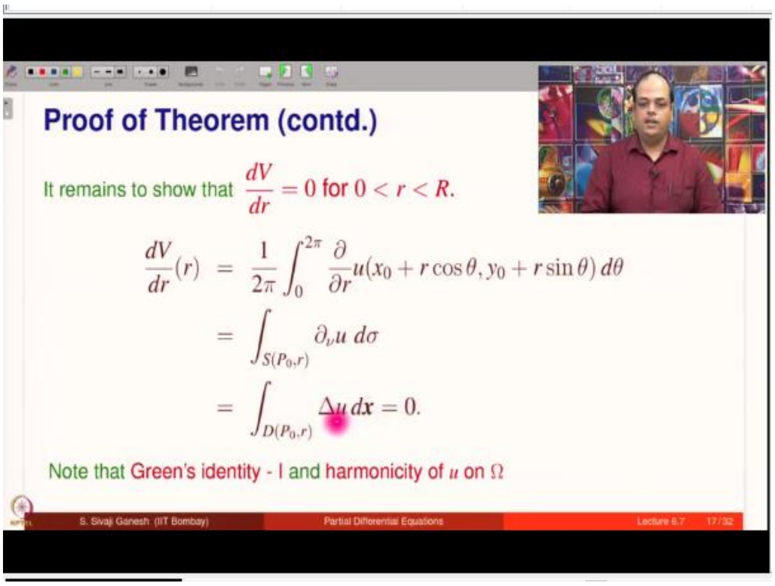


So, it remains to show that this derivative of V with respect to r is 0, let us compute. So, dV by dr is given by this. I have taken the differentiation inside the integral. That is allowed because u is a smooth function you just see the function and this is precisely this, this is the normal derivative of u on the boundary that is normal derivative of of u on the circle. Now, this is nothing but integral over the disk of Laplacian u dx and that is 0 because u is a harmonic function. So, we have used Green's identity I in passing from this to this and harmonicity of u in contributing that this integral is 0 because the integrand is C.

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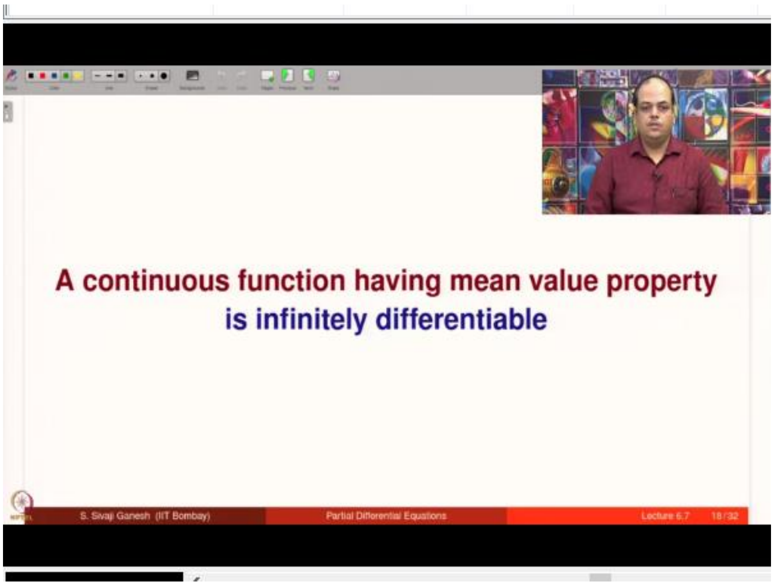


So, if you have a continuous function which has mean value property it is infinitely differentiable.

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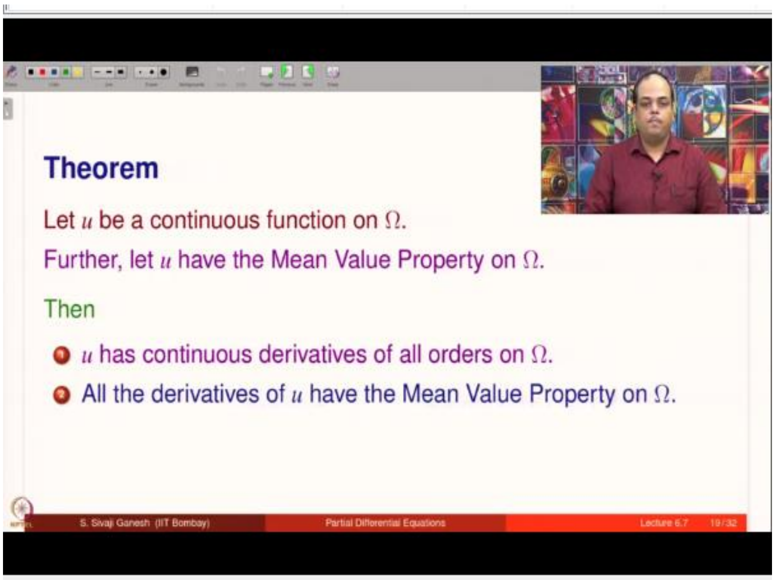


So, let u be a continuous function on omega. Omega is a subset of R 2. Let u have the mean value property on omega then u has continuous derivatives of all orders on omega and all the derivatives of u have the mean value property on omega. So, point 1 says use (()) (13:27) function and this says all the derivatives have a mean value property on omega.

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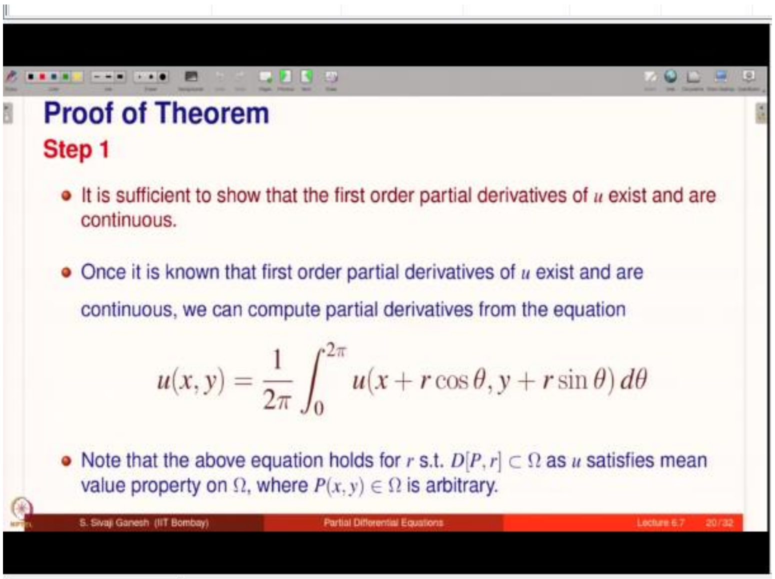


Step 1 it is sufficient to show that the first order partial derivative of u exists and are continuous. Once it is known that first order partial derivative of u exists under continuous, we can compute partial derivatives from this equation. This is known to hold because u has the mean value property. So, this holds. The question is whether I can differentiate this? I can if I know that the integrand is differentiable, that is what we are establishing that first order partial derivatives of u exists under continuous.

Therefore, differentiating this equation is allowed. Note that the above equation holds for all r such that this closed disk with center at P is inside omega as you satisfies mean value property on omega where P is an arbitrary point in omega. Mean value property holds on omega means, you take any closed disk which is contained in the domain omega then mean value property that is u at the center of the disk is given by the average. Let it be solid average or the surface average.

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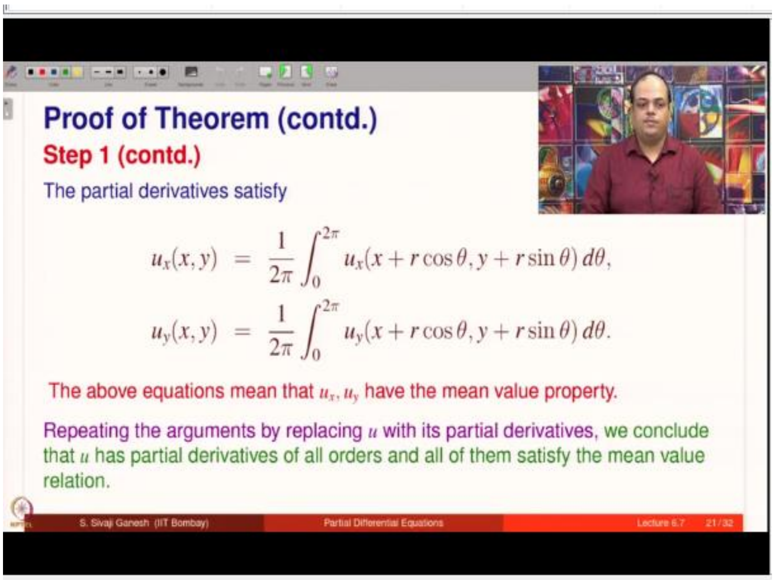


So, the partial derivatives satisfy u x of x, y equal to this are differentiated with respect to x and similarly u y. The above equations mean that you x, you y have the mean value property, because that this is exactly means that u x has the mean value property. Repeating the arguments by replacing u with its partial derivatives, we conclude that u has partial derivatives of higher orders and all of them satisfy the mean value property.

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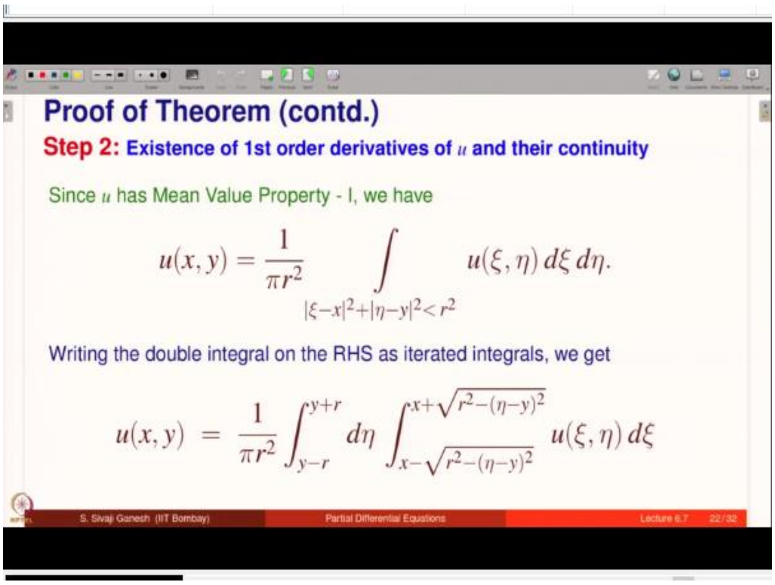


Now, let us show that first order derivatives in fact exist and are continuous. Since you have the mean value property, it has a mean value property I also therefore we have this expression for u at the center of this disk. Now, writing the double this integral on the disk, write it as iterated integrals, we get u of x, y equal to so this integral is precisely this integral.

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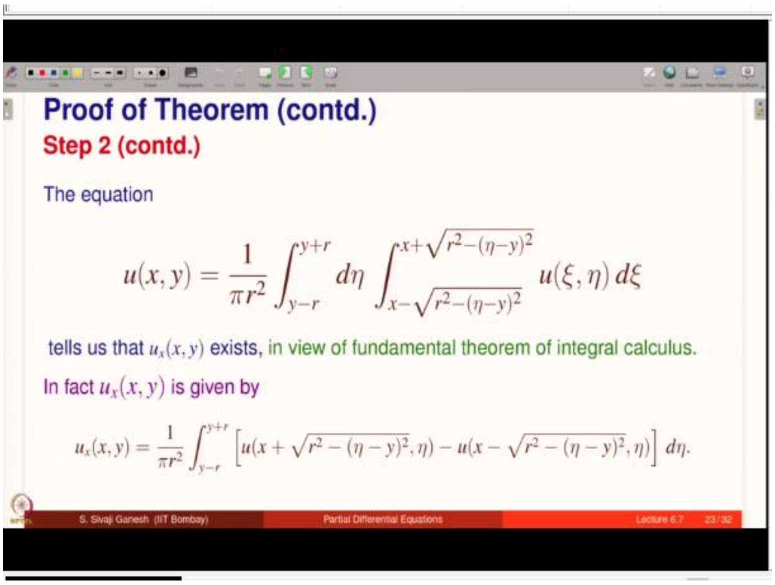


So, this equation tells us that u x of x, y exists how? X is appearing here inside and this integrand, imagine this is the first integral then it has an integrand which is this, there the dependence on the x is only through the limits and it is a nice dependence. So, by fundamental theorem of calculus this is differentiable and we can compute. So, when you compute and you differentiate this with respect to x you get this particular integrand here.

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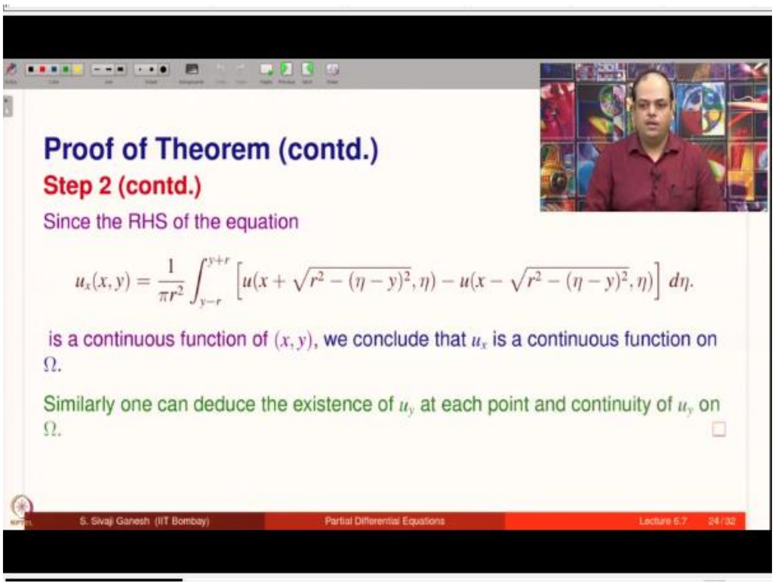


So, since the right hand side of this equation that is this is continuous function of x, y, it follows that u x is a continuous function of x, y. So, thus we are shown that u have a partial derivative with respect to x and u x is continuous function. Similarly, one can deduce the existence of u y at each point and the continuity of u y on omega.

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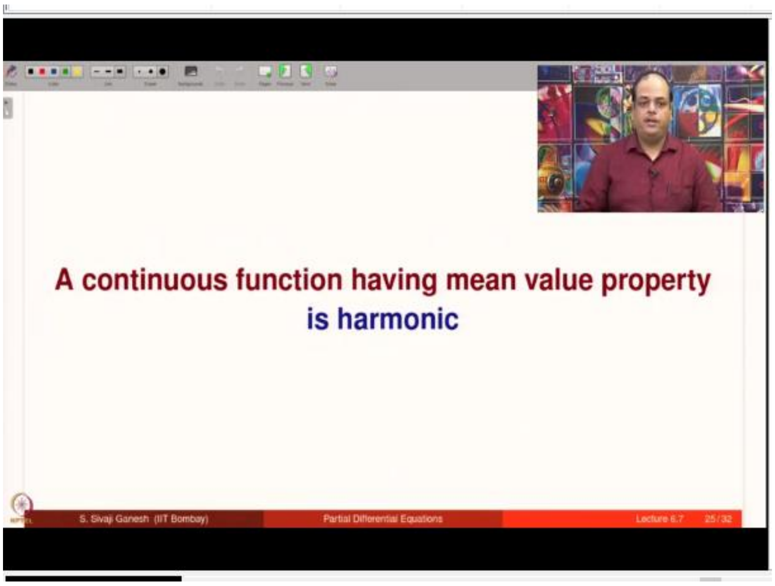


A continuous function having mean value property is indeed harmonic.

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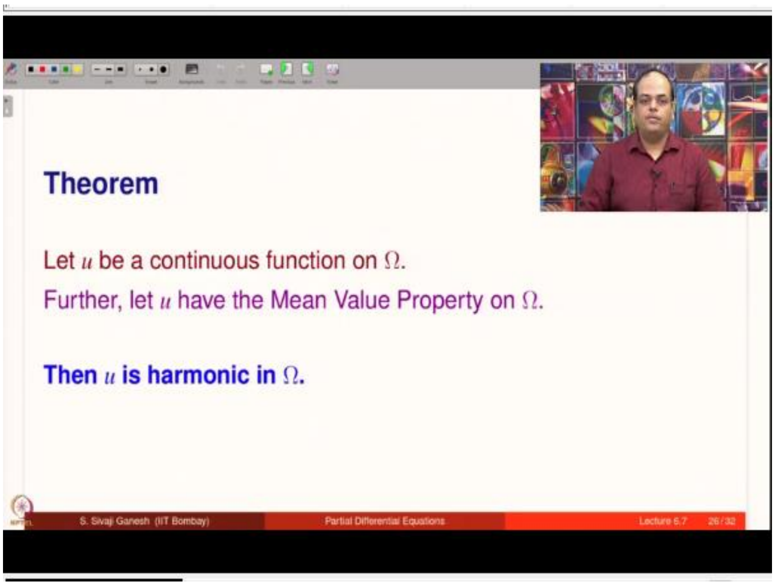


So, let u be a continuous function on omega. Assume that you also has a mean value property on omega then u is harmonic in omega. It means Laplacian u = 0 holds on omega.

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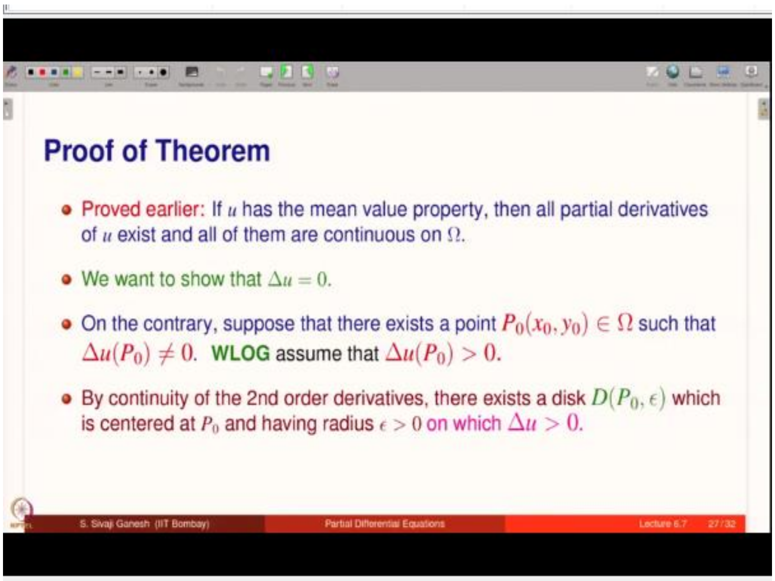


So, proof of the theorem, we have proved this earlier, if u is a continuous function and has a mean value property then all partial derivatives of u exists and all of them are continuous on omega; we have shown this. So, we want to show that Laplacian = 0 therefore, why we stated this is because Laplacian u should be meaningful, because we have only started with a continuous function.

So, as shown before u has all derivatives in particular secondary derivatives, therefore, Laplacian u is meaningful and we can think of showing that to be equal to 0. So, on the contrary assume that Laplacian u is not equal to 0 in omega that means there is a point P 0 in omega, where Laplacian u is not 0. Now, Laplacian u is a combination of second order partial derivatives of u, which are known to be continuous on omega. So, you have a continuous function namely Laplacian u, which is not 0 at a point P 0 therefore, in a disk are P 0 it will continue to be non-zero.

Without loss of generality, let us assume Laplacian u of P 0 is positive. What are the choices for Laplacian u of P 0 when it is non-zero it has to be positive or negative? So, let us assume it is positive, if it is negative exactly the same proof we can rewrite. Let us go ahead with Laplacian of u of P 0 as positive. By continuity of the second order derivatives, there is a disk D of positive radius epsilon centered at P 0 such that Laplacian u is greater than 0 on the disk.

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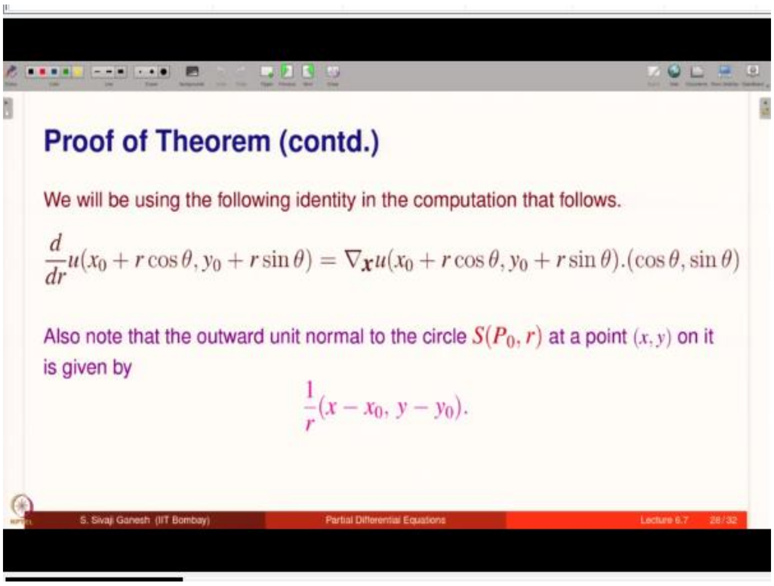


So, we will be using the following identity in the computation that follows d by dr of u of this quantity. By chain rule it is a gradient of u and then dot product with the derivative of the inside quantity with respect to r namely cos theta, sine theta. Also note that the outward unit normal to the circle at any point on it is given by x - x 0, y - y 0 by r. If you recall we have said this earlier as dou u by dou nu at a point on the boundary of the domain. That is at a point on the circle, this is the dou u by dou nu because this is a normal direction or the unit circle.

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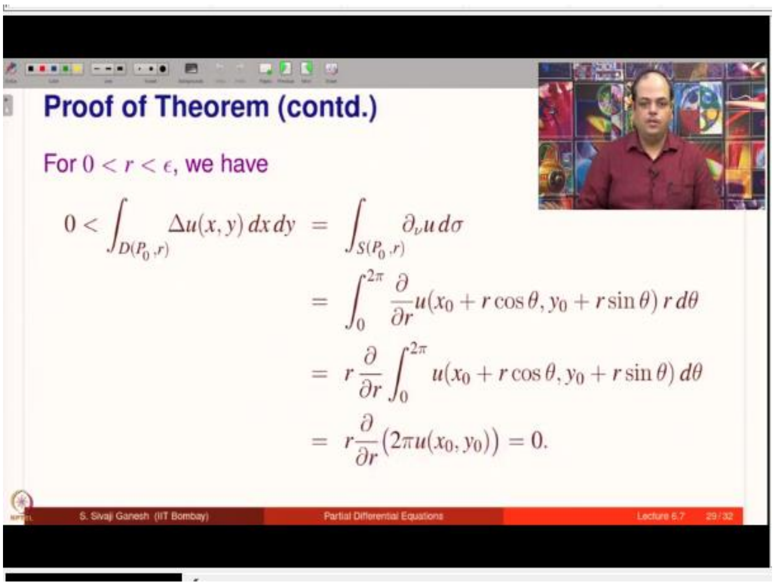


So, for r less than epsilon, we have this that integral over the disk of Laplacian u is positive because Laplacian nu at P 0 was positive and hence we found a disk of radius epsilon on Laplacian u states positive and this by Green's identity is equal to this quantity and this is precisely this and r comes out because integral is with respect to theta and dou by dou r also comes out what you have inside is precisely something related to mean value. And therefore, what you get is this quantity; this integral is 2 pi, u of x 0, y 0 which is 0 because this does not depend on r. So, derivative with respect to r is 0.

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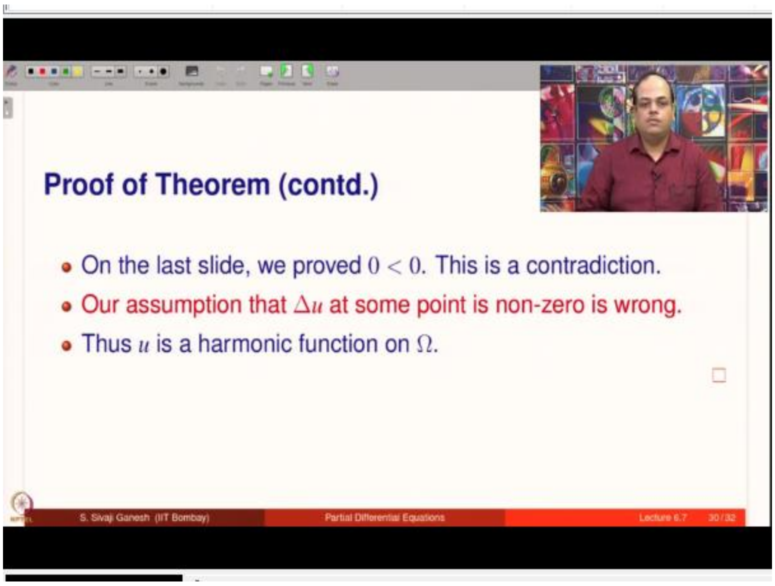


So, on the last slide what we have got is 0 is less than 0, it is a contradiction. Our assumption the Laplacian u at some point is non-zero is wrong. Therefore, u is a harmonic function.

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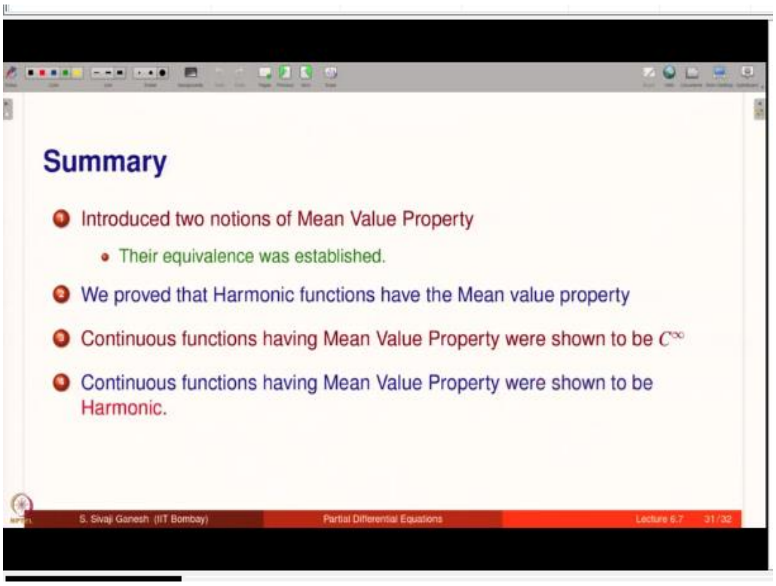


So, summarize today's lecture, we have introduced 2 notions of mean value property and their equivalence was established. We proved that harmonic functions have the mean value property. We proved that continuous functions having mean value property r C infinity and continuous functions having mean value property are in fact harmonic functions. Thank you.