Partial Differential Equations
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Lecture – 6.6
Tutorial 1 on Laplace Equation

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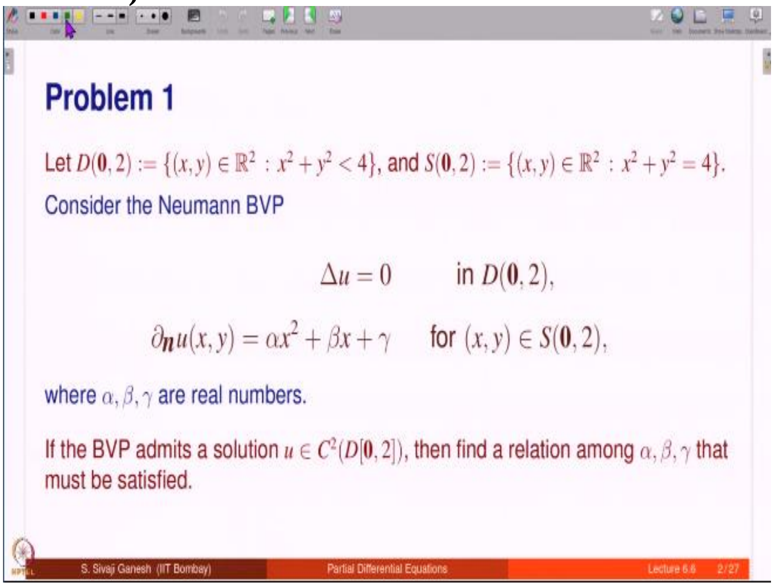

Welcome to tutorial 1 on Laplace equation. So, problem 1 is let D of 0, 2 denote the disk of radius to center at the origin and S of 0, 2 is a circle of radius to with center at the origin. Consider the Neumann boundary value problem Laplacian u = 0 in the disk and the normal derivative is prescribed as alpha x square + beta x + gamma for x, y belongs to S of 0, 2, where alpha, beta, gamma are real numbers.

Now, if the BVP admits a solution u which is C 2 of D 0, 2 close disk that means C 2 of closure of this open disk D 0, 2 then find a relation among alpha, beta, gamma that must be satisfied. So, this is an application of a Lemma that we have seen in lecture 6.1

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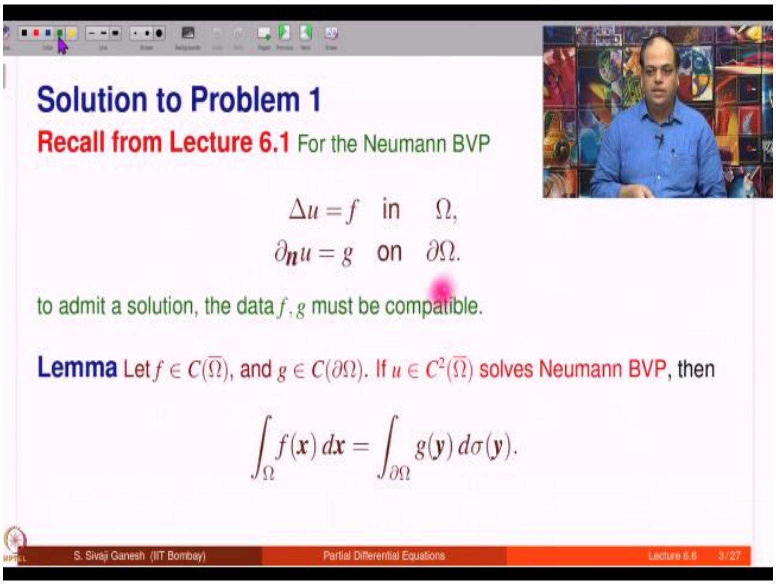

Recall from lecture 6.1 for the Neumann boundary value problem Laplacian u = f in omega dou n u = g on boundary of omega to admit a solution the data f and g must be compatible. What is that Lemma let f belongs to C of omega bar and g belong to C of boundary of omega if u belong to C 2 of omega bar solves Neumann boundary value problem which is here, then necessarily this integral of f over omega is equal to integral of g over boundary of omega.

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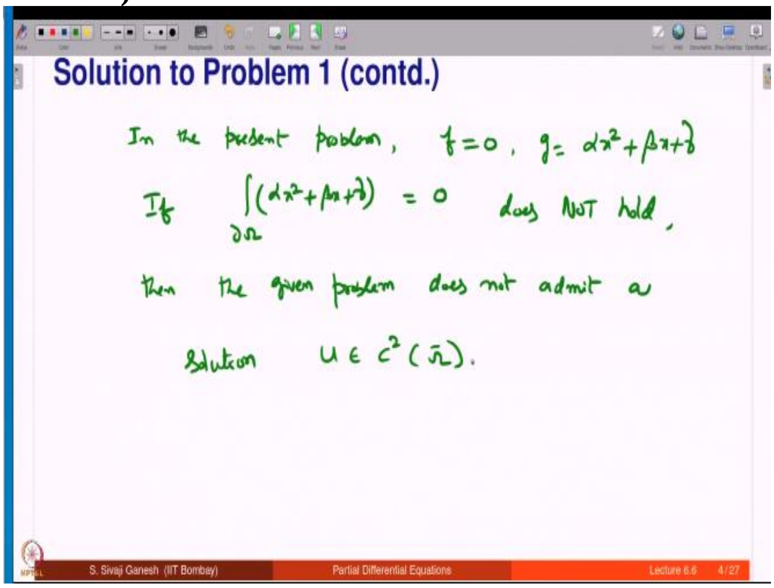

So, in the present problem f = 0 and g = alpha x square + beta x + gamma, so if alpha, beta, gamma are such that this integral over boundary of omega of alpha x square + beta x + gamma = 0 does not hold then the given problem does not admit a solution in C 2 for omega bar, but we are given that the boundary value problem has a solution nu which is in C 2 of omega bar therefore, this condition must be satisfied.

So, let us compute this integral if this does not hold then the given problem does not admit solution u belongs to C 2 of omega bar. Remember here that a solution of the Neumann problem is required to be only C 1 of omega bar but we have proved this compatibility condition and the assumption that u is C 2 omega bar that is the reason why we are assuming that u is C 2 of omega bar.

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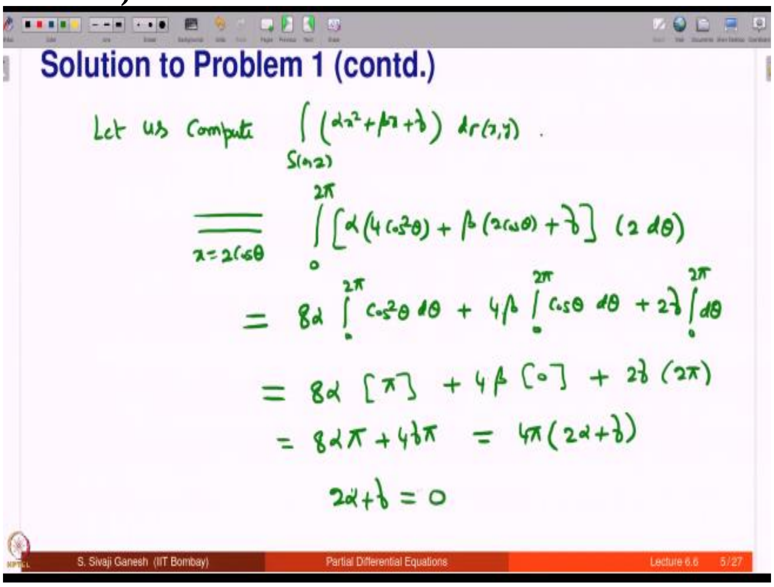

Let us compute the integral S of 0, 2 alpha x square + beta x + gamma d sigma of x, y, so x = 2 cos theta, y = 2 sin theta but y does not appear in our integral that will make this integral to be 0 to 2 pi alpha x is 2 cos theta. So, x square is 4 cos square theta + beta times 2 cos theta + gamma into the integration is 2 d theta, 2 is the radius of the circle, if the radius is R it will be R d theta. So, this is nothing but there is 4 alpha and there is a 2 here, so 8 alpha 0 to 2 pi cos square theta d theta + 4 beta 0 to 2 pi cos theta d theta + 2 gamma 0 to 2 pi d theta.

So that is equal to 8 alpha times pi + 4 beta times 0 because the average of cos theta on its period 0, 2 pi is 0 + 2 gamma into 2 pi. That is nothing but 8 alpha pi + 4 gamma pi which is equal to 4 pi if it a common I get 2 alpha + gamma. So, therefore, if u belongs to C 2 of omega bar is a solution, then definitely we must have 2 alpha + gamma = 0 is a necessary condition.

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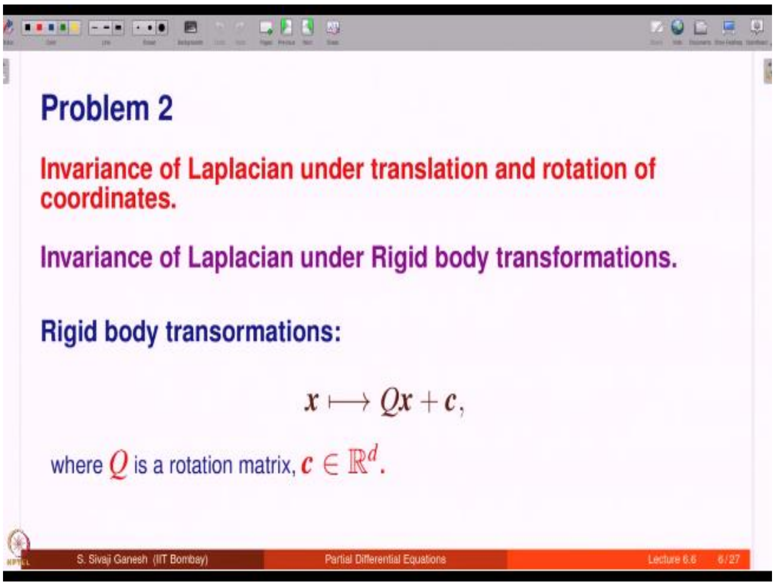

So, let us move on to problem 2 invariance of Laplacian under translation and rotation of coordinates. Invariance of Laplacian under rigid body transformations, what are rigid body transformations these are the maps x going to Qx + c, where Q is a rotation matrix and c is a fixed vector in R d. In other words, first x goes to Qx that means it is rotated and then you are translating the result and with a c so that is the rigid body transformations.

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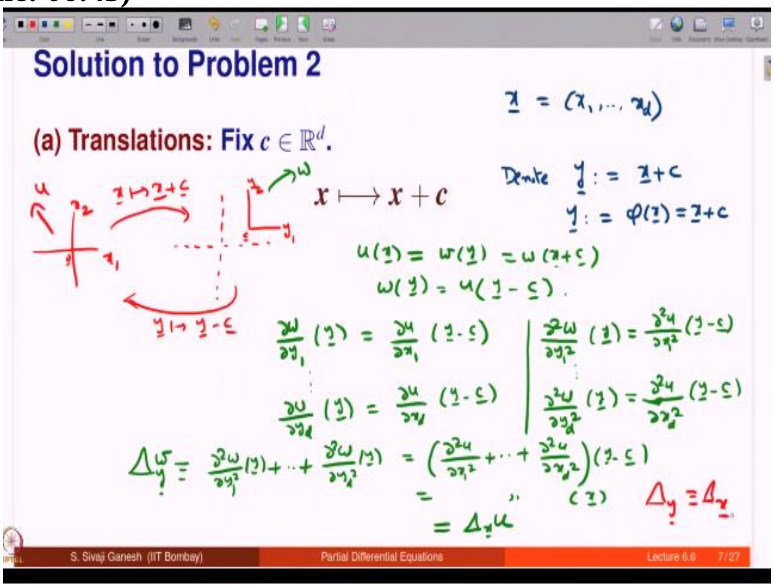

So, translations fix in R d x going to x + c is a translation, let us see that the Laplacian is invariant under this coordinate chain transformation. So, here x is nothing but x 1, x 2, x d is an element in R d. So, let us denote the new coordinates as y, actually we should be writing y is equal to some function of x which is x + c. Since we are in our experienced with change of coordinates already, I drop writing these phi I write y = x + c.

So, let us draw one picture so x 1, x 2 and you have a function u of x 1, x 2 and here you have this is y 1, y 2 this is a point c in your this coordinate system so it has origin has moved to c. And the mapping which connects this is x going to x + c and the inverse mapping is y going to y - c. So, let us write u of x bar = w of y bar so in terms of this new coordinates, let us write the function of w and this relation between them. So, actually, this is w of x + c and similarly, we can write w of y = u of y - c.

So, let us compute the derivatives and check that the Laplacian is invariant under change of coordinates which is occurring by translations. So, let us compute dou w / dou y 1 at the point y, that is dou u / dou x 1 at the point y - c. Similarly, dou w / dou y d at the point y is dou u / dou x d at the point y – c. Now, second our derivatives dou 2 w / dou y 1 square at the point y is once again dou 2 u / dou x 1 square at the point y - c. And same is true for all second order derivatives.

Now, let us write Laplacian these know how denote this Laplacian in y coordinates that means, let us write this acting on w that means it is dou 2 w / dou y 1 square y summed up to dou 2 w / dou y d square this by the above computations is nothing but dou 2 u / dou x 1 square I will write the argument at the end of x d square at the point y - c which is nothing but what is y - c in x. So that is nothing but Laplacian in the x coordinate acting on u. So, what we have got is Laplacian in the y coordinates, this operator is same as Laplacian in x coordinates, that is why Laplacian is said to be invariant.

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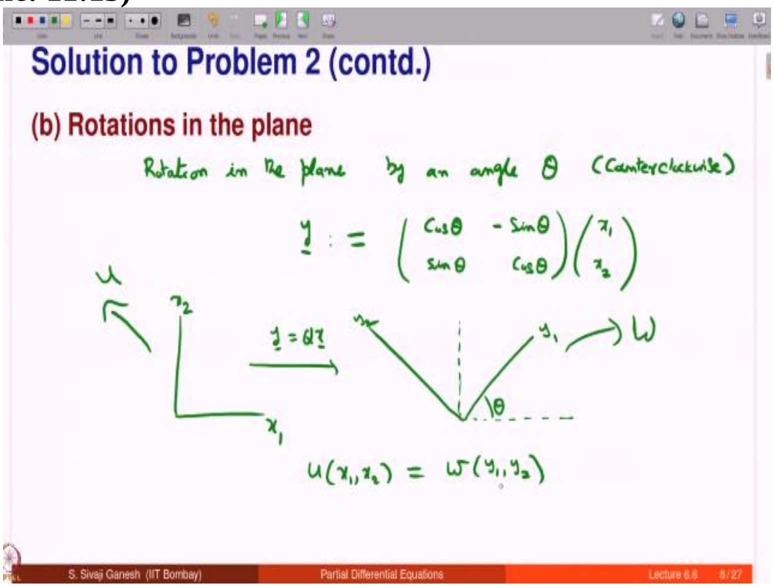

Now, let us show that Laplacian is invariant under rotations in the plane, so rotation in the plane by an angle theta, so say counter clockwise does not matter but the matrix that we are going to write represents such a rotation which is in the anti-clockwise direction. So, let us write the change of transformation cos theta - sin theta, sin theta cos theta acting on x 1, x 2 how do you check it is a counter clockwise rotation put 1, 0 and see what you get put 0, 1 and see what you get.

For example, if this is my x 1, x 2 direction, but new coordinates are like that. So, this is an angle theta this is y 1, y 2. So, if I have a function going out of this as u as before and w like that and from here to here we have this transformation y equal to let us write that matrix Q times x let Q denote this matrix which is here cos theta - sin theta, sin theta cos theta then we may write u of x 1, x 2 = w of y 1, y 2 we are going to differentiate this compute the Laplacian in both the coordinate systems and show that Laplacian is invariant.

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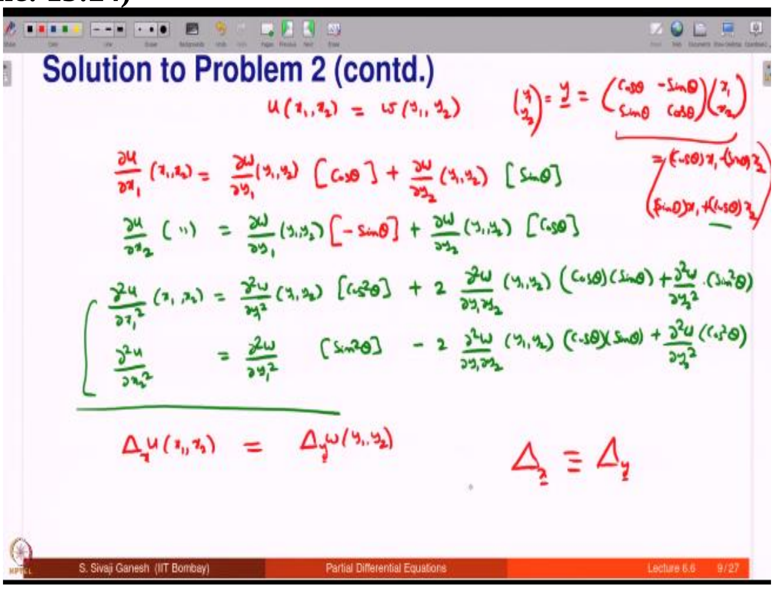

So, u of x 1, x 2 y 1, y 2 where y 1, y 2 are given in terms of x 1, x 2 by the matrix, so y = cos theta - sin theta, sin theta cos theta acting on x 1. So, this is y 1 y 2 now, let us differentiate. So, dou u / dou x 1 at the point x 1, x 2 by chain rule dou w / dou y 1 at the point y 1, y 2. So, y 1, y 2 is a shortcut for writing the center expression into derivative of y 1 with respect to x 1 which is cos theta.

Let us dou w / dou y 2 at the point y 1, y 2 into derivative of y 2 with respect to x 1 is sin theta for expand this what you get is cos theta into x 1 - sin theta into x 2 and the second component is sin theta x 1 + cos theta x 2. So, similarly dou u / dou x 2 I do not write arguments, but let me so now we get minus sin theta + dou w / dou y 2 into dou y 2 / dou x 2 which is cos theta.

So now, I write the expression for the second order derivatives check it dou 2 u / dou x 1 square at a point x 1, x 2 = dou 2 w / dou y 1 square at the point y 1, y 2 into cos square theta + 2 dou 2 w / dou y 1 dou y 2 at the point y 1, y 2 into cos theta into sin theta + dou 2 w / dou y 2 square into sin square theta. Similarly, dou 2 u / dou x dou x 2 square turns out to be dou 2 w / dou y 1 square into sin square theta - 2 times dou 2 w / dou y 1 y 2 into cos theta sin theta + dou 2 w / dou y 2 square into cos square theta.

So, summing these 2 equations, what we get is Laplacian in x coordinates on the left hand side equal to Laplacian in y coordinates of w on the right hand side. So that is Laplacian x Laplacian in y that means Laplacian is invariant under rotations in the plane.

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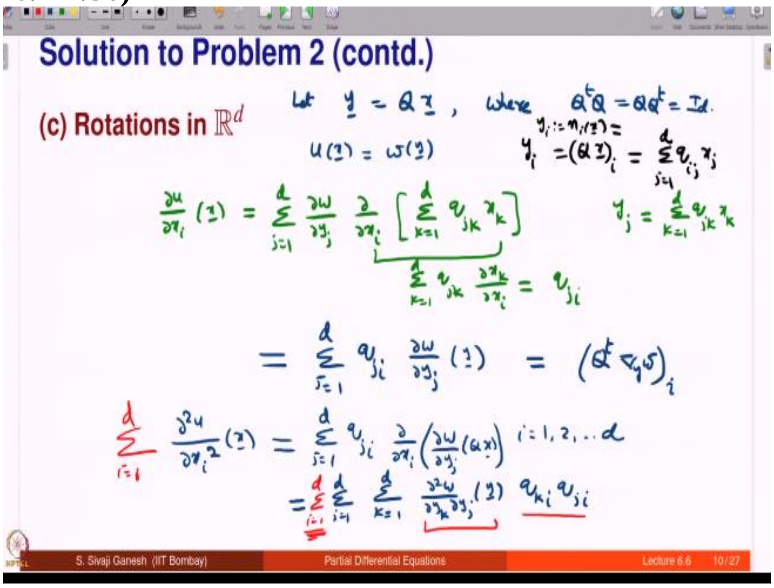

So, let us look at rotations in R d, d not necessary 2 so let y = Qx, Q is a rotation matrix so Q satisfies this relation Q transpose Q = Q Q transpose equal to the identity matrix before u of x = w of y, we compute Laplacian of u and Laplacian of w with respect to x and y coordinate system and check that they are the same. Let us express y i ith coordinate of y that is Qx and then ith coordinate of that. So, which turns out to be summation q ij x j, j = 1 to d, it is very important that we get used to summation notations like this.

Because the once dimension is about 2 or 3, it is almost impossible to write in a simpler manner. Therefore, we must get used to the summation. If you have any doubt you just try for y 1 y 2 y 3 and you can arrive yourself at this formula. As I told before, we should have ideally written y i = eta i of x which is equal to this expression here. But since we are in our experience, we are not writing that.

So, let us compute the derivatives dou u / dou x i at the point x, so that is dou w / dou y j dou y j / dou x i that is dou / dou x i of y j. What is y j from here this expression is for y i? So, y j means we have to change this notation, because this is anyway summation. So, y let us write y j let us k = 1 to d q jk x k. So, this is called a dummy index because you can just change j to k whatever you want because it is some up.

So, dou / dou x i of y j what is y j? Summation k = 1 to d q jk x k and then we are to sum over j that is the chain rule. So, please do this computation by yourself and then only proceed. So, this quantity is nothing but q j k is constants. So, dou / dou x i of the center quantity essentially is the same summation k = 1 to d q jk dou x k / dou x i, dou x k / dou x i = 1 if k = y otherwise it is 0 therefore, this quantity is nothing but it is nonzero only if k = i, in which case you get q ji.

Therefore these is equal to summation j = 1 to d q ji dou w / dou y j y at this point which if you want to see it as this Q transpose grad w with respect to y and ith component that is precisely dou u / dou x i. And this is true for all i. So, let us compute the second derivative dou 2 u / dou x i square at the point x that is equal to summation j = 1 to d q ji are constants. And we have to write dou / dou x i of this quantity dou w / dou y j at y but y is Qx by chain rule.

It turns out that this is equal to j = 1 to d summation a new summation will come k = 1 to d of dou 2 w / dou y k dou y j at the point Qx are equivalent k y into q ki q ji these expression you will get, but when we are interested in the Laplacian, we must put a summation here from i = 1 to d and here i = 1 to d. There are a lot of summations here, but they are all finite summations therefore, we can easily interchange the order in which we sum. What we do is we take the summation i = 1 to d and sum these up by freezing k and j I will write this again on the next slide.

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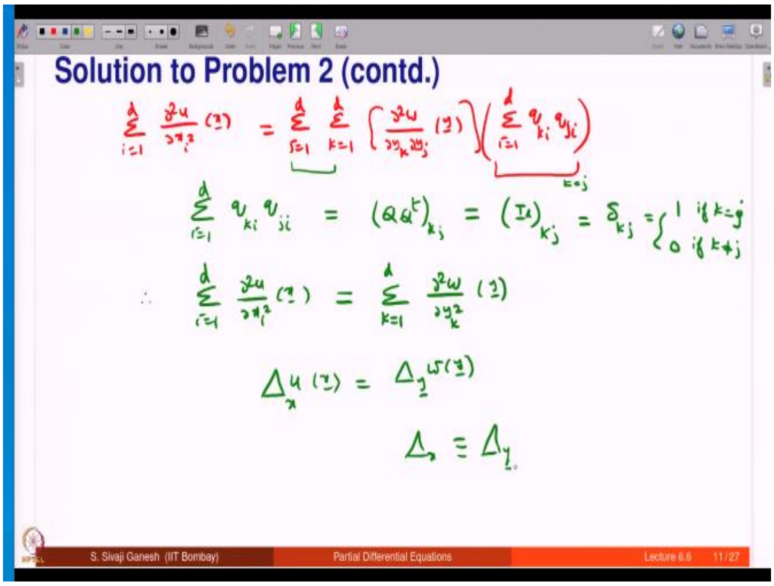

So, there is what we have is summation i = 1 to d of dou 2 u / dou x i square at the x the Laplacian and that equal to j = 1 to d k = 1 to d. Then we had dou 2 w / dou y j dou y k at the point y and summation i = 1 to d q ki q ji. We know something about this what is that we know that Q Q transpose = Q transpose Q equal to identity. So, this quantity i = 1 to d of q ki q ji is nothing but Q Q transpose this matrix under k j entry in that matrix that means, an element which is on the kth row and j column but Q Q transpose is identity.

So, identity matrix and kj entry this is a notation that we normally use delta k j it stands for 1 if k = j 0 if k is not equal to j. Therefore, what we obtain at the end is summation i = 1 to d dou2 u / dou x i square at the point x equal to so therefore, this summation survives only if k = j, therefore this summation will just become 1 summation with k = j you could use either k or j does not matter.

So, k = j in particular means it is dou y k dou j k that means dou 2 w / dou y k square the point y which is nothing but Laplacian in the x coordinates of u at a point x = Laplacian in y coordinates of w at the point y in other words Laplacian x Laplacian y are the same. So, the Laplacian is invariant under rotations in R d.

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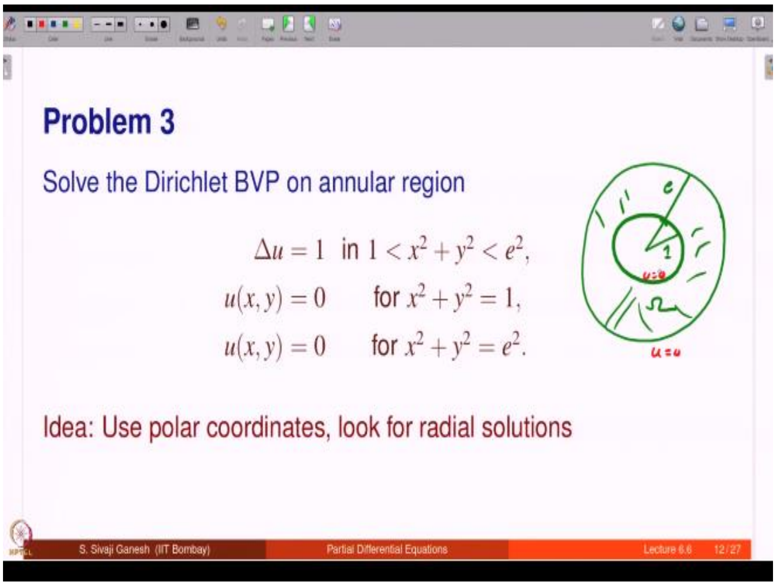

Let us move on to problem 3 solve the Dirichlet boundary value problem on annular region x square + y square is between 1 and e square in other words, look at circle of radius 1 is the inner boundary of the angular region and this one with radius e. So, this is our region omega here x square + y square will be bigger than 1 on less than e square and u is given to be 0 on the boundary. So, u = 0 here, u = 0 here also. So, the idea is to use polar coordinates and look for radial solutions.

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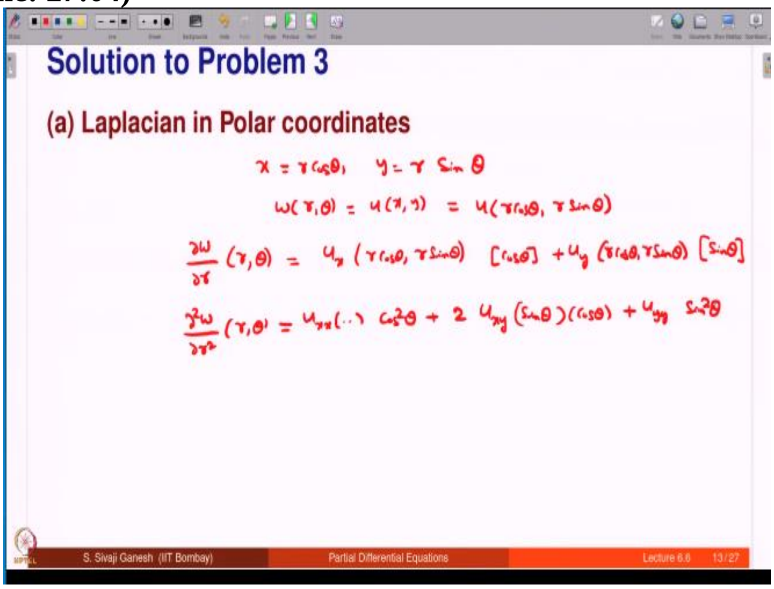

So, let us compute the Laplacian in polar coordinates first, so x = r cos theta, y = r sin theta so w of r theta = u of x, y. So, but actually x = r cos theta and y = r sin theta so it is u of r cos theta r sin theta. So, let us compute the derivatives dou w / dou r at the point r, theta that is nothing but u x that r cos theta r sin theta derivatives of x with respect to r which is cos theta. And u y at the point r cos theta r sin theta and derivatives of y with respect to r which is sin theta, so this is dou w / dou r.

So, if we compute dou 2 w / dou r square we get u xx I am not writing the argument it is r cos theta sin theta into cos square theta + 2 u xy sin theta cos theta + u yy into sin square theta this is dou 2 w / dou r square.

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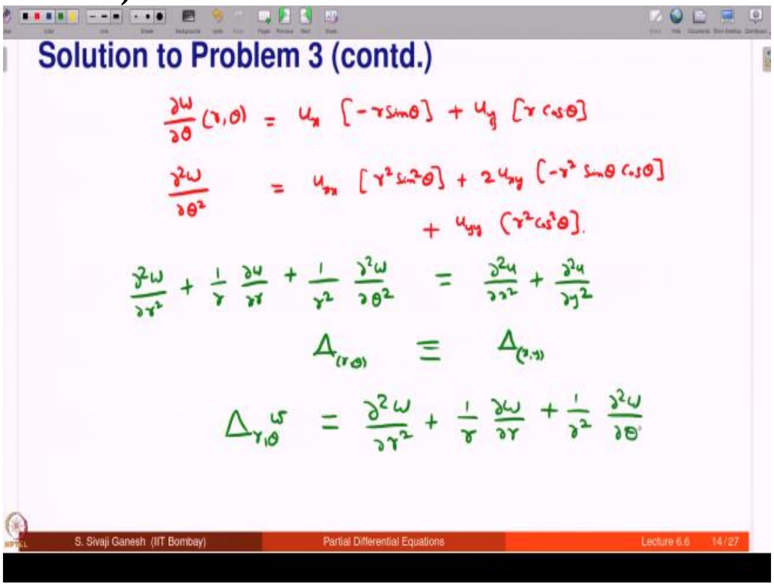

Let us compute dou w / dou theta at the point r, theta that will be u x into - r sin theta + u y into r cos theta. And dou 2 w / dou theta square = u xx into r square sin square theta + 2 u xy into - r square sin theta cos theta + u yy into r square cos square theta. Let us some this dou 2 w / dou r square + 1 / r dou w / dou r + 1 / x square dou 2 w / dou theta square that will turn out to be dou 2 u / dou x square + dou 2 u / dou y square. So, this is the Laplacian in the xy coordinates and therefore, this is the Laplacian in r, theta coordinates.

So, what is that Laplacian r, theta coordinates acting on a function w of r, theta is dou 2 w / dou r square + 1 / r dou w / dou r + 1 / r square dou 2 w / dou theta square.

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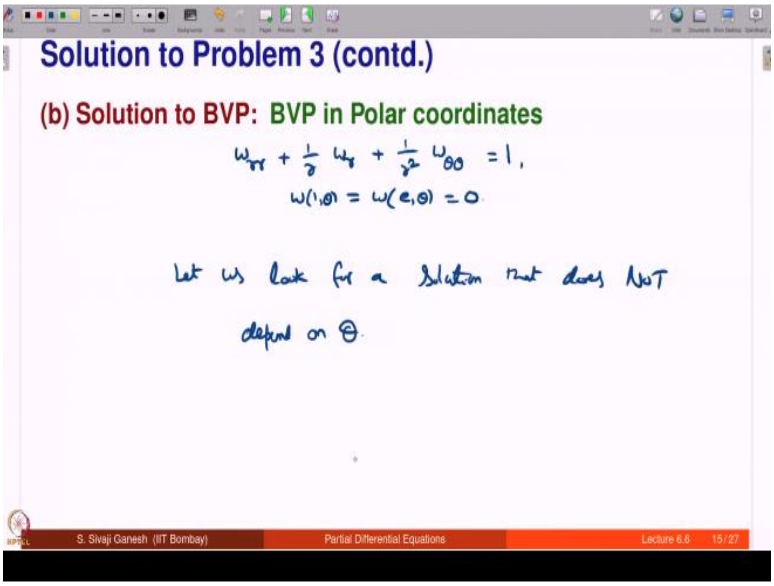

So, let us write the boundary value problem in the polar coordinates Laplacian = 1 has now become fragmented w rr + 1 / r w r + 1 / r square w theta theta = 1, w of 1, theta = w of e, theta is 0. So, let us look for a solution that does not depend on theta, if we are successful, we will end up with a solution to the boundary value problem. And we know that boundary value problem has a unique solution. Therefore, the solution we have found is a solution that is idea behind looking for a solution which does not depend on theta in case we are successful, it is fine if not we have to admit theta dependence and try to solve again.

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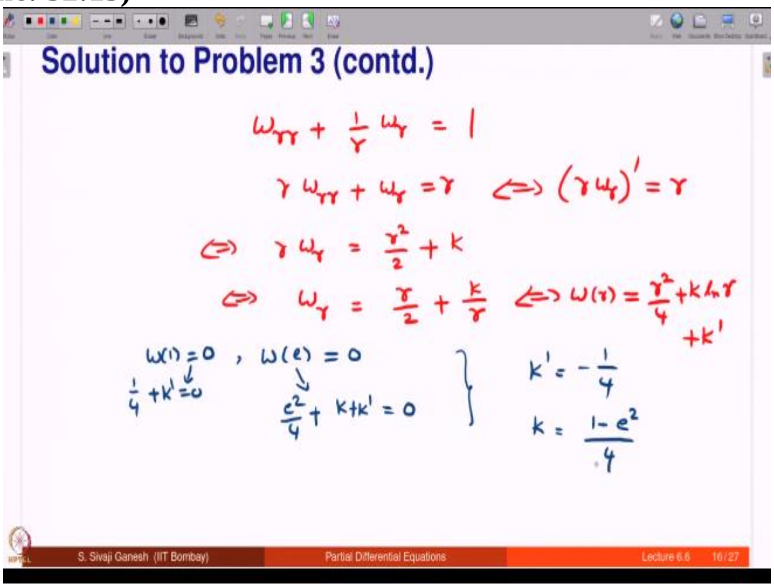

So, w rr + 1 / r w r = 1 so this equation can be rewritten as r w rr + w r = r that is if and only if r w r dash, dash is the derivative with respect to r = r. So that tells us we can solve for r w r which is r square / 2 plus a constant K and that implies that w r is r / 2 + K / r and from here we can again solve for w as a function of r only it does not depend on theta. So, I do not mention the theta dependence here. So, on integrating the right hand side what we get is r square / 4 + K log r plus another constant K dash.

We are determined what are K and K dash using the boundary conditions, what are the boundary conditions we have w of 1 = 0 and w of e = 0. So, w of 1 = 0 will give us 1 / 4 + K dash = 0 this will give us this and this will give us e square / 4 + K + K dash = 0. So, solving this system of equations we get K dash = - 1 / 4, K = 1 - e square / 4.

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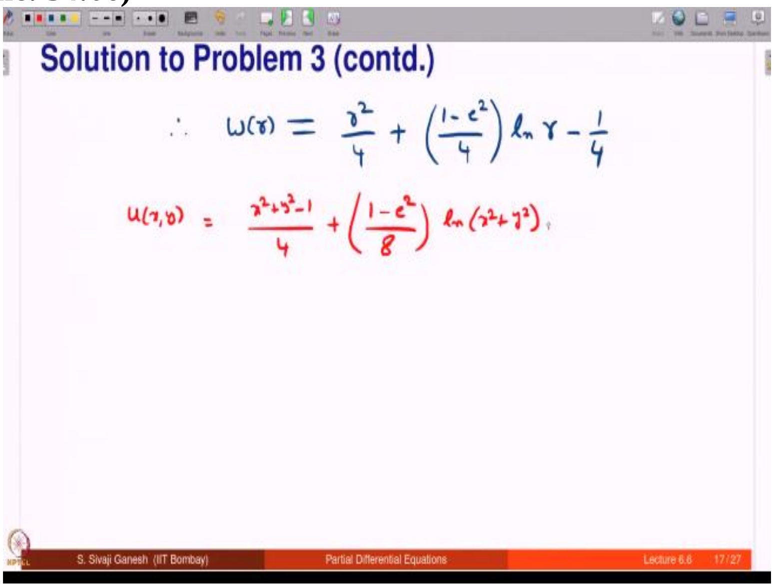

Therefore, w of r = r square / 4 + 1 - e square / 4 into log r - 1 / 4, so in terms of x and y coordinates u of x y = x square + y square - 1 / 4 + 1 - e square / 8 into log x square + y square. So, it is not necessary to resolve to separation of variables method all the time. One could also use separation of variables to solve this problem, so maybe that you can try.

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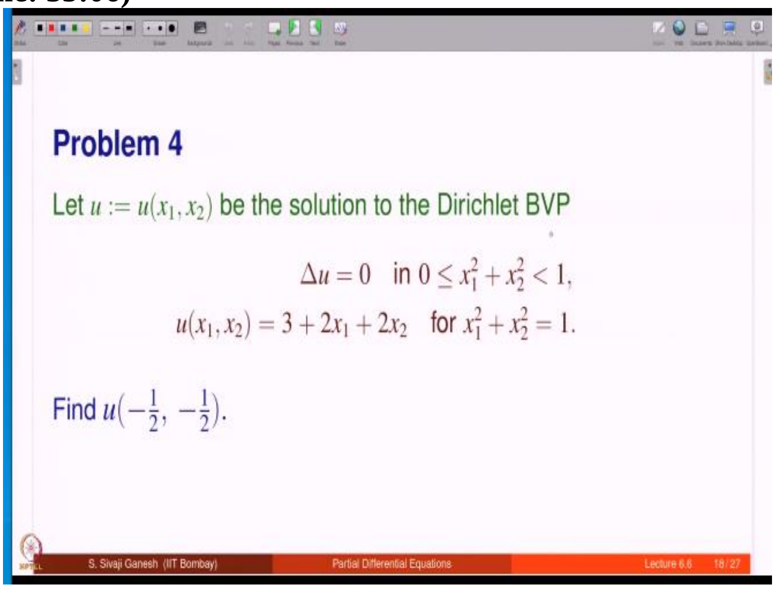

Let us move on to problem 4, let u be the solution to the Dirichlet boundary value problem in the unit disk Laplacian u = 0 in the unit disk and on the boundary of the unit disk u is given to be 3 + 2x 1 + 2x 2 and find u of minus half, minus half. So, how will you find one option is to explicitly solve this boundary problem and then evaluate at minus half, minus half. Another option is to use Poisson's formula indeed we are going to do that.

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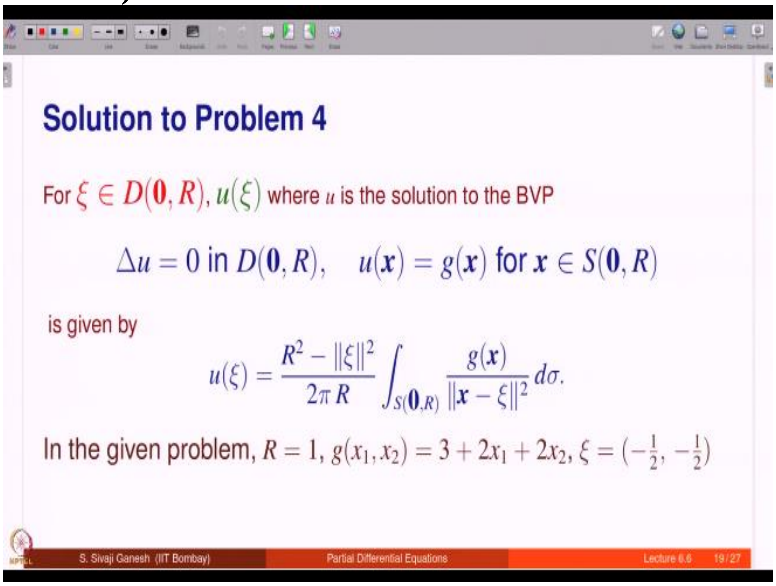

For Xi in D of 0, R u of Xi where u is a solution to this boundary value problem Laplacian u = 0 in the disk and u = g on the boundary that is given by Poisson's formula u of Xi = r square – norm Xi square / 2 pi R into integral over the circle gx by norm x - Xi square d sigma. The given problem R = 1, g = 3 + 2x 1 + 2x 2 and Xi = minus half, minus half.

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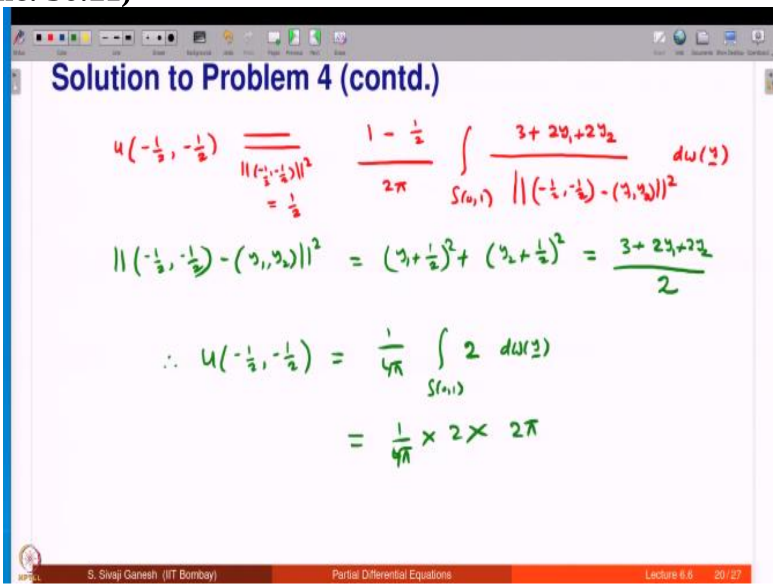

So, u of minus half, minus half equal to because what is the norm of minus half, minus half we need norm square, so that is equal to half so what we get is 1 – 1 / 2 divided by 2 pi into integral S of 0, 1 3 + 2y 1 + 2y 2 divided by norm of minus half, minus half – y 1, y 2 square d omega of y. So, let us evaluate what is the denominator in this integral so there is norm minus half, minus half – y 1, y 2 square is y 1 + 1 / 2 whole square + y 2 + 1 / 2 whole square which on simplification you will get 3 + 2y 1 + 2y 2 / 2.

Therefore, u of minus half, minus half is nothing but 1 / 4 pi integral of S of 0 1 into 2 d omega of y I have chosen simplest of the functions as Dirichlet data on the boundary, that is why it became very simple is 1 / 4 pi into 2 into the perimeter of unit circle, which is 2 pi so that is equal to 1.

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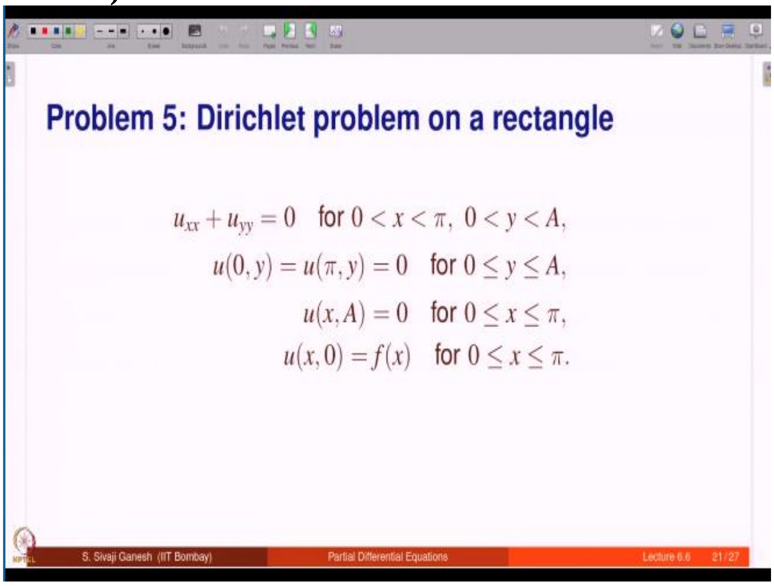

So, let us move on to problem number 5 it is a Dirichlet problem on a rectangle, so we have to solve Laplace equation in this rectangle and we are given boundary conditions only one of them is taken to be nonzero in case many of them are nonzero we are to split into sub problems where only one data is nonzero rest of them are 0 solve them and using linearity we can superpose and get a solution to the entire problem. So, let us solve this simpler problem, we are going to use separation of variables method to solve this BVP.

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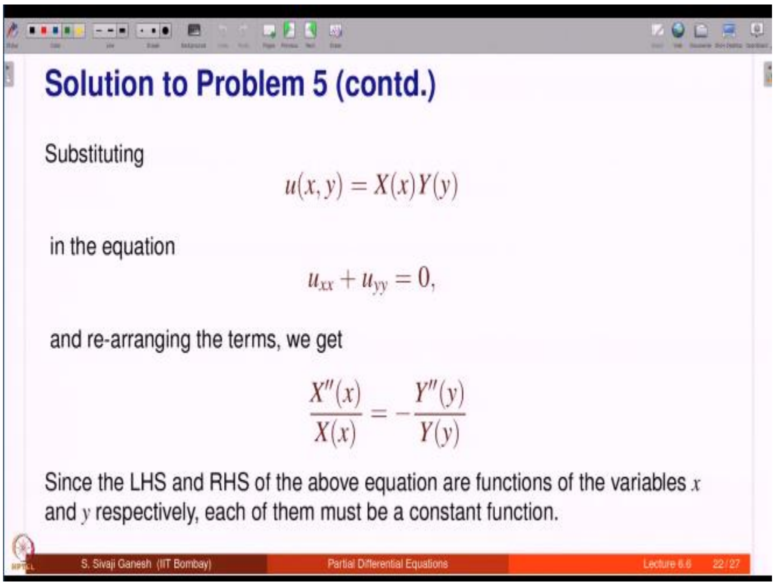

We are experts therefore; I just quickly go through the computations. So, we substitute u = X of x into Y of y and the Laplace equation and then rearranging the terms we get this now, we observe that the LHS depends only on X, RHS is a function of only Y and that is possible if and only if both of them are separately equal to a constant function.

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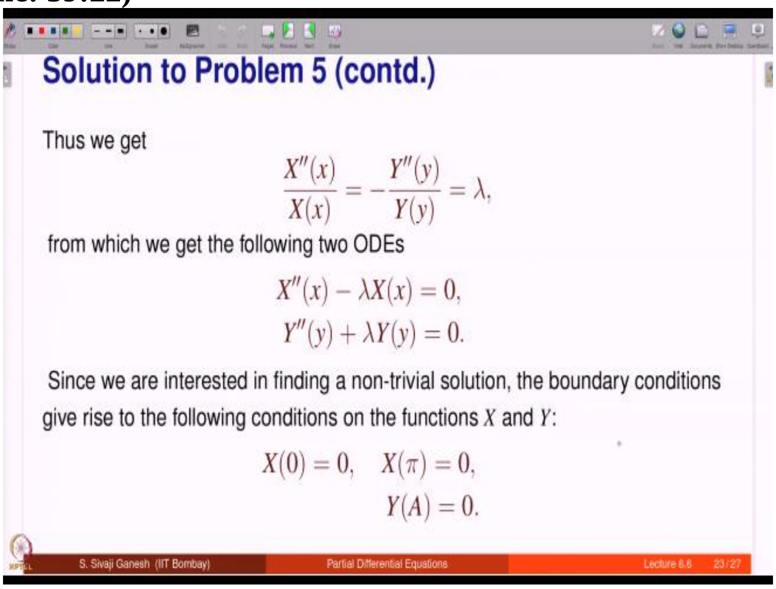

So, we put lambda here and from here we get the 2 ODEs we have to get the boundary conditions for these ODEs using the given boundary conditions we get this because we are not interested in finding solutions where either X pi = 0 or YA = 0 because it is of no help in trying to obtain solution to the given boundary problem. So, because of that we end up with these conditions for X and Y.

Now we see that for X we have 2 boundary conditions second order equation so we will start solving for X and find out those lambdas we admit non trivial solutions and for those lambdas we solve for Y use this initial condition and then multiply them and propose his superposition as a formal series solution.

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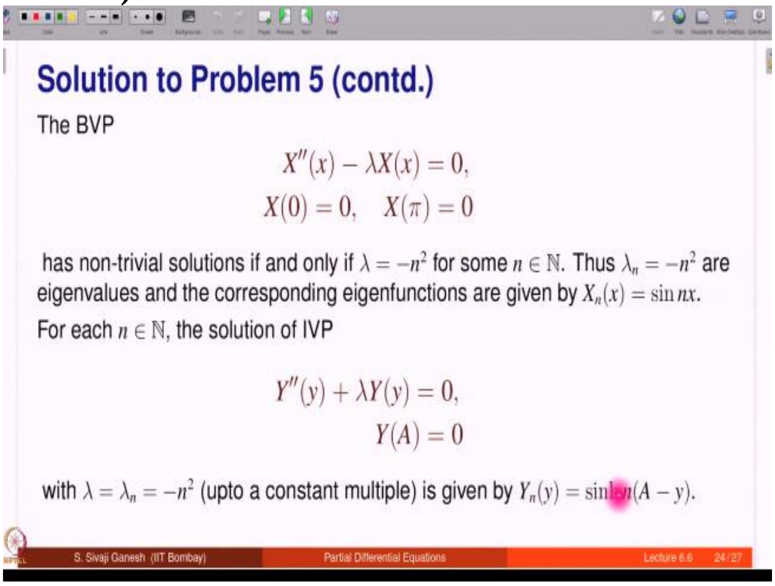

So, this boundary value problem for X has non trivial solutions if and only if lambda = - n square for some natural number. Therefore, lambda n = - n square are eigenvalues and corresponding eigenfunctions are sin nx. Now, for each n you solve this problem for Y, where lambda = lambda n = - n square solution is given by this up to a constant multiple.

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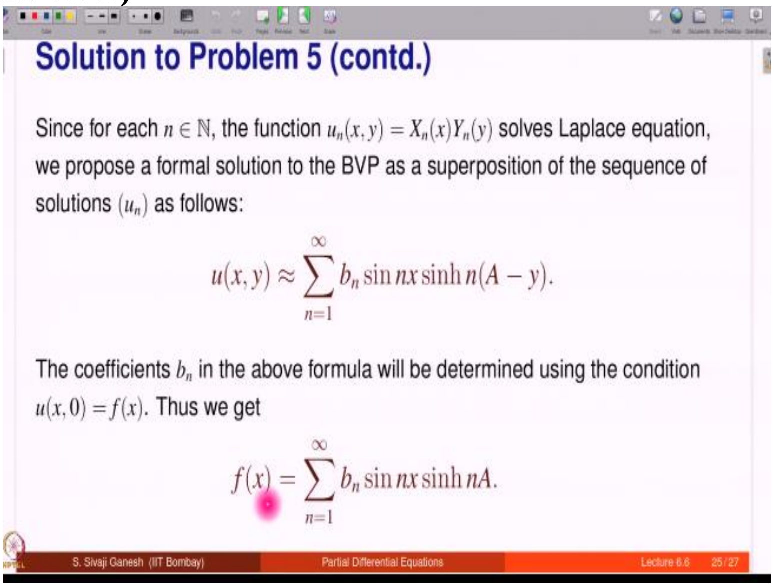

Now, for each n this product X n x into Y n y solves the Laplace equation so therefore, we propose a formal solution to the boundary problem as a superposition of u n like this. Now what is remaining is to find this b n and we have one more boundary condition which we have not used using that we will try to find b n and that is u x, o = fx. So, we get fx equal to this putting y = 0 we this reduces to this. So, we need to find b n so that fx is given by such a series.

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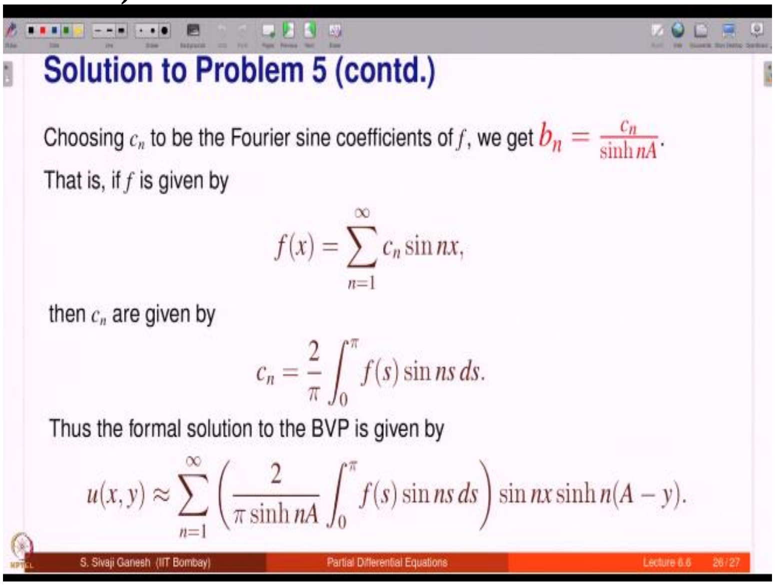

So, therefore choose c n to be the Fourier sine coefficients of f, we get b n = c n / sin hyperbolic nA. If f is given by the series then c n are given by this formula. We have already done this Fourier sine series earlier in the context of wave equation. So, please do the computations. Therefore, the formal solution is given by this expression. So, separation of variables method in principle you should be able to solve any problem because the basic idea is the same. Thank you.