

**Partial Differential Equations**  
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**Lecture – 6.5**  
**Dirichlet BVP on a Disk in  $\mathbb{R}^2$  for Laplace Equations**

Welcome to this lecture on a Dirichlet boundary value problem on a disk in  $\mathbb{R}^2$  for Laplace equation. The outline is as follows we solve the Dirichlet boundary value problem on a disk using separation of variables method and we read derive Poisson's formula in this situation for most part of this lecture is going to be computation. So, please stop at every slide and do the computations on your own.

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**Dirichlet BVP on a disk in  $\mathbb{R}^2$**

- Let  $D(\mathbf{0}, R)$  denote the disk having center at the origin, and radius  $R$ . That is,

$$D(\mathbf{0}, R) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$$

- Boundary of  $D(\mathbf{0}, R)$  is a circle, and is denoted by  $S(\mathbf{0}, R)$ .
- For a given function  $f \in C(S(\mathbf{0}, R))$ , we would like to solve the following Dirichlet BVP

$$\begin{aligned} \Delta u &= 0 && \text{in } D(\mathbf{0}, R), \\ u &= f && \text{on } S(\mathbf{0}, R). \end{aligned}$$

by Separation of variables method.

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So, Dirichlet boundary value problem on a disk in  $\mathbb{R}^2$  let  $D(0, R)$  denote the disk having center at the origin and radius  $R$  that is  $D(0, R)$  is set of all  $x, y$  in  $\mathbb{R}^2$  such that  $x^2 + y^2 < R^2$  boundary of  $D(0, R)$  is a circle and is denoted by  $S(0, R)$ . For a given function  $f$  which is continuous on  $S(0, R)$  that is on the circle we would like to solve the following Dirichlet boundary value problem.

What we end up solving using this method of separation of variables we will be able to justify only if this  $f$  is a little more nice. What we are going to see is that we need some more conditions

on  $f$  and through which we can justify the separation of variables method that we are going to discuss.

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**Does Separation of variables method work?**

- In separation of variables method, we look for solutions to  $\Delta u = 0$  having the separated form
 
$$u(x, y) = X(x) Y(y).$$
- We can substitute in  $\Delta u = 0$ , and get two ODEs, one for each of the functions  $X$  and  $Y$ .
- But the boundary condition is
 
$$u(x, y) = f(x, y) \text{ for } (x, y) \text{ s.t. } x^2 + y^2 = R^2.$$
- From here, we can not get any boundary conditions either for  $X$  or for  $Y$ .

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First of all does separation of variables method work. In separation of variables method we look for solutions to any equation in this context it is Laplacian equals 0 having this separated form  $u$  of  $x, y$  is a function of  $X$  into a function of  $Y$  we can substitute in Laplacian equal to 0 and get 2 ODEs one each for  $X$  and  $Y$  but the boundary condition is  $u$  of  $x, y = f$  of  $x, y$  given on the circle. From here we cannot get any boundary conditions for  $X$  or  $Y$ .

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**Does Separation of variables method work? (contd.)**

- Note the domain of a function of the form  $X(x) Y(y)$  is
 
$$\text{Domain of } X \times \text{Domain of } Y.$$
- Since domains of  $X$  and  $Y$  will be intervals, the domain of  $X(x) Y(y)$  will be a rectangle in  $xy$ -coordinate system.
- Thus separation of variables method will NOT work for disk.
- Thankfully, Disk is a Rectangle in Polar coordinate system!
- Now the equation  $\Delta u = 0$  needs to be transformed in to  $(r, \theta)$ -coordinate system.

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Note that the domain of a function which is in this product form  $X$  of  $x$  into  $Y$  of  $y$  is domain of  $X$  cross domain of  $Y$ . Because if  $X$  of  $x$  into  $Y$  of  $y$  makes sense for every tuple  $x, y$  such that  $X$

belongs to domain of X and Y belongs to domain of y. Since the domains of X and Y will be intervals because X and Y are functions of one variable the domain of the product will be a rectangle in x, y coordinate system.

Thus separate separation of variables method will not work for disk. Thankfully disk is a rectangle it looks surprising disk is not a rectangle but disk is a rectangle in some other coordinate system namely the polar coordinate system. Now the equation Laplacian equals 0 needs to be transformed into r theta coordinate system.

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**Laplacian in Polar coordinates**

The Dirichlet problem for the Laplace equation in polar coordinates is given by

$$\Delta_{(r,\theta)} v \equiv v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 \quad \text{for } 0 < r < 1, 0 \leq \theta \leq 2\pi,$$

$$v(R, \theta) = F(\theta) \quad \text{for } 0 \leq \theta \leq 2\pi,$$

where

$$F(\theta) = f(R \cos \theta, R \sin \theta)$$

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So, Laplacian in polar coordinates the Dirichlet problem for Laplace equation in polar coordinates is given by these 2 equation this is precisely the Laplacian in the polar coordinates. So, we have done enough exercises on how the PDE transforms under change of coordinates. So, I leave this as an exercise for you to check that Laplacian in polar coordinates is given by this operator which is here  $v_{rr} + 1/r v_r + 1/r^2 v_{\theta\theta}$ .

We are given u of x, y on the circle that will define a function F for theta in 0 to 2 pi as follows F theta is nothing but this is a given f R cos theta R sin theta in polar coordinates x = R cos theta y = R sin theta. So, this is how you get for points on the circle of radius R with center at origin.

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**Main steps in Separation of variables method**

**Step 1. Two families of ODEs obtained from Laplace equation.**

- Look for solutions of the Laplace equation in polar coordinates in the separated form  $v(r, \theta) = h(r)g(\theta)$ .
- The Laplace equation will give rise to two families of ODEs indexed by a single parameter  $\lambda$ : One for  $h$  and another for  $g$ .
- Equation for  $g$  will be supplemented with **periodic** boundary conditions.

$$g(0) = g(2\pi), \quad g'(0) = g'(2\pi) \text{ (why?)}$$

- At this stage, we can NOT infer any conditions for  $h$  or  $g$  from the Dirichlet boundary data.
- Thus Boundary data would be used at the very end of the procedure.

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So, what are the main steps in separation of variables method? Step 1 2 families of ODEs obtained from Laplace equation or any equation for which you are trying to do separation of variables method. Separation of variables method is not new to us we have done many exercises based on this for wave equation. So, look for solutions for the Laplace equation in polar coordinates in the separated form  $v$  of  $r$  theta as  $h$  of  $r$  into  $g$  of theta.

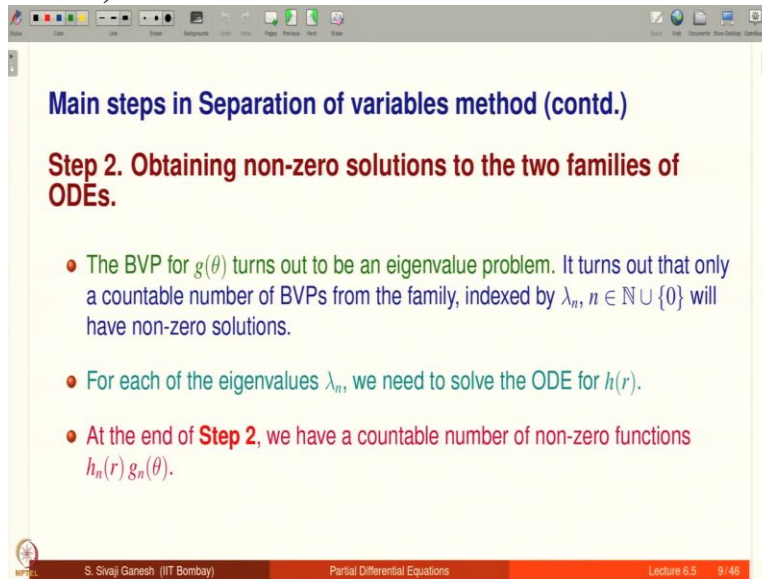
The Laplace equation will give rise to 2 families of ODEs indexed by a single parameter lambda one for  $h$  and one for  $g$ . Equation for  $g$  will be supplemented with a periodic boundary conditions which is  $g$  of  $0 = g$  of  $2\pi$  and  $g$  prime of  $0 = g$  prime of  $2\pi$ . The question is why? Why because what is our aim we want to solve Laplace equation in the disk that means the function should be continuous differentiable  $C^2$  etcetera.

But that we are trying to solve in terms of  $h$  of  $r$  and  $g$  of theta and  $g$  of theta. So, there should be some requirement on  $g$  that is precisely this. Please explore this further and then find out the precise reason which I have only indicated why this condition is at this stage we cannot infer any conditions for  $h$  or  $g$ . From the Dirichlet boundary data the boundary data will be used at the very end of the procedure.

If you recall for the wave equation also one equation we have solved a boundary value problem for the second equation we do not have boundary value problem we have if you want a partial

initial value problem and still there were some constant to be determined that we used the other initial data to fix those constants same thing here.

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**Main steps in Separation of variables method (contd.)**

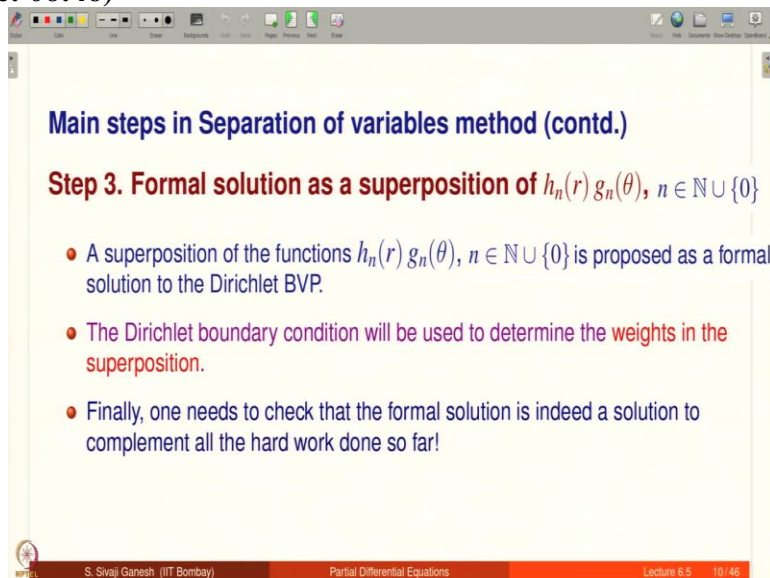
**Step 2. Obtaining non-zero solutions to the two families of ODEs.**

- The BVP for  $g(\theta)$  turns out to be an eigenvalue problem. It turns out that only a countable number of BVPs from the family, indexed by  $\lambda_n, n \in \mathbb{N} \cup \{0\}$  will have non-zero solutions.
- For each of the eigenvalues  $\lambda_n$ , we need to solve the ODE for  $h(r)$ .
- At the end of **Step 2**, we have a countable number of non-zero functions  $h_n(r) g_n(\theta)$ .

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So, step 2 obtaining non-zero solutions to the 2 families of ODEs the BVP for  $g$  of  $\theta$  turns out to be an eigenvalue problem. It turns out that only a countable number of such BVPs index will  $n$  belongs to  $n \cup 0$  we will have non-zero solutions. For each of the eigenvalues  $\lambda_n$  we need to solve the ODE for  $h$  of  $r$ . At the end of step 2 we have a countable number of non-zero functions  $h_n$  of  $r$  will be  $g_n$  of  $\theta$ .

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**Main steps in Separation of variables method (contd.)**

**Step 3. Formal solution as a superposition of  $h_n(r) g_n(\theta), n \in \mathbb{N} \cup \{0\}$**

- A superposition of the functions  $h_n(r) g_n(\theta), n \in \mathbb{N} \cup \{0\}$  is proposed as a formal solution to the Dirichlet BVP.
- The Dirichlet boundary condition will be used to determine the weights in the superposition.
- Finally, one needs to check that the formal solution is indeed a solution to complement all the hard work done so far!

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Step 3 is a proposal of a formal solution as a linear superposition of  $h_n(r)g_n(\theta)$  where  $n \in \mathbb{Z}$ . A superposition of the functions  $h_n(r)g_n(\theta)$  where  $n \in \mathbb{Z}$  is proposed as a formal solution to the Dirichlet boundary value problem. The Dirichlet boundary condition will be used to determine the weights in the superposition. Finally one needs to check that the formal solution is indeed a solution to complement all the hard work that we have done so far.

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**Step 1. Laplace equation gives rise to two ODEs**

Method of separation of variables looks for solutions of the form

$$v(r, \theta) = h(r)g(\theta),$$

where  $g(0) = g(2\pi), g'(0) = g'(2\pi)$ .

Substituting in the Laplace equation in polar coordinates

$$\Delta_{(r,\theta)} v \equiv v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0,$$

and then dividing both sides of the resultant equation with  $h(r)g(\theta)$  yields

$$\frac{h''(r)}{h(r)} + \frac{1}{r} \frac{h'(r)}{h(r)} + \frac{1}{r^2} \frac{g''(\theta)}{g(\theta)} = 0$$

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So, step 1 method of separation of variables looks for solutions of the form  $h(r)g(\theta)$  where  $g(0) = g(2\pi)$  and  $g'(0) = g'(2\pi)$  the periodic property for  $g$  and  $g'$  substituting in the Laplace equation in polar coordinates and then dividing both sides with  $h(r)g(\theta)$  gives us this equation.

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**Step 1. Laplace equation gives rise to two ODEs (contd.)**

Re-arranging terms in

$$\frac{h''(r)}{h(r)} + \frac{1}{r} \frac{h'(r)}{h(r)} + \frac{1}{r^2} \frac{g''(\theta)}{g(\theta)} = 0$$

yields

$$r^2 \frac{h''(r)}{h(r)} + r \frac{h'(r)}{h(r)} = -\frac{g''(\theta)}{g(\theta)}$$

- the LHS is a function of  $r$  only, while the RHS is a function of  $\theta$  only.
- Such an equation can hold if and only if both the functions are identically equal to a constant function.

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So, rearranging the terms in this equation gives us this equation or taking the  $g$  terms to the right side and multiplied with  $r$  square. So, I have got this now observe that LHS is a function of our only RHS is a function of theta only. Therefore such an equation can hold if and only if both functions are equal to a constant function.

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**Step 1. Laplace equation gives rise to two ODEs (contd.)**

It means that there exist  $\lambda \in \mathbb{R}$  such that

$$r^2 \frac{h''(r)}{h(r)} + r \frac{h'(r)}{h(r)} = -\frac{g''(\theta)}{g(\theta)} = \lambda$$

One of the tasks is to find all possible  $\lambda$ s coming from separated solutions.

This gives rise to two ODEs, which are given by

$$g''(\theta) + \lambda g(\theta) = 0,$$

$$r^2 h''(r) + r h'(r) = \lambda h(r)$$

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So, it means that there exists lambda in  $\mathbb{R}$  such that you have this equal to this and both of them equal to lambda one of the tasks is to find all possible lambdas coming from separated solutions this gives rise to 2 ODEs which are given by  $g'' + \lambda g = 0$   $r^2 h'' + r h' = \lambda h$ .

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**Step 1. Laplace equation gives rise to two ODEs (contd.)**

Thus we have the boundary value problem for  $g(\theta)$  given by

$$g''(\theta) + \lambda g(\theta) = 0,$$

$$g(0) = g(2\pi), \quad g'(0) = g'(2\pi).$$

The ODE for  $h(r)$  is given by

$$r^2 h''(r) + r h'(r) = \lambda h(r)$$

Dirichlet Boundary conditions to be used later.

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Thus we have the boundary value problem for  $g$  of  $t$  given by  $g'' + \lambda g = 0$  on this periodic boundary conditions and the ODE for  $h$  of  $r$  is given by  $r^2 h'' + r h' = \lambda h$  Dirichlet boundary conditions to be used later.

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**Step 2. Finding non-zero solutions to BVP**

The boundary value problem for  $g(\theta)$  is given by

$$g''(\theta) + \lambda g(\theta) = 0,$$

$$g(0) = g(2\pi), \quad g'(0) = g'(2\pi).$$

- The  $\lambda$ s for which the BVP admits a non-zero solution are called **eigenvalues** and the corresponding non-zero solutions are called **eigenfunctions**.
- Let us start our search for eigenvalues and eigenfunctions.

Note that  $\lambda \in \mathbb{R}$  can be zero, positive, or negative.

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Now let us try to solve this boundary value problem for  $g$ . So, there is a parameter  $\lambda$  so the  $\lambda$ s for which the BVP admits a non-zero solution are called Eigenvalues and the corresponding non-zero solutions are called Eigen functions. Let us start our search for Eigenvalues and Eigen functions. Note that the  $\lambda$  can be 0 positive or negative these are the only 3 possibilities for a real number.

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**Step 2. Finding non-zero solutions to BVP:  $\lambda$**

The BVP for  $g(\theta)$  becomes

$$g''(\theta) = 0,$$

$$g(0) = g(2\pi), \quad g'(0) = g'(2\pi).$$

- General solution of the ODE  $g'' = 0$  is given by  $g(\theta) = a\theta + b$ .
- Applying the boundary condition  $g(0) = g(2\pi)$ , we get  $a = 0$ .
- Thus  $\lambda = 0$  is an eigenvalue and eigenfunctions are constants  $g(\theta) = b$ .

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Let us solve the BVP with a lambda = 0 and see whether it has a non trivial solution or not. So, the BVP itself becomes this because the equation changes lambda = 0. So, the g term is dropped g double dash = 0. Now general solution of the ODE g double dash = 0 is a theta + b. Now apply these boundary conditions g of 0 = g of 2 pi if you put what we get is a = 0. Thus lambda = 0 is an Eigenvalue and Eigen functions are constants the second boundary condition does not give us any information because g prime is actually a = a so it is always true. So, essentially the first boundary condition gave us a = 0 therefore Eigen functions are g theta = b b is any constant.

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**Step 2. Finding non-zero solutions to BVP:  $\lambda > 0$**

Since  $\lambda > 0$ , we may write  $\lambda = \mu^2$  where  $\mu > 0$ .

The BVP for  $g(\theta)$  becomes

$$g''(\theta) + \mu^2 g(\theta) = 0,$$

$$g(0) = g(2\pi), \quad g'(0) = g'(2\pi).$$

- General solution of the above ODE is given by

$$g(\theta) = a \cos(\mu\theta) + b \sin(\mu\theta).$$

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Now let us look at boundary value problem for lambda positive so once lambda is positive we can write it as mu square for mu positive for definiteness are also because we are dealing with

the second order equation it is convenient to have somebody square here. So that is why mu square now general solution of the ODE is a combination of cosine and sine functions a cos theta + b sin mu theta.

Now we are determined the constants a and b such that these boundary conditions are satisfied and then ask if it is possible to choose the constants a and b at least one of them non-zero in which case we get Eigenvalues and Eigen functions.

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**Step 2. Finding non-zero solutions to BVP:  $\lambda > 0$**

- The function

$$g(\theta) = a \cos(\mu\theta) + b \sin(\mu\theta)$$

satisfies the boundary conditions  $g(0) = g(2\pi)$ ,  $g'(0) = g'(2\pi)$   
if and only if  $a, b$  satisfy the linear system

$$a = a \cos(2\mu\pi) + b \sin(2\mu\pi)$$
$$b\mu = -a\mu \sin(2\mu\pi) + b\mu \cos(2\mu\pi)$$

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Therefore this function here satisfies these boundary conditions. If and only if these 2 conditions are met I have just substituted into these expressions this formula so we get these 2 equations coming from these 2 conditions.

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**Step 2. Finding non-zero solutions to BVP:  $\lambda > 0$  (contd.)**

- The linear system on the last slide may be written as
 
$$\begin{pmatrix} 1 - \cos(2\mu\pi) & -\sin(2\mu\pi) \\ \sin(2\mu\pi) & 1 - \cos(2\mu\pi) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
- Note that the above system has non-trivial solutions **if and only if**  $\cos(2\mu\pi) = 1$ . This means that  $\mu \in \mathbb{N}$ .
- Thus we have a sequence of positive eigenvalues  $\lambda_n = n^2$ , indexed by  $n \in \mathbb{N}$ .
- For each  $n \in \mathbb{N}$ , eigenspace corresponding to  $\lambda_n$  is generated by the set
 
$$\{\cos(n\theta), \sin(n\theta)\}.$$

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Now it is always a good idea to write it as a linear system. So the 2 equations we have on the last slide let us write it as a linear system and we are looking for non-zero solutions of this system because that will give us Eigenvalues Eigen vectors it has non-zero solutions if and only if this determinant is 0 if the determinant is non-zero that implies a, b is 00. So that is not a situation we want.

So, therefore we want this determinant to be 0 which will give us  $\cos 2\mu\pi = 1$  that means  $\mu$  is a natural number. So, therefore we have a sequence of Eigenvalues what are the values their  $\mu$  square so  $\mu$  is  $\mathbb{N}$  here. So, any natural number so eigenvalues are squares of natural numbers and what are the Eigen functions? They are generated by  $\cos n\theta$  and  $\sin n\theta$  for each  $n$  is a linear combination of  $\cos n\theta$  and  $\sin n\theta$  that will be an Eigen function.

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**Step 2. Finding non-zero solutions to BVP:  $\lambda < 0$**

Since  $\lambda < 0$ , we may write  $\lambda = -\mu^2$  where  $\mu > 0$

The BVP for  $g(\theta)$  becomes

$$g''(\theta) - \mu^2 g(\theta) = 0,$$

$$g(0) = g(2\pi), \quad g'(0) = g'(2\pi).$$

- General solution of the above ODE is given by

$$g(\theta) = ae^{\mu\theta} + be^{-\mu\theta}.$$

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Now let us inquire into negative lambdas whether those be BVPs have a non trivial solutions or not. So, we can write lambda = - mu square where mu is positive and the equation is here these are the boundary conditions. So, general solution of the above ODE is a combination of exponentials, which is here and let us substitute these boundary conditions.

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**Step 2. Finding non-zero solutions to BVP:  $\lambda < 0$  (contd.)**

The function

$$g(\theta) = ae^{\mu\theta} + be^{-\mu\theta}$$

satisfies the boundary conditions  $g(0) = g(2\pi), \quad g'(0) = g'(2\pi)$

**if and only if**

$a, b$  satisfy the linear system

$$a + b = ae^{2\mu\pi} + be^{-2\mu\pi}$$

$$a\mu - b\mu = a\mu e^{2\mu\pi} - b\mu e^{-2\mu\pi}.$$

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We get these relations to understand whether this has an Android solution or not it is always a good idea to write it as a linear system which we do here.

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**Step 2. Finding non-zero solutions to BVP:  $\lambda < 0$  (contd.)**

- The linear system on the last slide may be written as
 
$$\begin{pmatrix} 1 - e^{2\mu\pi} & 1 - e^{-2\mu\pi} \\ 1 - e^{2\mu\pi} & -1 + e^{-2\mu\pi} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
- Note that the above system has non-trivial solutions **if and only if**  $e^{2\mu\pi} + e^{-2\mu\pi} = 2$ .
- This equation has the form  $\alpha + \frac{1}{\alpha} = 2$ , for which  $\alpha = 1$  is the only solution
- Since  $\mu > 0$ , it follows that  $e^{2\mu\pi} + e^{-2\mu\pi} \neq 2$ .
- It means that there are **No negative eigenvalues**

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Now we inquire into whether the determinant is 0 or non-zero. If it is 0 only trivial solutions that means no negative Eigenvalues. If determinant is 0 then we have nontrivial solutions then there are negative Eigenvalues. So, above system has nontrivial solution if and only if the determinant is 0 which comes out to be this  $e^{2\mu\pi} + e^{-2\mu\pi} = 2$  and it looks like a number plus 1 by that number equal to 2.

For which  $\alpha = 1$  is the only solution. Since  $\mu$  is positive it follows that this cannot be equal to 2 because  $\alpha = 1$  means  $\mu$  is 0 but  $\mu$  is not 0. Therefore this equation is not satisfied. It means that there are no negative Eigenvalues.

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**Summary on Eigenvalues and eigenfunctions**

The eigenvalues and corresponding eigenfunction  $n \in \mathbb{N} \cup \{0\}$ :

- $\lambda_0 = 0$  is an eigenvalue, with an eigenspace consisting of constant functions.
- $\lambda_n = n^2$  with eigenspace spanned by  $\{\cos(n\theta), \sin(n\theta)\}$ .

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So, let us summarize the Eigenvalues and Eigen functions to the BVP for  $g$  of  $\theta$ . They are indexed by  $n \in \mathbb{N} \cup \{0\}$ .  $\lambda_0 = 0$  is an eigenvalue with an Eigen space consisting of constant functions.  $\lambda_n = n^2$  with Eigen space spanned by  $\cos n\theta$  and  $\sin n\theta$ .

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**Step 2. Finding non-zero solutions to ODE for  $h(r)$**

Solve the ODE for  $h$

$$r^2 h''(r) + r h'(r) - \lambda h(r) = 0.$$

with  $\lambda = \lambda_n$  for each  $n \in \mathbb{N} \cup \{0\}$ .

- $\lambda_0 = 0$ : General solution to the above ODE is
 
$$h(r) = A \log r + B, \quad A, B \in \mathbb{R}$$
- Recall that our goal is to solve the BVP on the disk containing origin. In particular, we are looking for bounded solutions **ONLY**. Thus  $A = 0$ .
- Thus we obtain the following solution of Laplace equation:
 
$$v_0(r, \theta) = h_0(r)g_0(\theta) = \text{Constant} = B.$$

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Now it is time to solve the equation for  $h$  of  $r$  which is here for each  $\lambda = \lambda_n$  in natural numbers union  $\{0\}$ .  $\lambda_0 = 0$  general solutions to the above ODE is  $A \log r + B$  where  $A$  and  $B$  are real numbers. Recall that our goal is to solve the boundary value problem on the disk. In particular the function the solution that we are planning to get should be bounded at the origin it should be bounded on the disk but bounded at the origin in particular.

Therefore  $\log r$  term is not suitable for that therefore  $A$  must be 0 thus we have obtain the following solution of Laplace equation  $h$  of  $r$  into  $g$  of  $\theta$  remember  $g$  of  $\theta$  was also constant when  $\lambda_0 = 0$   $h$  of  $r$  is also constant because  $A$  is 0 there is only  $B$  here. So, we can take the constant to be 1 anyway we are going to take linear superposition of these numbers. So you keep it 1 or 2 is the same so keep it 1.

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**Step 2. Finding non-zero solutions to ODE for  $h(r)$  (contd.)**

For each  $n \in \mathbb{N} \cup \{0\}$ , the ODE for  $h$  with  $\lambda_n = n^2$  is given by

$$r^2 h''(r) + r h'(r) - \lambda h(r) = 0.$$

- The ODE is of Cauchy-Euler type, which reduces to a constant coefficient ODE by a change of variable  $s = \log r$ , and solving is left as an exercise.
- The general solution of the above ODE is given by

$$h_n(r) = Ar^n + Br^{-n}, \quad A, B \in \mathbb{R}.$$

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For each  $n$  in  $\mathbb{N}$  union  $0$  the ODE for  $h$  with  $\lambda_n = n^2$  is given by this equation. The ODE is of Cauchy Euler type which reduces to a constant coefficient ODE by change a variable  $s = \log r$  and solving his left hand exercise are nowadays people also know what kind of solutions are there for this equation they straightaway look for a solution of the form  $r$  power  $k$  and then you can solve like that also. So, the general solution of the above ODE is given by  $A r^n + B r^{-n}$  where  $A$  and  $B$  are real numbers.

**(Refer Slide Time: 16:39)**

**Step 2. Finding non-zero solutions to ODE for**

$$h_n(r) = Ar^n + Br^{-n}, \quad A, B \in \mathbb{R}.$$

- Recall that our goal is to solve the BVP on the disk containing origin. In particular, we are looking for bounded solutions **ONLY**. Thus  **$B = 0$** .
- Thus we obtain the following solution of Laplace equation:

$$v_n(r, \theta) = h_n(r)g_n(\theta) = r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

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Once again recall that our goal is to solve the BVP on the disk containing origin. Therefore we are looking for solutions which are bounded and hence this  $B$  should be  $0$  thus we obtain the following solution  $h_n r$  into  $g_n$  of  $\theta$  given by  $r^n$  from  $h_n$  and this is from  $g_n$ .

(Refer Slide Time: 17:06)

**Step 3. Proposing a formal solution to the Dirichlet BVP**

We propose a formal solution to the BVP, using 'superposition principle', by

$$v(r, \theta) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

The unknown coefficients  $a_0, a_n, b_n$  will be determined using the boundary condition  $u = f$ . This means

$$v(R, \theta) = F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

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Now let us propose a formal solution to the Dirichlet boundary value problem using superposition principle by  $v$  have our  $\theta = a$  naught / 2 + summation  $n = 1$  to infinity  $r$  power  $n$   $a_n \cos n \theta + b_n \sin n \theta$  if you recall this is a combination of the function 1 which is coming from  $h$  naught of  $r$  into  $g$  naught of  $\theta$  and this summation here is precisely  $h_n$  of  $r$  into  $g_n$  and  $f$   $\theta$ . So we are taking a combination the constants in the combination are taken  $a_n$  and  $b_n$ .

So, the unknown coefficients  $a_0, a_n$  and  $b_n$  has to be determined using the boundary condition because that is the only condition we have not used yet. That means  $v$  have  $R$  comma  $\theta$  that is  $r = R$  should be  $f$  of  $\theta$  and that should be equal to this expression with  $r = R$ .

(Refer Slide Time: 18:09)



**Step 3. On the coefficients  $a_0, a_n, b_n$**

$$v(R, \theta) = F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

Choose the constants  $R^n a_n (n \in \mathbb{N} \cup \{0\})$ ,  $R^n b_n (n \in \mathbb{N})$  as the Fourier coefficients of  $F(\theta)$ .

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} F(\theta) \cos(n\theta) d\theta, \quad b_n = \frac{1}{\pi R^n} \int_0^{2\pi} F(\theta) \sin(n\theta) d\theta.$$

This finishes the construction of a formal solution to the Dirichlet BVP.

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This is how we would like to find the constants  $a_n, b_n$ , a naught such that the function  $f$  of theta is given by this Fourier series. So, choose the constant  $R^n a_n$  into  $R^n b_n$  such that they are the Fourier coefficients of  $F$  of theta of course we have this expression for  $a_n$  and  $b_n$ . So, this finishes the construction of a formal solution to the Dirichlet BVP.

**(Refer Slide Time: 18:43)**

**Step 3. Formal solution to the Dirichlet BVP**

**A formal solution to the Dirichlet BVP is given by**

$$v(r, \theta) \approx \frac{1}{2\pi} \int_0^{2\pi} F(\tau) d\tau + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \left( \int_0^{2\pi} F(\tau) \cos(n(\tau - \theta)) d\tau \right)$$

**Is it a solution to the Dirichlet BVP?**

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So, formal solution to the Dirichlet BVP is given by this I have substituted the question  $c_n b_n$  into the series that we have proposed. So, now this has no  $a_n b_n$  everything in terms of  $f$  and this has a cos and sin terms inside. So, is it a solution to Dirichlet BVP that is a question now?

**(Refer Slide Time: 19:07)**

To show that the formal solution

$$v(r, \theta) \approx \frac{1}{2\pi} \int_0^{2\pi} F(\tau) d\tau + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \left( \int_0^{2\pi} F(\tau) \cos(n(\tau - \theta)) d\tau \right)$$

is a solution to Dirichlet BVP, we need to prove

- 1 The infinite series converges. Denote the sum by  $v(r, \theta)$ .
- 2 The function  $v(r, \theta)$  is a  $C^2$  function. We need to show that the infinite series can be differentiated two times, and the resultant series converge.
- 3  $v(R, \theta) = F(\theta)$  holds.

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To show that the formal solution given here is a solution we need to prove that the series here which is okay is just one term there is a series here that series converges. In other words this makes sense and defines a function. If such a thing happens call it  $v$  of  $r$  theta and then we have to show that  $v$  of  $r$  theta is a  $C^2$  function we need to show that infinite series can be differentiated 2 times and the resultant series converge.

Then we have to show that the boundary condition is satisfied  $v$  of  $r$  theta =  $f$  theta holds these 3 things we have to do. So, from now onwards you can just see the video once and you can forget the details because some of the details have not mentioned very clearly. For whichever given you a reference but we should be happy that we have got a formal series here rigorous analysis of course I have given from now onwards.

**(Refer Slide Time: 20:08)**

**On the convergence of the series**

Questions of convergence of the infinite series appearing in the formal solution

$$v(r, \theta) \approx \frac{1}{2\pi} \int_0^{2\pi} F(\tau) d\tau + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \left( \int_0^{2\pi} F(\tau) \cos(n(\tau - \theta)) d\tau \right)$$

are better discussed in terms of its original *avataar*, which is

$$v(r, \theta) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)),$$

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} F(\theta) \cos(n\theta) d\theta, \quad b_n = \frac{1}{\pi R^n} \int_0^{2\pi} F(\theta) \sin(n\theta) d\theta.$$

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So, questions of convergence of the infinite series appearing in this formal solution namely this one are better handled using its original avatar which is this very a n has this formula b n has this formula.

**(Refer Slide Time: 20:24)**

**On the convergence of the series (contd.)**

On formally differentiating the formal solution

$$v(r, \theta) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)),$$

we get

$$v_r(r, \theta) \approx \sum_{n=1}^{\infty} n r^{n-1} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

$$v_{rr}(r, \theta) \approx \sum_{n=1}^{\infty} n(n-1) r^{n-2} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

$$v_{\theta\theta}(r, \theta) \approx - \sum_{n=1}^{\infty} n^2 r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

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So, I am formally differentiating the formal solution we get v r to be like this v rr to be like this v theta, theta should be like this. So, every time you differentiate you pick up some n so r n n is coming here, here n into n - 1 that is n square term. So, in second order do you see how n square terms and in first order derivative equal to n terms when you are no derivative there is no n as a coefficient here of course r power n cos n theta they are all there.

**(Refer Slide Time: 20:58)**

Substituting the expressions for  $v_r$ ,  $v_{rr}$ ,  $v_{\theta\theta}$ , from the previous slide, we conclude that  $v(r, \theta)$  satisfies

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0$$

It remains to prove that all the series are convergent, and the boundary condition is satisfied.

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So, substituting these expressions for  $v_r$ ,  $v_{rr}$ ,  $v_{\theta\theta}$  which are given here formally you add them up with the corresponding weights we are which weights the one which are in the Laplace operator this one so  $v_{rr}$  plus multiply  $v_r$  with  $1/r$   $v_{\theta\theta}$  with  $1/r^2$  you see that you get 0. So, formally we have shown that when the series on the previous slide makes sense they add up and in this way to get that  $v$  of  $r$   $\theta$  solution to a Laplace equation.

So, it remains to prove that all the series are convergent and of course the boundary condition is satisfied that we have to prove so once the series are convergent we have just shown that Laplacian  $v = 0$  is satisfying.

**(Refer Slide Time: 21:51)**

**Observation**

Convergence of the series

$$v_{rr}(r, \theta) \approx \sum_{n=1}^{\infty} n(n-1)r^{n-2}(a_n \cos(n\theta) + b_n \sin(n\theta))$$

guarantees the convergence of

$$v(r, \theta) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

$$v_r(r, \theta) \approx \sum_{n=1}^{\infty} n r^{n-1} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

$$v_{\theta\theta}(r, \theta) \approx - \sum_{n=1}^{\infty} n^2 r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

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So, convergence of the series for  $r < R$  guarantees the convergence of the rest of the things because of the estimate that we are going to do. For this reason, I have explained that there is a  $n^2$  here, there is no  $n$  here, there is  $n$  here, there is  $n^2$  here. So, this and this we have similarly, whereas these behave slightly better, in fact, more than better than the second order derivatives.

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**On the convergence of the series (contd.)**

$$|n(n-1)r^{n-2}(a_n \cos(n\theta) + b_n \sin(n\theta))| \leq n^2 r^{n-2} (|a_n| + |b_n|)$$

$$\leq 2 \frac{n^2}{R^2} \left(\frac{r}{R}\right)^{n-2}$$

in view of

$$R^n a_n = \frac{1}{\pi} \int_0^{2\pi} F(\theta) \cos(n\theta) d\theta, \quad R^n b_n = \frac{1}{\pi} \int_0^{2\pi} F(\theta) \sin(n\theta) d\theta.$$

The series  $\sum_{n=1}^{\infty} 2 \frac{n^2}{R^2} \left(\frac{r}{R}\right)^{n-2}$  is convergent for all  $r < R$

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Look at this estimate, this is the general term in the series for  $r < R$  that is less than or equal to  $n^2 r^{n-2}$ . This is  $n^2 - n$  that is less than or equal to  $n^2 r^{n-2}$  as it is and  $\cos n\theta$ ,  $\sin$  and  $\theta$  modulus is less than or equal to 1. So, you have  $|a_n| + |b_n|$  and that is less than or equal to this quantity in view of these relations. So, please do this computation by yourself. This series is convergent by, let us say, ratio test for  $r < R$ .

(Refer Slide Time: 22:59)

**On the convergence of the series (contd.)**

- It follows that the series in the definition of  $v(r, \theta)$  along with the series obtained after differentiating the one for  $v$  are all **uniformly convergent for  $r \leq R_0$**  for every  $R_0 < R$ .
- Therefore term-by-term differentiation of the series for  $v(r, \theta)$  is valid.
- Our proof can be continued to show that  $v(r, \theta)$  is an infinitely differentiable function in  $D(0, R)$ .

**Reference:** H.F. Weinberger, A first course in Partial differential equations with complex variables and transform methods, Dover, 1965.

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So, it follows that the series in the definition of  $v$  of  $r$  theta along with the series obtained after differentiating the one for  $v$  of  $r$  all uniformly convergent for all  $r$  less than or equal to  $R$  naught where  $R$  naught is strictly less than  $R$  and for every such  $R$  naught therefore term by term differentiation of a series for  $v$  of  $r$  theta is valid our proof can be continued to show that  $v$  is in fact an infinitely differentiable function in the open disk  $D$  of  $0$  comma  $R$ . Reference is a book by Weinberger on PDE with complex variables and transfer methods I have taken this material from this book all these details you can see in more details in this book.

**(Refer Slide Time: 23:50)**

**Checking the Boundary Condition**

- We will use **Weak maximum principle** now.
- We need to assume conditions on  $F$  so that Fourier series for  $F$

$$F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

converges to  $F$  uniformly.

- A sufficient condition:**  $F$  is continuous, periodic, and  $F'$  is square-integrable on  $[0, 2\pi]$ .
- Let  $s_k$  denote the  $k^{\text{th}}$  partial sum in the series of  $F$ . That is,

$$s_k(\theta) = \frac{a_0}{2} + \sum_{n=1}^k R^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

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So, we will use the maximum principle to show that the boundary condition is satisfied. So, you can think this is an application of maximum principle. So, we need to assume conditions on  $F$

which guarantee that this Fourier series on the right hand side converges to  $F$  uniformly that is needed a sample condition is this  $F$  is continuous periodic  $F$  dash is square integrable on  $0, 2\pi$  of course there are many more such a sufficient conditions. Let  $s_k$  denote the  $k$ th partial sum in the series for  $F$  that is this you will truncate this after  $k$  terms so you have this.

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**Checking the Boundary Condition (contd.)**

- For  $k > l$ ,  $s_k(\theta) - s_l(\theta)$  is given by
 
$$s_k(\theta) - s_l(\theta) = \sum_{n=l+1}^k R^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$
- Let  $v_k$  denote the  $k^{\text{th}}$  partial sum in the series of  $v$ . For  $k > l$ 

$$v_k(r, \theta) - v_l(r, \theta) = \sum_{n=l+1}^k r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$
 is a harmonic function and equals  $s_k(\theta) - s_l(\theta)$  when  $r = R$ .

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So for  $k$  bigger than  $l$   $s_k \theta - s_l \theta$  is given by this let  $v_k$  denote the  $k$  partial sum in the series of  $v$  for  $k$  bigger than  $l$   $v_k - v_l$  has this expression it is a harmonic function because each of the inside terms is harmonic function is a finite sum and therefore it is a harmonic function and it is equals  $s_k \theta - s_l \theta$  on the boundary when  $r = R$ .

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**Checking the Boundary Condition (contd.)**

- Note that  $v_k(r, \theta) - v_l(r, \theta)$  is  $C^2$  on the disk and continuous on the boundary of the disk.
- By **Weak maximum principle**,
 
$$\max_{0 \leq r \leq R, \theta \in [0, 2\pi]} |v_k(r, \theta) - v_l(r, \theta)| \leq \max_{\theta \in [0, 2\pi]} |s_k(\theta) - s_l(\theta)|.$$
- Since  $s_k$  is a Cauchy sequence, we conclude that  $v_k$  is a Cauchy sequence (in the sup-norm of the space  $C(D[0, R])$ ).
- Thus  $v_k$  converges uniformly in  $D[0, R]$ .  $v$  is continuous upto the boundary of the disk.
- $v(R, \theta) = F(\theta)$  is satisfied. □

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And  $v_k - v_l$  is a  $C^2$  function on the disk and it is continuous up to the boundary of the disk. In fact  $v_k - v_l$  is  $C^\infty$  on  $\mathbb{R}^2$  so by weak maximum principle we have this maximum in the domain or omega closure is less than or equal to maximum on the boundary since  $s_k$  is a Cauchy sequence in  $C$  of  $[0, 2\pi]$  so and it converges to  $F$  that is why it is a Cauchy sequence and hence  $v_k$  will be a Cauchy sequence in this domain  $0 < r \leq R$   $\theta$  belongs to  $[0, 2\pi]$  which is the closed disk.

So, therefore  $v_k$  converges uniformly the closed disk and hence the limit has to be a continuous function which is continuous after the boundary by the property that it is continuous on  $D$  of  $[0, R]$   $v$  is continuous and  $v$  of  $R, \theta = F(\theta)$  is satisfied.

**(Refer Slide Time: 26:05)**

**Poisson's formula**

The summation and integral can be interchanged in

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} F(\tau) d\tau + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \left( \int_0^{2\pi} F(\tau) \cos(n(\tau - \theta)) d\tau \right)$$

since the series is uniformly convergent for  $r \leq R$

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Now let us re derived Poisson's formula the summation and integral can be interchange in this expression since a series is uniformly convergent for  $r \leq R$ .

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**Poisson's formula (contd.)**

$$\begin{aligned}
 v(r, \theta) &= \frac{1}{\pi} \int_0^{2\pi} F(\tau) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n(\tau - \theta)) \right] d\tau \\
 &= \frac{1}{\pi} \int_0^{2\pi} F(\tau) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \left[ \frac{e^{in(\tau-\theta)} + e^{-in(\tau-\theta)}}{2} \right] \right] d\tau \\
 &= \frac{1}{2\pi} \int_0^{2\pi} F(\tau) \left[ 1 + \frac{\frac{r}{R}e^{i(\tau-\theta)}}{1 - \frac{r}{R}e^{i(\tau-\theta)}} + \frac{\frac{r}{R}e^{-i(\tau-\theta)}}{1 - \frac{r}{R}e^{-i(\tau-\theta)}} \right] d\tau
 \end{aligned}$$

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So, this is equal to this in terms of the complex exponentials and on simplification it is the term in the brackets is like this.

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**Poisson's formula (contd.)**

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} F(\tau) \left[ 1 + \frac{\frac{r}{R}e^{i(\tau-\theta)}}{1 - \frac{r}{R}e^{i(\tau-\theta)}} + \frac{\frac{r}{R}e^{-i(\tau-\theta)}}{1 - \frac{r}{R}e^{-i(\tau-\theta)}} \right] d\tau$$

The term within square brackets in the above integral is of the form

$$1 + \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{1 - |z|^2}{1 - (z + \bar{z}) + |z|^2}$$

Thus we get

$$v(r, \theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \left[ \frac{F(\tau)}{R^2 - 2rR \cos(\tau - \theta) + r^2} \right] d\tau.$$

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Now the term within the square brackets here is looking like  $1 + 1 / 1 - z + z \text{ bar} / 1 - z \text{ bar}$  which can be computed to be this. Please do this computation. So, now I am going to substitute what is my  $z$  that is  $r / R e^{i \tau - \theta}$  and I get this expression.

**(Refer Slide Time: 27:00)**

**Poisson's formula (contd.)**

Solution to Dirichlet BVP in polar coordinates is

$$v(r, \theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \left[ \frac{F(\tau)}{R^2 - 2rR \cos(\tau - \theta) + r^2} \right] d\tau.$$

In  $xy$ -coordinates, the above formula transforms to

$$u(x) = \frac{R^2 - \|x\|^2}{2\pi} \int_{S(\mathbf{0}, R)} \frac{f(y)}{\|x - y\|^2} dy.$$

Poisson's formula is valid on the disk  $D(\mathbf{0}, R)$ .

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So, solution to Dirichlet boundary value problem in polar coordinates is given by this for  $r$  strictly less than  $r = R$ . This is not meaningful you should not do that  $R$  less than  $r$ . In  $x, y$  coordinates the; above formula transforms to the well known Poisson's formula it is valid on the disk  $D$  of  $0, R$ .

**(Refer Slide Time: 27:28)**

**Summary**

- Dirichlet BVP on a disk is solved using separation of variables method.
  - Knowing that disk is a rectangle in polar coordinates, the BVP was transformed to polar coordinates.
  - The problem becomes suitable for using separation of variables method.
- Since the series expansion could be summed up, Poisson's formula was obtained as a by-product.

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Let us summarize what we did in this lecture Dirichlet boundary value problem on a disk is solved using separation of variables method knowing that disk is a rectangle in polar coordinates that BVP was transformed to polar coordinates. The problem becomes suitable for using separation of variables method. Since the series expansion could be summed up Poisson's formula was obtained as a by-product. Thank you.