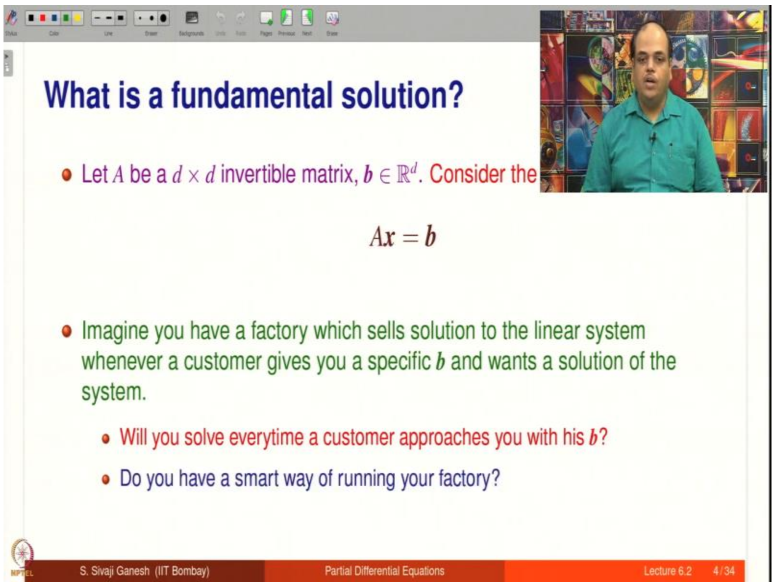
Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 6.2
Laplace Equations

Welcome in this lecture we are going to discuss fundamental solutions in R d further Laplacian. The outline of the lecture is as follows. First we introduce the idea of a fundamental solution then we move on to find fundamental solutions for Laplace operator in R d and then we study some properties of fundamental solutions. Fundamental solution what is it why is it fundamental.

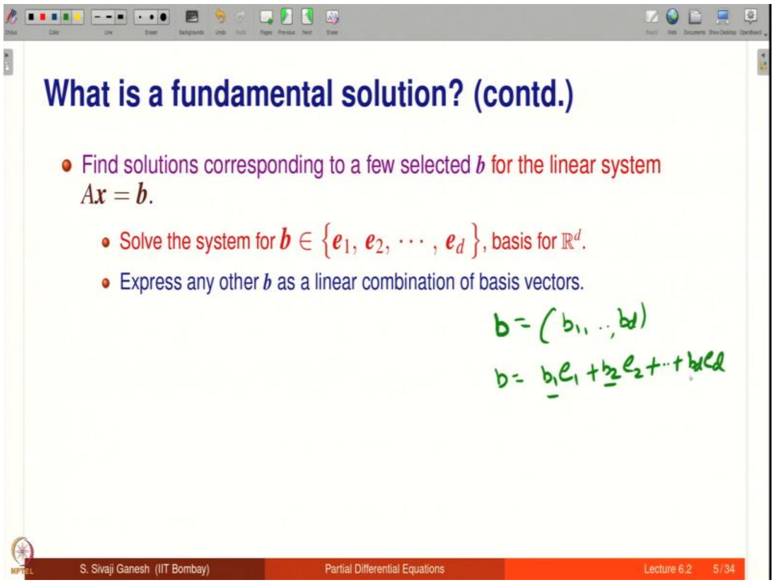
(Refer Slide Time: 00:49)



So, what is a fundamental solution? Let us start with a matrix analogy let A be a d / d invertible matrix and b be a vector in R d. Consider the linear system A x = b, b is given you want to find solution for x since A is invertible it has exactly one solution we know that. Imagine you have a factory which sells solutions to the linear system Ax = b whenever a customer gives you a specific b, you will give him x to the customer.

Will you solve every time a customer approaches you with his b that means whenever a customer comes and gives you b you try to go and find solution for x by your own method how to solve the system. And as a customer various you have to solve the system again and again will you do that? Or do you have a smart way of running you are factoring.

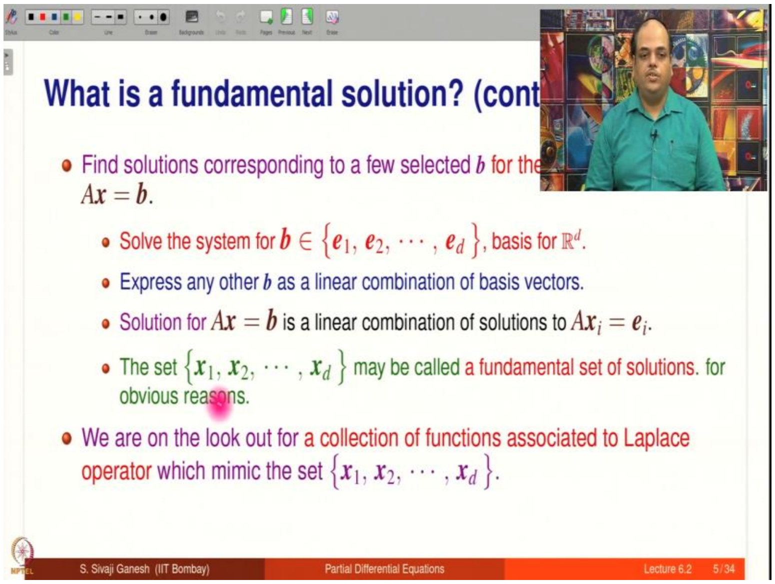
(Refer Slide Time: 01:49)



So, what is the fundamental solution? Find solutions corresponding to a few selected b for the linear system Ax = b solve a system for b in e 1, e 2 up to e d that means for b = e 1 you solve b = e 2 you Ax = e 2, similarly you solve up to Ax = e d that means d times you solve this system. What is e 1, e 2, e d it is the basis for R d. Let us say we take the standard order basis e 1, e 2, e d for R d.

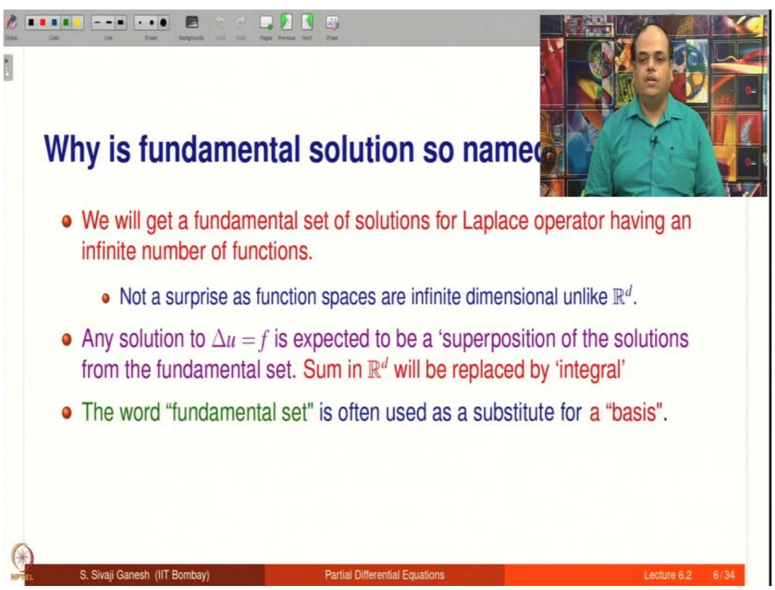
So, e 1will be the d tuple where the first component is 1 rest of them are 0 e 2 is the second component is 1 rest of them 0 and similarly e d is the dth component is 1 and first d - 1 components are 0 we know that this is the standard order basis for R d. So, for each of the basic elements you solve Ax = b. Express any other b which the customer gives you as a linear combination of these basis vectors which is very easy. If b = b 1, b 2, b d then b is nothing but b 1 e 1+ b 2 e 2 up to b d e d, b is vector like this then b is nothing but b 1 e 1 + b 2 e 2. So, you know the readily what are the coefficients which are appearing in this combination.

(Refer Slide Time: 03:29)



Solution for Ax = b is a linear combination of the solutions x i's which are solving Ax i = e i. So, the set x 1, x 2, x d may be called a fundamental set of solutions in the context of Ax = b for obvious reasons. We are on the lookout for a collection of functions associated to the Laplace operator which mimic this set x 1, x 2, x d in the case of Ax = b.

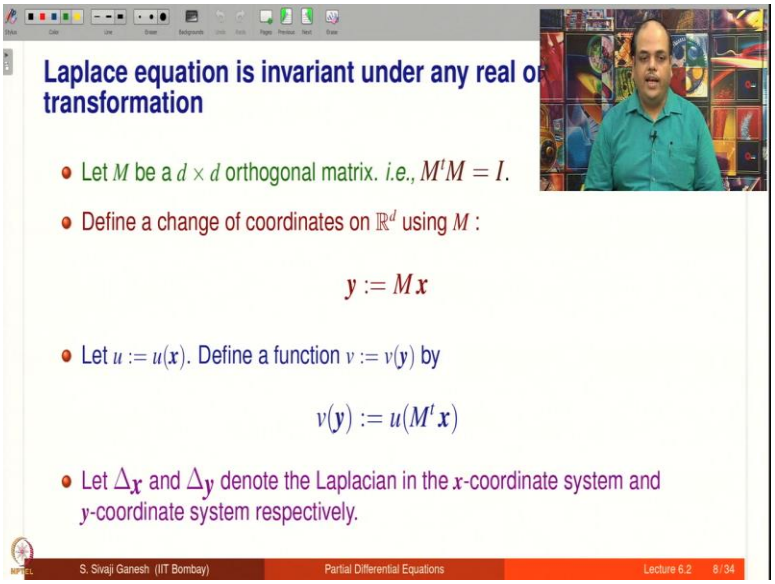
(Refer Slide Time: 04:08)



So, why is fundamental solution so named? We will get a fundamental set of solutions for Laplace operator having an infinite number of functions. So, the set we are going to get for Laplace operator will consist of infinite number of elements. Unlike the case of linear system Ax = b where it had only d number of elements. It is not a surprise as function spaces are infinite dimensional unlike R d which is finite dimensional

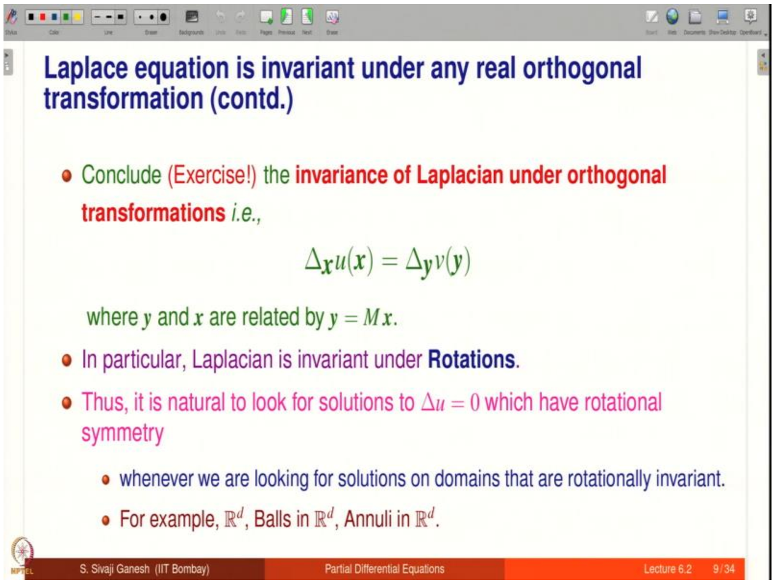
Any solution to Laplacian u = f is expected to be a superposition of the solutions from the fundamental set. Sum in R d will be replaced by an integral we are going to see this. The word fundamental set is often used as a substitute for a basis.

(Refer Slide Time: 05:07)



So, fundamental solution for Laplace operator in R d Laplace equation is invariant under any real orthogonal transformation what does that mean? Let M be a d / d orthogonal matrix that is M transpose M equal to identity matrix define a change of coordinates on R d using this orthogonal matrix M by this set y = M x x is your original coordinate system you are introducing new coordinate y, y = Mx let u be denoted by u of x. Define a function v a function of y by this v of y = u of M transpose x. Let delta x and delta y denote a Laplacian in the x coordinate system and y coordinate system respectively.

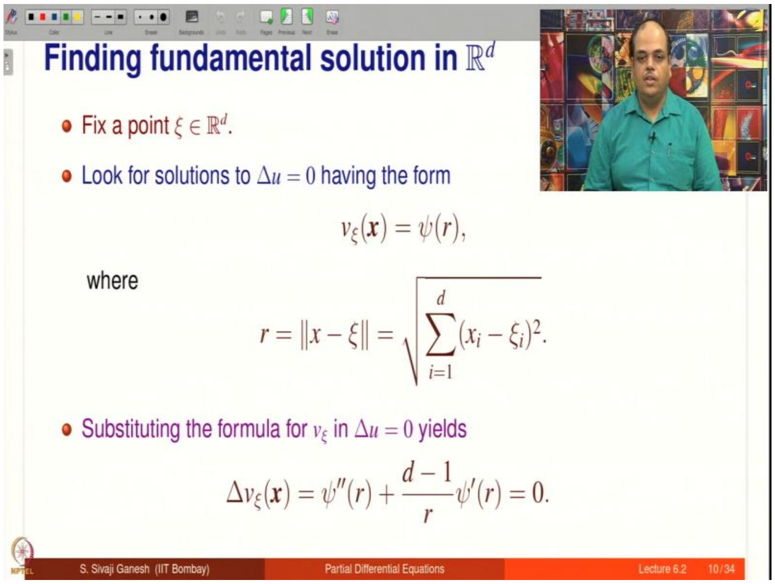
(Refer Slide Time: 06:09)



Conclude so this is going to be an exercise we have done enough exercises on change of variables and how it affects a PDE, how the PDE gets transformed under change of variables conclude the invariance of Laplacian under orthogonal transformations that is Laplacian with respect to x variables of u is same as Laplacian with respect to y variable of the function v where y and x are related by y = Mx in particular Laplacian is invariant under rotations.

Thus it is natural to look for solutions to Laplacian u = 0 which have rotational symmetry whenever we are looking for solutions and domains with themselves have this rotational symmetry that is rotationally invariant. For example R d trivially balls in R d and annular regions in R d.

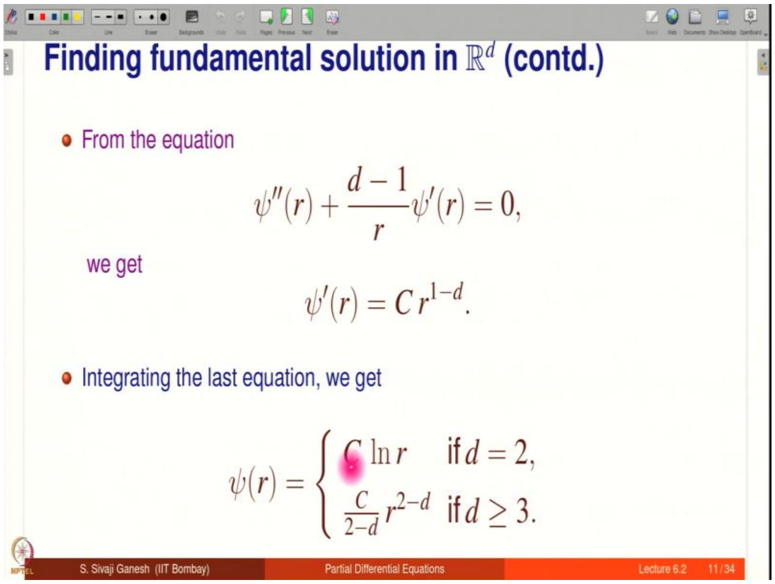
(Refer Slide Time: 07:16)



So, finding fundamental solution in R d how do we do that? Fix a point Xi in R d. Look for solutions to a Laplacian u = 0 having this form that v Xi of x because psi is fixed. So, for every fixed Xi in R d we are going to find solution v Xi of x = psi of r, this is already suggests we are going to find as many functions as the elements in R d. So, look for solutions to Laplacian equals to 0 having this form v Xi x = psi of r.

What is r? r is nothing but norm x - Xi that is a distance from x to the fixed point Xi which is given by this formula of course this is a Euclidean r. Therefore it is equal to square root of i = 1 to d x i - Xi i square substituting the formula for v Xi in Laplacian u = 0 yields Laplacian v Xi of x = psi double dash of r + d -1 / r into psi prime of r and that is equal to 0 this is what we want. Therefore finding v Xi is same as finding psi and psi satisfies this ODE. So, we need to solve this ODE this is a second order ODE with variable coefficient but it is simple variable equation so it is very easy to solve.

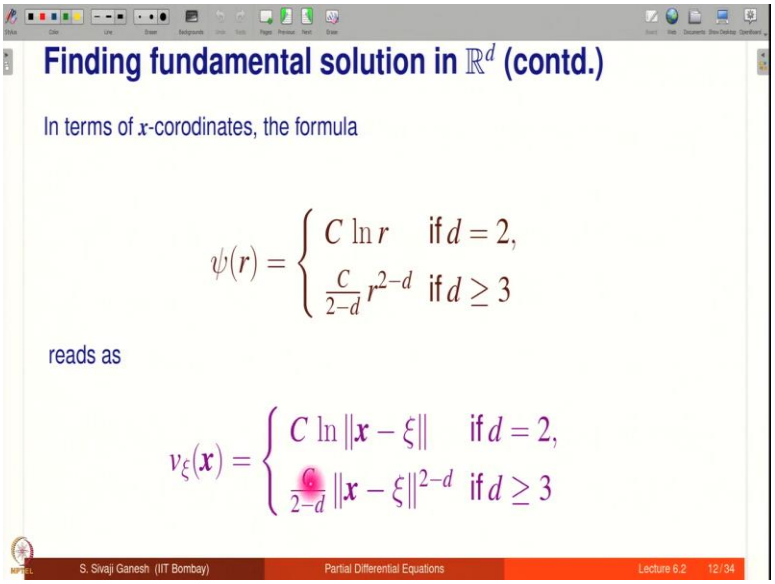
(Refer Slide Time: 08:42)



So, from this equation psi double dash of r + d - 1 / r into psi dash of r = 0 which was obtained on the last slide we get psi dash of r equal to constant times r power 1 - d because there is no term psi in this equation without derivative you said psi dash of r equals some g of r then this will be a first order ODE you can solve that and you get this expression. So, therefore psi dash of r equal to constant times r power 1 - d.

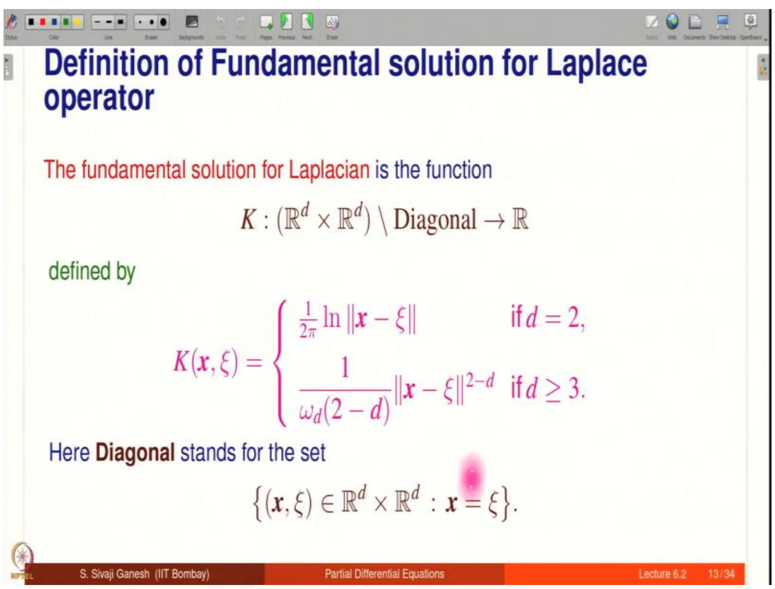
Integrating the last equation we get psi of r = C times log r if d = 2 and C / 2 - d r power 2 - d if d is greater than or equal to 3. So, therefore the form looks different in dimension 2 and dimensions bigger than or equal to 3. This is the reason why we will be considering d = 2 separately and d greater then equal to 3 separately in our analysis in the next 2 lectures.

(Refer Slide Time: 09:48)



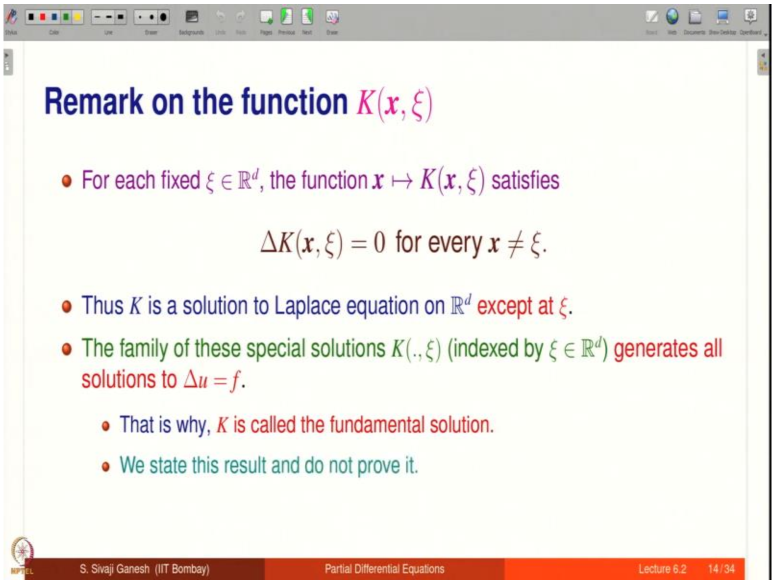
So, in terms of x coordinates v Xi of x is C times log r is norm x – Xi, so substitute r = norm x - Xi we get these expressions v Xi of x.

(Refer Slide Time: 10:04)



So, now we are ready to define what is called fundamental solution for Laplacian, our fundamental solution for the Laplace operator in R d. The fundamental solution for Laplacian is this function K it is a mapping from R d cross R d minus diagonal you are removing a set from R d cross R d a certain set which you will define 2 R defined by exactly the same formula as before. So, we have to simply, mentioned what is the diagonal? Diagonal stands for all those x Xi in R d cross R d such that x = Xi.

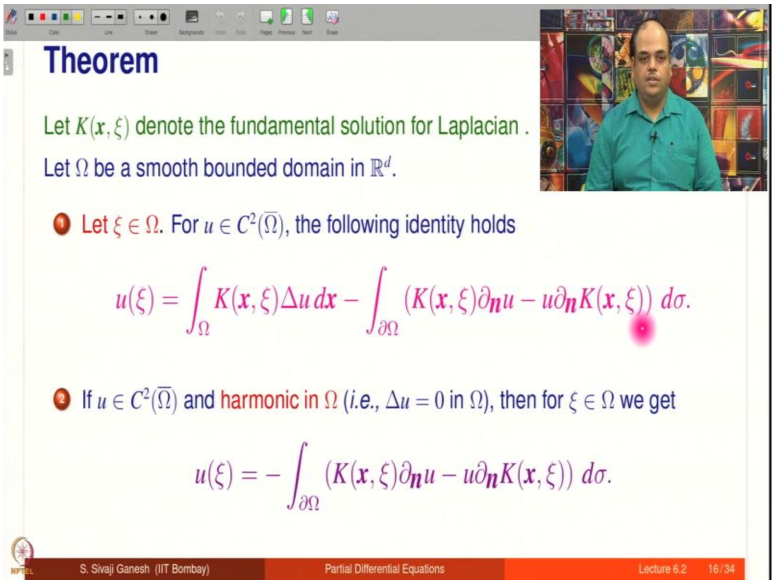
(Refer Slide Time: 10:46)



So, remark on the function K of x, Xi for each fixed Xi in R d, the function x going to K of x, Xi satisfies Laplacian K of x, Xi = 0 for every x different from Xi when x = Xi there is a problem it is not defined K is not defined but for any other x Laplacian K of x, Xi = 0 thus K the solution to Laplace equation on R d except for this Xi, the family of these special solutions that is the families indexed by Xi in R d.

This family generates all solutions to a Laplacian u = f that is why K is called the fundamental solution. Now compare the analogy that we are given in the case of system of linear equations. Fundamental set there would finitely many there x 1, x 2, x d here we have this family of functions indexed by Xi in R d. We state this result and we do not prove the result. Let us look at some properties of fundamental solutions.

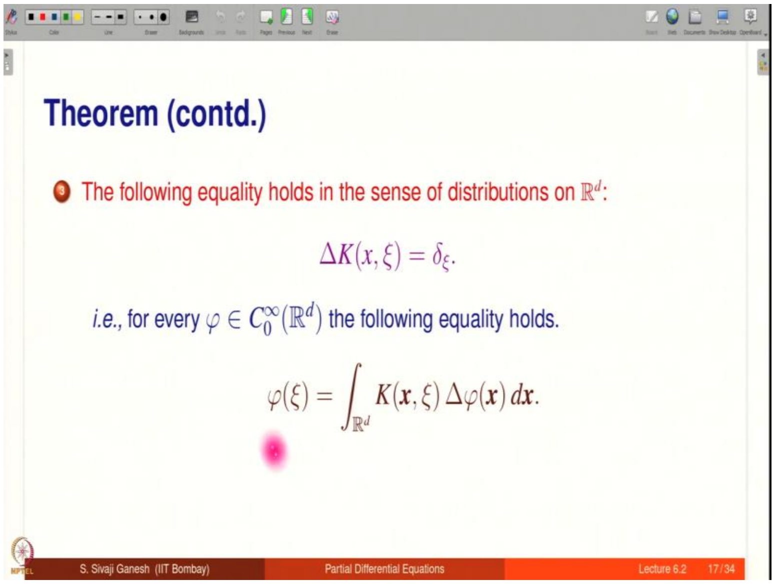
(Refer Slide Time: 12:06)



That is a theorem. Let K of x, Xi denote the fundamental solution for Laplacian we have already defined this on an earlier slide. Let omega be a smooth bounded domain in R d. Let Xi belongs to omega for u belonging to C 2 of omega bar the following identity holds that is u of Xi = integral or omega of K x, Xi Laplacian u dx minus integral over boundary of omega K dou n u - u dou n K x, Xi d sigma.

If u is C 2 of omega bar and harmonic in omega that means Laplacian u = 0 then the first term will drop out then you have only this term. Then for Xi I omega we get u Xi equal to this integral which is the second term here. So, once you show 1 2 follows immediately.

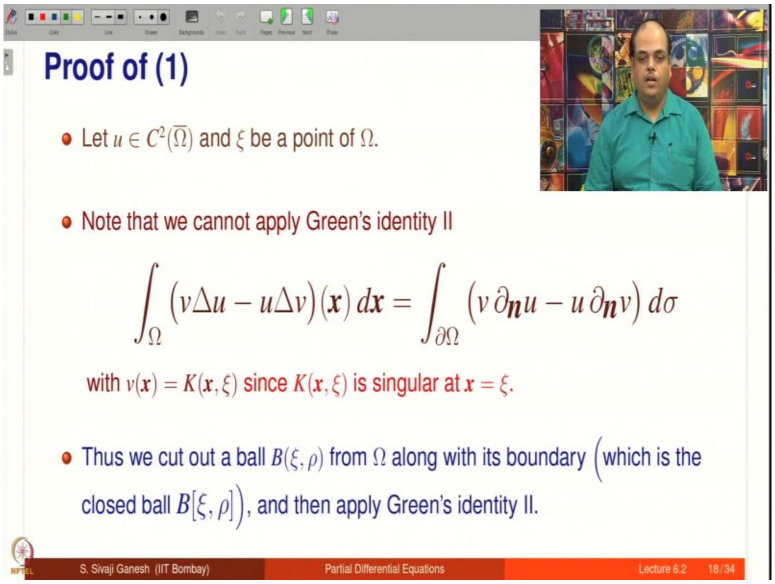
(Refer Slide Time: 13:07)



And the following quality holds in a sense of distributions and R d that is a Laplacian K of x, Xi = delta Xi what we Xi Laplacian K of x, Xi = 0 whenever x is not equal to Xi. Now there is always this question what happens at x = Xi? So, that is the effect here delta Xi comes in delta Xi the Dirac delta in case you do not know this you can ignore I am going to explain what this means.

This means that for every phi in C 0 infinity of R d the following equality holds, so loosely speaking multiply with phi and integrate integral phi delta will give you phi of Xi and here you do integration by parts transfer the Laplacian from K to phi and you get this. So, phi Xi equal to integral over R d of K of x, Xi delta phi of x dx.

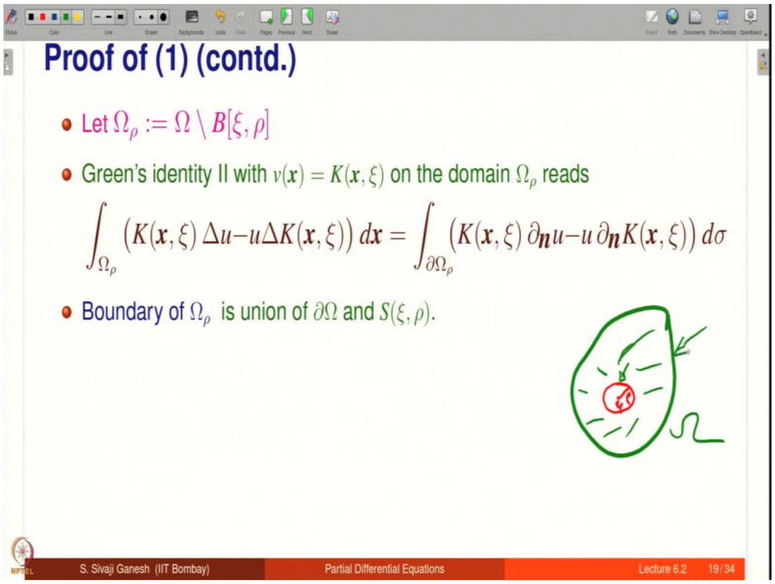
(Refer Slide Time: 14:10)



Proof of 1, let u belongs to C 2 of omega bar and Xi be a point of omega note that we cannot apply Green's identity II directly with v = K x Xi we would like to do that but we cannot do that why because K is singular at x = Xi there is trouble for K at x = Xi. And here if you are trying to use v = K you have a Laplacian K that will not be integrable. So, there will be such problems so we will not do that.

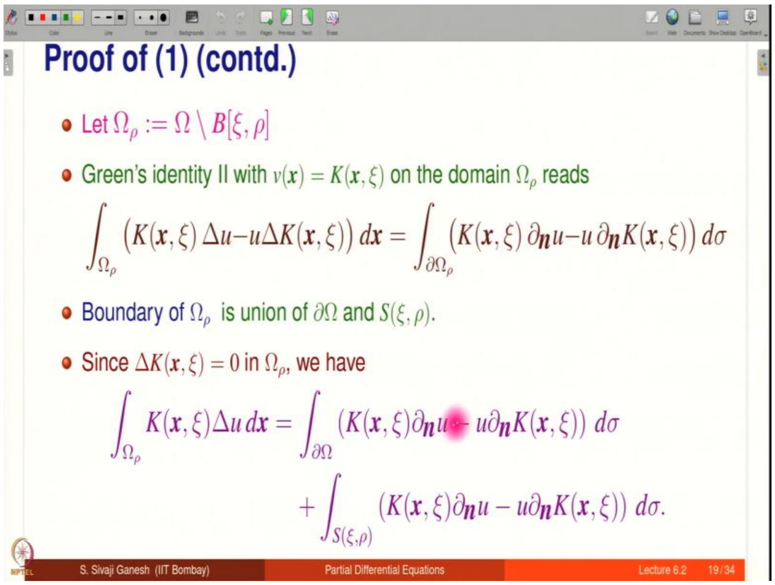
What we will do is? We somehow remove these points. So, we cut out a ball B of Xi rho from omega then everything along with its boundary, cutting a ball along with his boundary means cutting this closed ball. Recall this is the notation we were using B closed Xi, rho means it is all those points which are at a distance less than or equal to rho from the points Xi, here it is strictly less than for the open ball, this is a closed ball. And then we will apply Green's identity II.

(Refer Slide Time: 15:27)



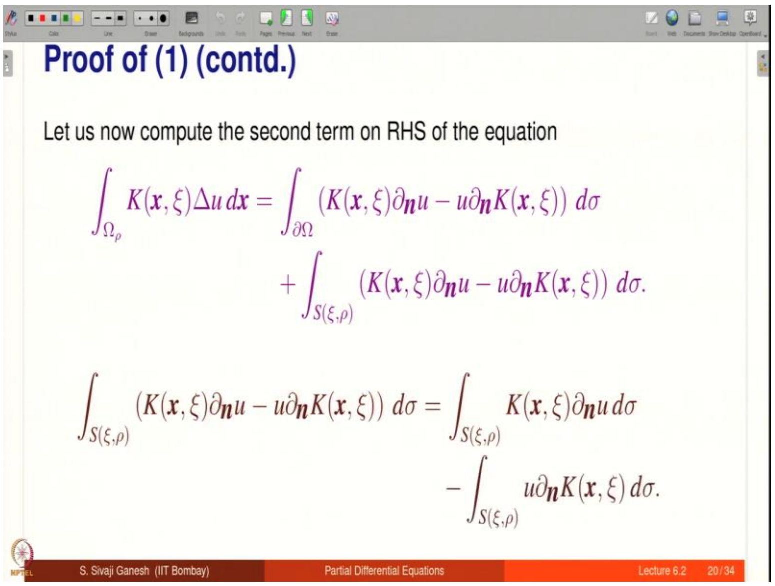
So, let omega rho B omega minus the closed ball Green's identity II with v = K of x, Xi on the domain omega rho reeds as this is exactly Green's identity II I have just put v = K and then instead of omega I am doing an omega rho. Boundary of omega rho is a union of boundary of omega and S Xi, rho. For example this is omega, this is Xi radius rho. So, I am removing this. So, this is my domain where is the domain? This is the domain. So, this domain has 2 boundaries one is this boundary and one is this boundary.

(Refer Slide Time: 16:27)



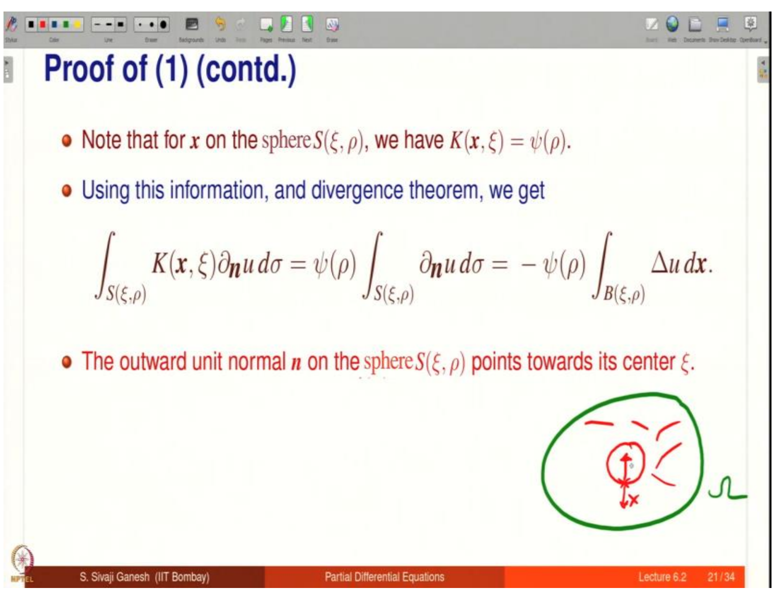
Since Laplacian K is 0 for x different from Xi, now in omega rho there is no Xi, Xi is taken out therefore this is 0 and hence this term drops out. So, what we have is the first term on the LHS is equal to this quantity and boundary consists of 2 parts. So, I have inputted that one is boundary of omega other one is S of Xi, rho this is sphere. Now let us look at this term and try to simplify this term because assertion 1 contains this term, this term and not this term but a simplified version of this. So, let us look at the second term.

(Refer Slide Time: 17:19)



Let us comes to the second term on RHS of this equation this equal to this is the first term here minus the second term. So, let us address each of them separately.

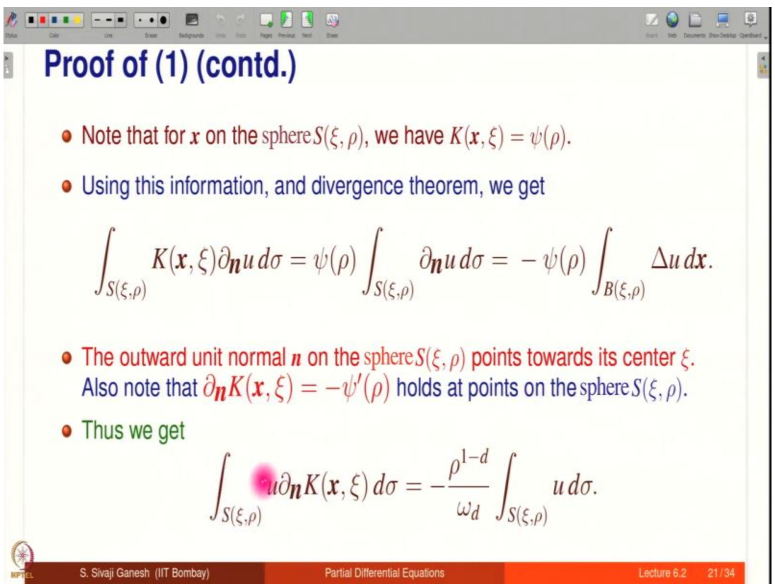
(Refer Slide Time: 17:39)



Note that for x on the sphere S Xi, rho we have K x Xi = psi of rho using this information and divergence theorem we get K dou n u = K x Xi rho, so it comes out it does not depend on the integration variable because K is constant. So, the Xi of rho that comes out and integral of dou n u or S Xi, rho this is where we apply divergence theorem and we get in terms of Laplacian. So, minus Xi of rho integral over the ball Laplacian u dx.

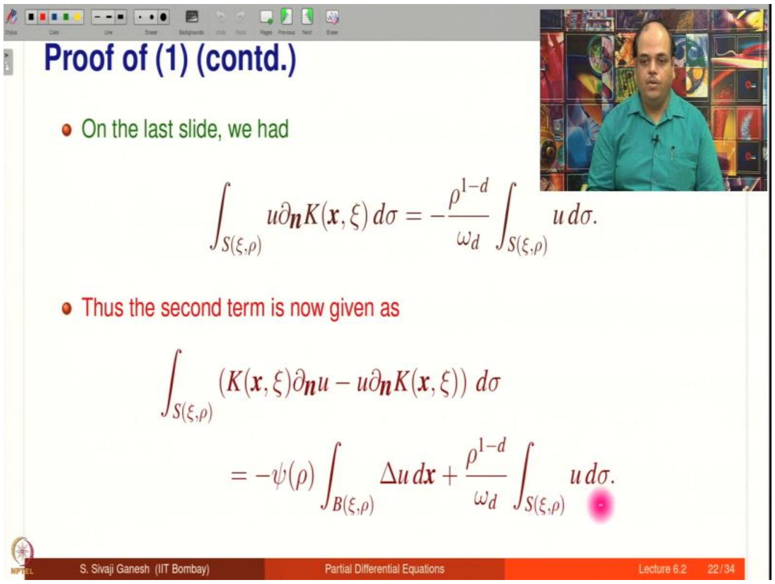
The outward normal n on the sphere points towards its center Xi, let us see our picture this is our omega and inside that we have removed a ball, our domain is really this one if you take a point here normal if you take this side it is the inside pointing normal. So, this is not the one, so this is the one which is outside pointing outward point. So, therefore this is towards the center of this ball.

(Refer Slide Time: 18:57)



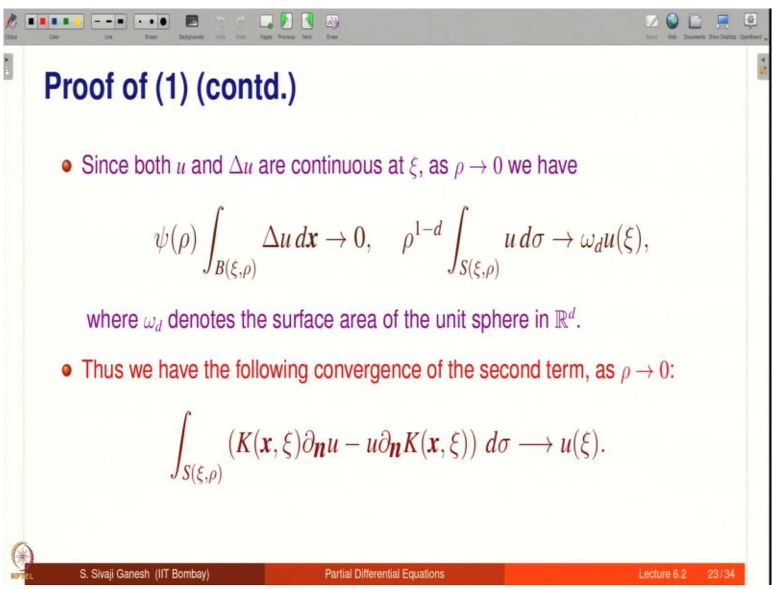
Also note that dou n K of x Xi is nothing but minus psi dash of rho holds at points on the sphere S of Xi, rho thus we get integral of u dou n K = - rho power 1 - d / omega d integral over the sphere u d sigma.

(Refer Slide Time: 19:24)



So, on the last slide we have proved this equality. Thus the second term now is given by minus Xi of rho integral over the ball of Laplacian plus rho power 1 - d / omega d integral over the sphere of u d sigma.

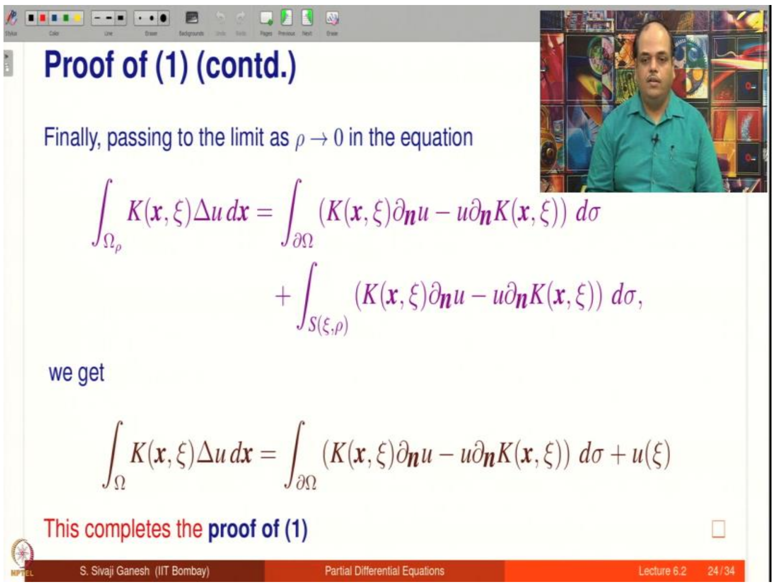
(Refer Slide Time: 19:46)



Since both you u and Laplacian u are continuous at Xi we have assumed C 2 of omega bar as rho goes to 0 we have psi of rho into integral over the ball of radius rho of Laplacian goes to 0 because modulus of psi of rho into this integral term is less than or equal to M times psi of rho into the volume of this ball, what is M? M is a bound for modulus of Laplacian u. Now psi of rho is like rho power d - 2.

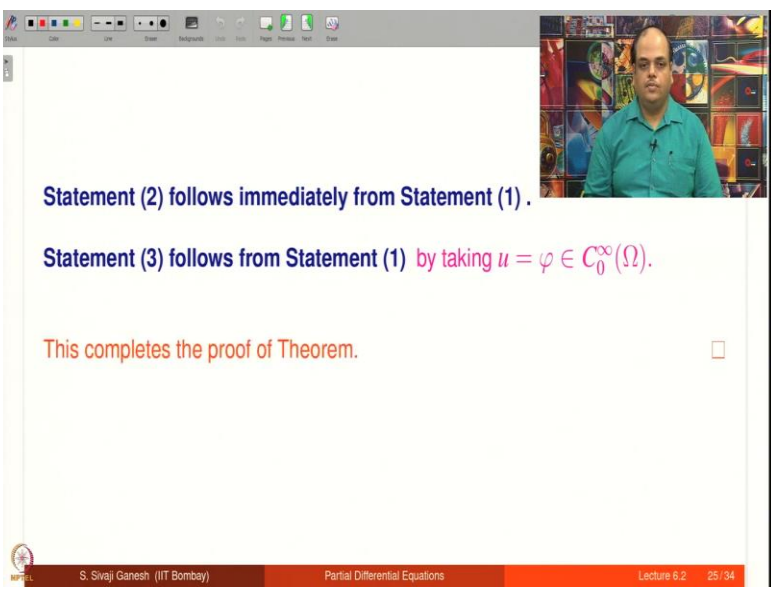
Whereas the volume of the ball is like rho power d therefore their product will behave like rho square. So, therefore as rho goes to 0 this term goes to 0, rho power 1 - d integral of this sphere will go to the omega d into u of Xi where omega d denotes the surface area of the unit sphere in R d, please check these assertions by yourself. Thus we have the following convergence of the second term as rho goes to 0, this is the second term this goes to u of Xi because the first term went to 0 second term went to u Xi.

(Refer Slide Time: 21:06)



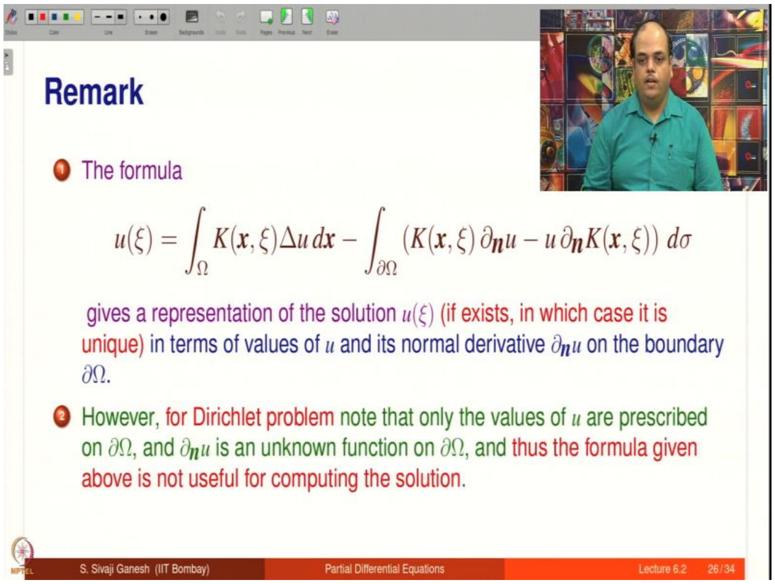
Finally pass into the limit as rho goes to 0 in this equation we get this equation, this completes the proof of 1 u of Xi equal to this integral minus this integral this is what stated in 1.

(Refer Slide Time: 21:29)



As mentioned before statement 2 follows immediately from statement 1, statement 3 follows from statement 1 by taking u = phi which is C 0 infinity of omega, this completes the proof of this theorem.

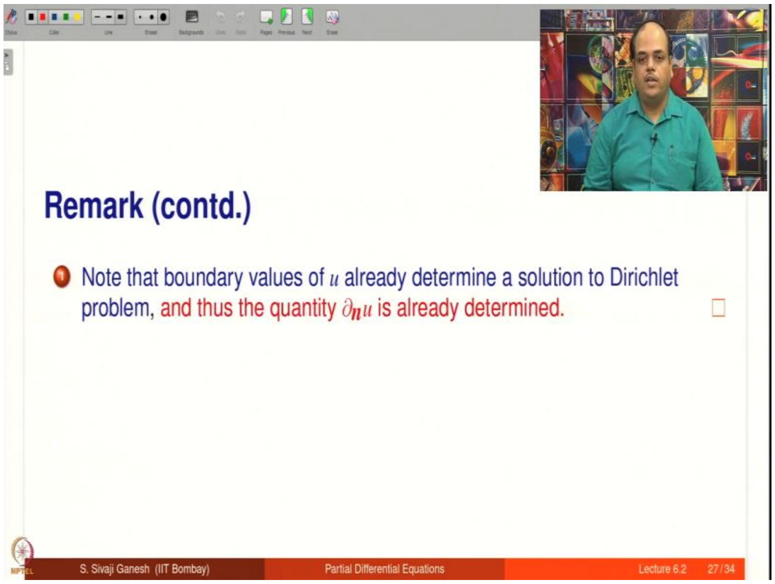
(Refer Slide Time: 21:50)



Remark this formula we have just proved this is assertions 1 gives a representation of the solution you want to know u of Xi it gives in terms of this K Laplacian u is if you are solving Laplacian u = f this is known, if Laplacian u = 0 this term is not there. So, these are known term K is already known. But this second term involves dou n u as well as u if you are solving Dirichlet problem u is known but this is not known.

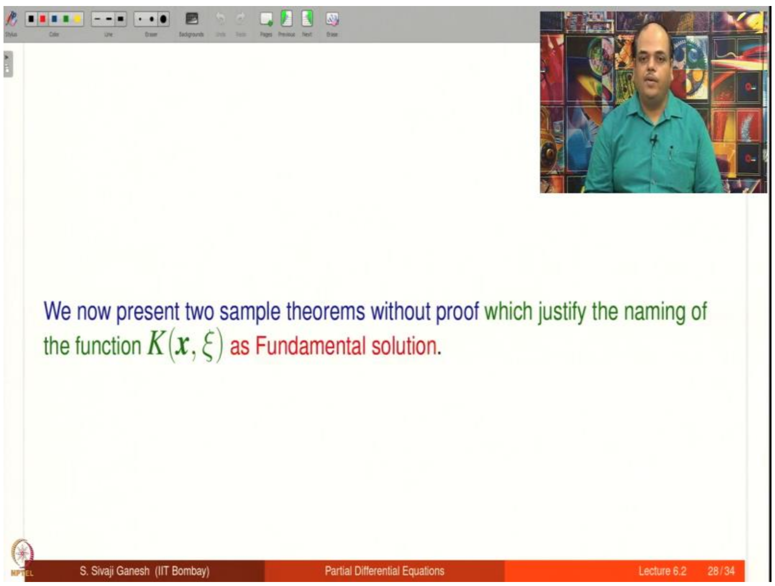
If you are solving Neumann problem dou n u is known on the boundary but u is not known. Let us discuss this point a little bit. Of course this represents a solution if it exists of course we know that if solution exists it is going to be unique we already proved that. So, this formula is a representation for u of Xi in terms of values of u on values of dou n u on the boundary of omega. However for Dirichlet problem note that only the values of u are prescribed on boundary of omega that means only this term is known. And this is not known. And there is a formula given a boy is not useful for computing this solution.

(Refer Slide Time: 23:19)



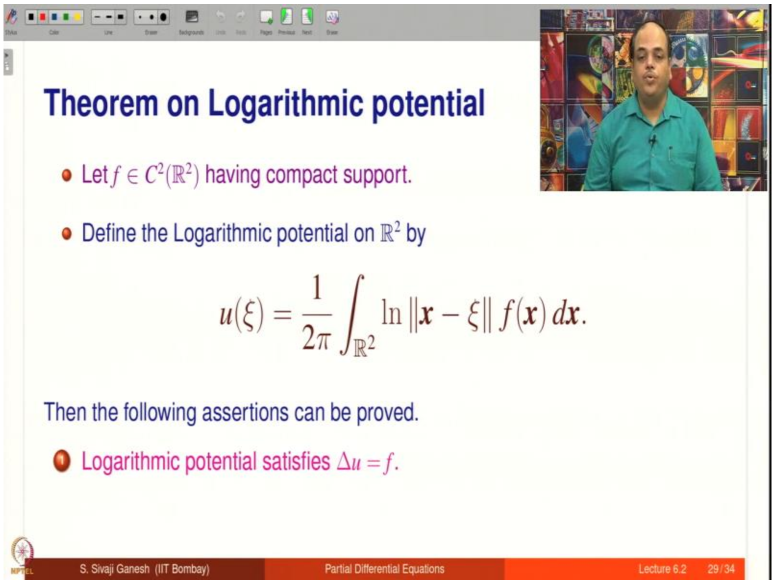
Note that the boundary values of u already determined a solution to Dirichlet problem and thus the quantity dou n u is not only not known it is already determined.

(Refer Slide Time: 23:32)



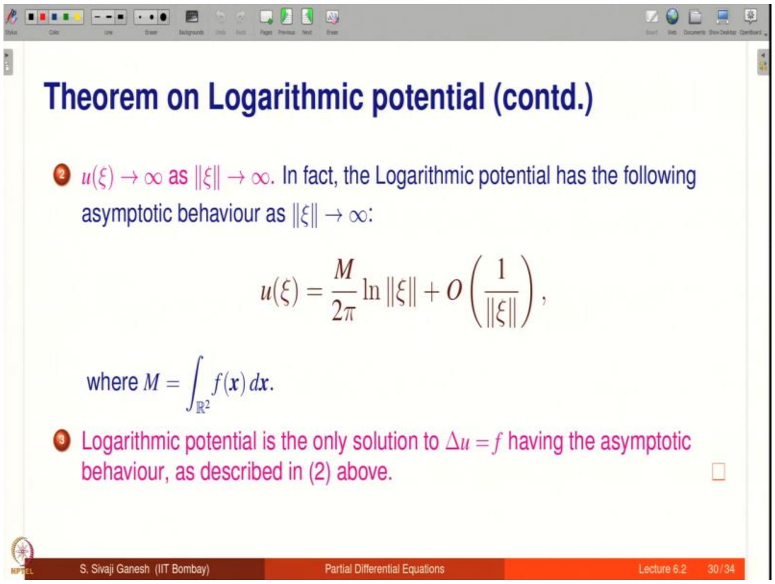
We now present 2 sample theorems without proof which justify the naming of K of x Xi as a fundamental solution.

(Refer Slide Time: 23:43)



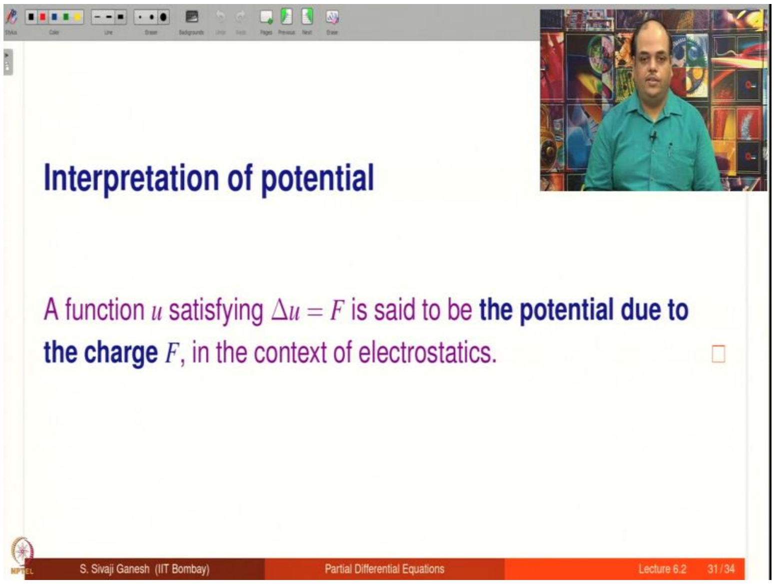
Theorem on logarithmic potential naming will be obvious once we state the theorem, let f be a C 2 function define R 2 having compact support define the logarithmic potential on R 2 / u of Xi = 1 / 2 pi integral over R 2 ln of norm x - Xi fx dx, then the following assertions can be proved. Of course we have not proven that is why I have stated as the following assertion can be proved logarithmic potential satisfies Laplacian u = f that means this formula is a solution to the Poisson's equation.

(Refer Slide Time: 24:26)



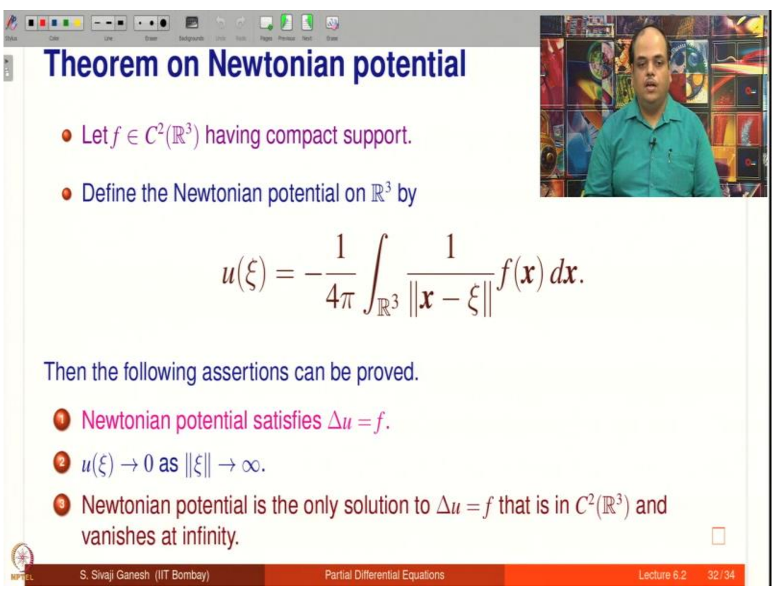
And u of Xi goes to infinity as norm Xi goes to infinity. In fact we have the following asymptotic behavior of the logarithmic potential at infinity, u of Xi = M / 2 pi log norm Xi + O of 1 / norm Xi where M equal to integral of f over R 2 is a finite quantity because f is assumed to be compact support. So, integral is finite. Logarithmic potential is only solution to Laplacian u = f having the asymptotic behavior as mentioned in 2 above.

(Refer Slide Time: 25:08)



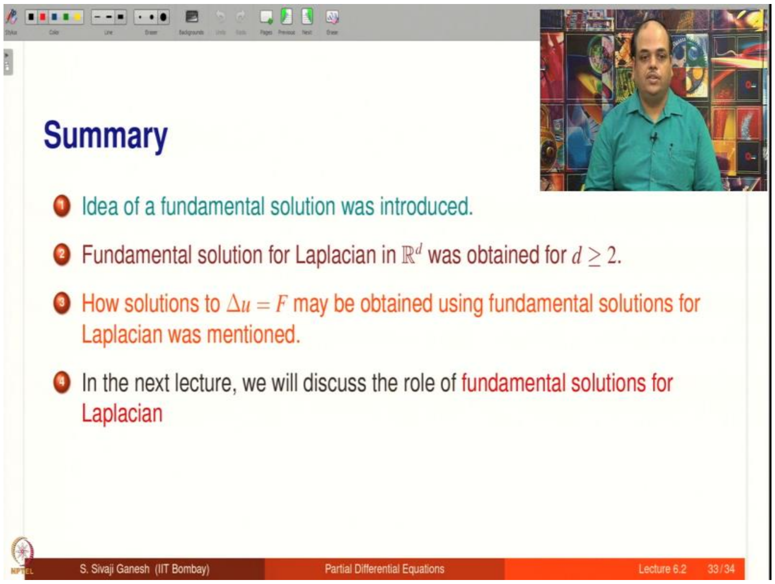
So, interpretation of potential a function u satisfying Laplacian u = F is said to be the potential due to the charge F in the context of electrostatics.

(Refer Slide Time: 25:23)



Theorem on Newtonian potential let f belongs to C 2 of R 3 having complex support define the Newtonian potential and R 3 / u Xi = - 1 / 4 pi integral over R 3 of f x / norm x - Xi dx. Then the following assertion can be proved Newtonian potential satisfies Laplacian u = f u Xi goes to 0 as norm Xi goes to infinity. Newtonian potential is the only solution to a Laplacian u = f that is in C 2 R 3 and vanishes at infinity.

(Refer Slide Time: 26:04)



So, let us summarize what we did in this lecture. Idea of a fundamental solution was introduced. Fundamental solution for Laplacian in R d was obtained for d greater than or equal to 2. How solutions to Laplacian u = f may be obtained using fundamental solutions for Laplacian was mentioned the 2 theorems. In the next lecture we will discuss the role of fundamental solutions for Laplacian in determining solutions to Dirichlet boundary value problem. Thank you.