**Partial Differential Equations**
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**Lecture – 2.4**
**First Order Partial Differential Equations**
**Lagrange's Method for Quasilinear Equations**

In the last lecture, we have discussed some methods to find general solutions to linear and semilinear equations. In this lecture, we take the method one more step forward namely to deal with Quasilinear equations. We are going to describe Lagrange's method for finding general solutions to Quasilinear equations.

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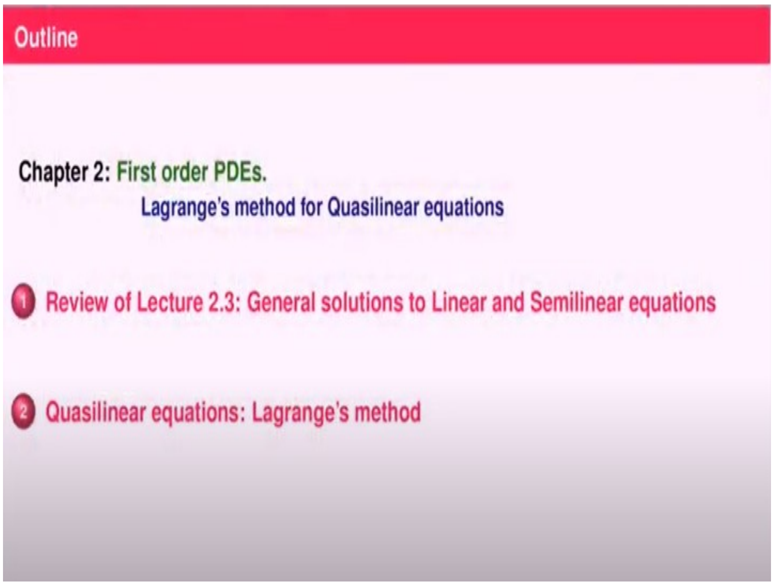


We start with a brief review of the last lecture lecture 2.3, where general solutions to linear and semilinear equations have been obtained, basically a method was described. Now, we are going to describe in today's class Lagrange's method for finding general solutions to Quasilinear equations. How the ideas from linear and semilinear can be extended to the case of Quasilinear equations? Whether it is possible or not, we are going to see.

And which part of this method for linear and semilinear can be extended, which cannot be we will see that as well.

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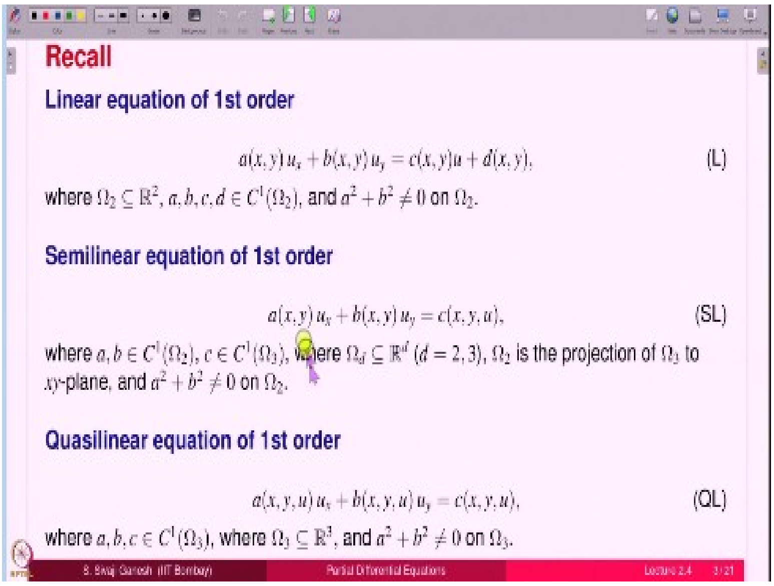


So, let us recall just to reinforce the notations, L stands for the linear equation $a(x,y)u_x$ + $b(x,y)u_y$ = c (x, y )u + d (x, y). SL stands for semilinear equation, where the left hand side is same as that of the linear equation. That means, the manner in which the first order partial derivatives appear is exactly the same. However, the right hand side can depend nonlinearly on c ( x, y ).

Now, QL stands for Quasilinear equations. Here, the coefficients of $u_x$ and $u_y$ can depend on u also. So, $a(x,y)u_x$ + $b(x,y)u_y$ = c (x, y )u . And hypotheses that we will be working under is a b c are $\mathbb{C}^1$ functions on omega 3. Omega 3 is a subset of $\mathbb{R}^3$, open subset of $\mathbb{R}^3$. And as usual, we require that both a and b do not vanish simultaneously at each of the points of omega 3. So, this is Quasilinear equation.

As you observed, the first order partial derivatives, the coefficients in both L and SL, they depend only on x and y. Whereas, in Quasilinear equations, they depend on u as well. This is the difference between semilinear, linear and Quasilinear equations.

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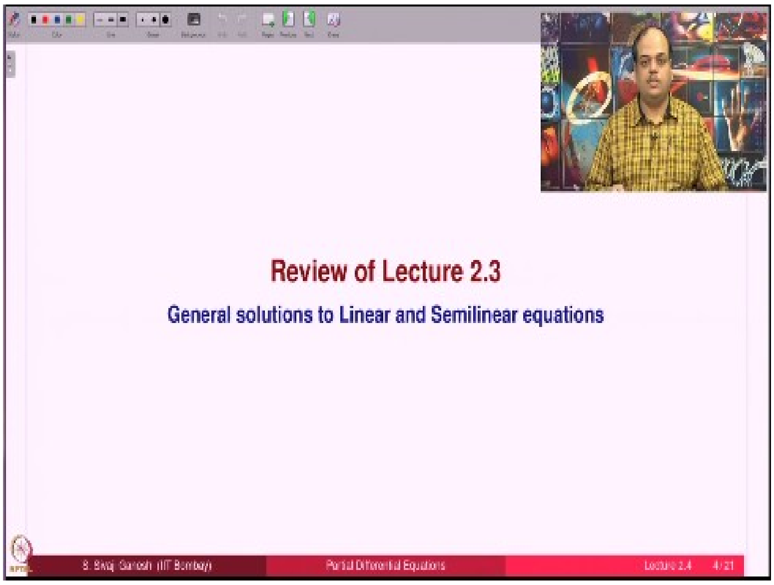


So, let us review lecture 2.3 now, where we have obtained general solutions to linear and semilinear equations, basically a method was described. Now, it is up to us whether we are able to implement it or not that is silent upon but it is possible that is what the method says.

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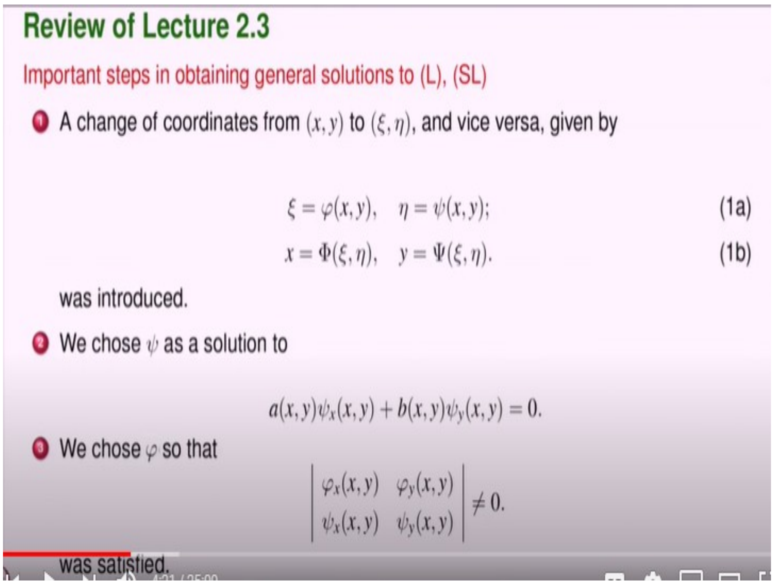


What are the important steps in obtaining general solutions? There is a change of coordinates involved x y to $\xi$ $\eta$. Because, in L and SL, both $u_x$ and $u_y$ appear, so we thought that we will eliminate one of those partial derivatives. But you know, $u_x$ and $u_y$ are appear, x and y coordinates are there, coefficients a b are given, we cannot suddenly make it 0. Then we thought let us change coordinates and then find out if there is a possibility of finding change of coordinates where after the transforming the PDE.

The new PDE in the new coordinates will feature only one derivative, maybe with respect to $\xi$ or $\eta$only one of them. That was the idea. Therefore, there is a change of coordinates in the background. Remember change of coordinates works in x y plane. That is in omega 2. Some subset of omega 2, there we have both x y and $\xi$ $\eta$ coordinates. And then we choose a $\psi$ to be a solution of this linear homogeneous PDE. And then we choose $\varphi$ such that this Jacobian

$$\begin{vmatrix} \varphi_x(x,y) & \varphi_y(x,y) \\ \psi_x(x,y) & \psi_y(x,y) \end{vmatrix} \neq 0.$$ is never 0.

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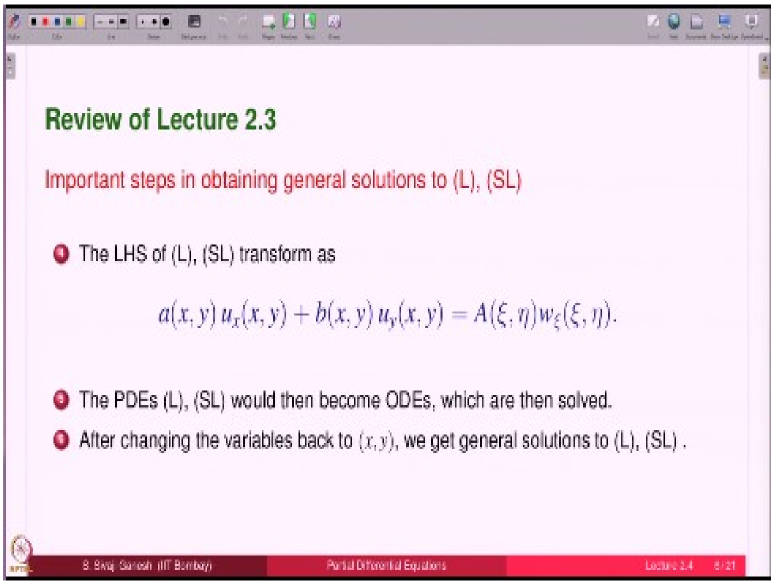


Then the LHS of L and SL which is this now becomes *to* $a(x,y)\, u_x(x,y) + b(x,y)\, u_y(x,y) = A(\xi,\eta) w_\xi(\xi,\eta)$.this, it features only the derivative with respect to $\psi$. And the PDEs will then become ODEs, which are then solved that was the idea. Now, after changing the variables back to x y from $\xi$ $\eta$ , we get the solutions as functions of x and y.

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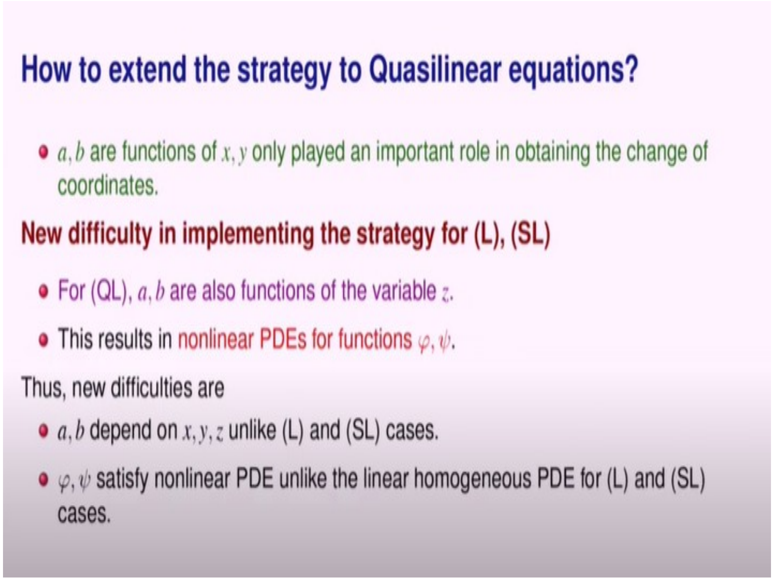


Now, how to extend the strategy to Quasilinear equations? a b are functions of x y only. They played an important role in obtaining change of coordinates. Therefore, the new difficulty in implementing this strategy is that for Quasilinear equations, a b are also functions of the variable set. So, this results in nonlinear PDEs for $\psi$ equivalent to $\varphi$, because we have as we observed, the equations are the same and only one of them is useful.

So, we can only find one function. Because, if you choose both $\varphi$ and $\psi$ as solutions are the same equation, then we will get the Jacobian will be 0. So, that is the reason we choose $\psi$ first. And then we choose $\varphi$ phi as anybody says that the Jacobian is not 0. So, now, it is clear the new difficulties, a b depend on x y z unlike L and SL.

This we are mentioning for the completeness sake in the sense that $\varphi$ $\psi$'s satisfy nonlinear **(05:56)** useless because change of coordinates, now, had to be done in x y z coordinate not in x y but the PDEs only x y variables. So, therefore, we do not expect the change of coordinates can be done before in what other way we can extend the strategy. Let us discuss that now.

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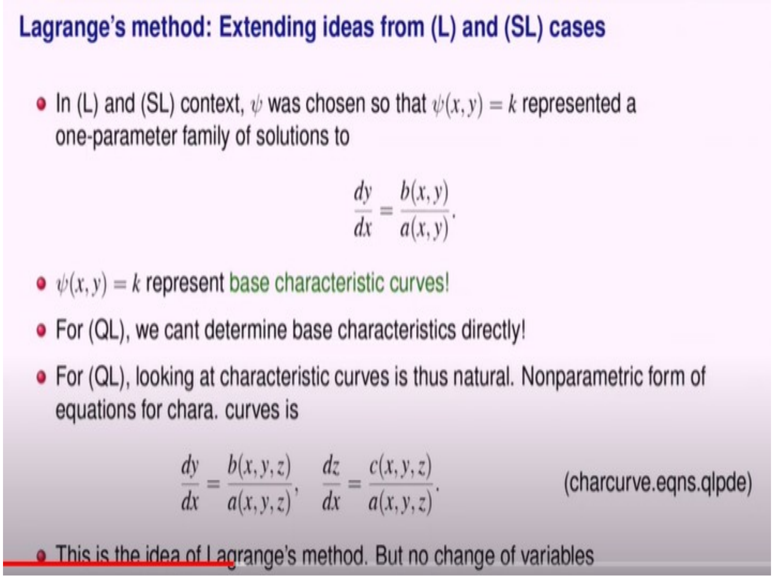


In L and SL, $\psi$ choosen such that $\psi$ equal to k represented a one parameter family of solutions to $\frac{dy}{dx} = \frac{b}{a}$, this what we did right. Now, in the Quasilinear case what happens is b ( x y z), a (x y z )will come. So, necessarily we have to bring in some equation for z. We will come to that. So, $\psi$ ( x , y )= k represent base characteristic curves,

$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$is the equation for base characteristic curves after eliminating the parameter t.

Therefore, for Quasilinear equations, we cannot determine base characterised directly that we know because of the presence of z in both a and b. So, for Quasilinear equations, looking at characteristic curves is thus natural. So, $\frac{dy}{dx} = \frac{b(x,y,z)}{a(x,y,z)}, \quad \frac{dz}{dx} = \frac{c(x,y,z)}{a(x,y,z)}$.this is the way we are going to extend ideas. We are saying that for linear and semilinear looking at base characteristic curves helped us in obtaining a general solution to L and SL. For QL, we have to look at full characteristics.

So, nonparametric form of the equations for characteristic curves, these are after eliminating the parameter t is $\frac{dy}{dx} = \frac{b}{a}, \frac{dz}{dx} = \frac{c}{a}$. This is the idea of Lagrange's method but no change of variables as expected. So, let us now describe Lagrange's method for Quasilinear equations.

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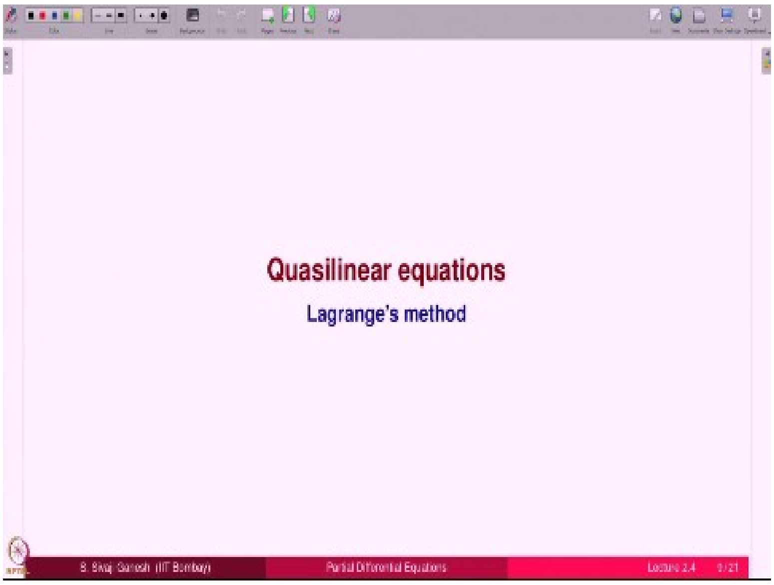


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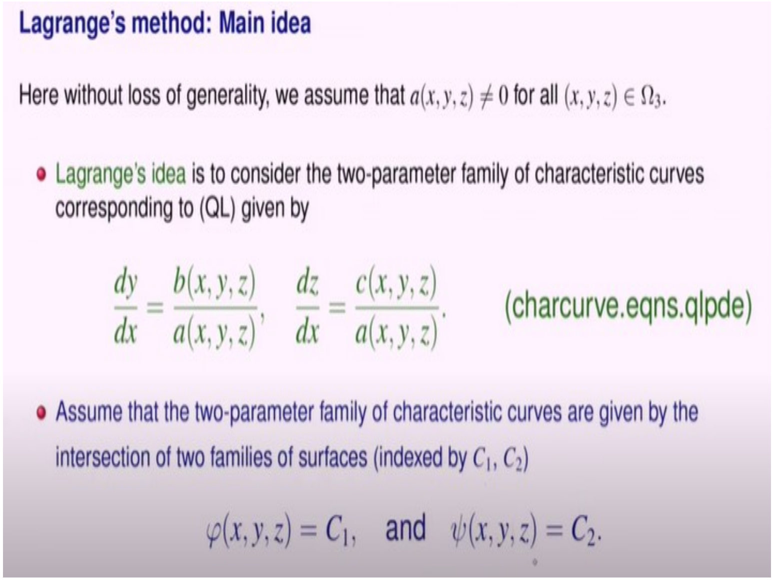


So, Lagrange's method the main idea is this, before we present that we will assume that a is never 0 on omega 3. So, Lagrange's idea is to consider the 2 parameter family of characteristic curves corresponding to Quasilinear equation given by $\frac{dy}{dx} = \frac{b(x,y,z)}{a(x,y,z)}, \quad \frac{dz}{dx} = \frac{c(x,y,z)}{a(x,y,z)}$.these are the equations of the characteristic curves. Assume that the 2 parameter family of characteristic curves are given by intersection of 2 families of surfaces $\varphi(x,y,z) = C_1, \quad \text{and} \quad \psi(x,y,z) = C_2.$

This is the assumption. We will see an examples that we can get this. So, finding the really the $\varphi$ , $\psi$ is the most challenging part of this Lagrange's method.

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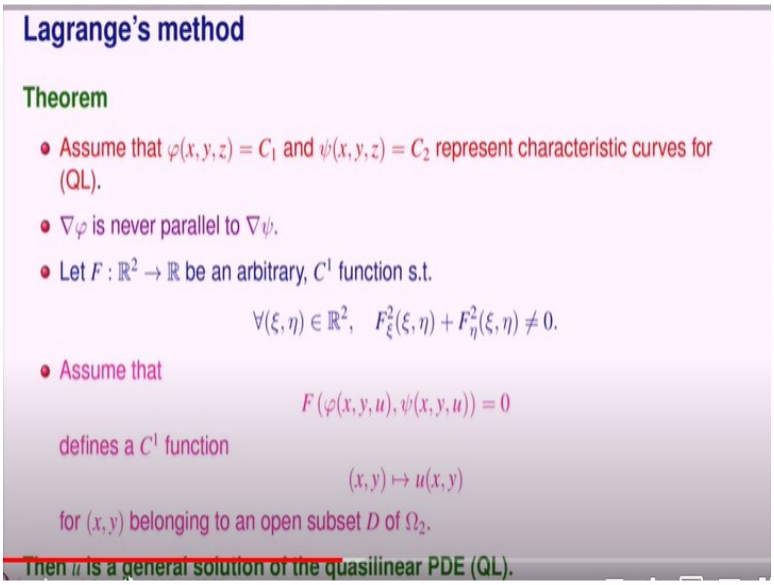


Let us take the theorem of the Lagrange's method. Assume that $\varphi(x,y,z) = C_1$, and $\psi(x,y,z) = C_2$. represent characteristic curves for QL. And $\nabla\varphi$ is never parallel to $\nabla\psi$. Although x y z which satisfy both these equations, at those points $\nabla\varphi$ is never parallel to $\nabla\psi$.

Let F be an arbitrary $C_1$ function defined in $\mathbb{R}^2$, such that $\forall(\xi,\eta) \in^2, \quad F_\xi^2(\xi,\eta) + F_\eta^2(\xi,\eta) \neq 0$.In other words, the gradient of phi sorry gradient of F is never 0 at every point in $\mathbb{R}^2$because it is domain of F okay. If you do not have $\mathbb{R}^2$here and some domain of F, then we require this for every $\xi$ $\eta$ belonging to domain of F. F ( $\xi$ , $\eta$ ) $notequalto$ 0,0. They do not simultaneously vanish.

Assume that $F(\varphi(x,y,u), \psi(x,y,u)) = 0$, this equation defines a C 1 function. x y mapping to u of x y. That means this is an implicit equation for u and that can be solved. u can be solved in terms of the other 2 variables, for x y belonging to some open subset of omega 2. As usual we do not demand that everything should happen on omega 2 because typically these the existence of results which deal with such assertions is implicit function theorem that is once again a local theorem.

So, for x y belonging to an open set, we demand that $F(\varphi(x,y,u), \psi(x,y,u)) = 0$, gives rise to a function u (x, y). If you have $\varphi$ , $\psi$ or $\mathbb{C}^1$, automatically u will become $\mathbb{C}^1$. That is we are discussing now, whether you can always assert this that is implicit function theorem, but as far as this theorem is concerned, we assume that such a thing exists. Then u is a general solution of the Quasilinear PDE QL.

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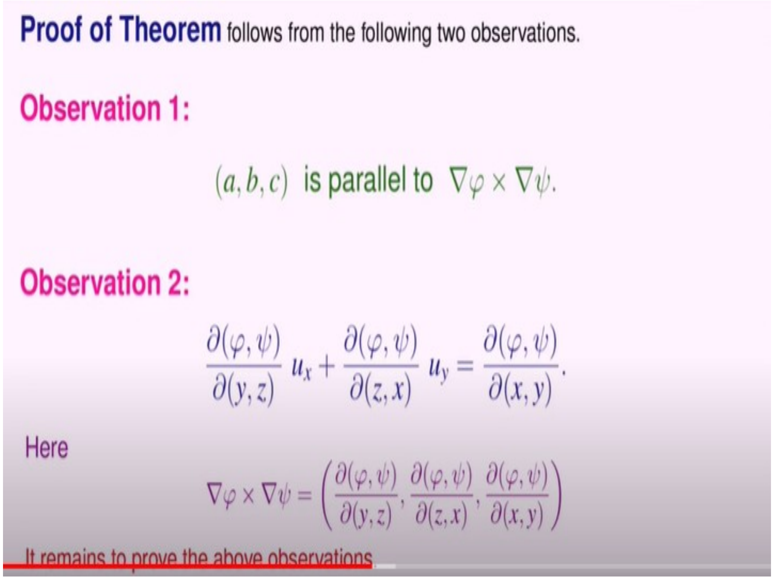


First observation is a b c is parallel to $\nabla\varphi \times \nabla\psi$.

Observation 2 is such a PDE satisfied. $\frac{\partial(\varphi,\psi)}{\partial(y,z)} u_x + \frac{\partial(\varphi,\psi)}{\partial(z,x)} u_y = \frac{\partial(\varphi,\psi)}{\partial(x,y)}$.

$$\nabla\varphi \times \nabla\psi = \left(\frac{\partial(\varphi,\psi)}{\partial(y,z)}, \frac{\partial(\varphi,\psi)}{\partial(z,x)}, \frac{\partial(\varphi,\psi)}{\partial(x,y)}\right).$$

In other words, this is $\psi$. This is the first component; this is the second component and this RHS is the third component. If this is parallel to a b c, what does that mean? This vector is some constant times a b c. Therefore, that let us say alpha a b c.

So, you can substitute these alpha a, alpha b and alpha c; cancel alpha, what you get is a u x plus b u y equal to c. Therefore, these 2 observations proved the theorem. So, therefore, the proof of the theorem follows from the following 2 observations. So, what remains to prove is these 2 observations.

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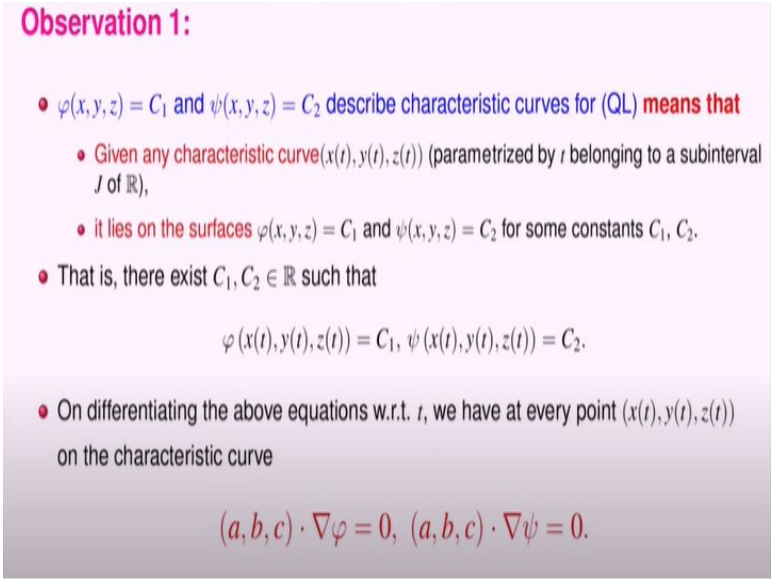


Observation 1: what is observation 1? a b c is parallel to $\nabla\varphi \times \nabla\psi$. So, $\varphi$ = C $_1$ and $\psi$ = C $_2$ describe characteristics curves for QL, these assumptions. What does it mean? It means that given any characteristic curve x t, y t, z t parameterize by t belonging to some interval J of R, x t, y t, z t satisfies both the equations. What are the equation? $\varphi\left(x(t), y(t), z(t)\right) = C_1,\ \psi\left(x(t), y(t), z(t)\right) = C_2$., it lies on the surfaces.

That is the meaning of saying, this and this together describe characteristic curve. It means if characteristic curve, they lie on the intersection of these 2 surfaces. So, it means that there is a $C_1$ and $C_2$ such that $\varphi\left(x(t), y(t), z(t)\right) = C_1,\ \psi\left(x(t), y(t), z(t)\right) = C_2$. Now, once we have this, we differentiate with respect to t both these equations. And we will $(a,b,c)\cdot\nabla\varphi = 0,\ (a,b,c)\cdot\nabla\psi = 0.$

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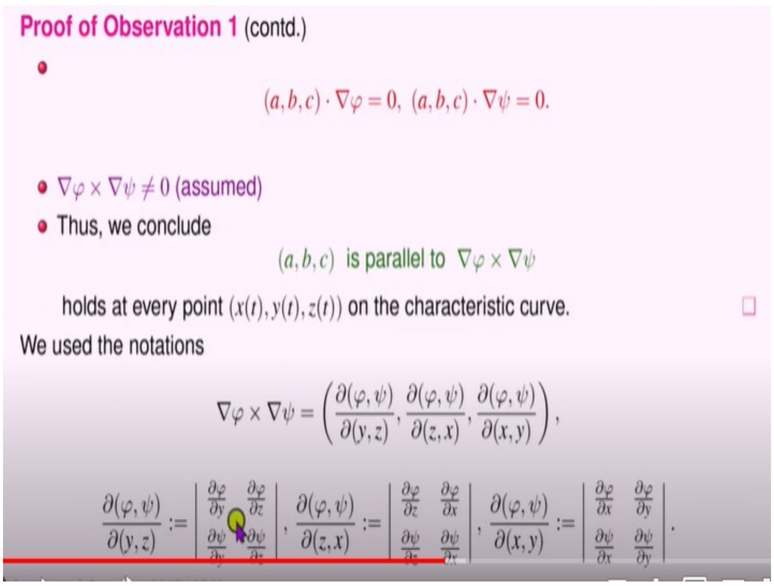

So, $(a,b,c)\cdot\nabla\varphi=0,\ (a,b,c)\cdot\nabla\psi=0$ this is what we have,

but $\nabla\varphi\times\nabla\psi\neq 0$ that is what we have assumed.

Therefore, (a b c )must be parallel to $\nabla\varphi\times\nabla\psi$ .

$$\nabla\varphi\times\nabla\psi=\left(\frac{\partial(\varphi,\psi)}{\partial(y,z)},\frac{\partial(\varphi,\psi)}{\partial(z,x)},\frac{\partial(\varphi,\psi)}{\partial(x,y)}\right)$$

$$\frac{\partial(\varphi,\psi)}{\partial(y,z)}:=\begin{vmatrix}\frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z}\\ \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z}\end{vmatrix},\ \frac{\partial(\varphi,\psi)}{\partial(z,x)}:=\begin{vmatrix}\frac{\partial\varphi}{\partial z} & \frac{\partial\varphi}{\partial x}\\ \frac{\partial\psi}{\partial z} & \frac{\partial\psi}{\partial x}\end{vmatrix},\ \frac{\partial(\varphi,\psi)}{\partial(x,y)}:=\begin{vmatrix}\frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y}\\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y}\end{vmatrix}$$

If a vector is orthogonal to 2 vectors, let us say u dot v is 0; u dot w is zero, then u is parallel to v cross w, if v cross w is nonzero. So, we have used that we conclude that a b c is parallel to grad phi cross grad psi. This holds at every point x t, y t, z t on the characteristic curve.

So, this is the notations we have used, this stands for this particular data map. This is the Jacobian of phi psi with respect to the variables y z and so on.

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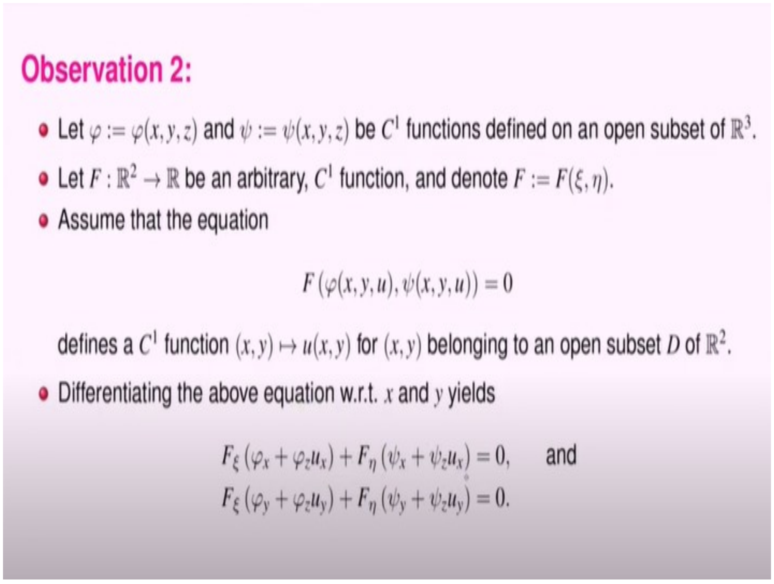


Observation 2: What is observation 2? Observation 2 is such an equation is satisfied, this equation is satisfied. Let $\varphi$ $\psi$ be $\mathbb{C}^1$functions. This is how we write. $\varphi = \varphi$ ( x y z ) ; just to denote the variables for the function $\varphi$ . Later on we are going to substitute in place of z, u ( x ,y )but this is a function of 3 variables. So, it is definedon open subset of $\mathbb{R}^3$ . It is very much useful when we apply chain rule so that we do not have confusion.

So, F from $\mathbb{R}^2$ to R be an arbitrary $\mathbb{C}^1$ function and we denote this by F is F $F := F(\xi, \eta)$ . Assume that this equation $F\left(\varphi(x,y,u), \psi(x,y,u)\right) = 0$ defines this $\mathbb{C}^1$ function that was part of the assumption.

Now differentiate this equation with respect to x and y. So, if you want to differentiate this equation with respect to x; x is present in both of them. Therefore, differentiate F with respect to xi At this point, $\varphi$ , $\psi$ which the arguments are not written.

Then differentiate this with respect to x, but here x appears here,

$F_\xi\left(\varphi_x + \varphi_z u_x\right) + F_\eta\left(\psi_x + \psi_z u_x\right) = 0$ and

$F_\xi\left(\varphi_y + \varphi_z u_y\right) + F_\eta\left(\psi_y + \psi_z u_y\right) = 0$

in the first coordinate and also the third coordinate. So, differentiate $\varphi$ with respect to x plus x with respect to x is 1 that is why it is phi x, plus phi with respect to third one, the third variable we are denoted by z. So, it will be phi z into u is there. Differentiate u with respect to x so, you get u x. Similarly, u differentiate here also, you get F eta because the second variable of F was called eta, so, F eta.

At this point, into derivative of psi with respect to x plus derivative of psi with respect to the z variable into derivative of u with respect to x that will give you the first equation here. Similarly, we get the second equation if you differentiate this with respect to y. So, here we should have written u (x, y ) and here u (x, y ) that is the meaning here. It defines the $\mathbb{C}^1$ function that means, I can substitute here u (x, y ) , u (x, y ) And that equation will be valid on a subset D of $\mathbb{R}^2$.

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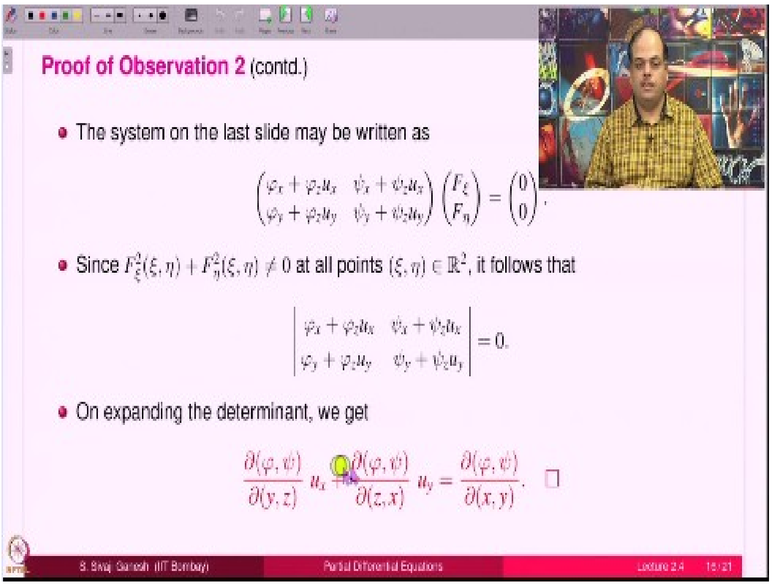

So, the system on the last slide,

$$\begin{pmatrix} \varphi_x + \varphi_z u_x & \psi_x + \psi_z u_x \\ \varphi_y + \varphi_z u_y & \psi_y + \psi_z u_y \end{pmatrix} \begin{pmatrix} F_\xi \\ F_\eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

it is a linear system for Fxi and F eta F xi and F eta. So, you write F xi F eta is one column.

$F_\xi^2(\xi,\eta) + F_\eta^2(\xi,\eta) \neq 0$ at all points $(\xi,\eta) \in^2$

$$\begin{vmatrix} \varphi_x + \varphi_z u_x & \psi_x + \psi_z u_x \\ \varphi_y + \varphi_z u_y & \psi_y + \psi_z u_y \end{vmatrix} = 0.$$

Then this is the first row a 1,1; first 1,1 entry; 1,2 entry; 2,1 entry; 2,2 entry. This is what we have. Now, we have assumed that both Fxi and F eta cannot be 0 simultaneously, which means, this system has a nontrivial solution, a nonzero solution is homogeneous system, it has a nonzero solution, it means that the determinant of this is 0. And if we expand these what we get.

$$\frac{\partial(\varphi,\psi)}{\partial(y,z)}\,u_x + \frac{\partial(\varphi,\psi)}{\partial(z,x)}\,u_y = \frac{\partial(\varphi,\psi)}{\partial(x,y)}.$$

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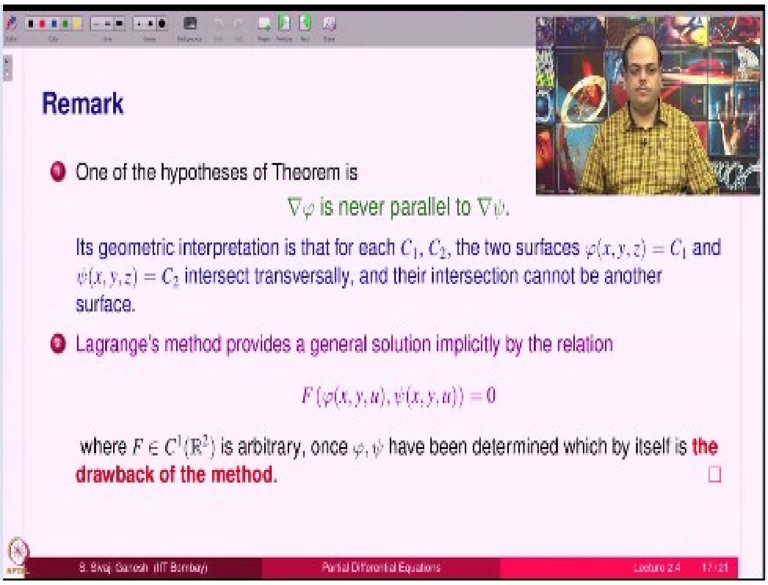


So, one of the hypotheses of the theorem is

$\nabla\varphi$ is never parallel to $\nabla\psi$.

What does that mean? What is the geometric interpretation? For each $C_1$ $C_2$, the surfaces $\varphi(x,y,z) = C_1$and $\psi(x,y,z) = C_2$ intersect transversally. And their intersection cannot be another surface because normals do not have the same direction, normals are different directions. Therefore, the tangent planes will have different directions at every point which is common to both the surfaces $\varphi = C_1$ and $\xi = C_2$.

At such a point, look at the tangent plane for $\varphi = C_1$ and tangent plane for $\xi = C_2$, they do not coincide at all that is the assumption we aremaking.Then $\nabla\varphi$ is never parallel to $\nabla\psi$. anywhere. So, Lagrange's method provides a general solution implicitly by this relation

$F\left(\varphi(x,y,u),\psi(x,y,u)\right)=0$

because we assume that you can be solved for in terms of x y. In such a case, we define that to be a general solution.

Where F is arbitrary function, we are just make sure that $\varphi$and $\xi$ take values in the domain of F. If we are taking F in $\mathbb{R}^2$, then we do not have to take even that much care. Once $\varphi$ and $\xi$ have been determined, this is where the cache pages. This itself is the drawback of the method, finding $\varphi$ and $\psi$ .

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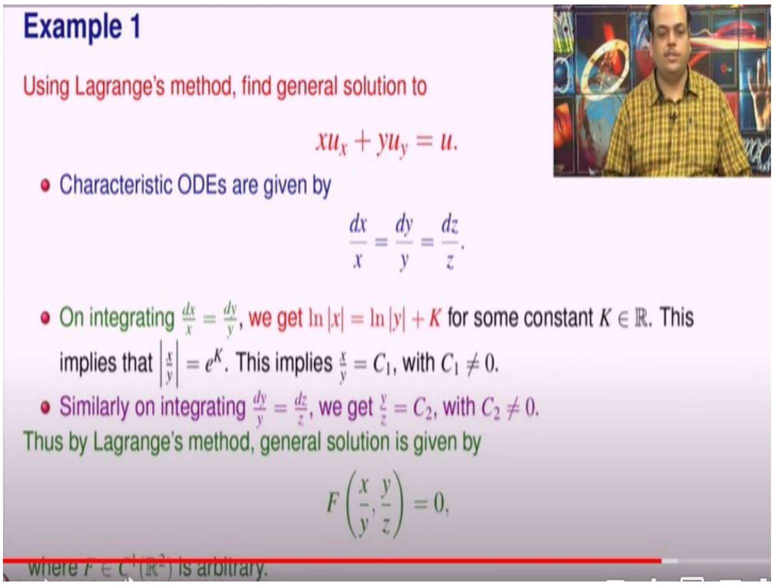


Let us look at an example. Using Lagrange's method, find general solutions to solution to PDE $xu_x + yu_y = u$.

Actually this is a linear equation. Okay characteristic ODEs are given by, this is another way of writing dx by dt equal to x, dy by dt equal to y, dz by dt equal to z or after since t is not present, there is another way of writing $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$.

So, you integrate $\frac{dx}{x} = \frac{dy}{y}$ , you get $\ln|x| = \ln|y| + K$. $K$ for some constant. And that gives us that $\left|\frac{x}{y}\right| = e^K$. And that implies $\frac{x}{y} = C_1$ where $C_1$ is nonzero, because e power k will never be 0 right, no matter what k is equal, k is never 0. So, $\frac{x}{y} = C_1$ Similarly, an integrating the other equation $\frac{dy}{y} = \frac{dz}{z}$ , we could choose any combination, we just wanted to 2 functions $\varphi$ and $\psi$ .

So, this is one solution $\frac{x}{y}$ equal constant. It is a solution of this particular equation $\frac{dx}{x} = \frac{dy}{y}$ .

Now, we will look $\frac{dy}{y} = \frac{dz}{z}$, we get similarly y by z equal to constant. So, now, what we want is $\varphi$ and $\psi$ so, that $\varphi = C_1$ , $\psi = C_2$ together represent characteristic curves. And now, by Lagrange's method says F ( $\varphi$ , $\psi$ )= 0. So, therefore, this is represents a general solution.

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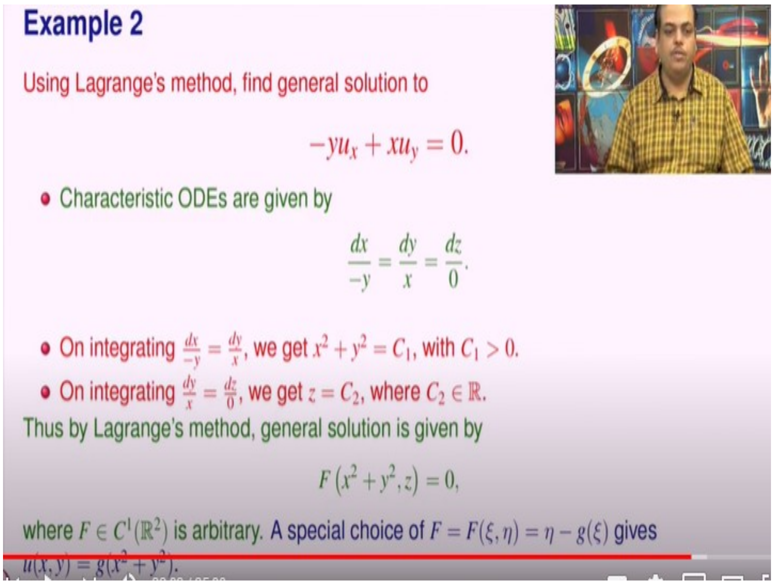


Let us look at another example. $-yu_x + xu_y = 0.$

Unfortunately, this is also linear. We will discuss some Quasilinear equation in the tutorials. So, characteristic ODEs are give

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{0}.$$

.Now, on integrating this equation, we get

$x^2 + y^2 = C_1$ . $C_1$ positive. And integrating this we get $z=C_2$. Therefore, Lagrange's method says general solution is given $F\left(x^2+y^2, z\right)=0,$ by; F is arbitrary.

If I take a specific choice of F, which is $F=F(\xi,\eta)=\eta-g(\xi)$,
what I get is $u(x,y)=g(x^2+y^2)$ . In other words, this solution if you observe, these constant on each circle with centre at origin $x^2 + y^2$ is constant, positive constant will give u is constant on all circles with centre at origin. Please note that if you have integrated this differently from what is done here, you will get F of some other 2 functions equal to 0.

And those 2 functions, these 2 functions will be related somehow. Okay, they will be functionally dependent. I do not want to use these terms and discuss this further. But, as long as you get $\varphi$=constant and $\psi$ = constant ,representing characteristic curves, capital F ( $\varphi$ , $\psi$ )=0 will always represent general solution provided we can solve for z in terms of x and y. For example, if I chose this F, of course, this is solvable u equal to g ( $x^2 + y^2$ ) where g is arbitrary function.

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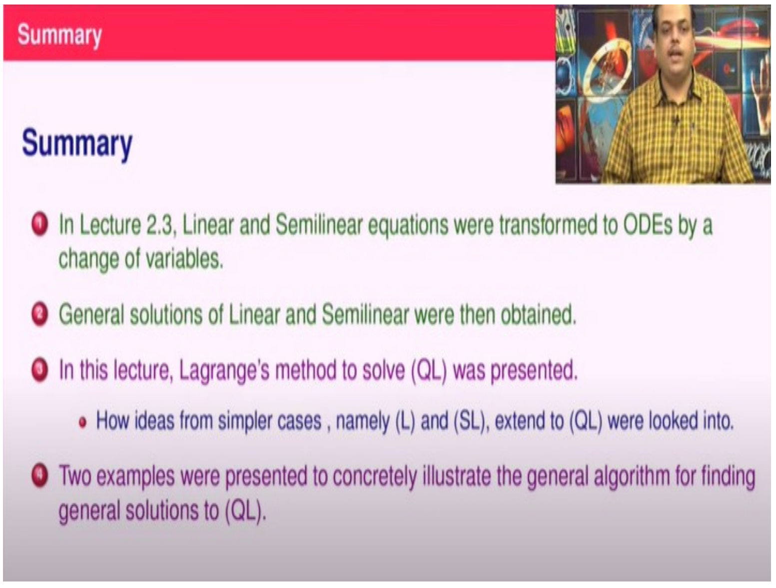


So, let us summarise what we did in this lecture. In lecture 2.3, linear and semilinear equations were transformed to ODEs by a change of variables, general solutions were obtained. In this lecture Lagrange's method to solve QL was presented, how ideas from simpler cases of L and SL extend to QL were looked into. And we are presented 2 examples to concretely illustrate the general algorithm, which is given for finding general solutions to QL.

And we cannot do this kind of general solutions to nonlinear equations. So, beyond QL, our theory does not go. So, to solve even such equations, we will try to solve using what is called method of characteristics. In the next few lectures, we will be presenting the method of characteristics and its preliminaries initially and then we use that method to solve Quasilinear equations first and then once again, extend those ideas to solve general first order nonlinear equations. Thank you.