Partial Differential Equations
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Lecture – 6.5
Dirichlet BVP on a Disk in R 2 for Laplace Equations

Welcome to this lecture on a Dirichelt boundary value problem on a disk in R 2 for Laplace equation. The outline is as follows we solve the Dirichelt boundary value problem on a disk using separation of variables method and we read derive Poisson's formula in this situation for most part of this lecture is going to be computation. So, please stop at every slide and do the computations on your own.

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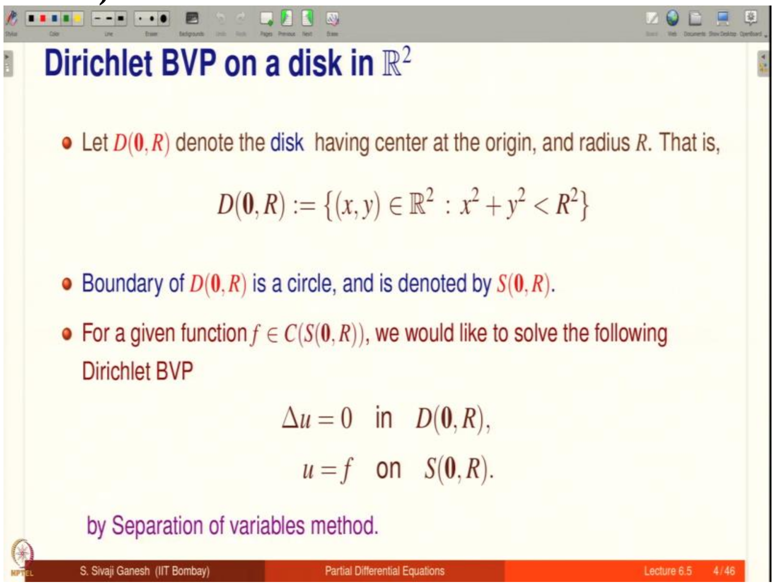

So, Dirichlet boundary value problem on a disk in R 2 let D of 0, R denote the disk having center at the origin and radius R that is D of 0, R is set of all x, y in R 2 such that x square + y square is less than R square boundary of D of 0, R is a circle and is denoted by S of 0, R. For a given function f which is continuous on S of 0, R that is on the circle we would like to solve the following Dirichlet boundary value problem.

What we end up solving using this method of separation of variables we will be able to justify only if this f is a little more nice. What we are going to see is that we need some more conditions on f and through which we can justify the separation of variables method that we are going to discuss.

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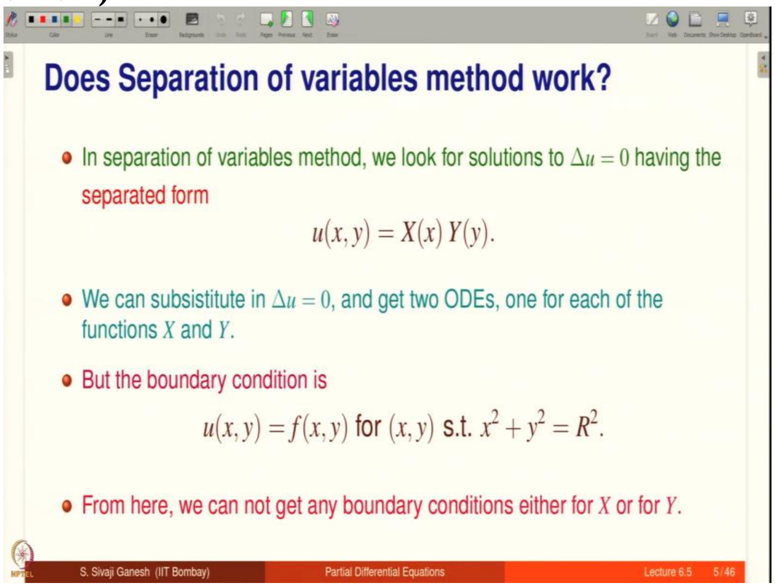

First of all does separation of variables method work. In separation of variables method we look for solutions to any equation in this context it is Laplacian equals 0 having this separated form u of x, y is a function of X into a function of Y we can substitute in Laplacian equal to 0 and get 2 ODEs one each for X and Y but the boundary condition is u of x, y = f of x, y given on the circle. From here we cannot get any boundary conditions for X or Y.

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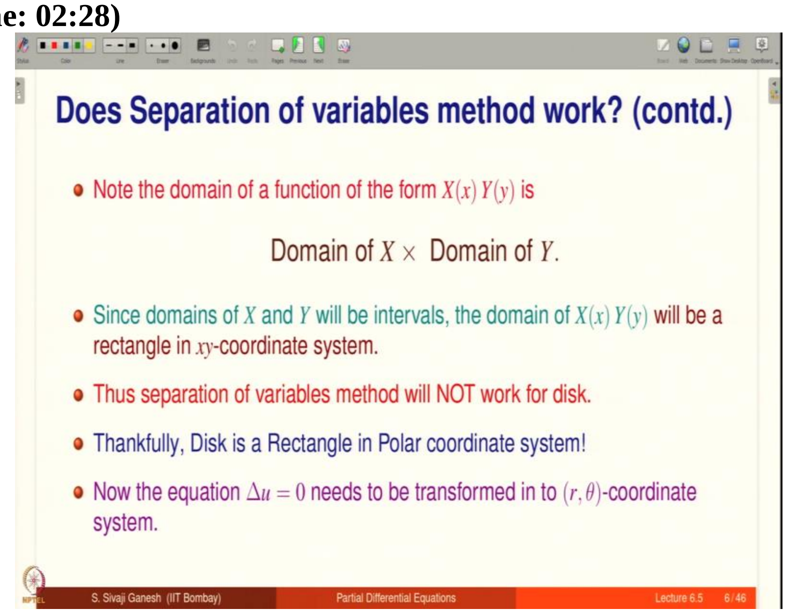

Note that the domain of a function which is in this product form X of x into Y of y is domain of X cross domain of Y. Because if X of x into Y of y makes sense for every tuple x, y such that X belongs to domain of X and Y belongs to domain of y. Since the domains of X and Y will be intervals because X and Y are functions of one variable the domain of the product will be a rectangle in x, y coordinate system.

Thus separate separation of variables method will not work for disk. Thankfully disk is a rectangle it looks surprising disk is not a rectangle but disk is a rectangle in some other coordinate system namely the polar coordinate system. Now the equation Laplacian equals 0 needs to be transformed into r theta coordinate system.

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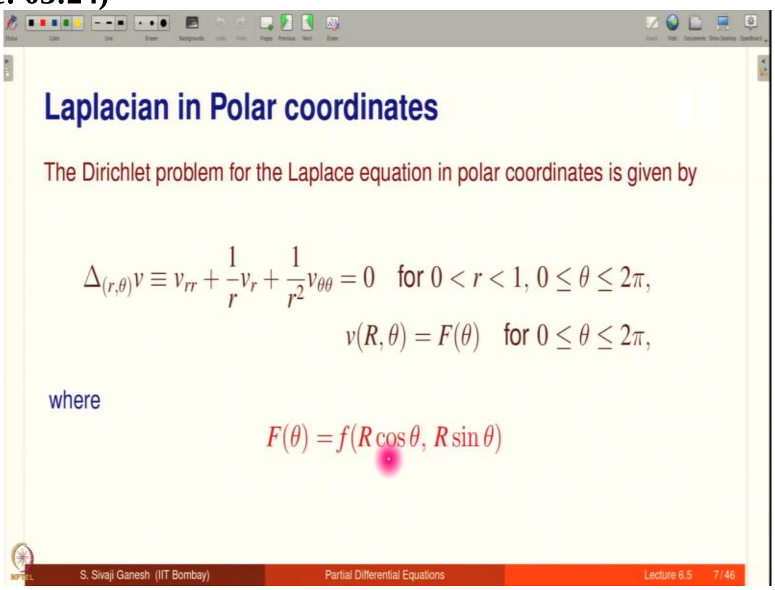

So, Laplacian in polar coordinates the Dirichlet problem for Laplace equation in polar coordinates is given by these 2 equation this is precisely the Laplacian in the polar coordinates. So, we have done enough exercises on how the PDE transforms under change of coordinates. So, I leave this as an exercise for you to check that Laplacian in polar coordinates is given by this operator which is here v rr + 1 / v vr + 1 / r square v theta, theta.

We are given u of x, y on the circle that will define a function F for theta in 0 to 2 pi as follows F theta is nothing but this is a given f R cos theta R sin theta in polar coordinates x = R cos theta y = R sin theta. So, this is how you get for points on the circle of radius R with center at origin.

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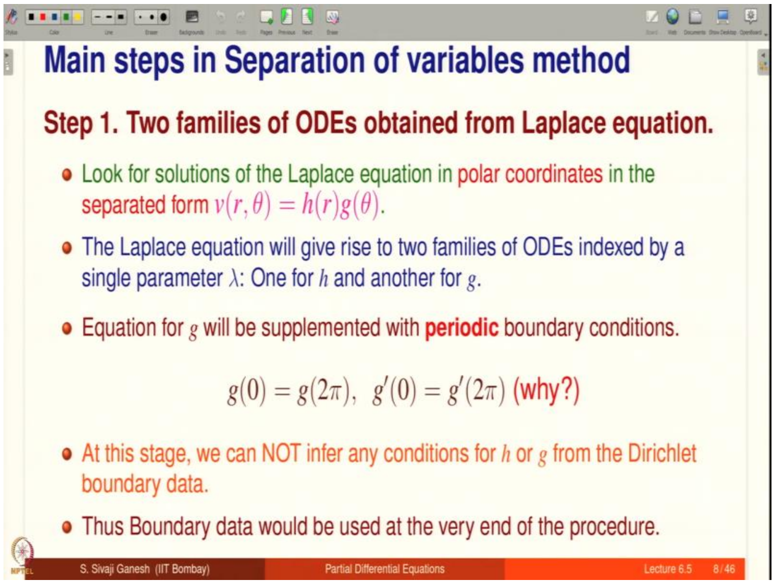

So, what are the main steps in separation of variables method? Step 1 2 families of ODEs obtained from Laplace equation or any equation for which you are trying to do separation of variables method. Separation of variables method is not new to us we have done many exercises based on this for wave equation. So, look for solutions for the Laplace equation in polar coordinates in the separated form v of r theta as h of r into g of theta.

The Laplace equation will give rise to 2 families of ODEs indexed by a single parameter lambda one for h and one for g. Equation for g will be supplemented with a periodic boundary conditions which is g of 0 = g of 2 pi and g prime of 0 = g prime of 2 pi. The question is why? Why because what is our aim we want to solve Laplace equation in the disk that means the function should be continuous differentiable C 2 etcetera.

But that we are trying to solve in terms of h of r and g of theta and g of theta. So, there should be some requirement on g that is precisely this. Please explore this further and then find out the precise reason which I have only indicated why this condition is at this stage we cannot infer any conditions for h or g. From the Dirichlet boundary data the boundary data will be used at the very end of the procedure.

If you recall for the wave equation also one equation we have solved a boundary value problem for the second equation we do not have boundary value problem we have if you want a partial initial value problem and still there were some constant to be determined that we used the other initial data to fix those constants same thing here.

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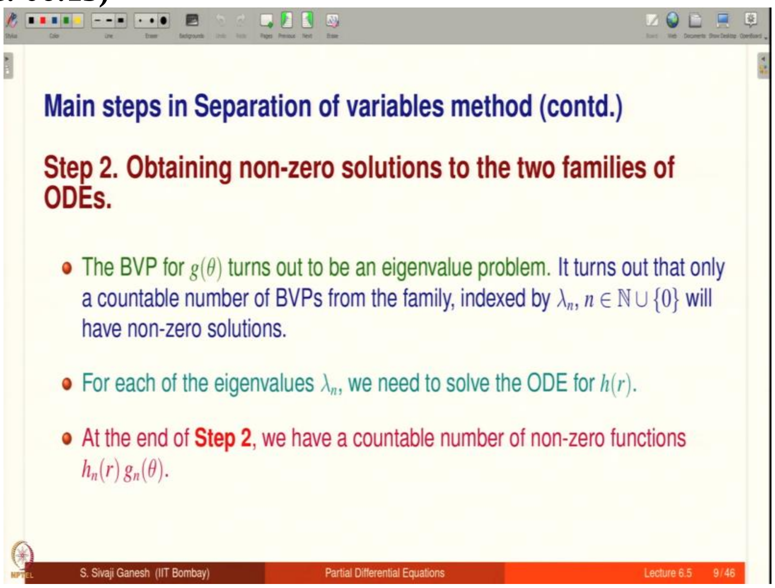

So, step 2 obtaining non-zero solutions to the 2 families of ODEs the BVP for g of theta turns out to be an eigenvalue problem. It turns out that only a countable number of such BVPs index will n belongs to n union 0 we will have non-zero solutions. For each of the eigenvalues lambda n we need to solve the ODE for h of r. At the end of step 2 we have a countable number of non-zero functions h n of r will be g n of theta.

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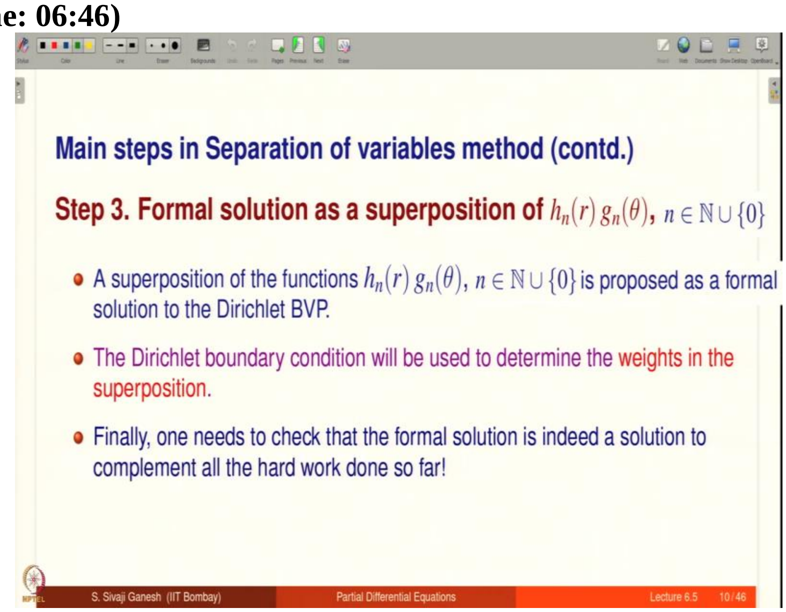

Step 3 is a proposal of a formal solution as a linear superposition of h n of r g n of theta n belongs to n union 0. A superposition of the functions h n r g n theta n belongs to n union 0 is proposed as a formal solution to the Dirichlet boundary value problem. The Dirichlet boundary condition will be used to determine the weights in the superposition. Finally one needs to check that the formal solution is indeed a solution to complement all the hard work that we have done so for.

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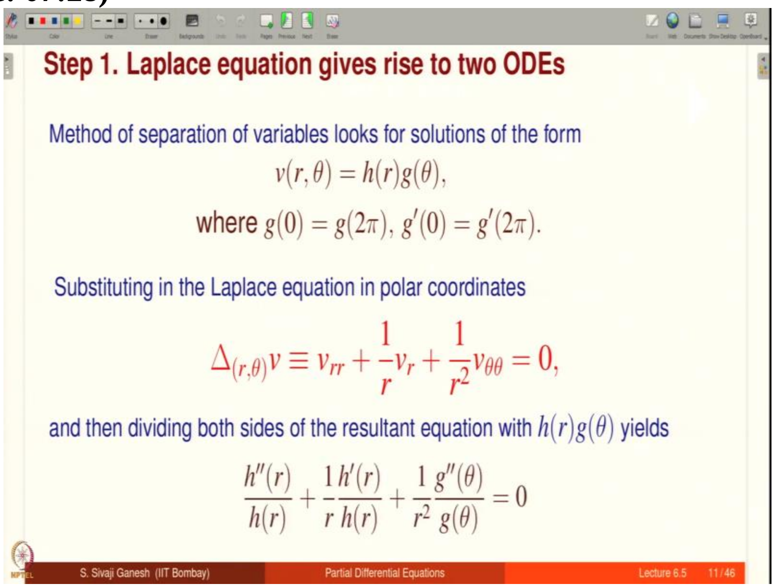

So, step 1 method of separation of variables looks for solutions of the form h of t into g of theta where g of 0 = g of 2 pi and g prime of 0 = g prime of 2 pi the periodic property for g and g dash substituting in the Laplace equation in polar coordinates and then dividing both sides with h of r into g of theta gives us this equation.

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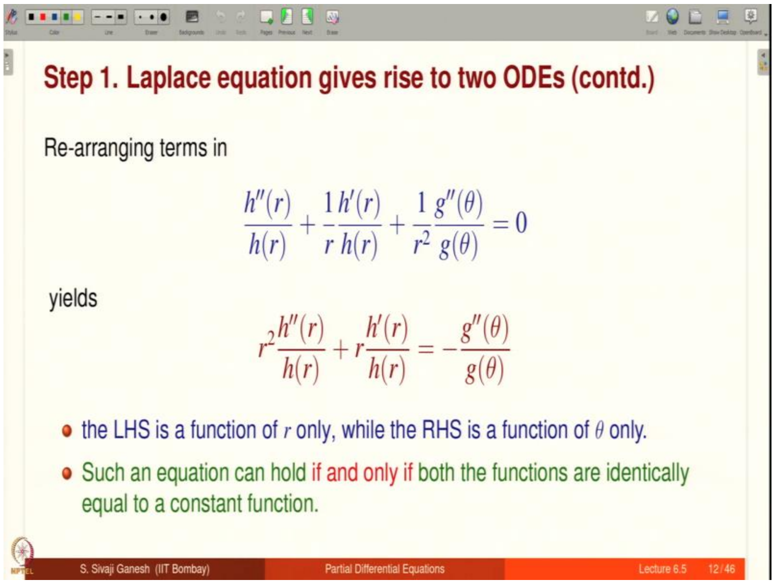

So, rearranging the terms in this equation gives us this equation or taking the g terms to the right side and multiplied with r square. So, I have got this now observe that LHS is a function of our only RHS is a function of theta only. Therefore such an equation can hold if and only if both functions are equal to a constant function.

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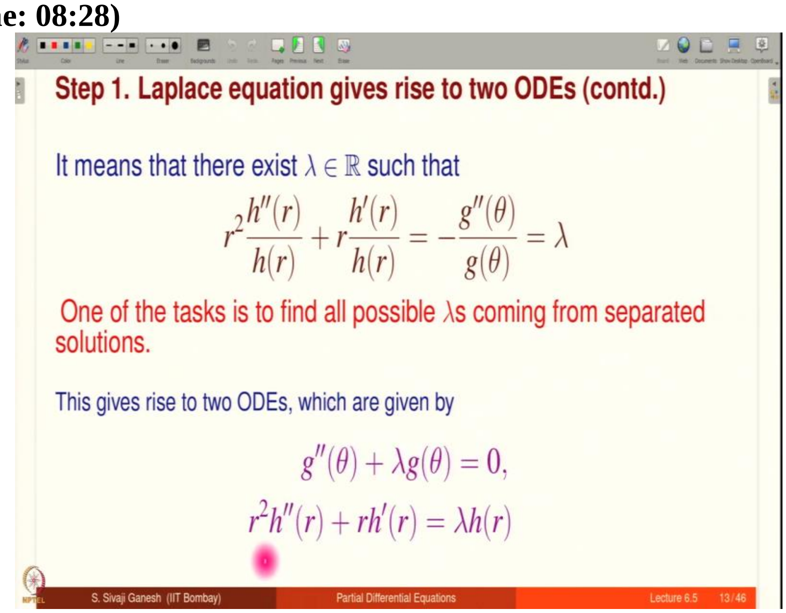

So, it means that there exists lambda in R such that you have this equal to this and both of them equal to lambda one of the tasks is to find all possible lambdas coming from separated solutions this gives rise to 2 ODEs which are given by g double dash + lambda g = 0 r square h double dash + r h dash = lambda h.

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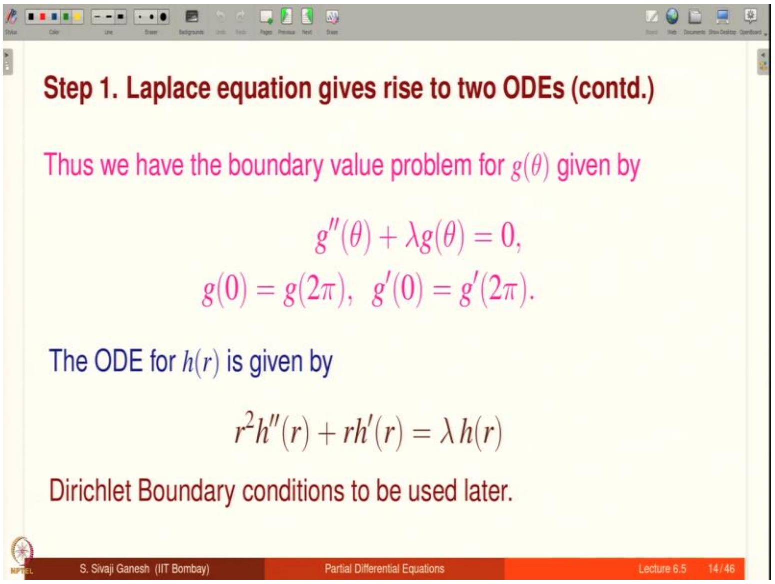

Thus we have the boundary value problem for g of t given by g double dash + lambda g = 0 on this periodic boundary conditions and the ODE for h of r is given by r square h double dash + r h dash = lambda h Dirichlet boundary conditions to be used later.

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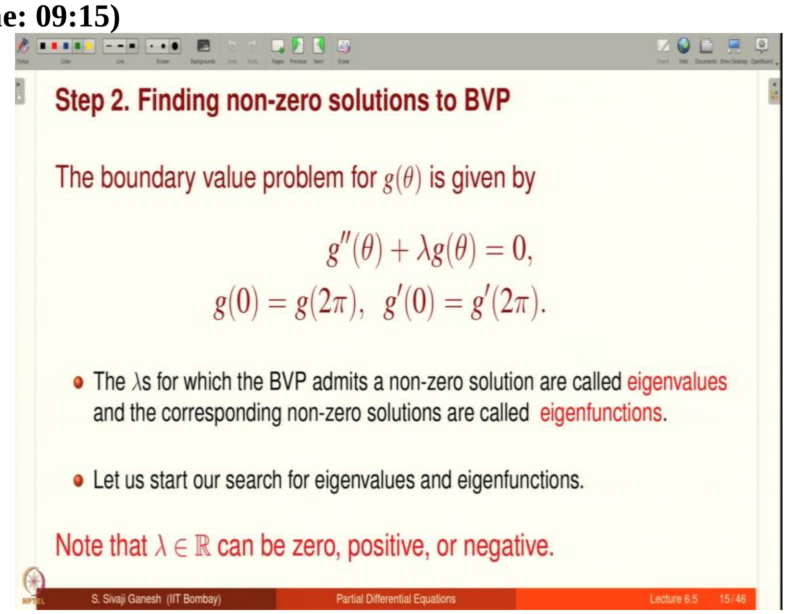

Now let us try to solve this boundary value problem for g. So, there is a parameter lambda so the lambdas for which the BVP admits a non-zero solution are called Eigenvalues and the corresponding non-zero solutions are called Eigen functions. Let us start our search for Eigenvalues and Eigen functions. Note that the lambda can be 0 positive or negative these are the only 3 possibilities for a real number.

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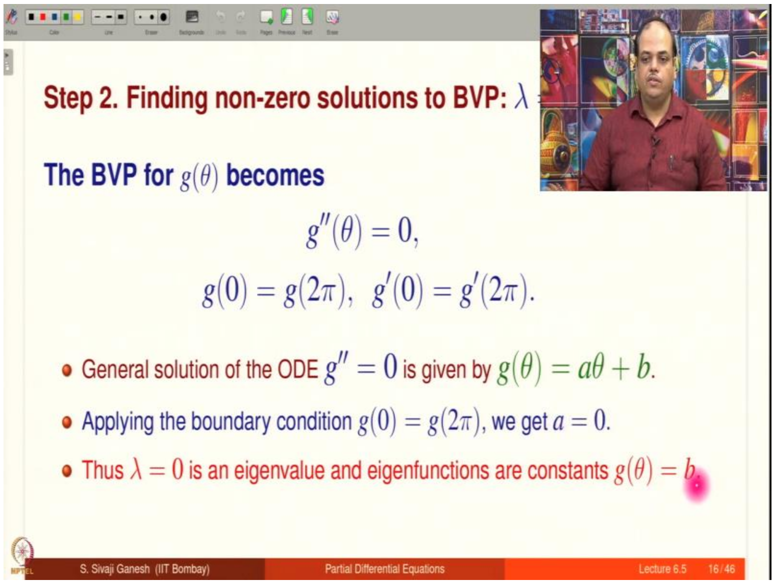

Let us solve the BVP with a lambda = 0 and see whether it has a non trivial solution or not. So, the BVP itself becomes this because the equation changes lambda = 0. So, the g term is dropped g double dash = 0. Now general solution of the ODE g double dash = 0 is a theta + b. Now apply these boundary conditions g of 0 = g of 2 pi if you put what we get is a = 0. Thus lambda = 0 is an Eigenvalue and Eigen functions are constants the second boundary condition does not give us any information because g prime is actually a = a so it is always true. So, essentially the first boundary condition gave us a = 0 therefore Eigen functions are g theta = b b is any constant.

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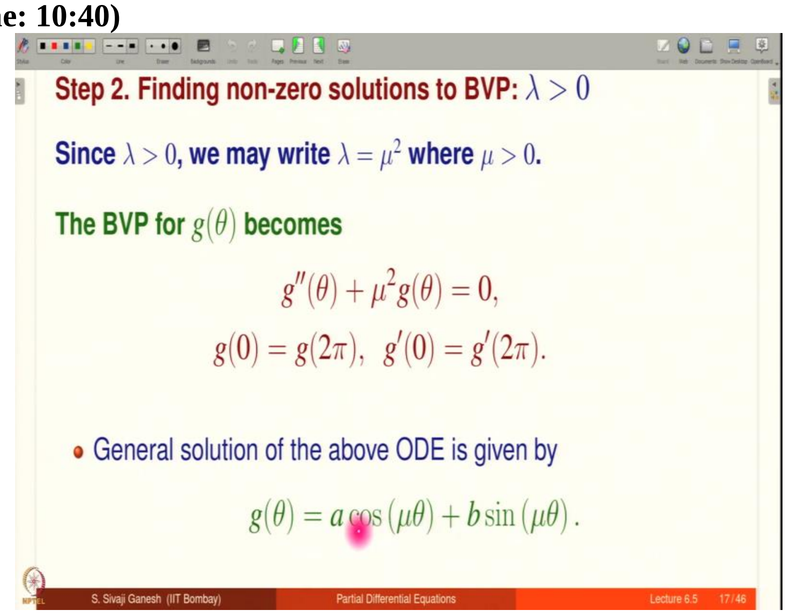

Now let us look at boundary value problem for lambda positive so once lambda is positive we can write it as mu square for mu positive for definiteness are also because we are dealing with the second order equation it is convenient to have somebody square here. So that is why mu square now general solution of the ODE is a combination of cosine and sine functions a cos theta + b sin mu theta.

Now we are determined the constants a and b such that these boundary conditions are satisfied and then ask if it is possible to choose the constants a and b at least one of them non-zero in which case we get Eigenvalues and Eigen functions.

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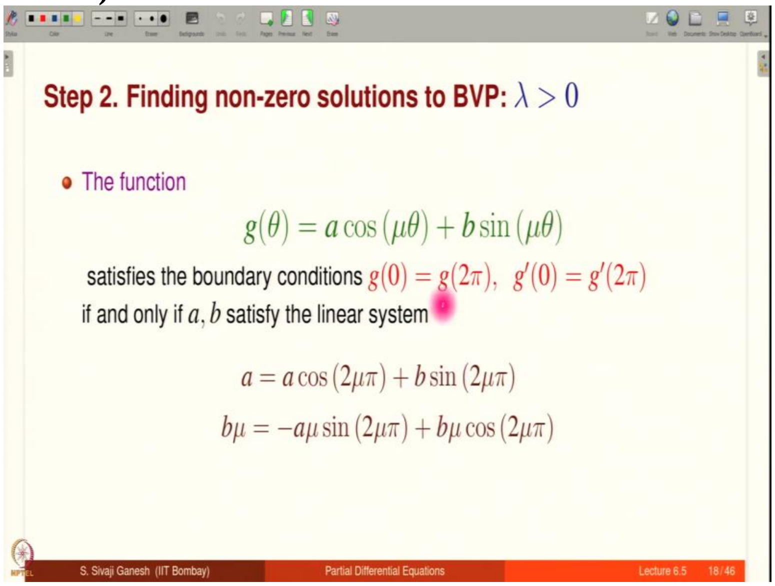

Therefore this function here satisfies these boundary conditions. If and only if these 2 conditions are met I have just substituted into these expressions this formula so we get these 2 equations coming from these 2 conditions.

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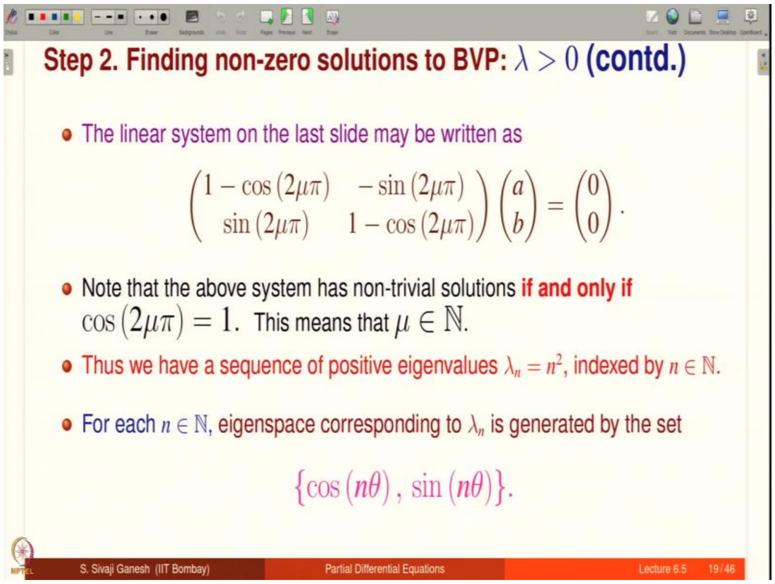

Now it is always a good idea to write it as a linear system. So the 2 equations we have on the last slide let us write it as a linear system and we are looking for non-zero solutions of this system because that will give us Eigenvalues Eigen vectors it has non-zero solutions if and only if this determinant is 0 if the determinant is non-zero that implies a, b is 00. So that is not a situation we want.

So, therefore we want this determinant to be 0 which will give us cos 2 mu pi = 1 that means mu is a natural number. So, therefore we have a sequence of Eigenvalues what are the values their mu square so mu is N here. So, any natural number so eigenvalues are squares of natural numbers and what are the Eigen functions? They are generated by cos n theta and sin n theta for each n is a linear combination of cos n theta and sin n theta that will be an Eigen function.

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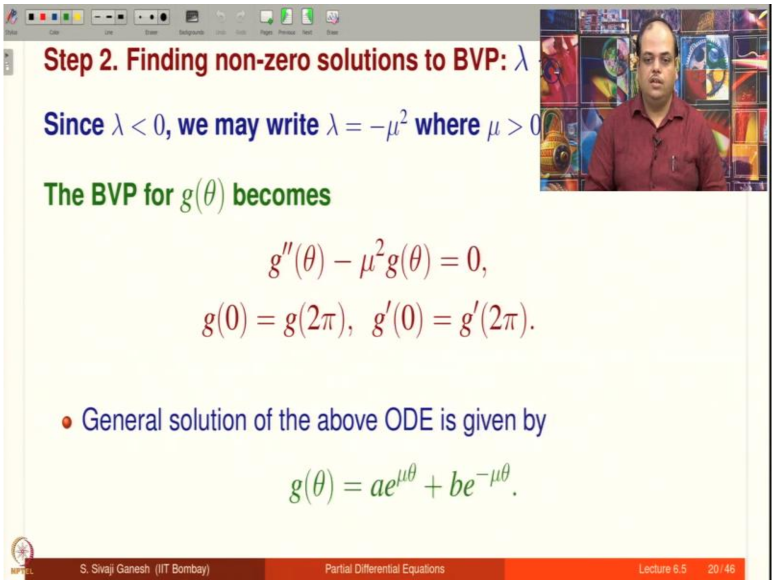

Now let us inquire into negative lambdas whether those be BVPs have a non trivial solutions or not. So, we can write lambda = - mu square where mu is positive and the equation is here these are the boundary conditions. So, general solution of the above ODE is a combination of exponentials, which is here and let us substitute these boundary conditions.

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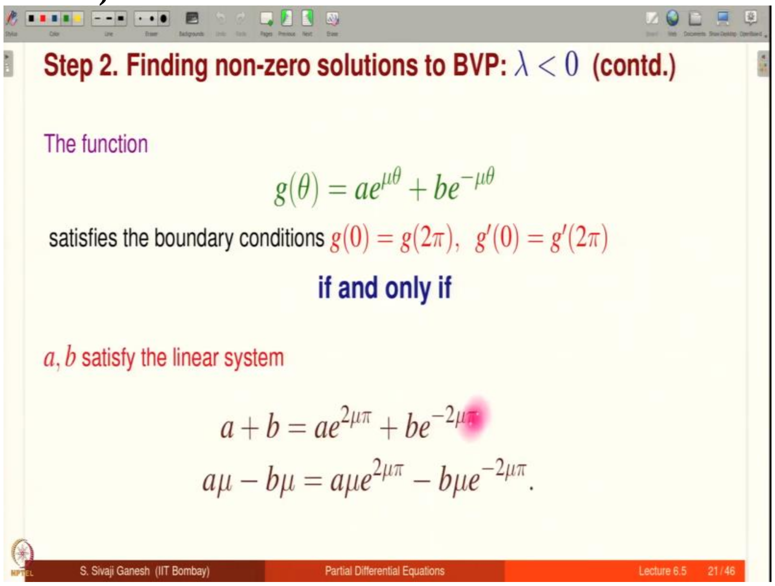

We get these relations to understand whether this has an Android solution or not it is always a good idea to write it as a linear system which we do here.

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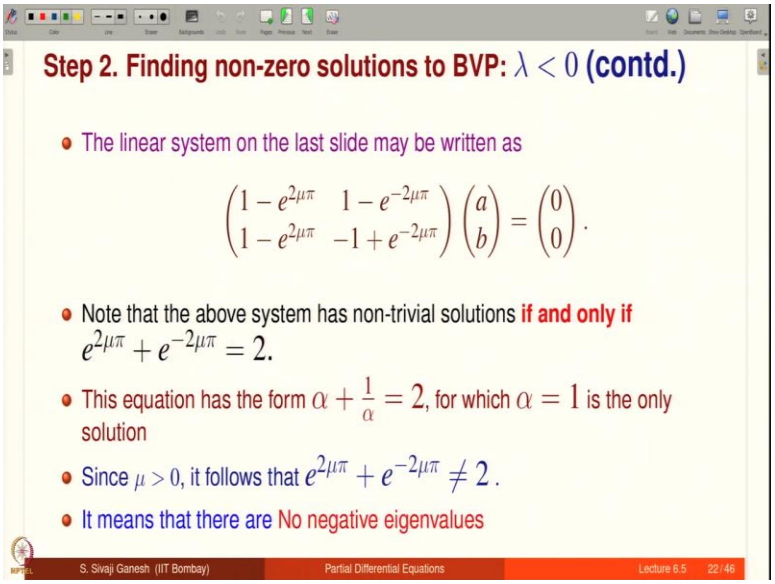

Now we inquire into whether the determinant is 0 or non-zero. If it is 0 only trivial solutions that means no negative Eigenvalues. If determinant is 0 then we have nontrivial solutions then there are negative Eigenvalues. So, above system has nontrivial solution if and only if the determinant is 0 which comes out to be this e power 2 mu pi + e power - 2 mu pi = 2 and it looks like a number plus 1 by that number equal to 2.

For which alpha = 1 is the only solution. Since mu is positive it follows that this cannot be equal to 2 because alpha = 1 means mu is 0 but mu is not 0. Therefore this equation is not satisfied. It means that there are no negative Eigenvalues.

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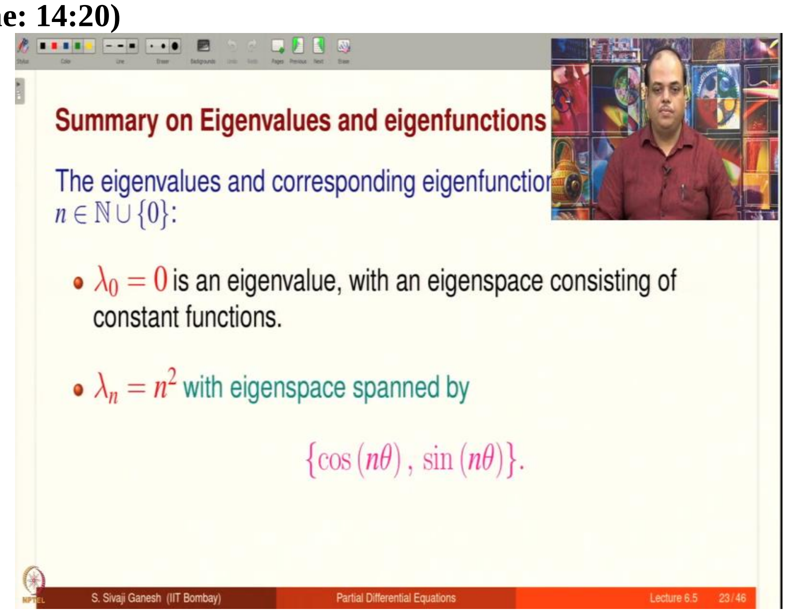

So, let us summarize the Eigenvalues and Eigen functions to the BVP for g of theta. They are indexed by n union 0 natural numbers union 0 lambda naught = 0 is an eigenvalue with an Eigen space consisting of constant functions lambda n = n square with Eigen space spanned by cos n theta and sin n theta.

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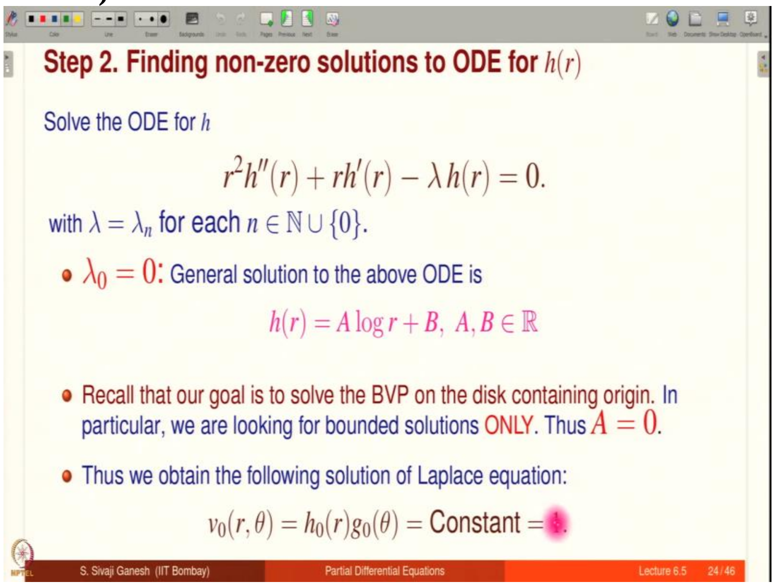

Now it is time to solve the equation for h of r which is here for each lambda = lambda n n in natural numbers union 0 lambda naught = 0 general solutions to the above ODE is A log r + B where A and B are real numbers. Recall that our goal is to solve the boundary value problem on the disk. In particular the function the solution that we are planning to get should be bonded at the origin it should be bonded on the disk but bonded at the origin in particular.

Therefore log r term is not suitable for that therefore A must be 0 thus we have obtain the following solution of Laplace equation h naught of r into g naught of theta remember g naught theta was also constant when lambda naught = 0 h naught afar is also constant because a is 0 there is only b here. So, we can take the constant to be 1 anyway we are going to take linear superposition of these numbers. So you keep it 1 or 2 is the same so keep it 1.

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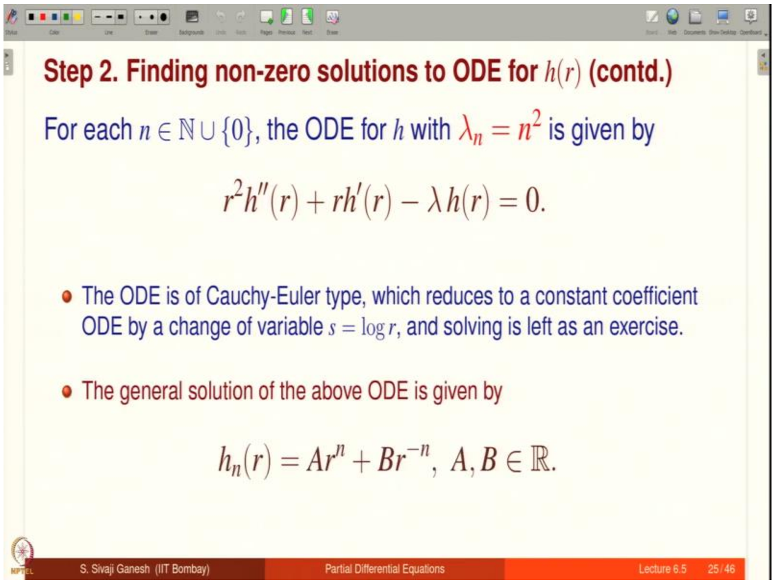

For each n in N union 0 the ODE for h with lambda n = n square is given by this equation. The ODE is of Cauchy Euler type which reduces to a constant coefficient ODE by change a variable s = log r and solving his left hand exercise are nowadays people also know what kind of solutions are there for this equation they straightaway look for a solution of the form r power k and then you can solve like that also. So, the general solution of the above ODE is given by A r power n + B r - n where A and B are real numbers.

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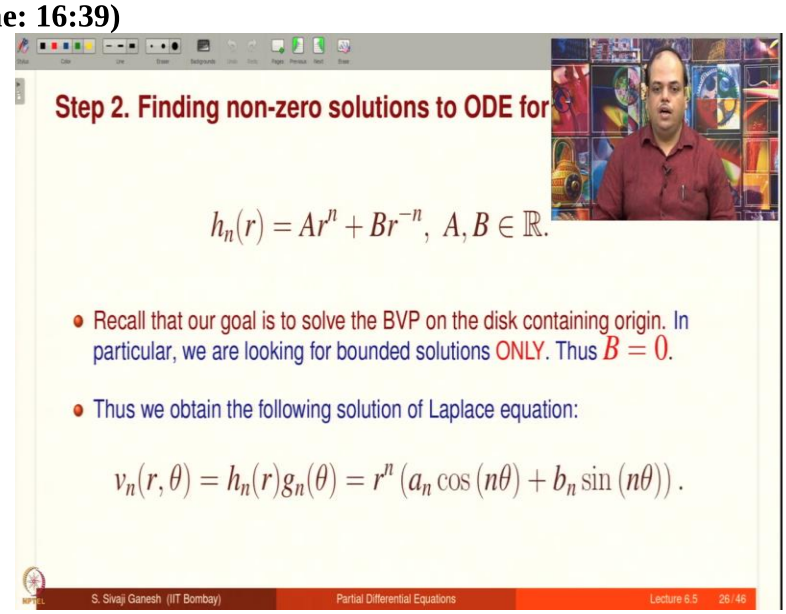

Once again recall that our goal is to solve the BVP on the disk containing origin. Therefore we are looking for solutions which are bounded and hence this B should be 0 thus we obtain the following solution h n r into g n of theta given by r power n from h n and this is from g n.

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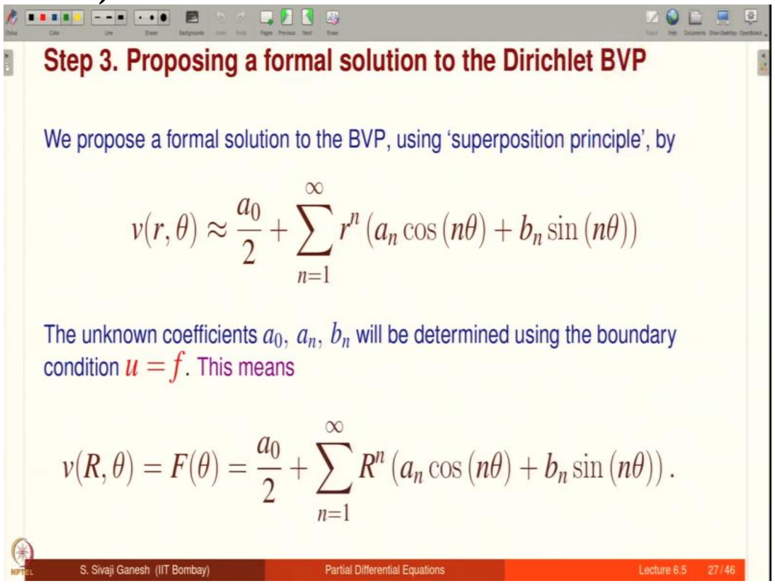

Now let us propose a formal solution to the Dirichlet boundary value problem using superposition principle by v have our theta = a naught / 2 + summation n = 1 to infinity r power n a n cos n theta + b n sin n theta if you recall this is a combination of the function 1 which is coming from h naught of r into g naught of theta and this summon here is precisely h n of r into g n and f theta. So we are taking a combination the constants in the combination are taken a n and b n.

So, the unknown coefficients a 0, a n and b n has to be determined using the boundary condition because that is the only condition we have not used yet. That means v have R comma theta that is r = R should be f of theta and that should be equal to this expression with r = R.

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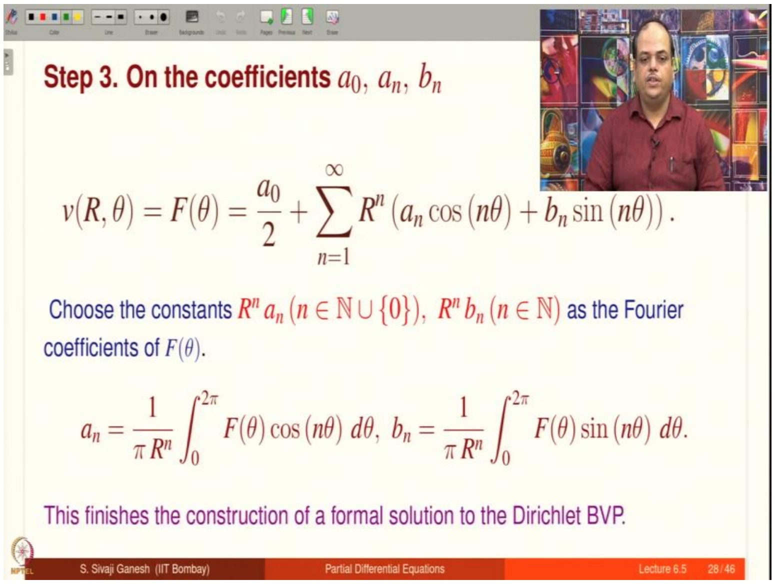

This is how we would like to find the constants a n, b n, a naught such that the function f of theta is given by this Fourier series. So, choose the constant R power n a n into R power n b n such that they are the Fourier coefficients of F of theta of course we have this expression for a n and b n. So, this finishes the construction of a formal solution to the Dirichlet BVP.

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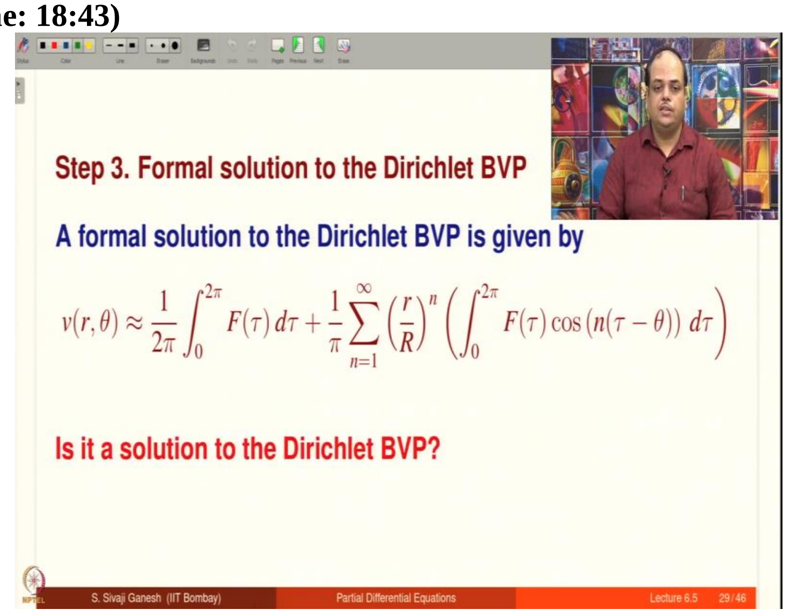

So, formal solution to the Dirichlet BVP is given by this I have substituted the question c n b n into the series that we have proposed. So, now this has no a n b n everything in terms of f and this has a cos and sin terms inside. So, is it a solution to Dirichlet BVP that is a question now?

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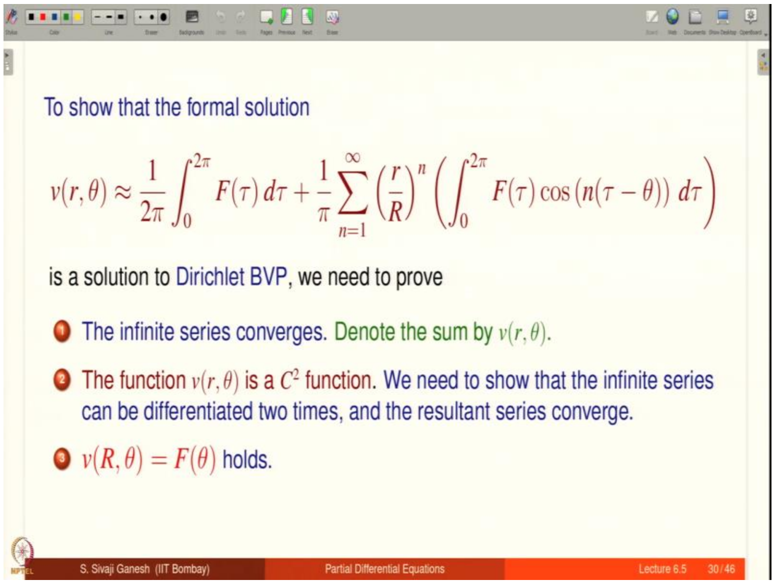

To show that the formal solution given here is a solution we need to prove that the series here which is okay is just one term there is a series here that series converges. In other words this makes sense and defines a function. If such a thing happens call it v of r theta and then we have to show that v of r theta is a C 2 function we need to show that infinite series can be differentiated 2 times and the resultant series converge.

Then we have to show that the boundary condition is satisfied v of r theta = f theta holds these 3 things we have to do. So, from now onwards you can just see the video once and you can forget the details because some of the details have not mentioned very clearly. For whichever given you a reference but we should be happy that we have got a formal series here rigorous analysis of course I have given from now onwards.

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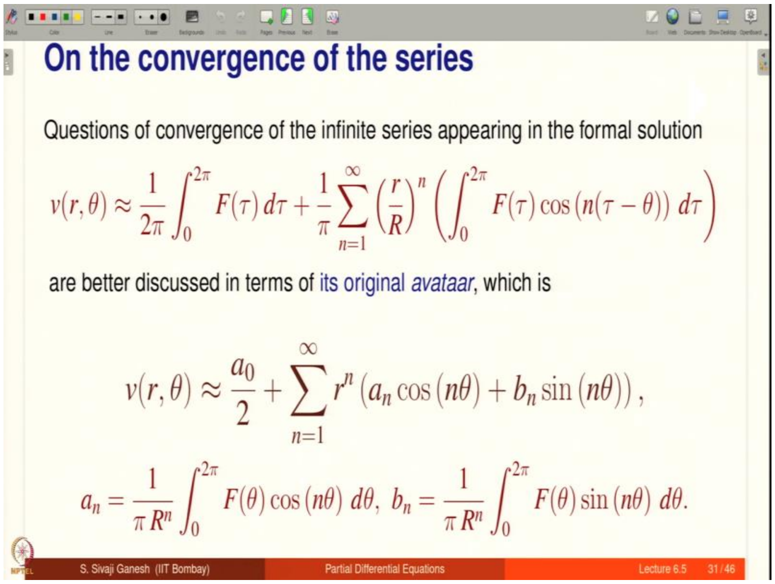

So, questions of convergence of the infinite series appearing in this formal solution namely this one are better handled using its original avatar which is this very a n has this formula b n has this formula.

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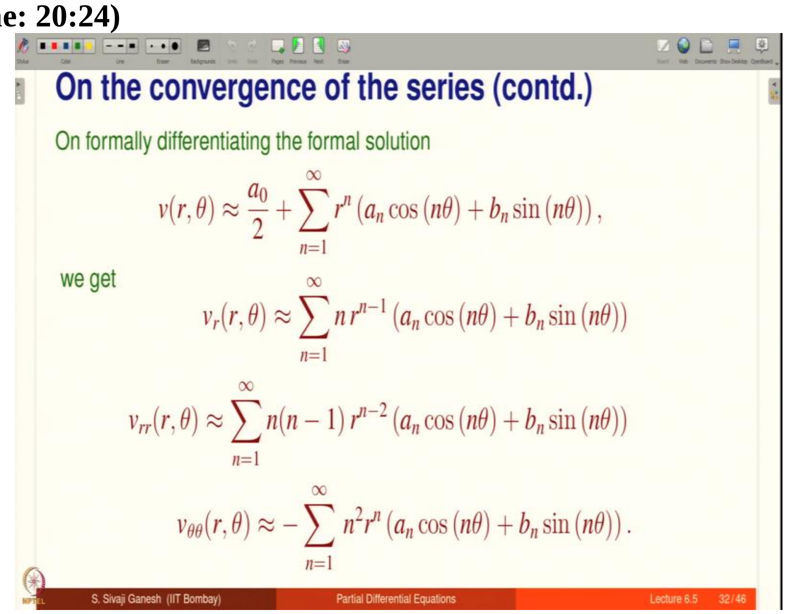

So, I am formally differentiating the formal solution we get v r to be like this v rr to be like this v theta, theta should be like this. So, every time you differentiate you pick up some n so r n n is coming here, here n into n - 1 that is n square term. So, in second order do you see how n square terms and in first order derivative equal to n terms when you are no derivative there is no n as a coefficient here of course r power n cos n theta they are all there.

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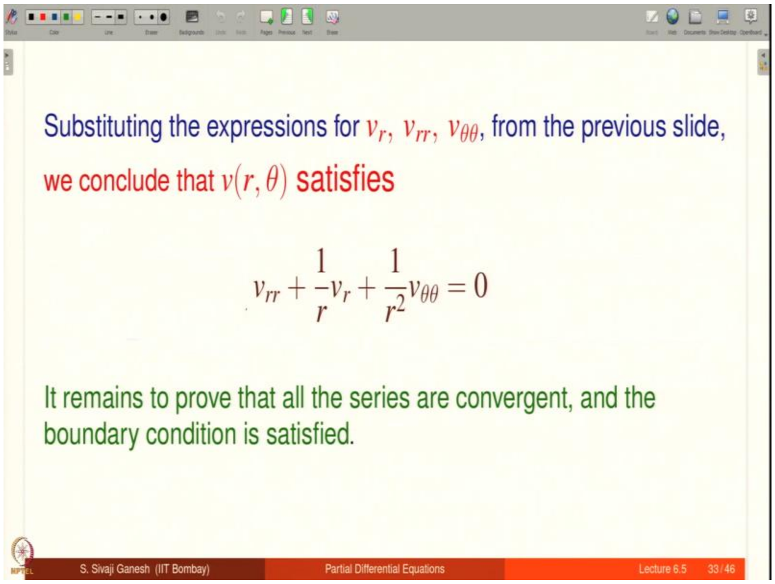

So, substituting these expressions for v r, v rr, v theta, theta which are given here formally you add them up with the corresponding weights we are which weights the one which are in the Laplace operator this one so v rr plus multiply v r with 1 / r v theta with 1 / r square you see that you get 0. So, formally we have shown that when the series on the previous slide makes sense they add up and in this way to get that v of r theta solution to a Laplace equation.

So, it remains to prove that all the series are convergent and of course the boundary condition is satisfied that we have to prove so once the series are convergent we have just shown that Laplacian v = 0 is satisfying.

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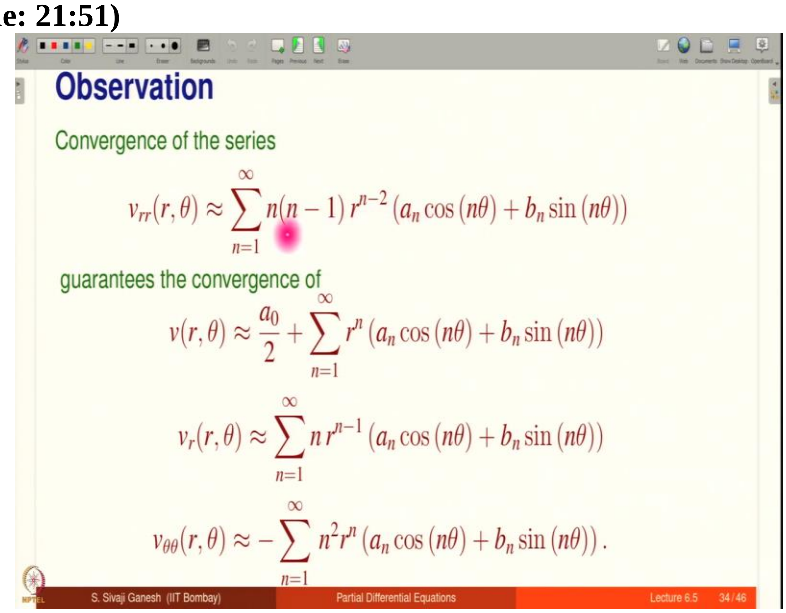

So, convergence of the series for v rr guarantees the convergence of the rest of the things because of the estimate that we are going to do prove for this reason I have explained that there is a n square here there is no n here there is n here there is n square here. So, this and this we have similarly whereas these behave slightly better in fact more than better than the second order derivatives.

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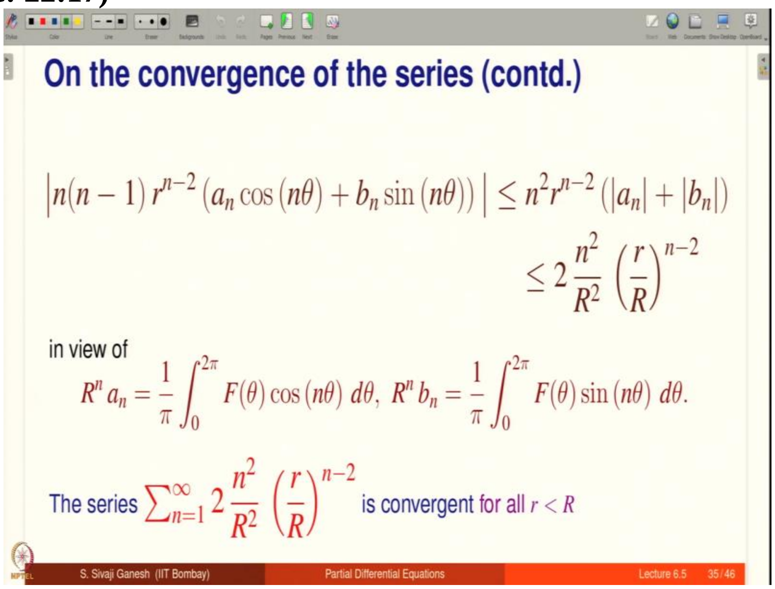

Look at this estimate this is the general term in the series for v rr that is less than or equal to these n into n - 1 that is n square - n that is less than or equal to n square r power n - 2 as it is and cos n theta sin and theta modulus is less than or equal to 1. So, you have mod a n + mod b n and that is less than or equal to this quantity in view of this relations. So, please do this computation by yourself. This series is convergent by let us say ratio test for r less than R.

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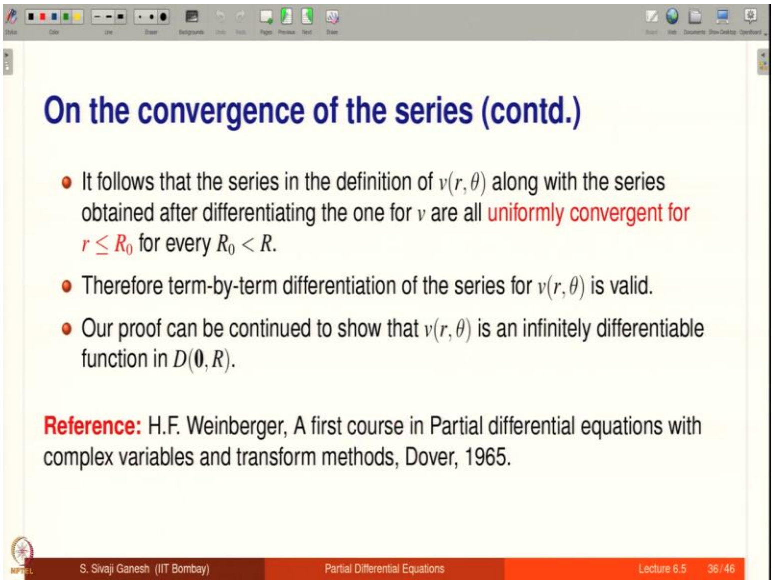

So, it follows that the series in the definition of v of r theta along with the series obtained after differentiating the one for v of r all uniformly convergent for all r less than or equal to R naught where R naught is strictly less than R and for every such R naught therefore term by term differentiation of a series for v of r theta is valid our proof can be continued to show that v is in fact an infinitely differentiable function in the open disk D of 0 comma R. Reference is a book by Weinberger on PDE with complex variables and transfer methods I have taken this material from this book all these details you can see in more details in this book.

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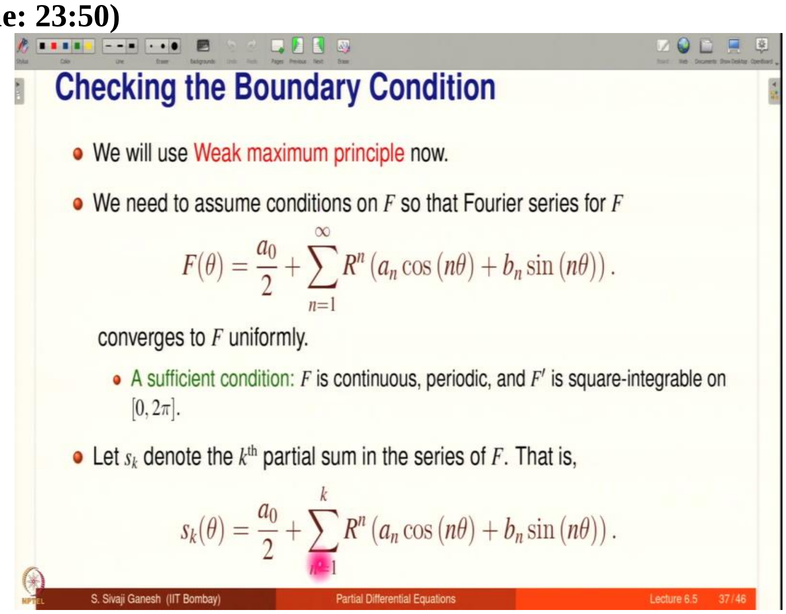

So, we will use the maximum principle to show that the boundary condition is satisfied. So, you can think this is an application of maximum principle. So, we need to assume conditions on F which guarantee that this Fourier series on the right hand side converges to F uniformly that is needed a sample condition is this F is continuous periodic F dash is square integrable on 0, 2 pi of course there are many more such a sufficient conditions. Let s k denote the kth partial sum in the series for F that is this you will truncate this after k terms so you have this.

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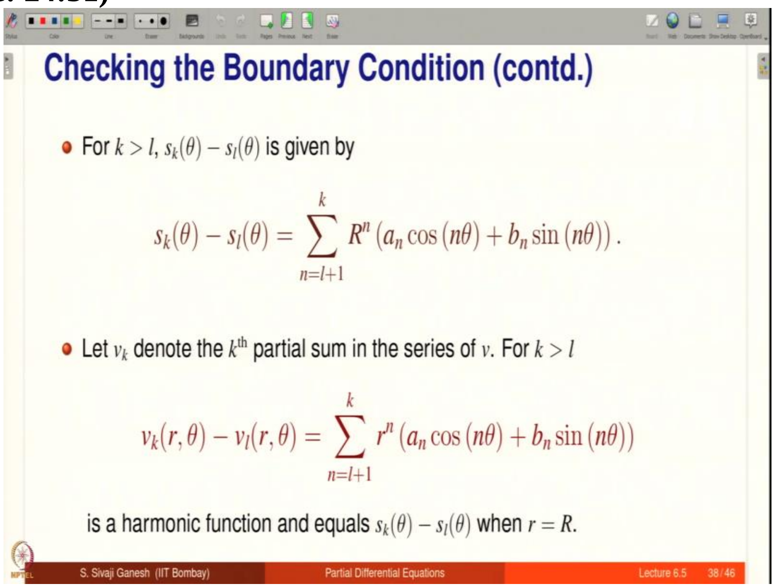

So for k bigger than l s k theta - s l theta is given by this let v k denote the k partial sum in the series of v for k bigger than l v k - v l has this expression it is a harmonic function because each of the inside terms is harmonic function is a finite sum and therefore it is a harmonic function and it is equals s k theta - s l theta on the boundary when r = R.

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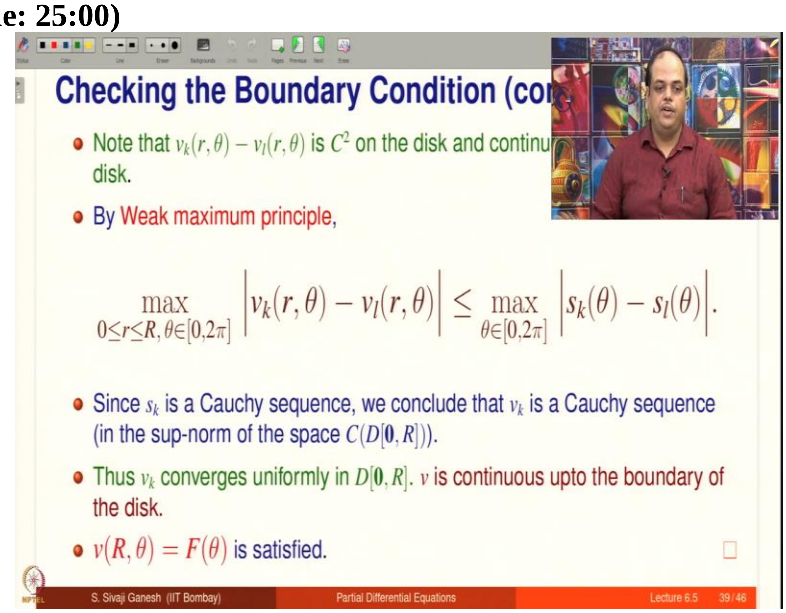

And v k - v l is a C 2 function on the disk and it is continuous up to the boundary of the disk. In fact v k - v l is C infinity on R 2 so by weak maximum principle we have this maximum in the domain or omega closure is less than or equal to maximum on the boundary since s k is a Cauchy sequence in C of 0 2pi so and it converges to F that is why it is a Cauchy sequence and hence v k will be a Cauchy sequence in this domain 0 less than r less than or equal to R theta belongs to 0 2 pi which is the closed disk.

So, therefore v k converges uniformly the closed disk and hence the limit has to be a continuous function which is continuous after the boundary by the property that it is continuous on D of 0, R v is continuous and v of R theta = F theta is satisfied.

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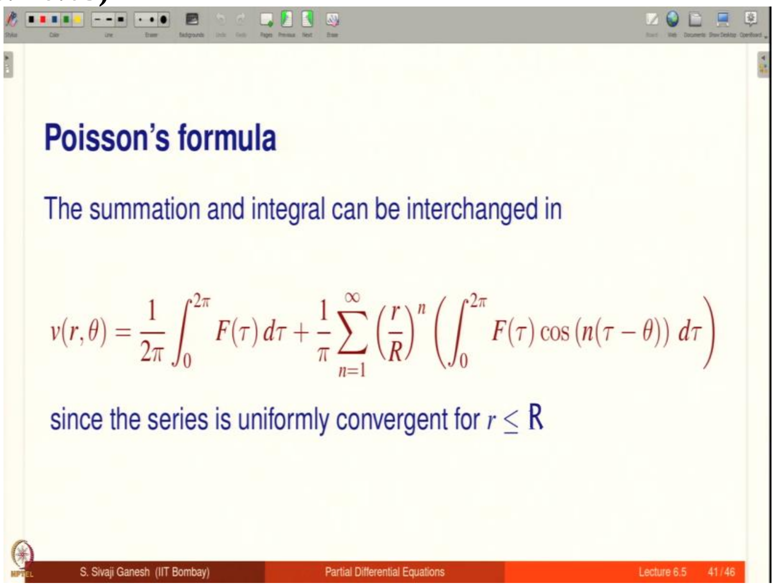

Now let us re derived Poisson's formula the summation and integral can be interchange in this expression since a series is uniformly convergent for r less than or equal to R.

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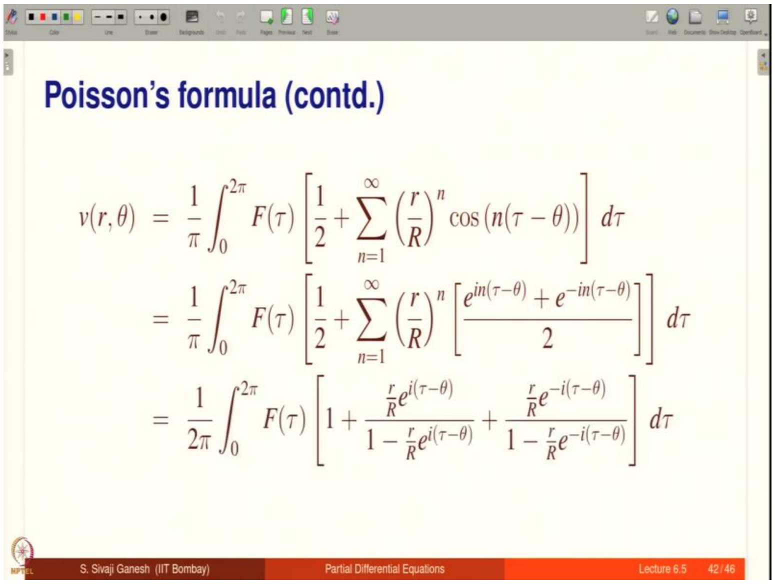

So, this is equal to this in terms of the complex exponentials and on simplification it is the term in the brackets is like this.

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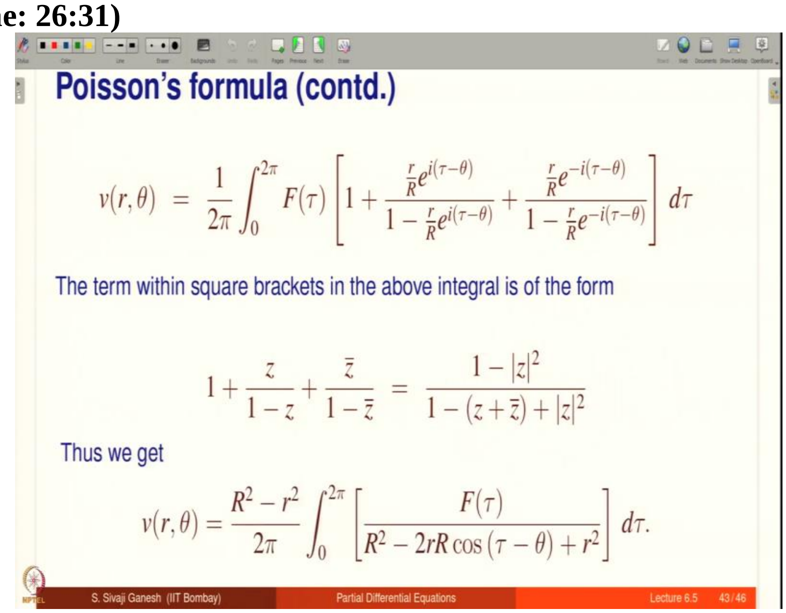

Now the term within the square brackets here is looking like 1 + 1 / 1 - z + z bar / 1 - z bar which can be computed to be this. Please do is this computation. So, now I am going to substitute what is my z that is r / R e power i tau - theta and I get this expression.

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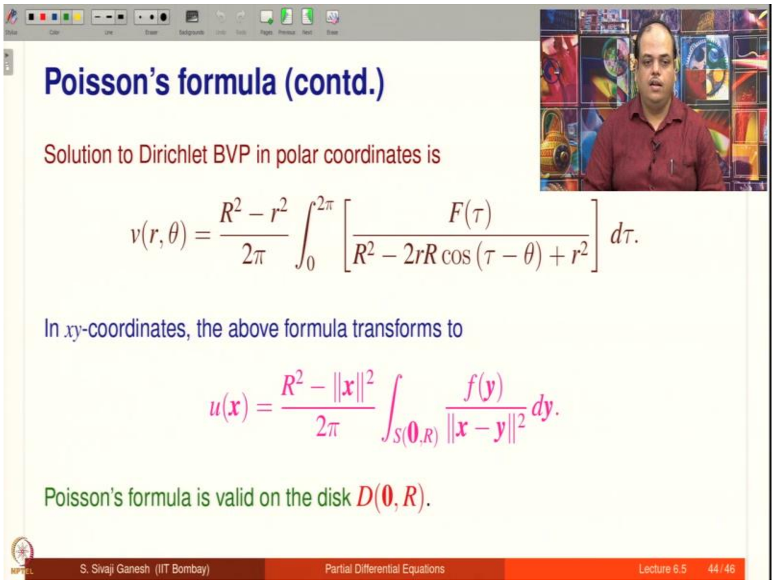

So, solution to Dirichlet boundary value problem in polar coordinates is given by this for r strictly less than r R = r this is not meaningful you should not do that R less than r. In x, y coordinates the; above formula transforms to the well known Poisson's formula it is valid on the disk D of 0, R.

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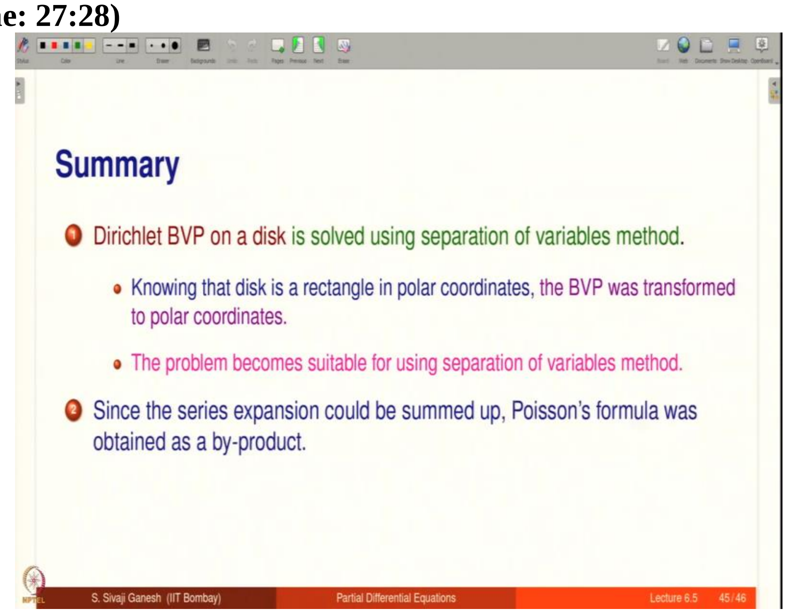

Let us summarize what we did in this lecture Dirichlet boundary value problem on a disk is solved using separation of variables method knowing that disk is a rectangle in polar coordinates that BVP was transformed to polar coordinates. The problem becomes suitable for using separation of variables method. Since the series expansion could be summed up Poisson's formula was obtained as a by-product. Thank you.