Partial Differential Equations
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Lecture – 6.1
Laplace Equation – Associated Boundary Value Problems

Welcome to this lecture, in this lecture, we are going to discuss boundary value problems for Laplace equation, the outline of the lecture is as follows. First we derive what are called Green's identities which are derived from the divergence theorem and then we introduce boundary value problems associated to Laplace equation and study the uniqueness properties of solutions to those boundary value problems.

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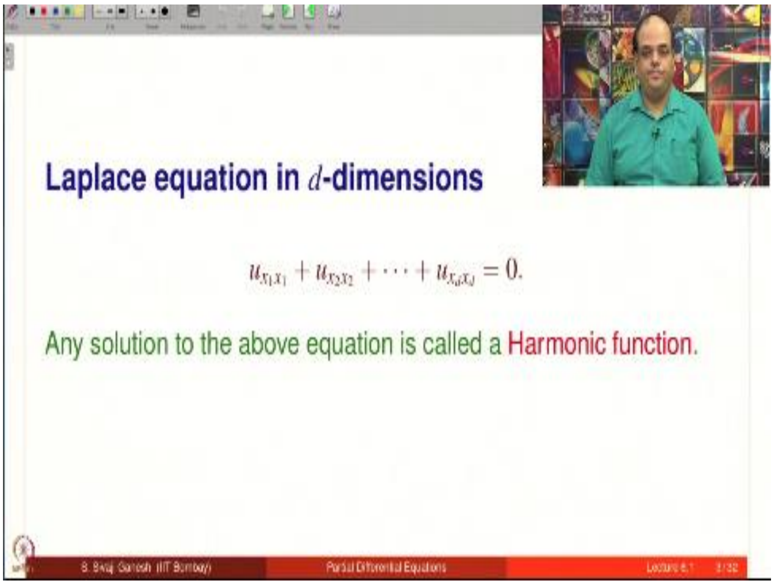

So, Laplace equation in d space dimensions there are only space dimensions here, there is unlike the wave equation. So, Laplace equation in d dimensions is given by u x 1 x 1 + u x 2 x 2 + up to u x d x d = 0, the operator on the left hand side is called Laplacian u on denoted by capital delta u, that is the standard notation for a Laplacian. This is nothing but the trace of the hessian matrix of the function u, hessian in matrix recall is the matrix of second derivatives of u. So, this is the trace of that. That means sum of the diagonal terms. Any solution to the above equation is called Harmonic function.

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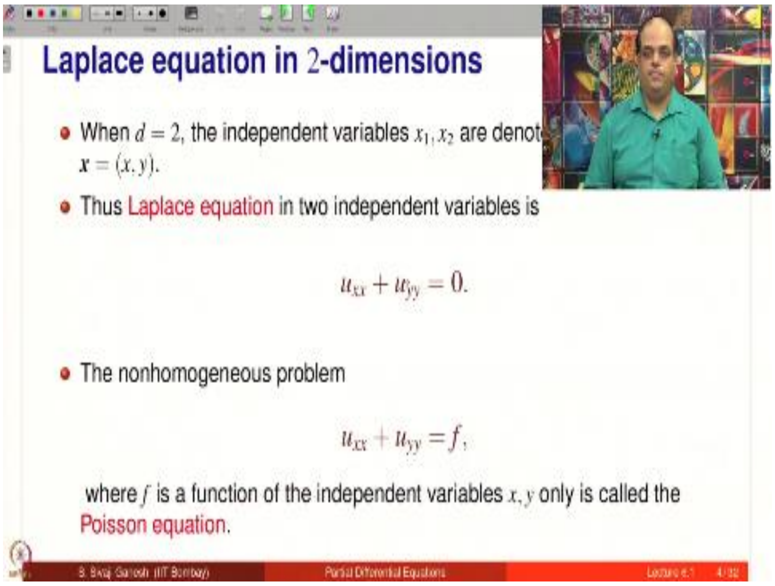

Laplace equation in 2 dimensions we are going to deal with in some of the lectures which come later on. When d = 2 the independent variables x 1 x 2 are denoted by x y, that is a standard practice and we write boldface x sometimes that denotes the triple x, y in R 2. This Laplace equation in 2 independent variables is nothing but u xx + u yy = 0. The non homogeneous problem where there is a right hand side not 0 functions, but the general function f, this equation is called Poisson equation.

So, in particular, if f = 0 get back this equation, so we the difference between Poisson equation and Laplace equation is kind of blurred for us, we simply call anything as a Laplace equation. Actually Laplacian is this operator which is on the left side of both the equations, here it is a homogeneous equation, here is a non homogeneous equation and the non homogeneous equation is usually called Poisson equation. So, let us discuss, what are the Green's identities and how to derive them

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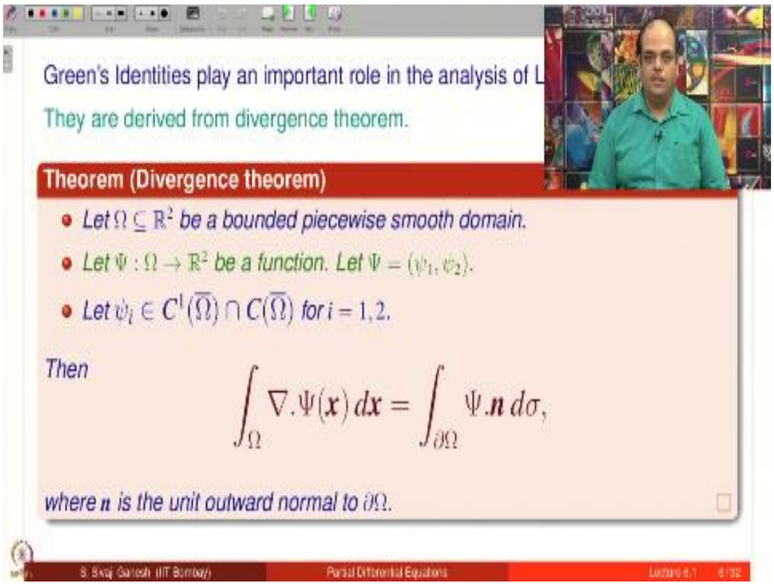

Green's identities plays an important role in the analysis of Laplace equation, they are derived from divergence theorem. So, what is divergence theorem? It is stated for a domain omega which is bounded and piecewise smooth domain. We are going to use in this form, let psi be a function which is our 2 valued function that means, psi has 2 component functions psi 1 and psi 2, it is defined on this domain omega. Omega is assumed to be bonded and piecewise smooth domain we will come to that as soon as we see the formula.

And we assume that the psi i has this property that it is C 1 of omega bar intersection C of omega bar it means the derivative is continuous up to the boundary of omega. So, omega closure involves omega union boundary for omega. That means, the derivative of this function psi i is meaningful on the boundary of omega also and C of omega bar means the functions are meaningful on the boundary of omega. In fact, the functions psi i's are continuous on omega bar.

Then we have this integral **o**mega domain integral on the right hand side you have a integral on the boundary. So, this is divergence of psi dx equal to integral over boundary of omega psi dot n d sigma. What is n? n is a unit outward normal to boundary of omega and what is d sigma? It is the surface measure which is coming to the boundary of omega from the omega from the usual measure on omega. So, in order that the right hand side is meaningful, we need to assume that domain is piecewise smooth domain. We will not elaborate further on this, because you will have learned this already in course on Multivariable Calculus.

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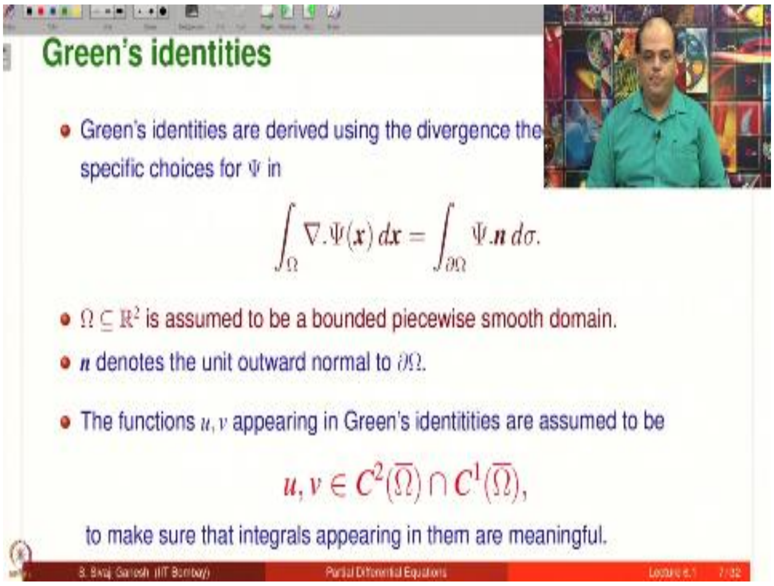

So, Green's identities they are derived using the divergence theorem and making specific choices for psi in this conclusion of the divergence theorem. So, choosing different psi's will give you different identities. And omega is assumed to be a bounded piecewise smooth domain, n denotes the unit outward normal to boundary omega, the functions u v appearing in Green's identities are assumed to be of this type C 2 of omega bar intersection C 1 of omega bar. We have already seen the space on the previous slide.

This means that the functions, the second derivatives are also continuous on omega bar that is the meaning of C 2 of omega bar. To make sure that the integrals appearing in them are meaningful, first of all this integral to make sense psi x inside the integrand is continuous on omega bar. And that is guaranteed if psi is C 1 of omega bar later on we are going to see an integral which feature second derivatives as the integrand that is what we are assuming C 2 of omega bar it will be very clear from the context. These hypotheses are made, so that it is varied for all Green's identities.

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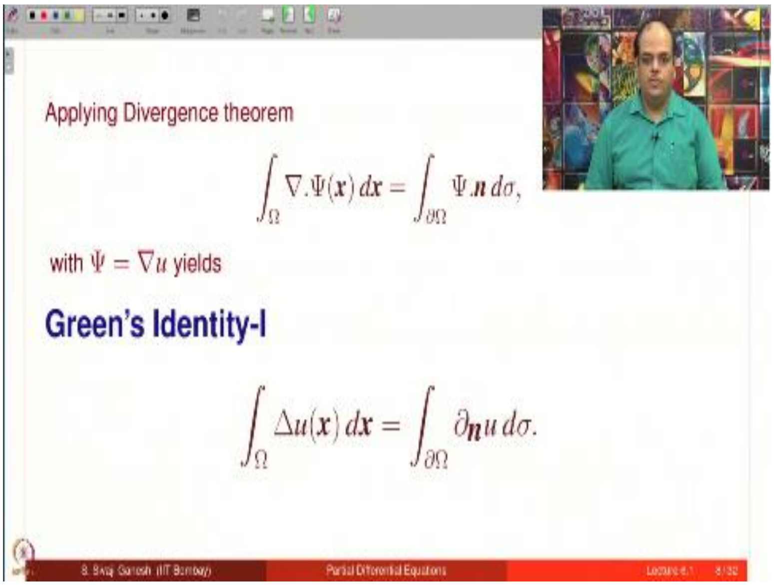

For some of them, we do not require all the hypothesis. So, now, if you apply a divergence theorem with psi = grad u, what we get is called Green's identity of 1. So, divergence of psi is now, divergence of grad u that is Laplacian u and grad u dot n is precisely the normal derivative of u dou n u, so, this is Green's identities one.

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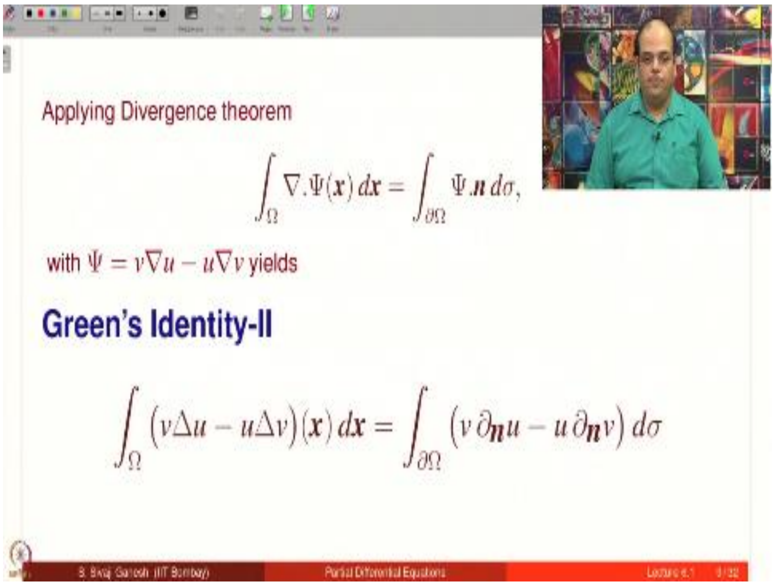

Now, if you apply divergence theorem with this choice of psi, we get Green's identities 2. So, it is integral over omega v Laplacian u - u Laplacian v of x dx = an integral on the boundary v dou n u - u dou n v d sigma.

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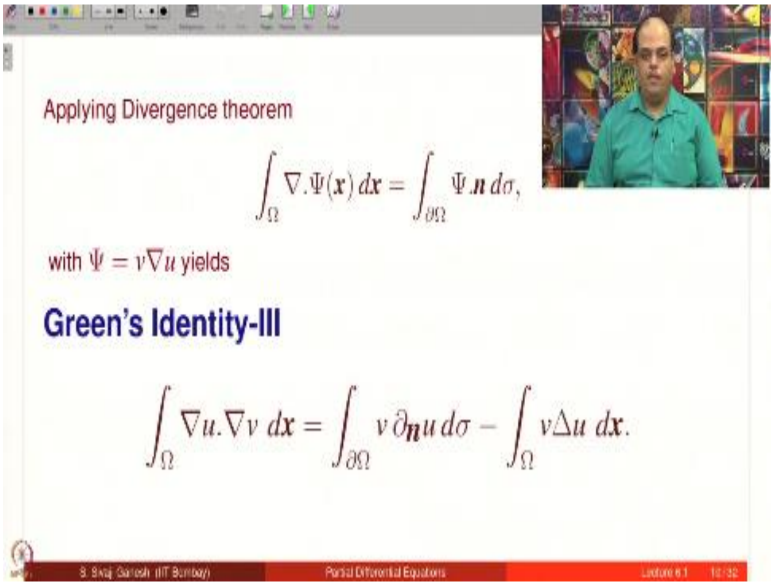

If you apply divergence theorem with psi = v grad u, you get Green's identities 3, which is integral over omega of grad u dot grad v dx = integral over boundary of omega v dou n u d sigma - integral over omega v Laplacian u dx. In fact, this identity we have used many times before which we call it as integration by Poisson's formula. If you notice here the derivative on v shifting to the derivative on u on grad u, which is giving you a Laplacian u, grad when it goes to the grad u, it becomes divergence. So, you get v Laplacian u and here is the boundary term. So, let us now discuss some of the boundary value problems for Laplace equation.

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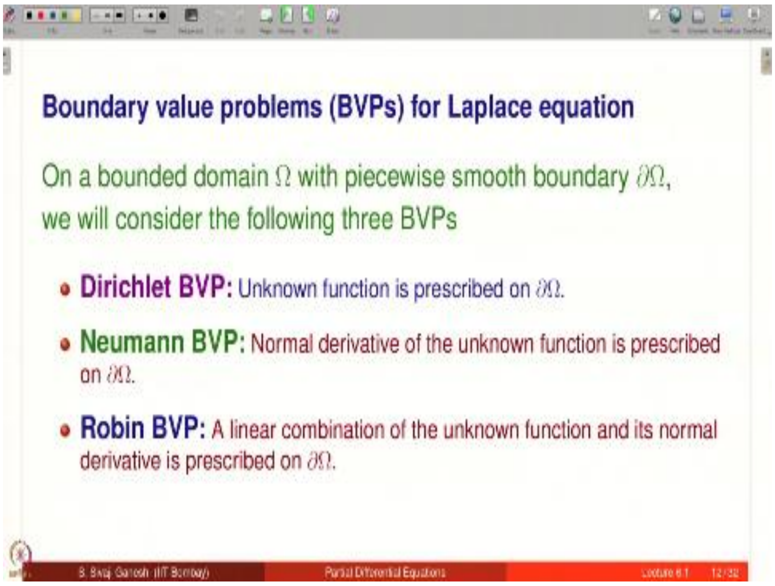

On a bounded domain omega with piecewise smooth boundary of omega, it is the boundary of omega is denoted by dou omega, we will consider the following 3 boundary value problems, what are them? Dirichlet boundary value problem unknown function is prescribed on boundary of omega here and then we have a Neumann boundary value problem in which normal derivative of the unknown function is prescribed on boundary of omega.

Then we have a Robin boundary value problem, which sometimes is also called a third boundary value problem, in this a mix of the unknown function and it is normal derivative are prescribed. So, a linear combination of the unknown function and it is normal derivative is prescribed on boundary of omega.

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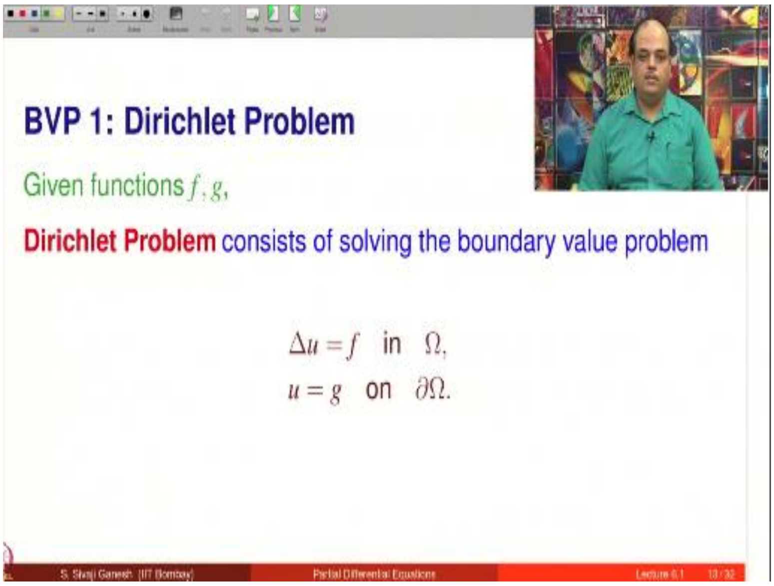

So, let us see what is the Dirichlet boundary value problem? So, this is BVP 1 boundary value problem 1 Dirichlet problem. Given functions f and g, Dirichlet problem consists of solving the boundary value problem, it consists of solving the Poisson's equation Laplacian u = f in omega and u = g on the boundary of omega. That means, the unknown function should agree with the pre prescribed function g on the boundary of omega and Laplacian u should coincide with the prescribed function f in omega is what we want.

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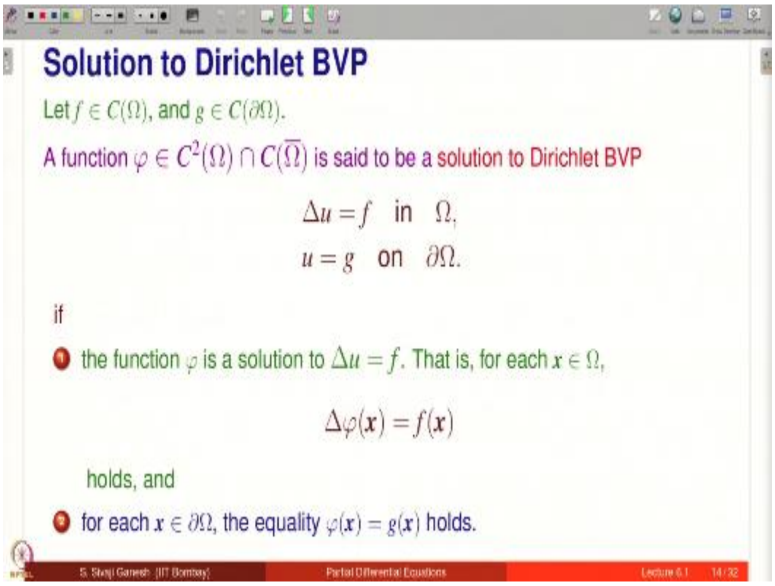

What is the meaning of a solution to Dirichlet boundary value problem? Here we have to start assuming something on the data, let f be a continuous function and g also be a continuous function on omega and on boundary of omega respectively, a function phi which is C to omega intersection C of omega bar. You will actually see why this hypothesis coming here on phi will be very clear once you see the definition is said to be a solution to a boundary value problem. If Laplacian u = f in omega and u = g on boundary of omega.

This is the statement of the Dirichlet boundary value problem. If the function phi is a solution to Laplacian u = f, it means that for each x in Omega Laplacian phi of x should be equal to f of x. So, in order that the left hand side makes sense, we are assuming phi C 2 of omega that is phi C 2 of omega. Now, there is a second condition that is a boundary condition for each x in boundary of omega phi of x should be equal to g of x.

If phi is continuous up to boundary that is C of omega bar, then values of phi for points x, which are in the boundary is meaningful, and asking that is equal to g of x is meaningful. So, that is the reason why we have to put these spaces naturally.

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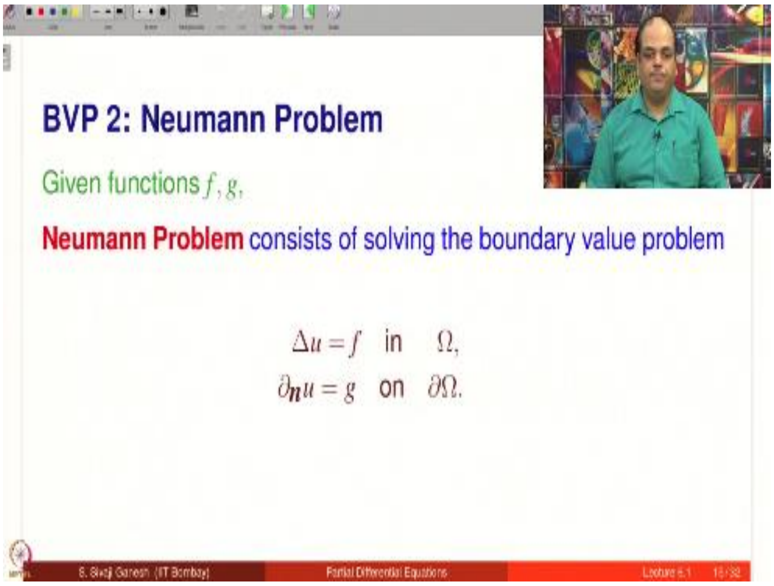

Now, let us look at the second boundary value problem called Neumann problem. So, here given functions f and g. Neumann problem consists of solving the boundary value problem, Laplacian u = f in omega, and the normal derivative equal to g on boundary of omega, g and f are prescribed, they are given to us.

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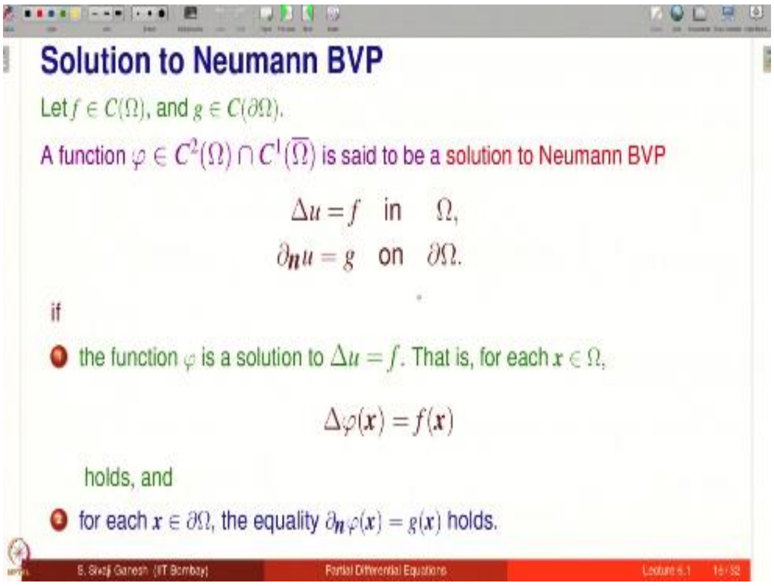

What is the meaning of a solution to the Neumann boundary value problem, assume that f is continuous on omega and g is continuous on boundary of omega. A function phi which is C 2 of omega intersection C 1 of omega bar he said to be a solution to Neumann boundary value problem. If the function phi is a solution to Laplacian u = f, that is why we are going to assume phi C 2 of omega because of this Laplacian phi at every point x in omega should be equal to f of x.

Now, we have the boundary condition which involves the normal derivative. Normal derivative is nothing but gradient u dot n. It is a directional derivative in the direction of the normal. For that reason, we need C 1 of omega bar that means derivatives are also continuous up to boundary and hence this is meaningful. So, dou n phi of x is meaningful, because phi is C 1 of omega bar. And hence, we can ask that it should be equal to gx, that is why the notion of solution to Neumann problem is like this phi should be in this space.

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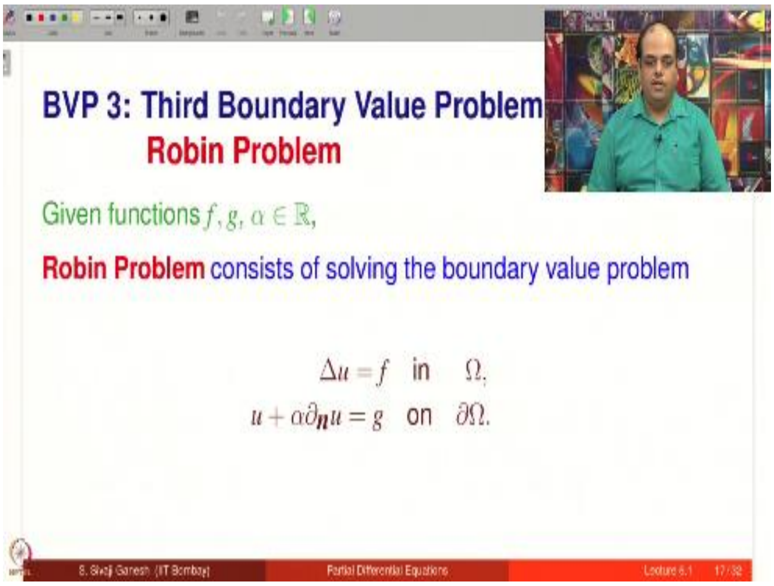

Now, let us look at the third boundary value problem, which is also known as Robin problem. Given functions f g, and alpha is a real number. Robin problem consists of solving the boundary value problem equation is Poisson’s equation in omega, boundary condition is a combination of u and the normal derivative of u, u + alpha times dou n u = g on boundary of omega.

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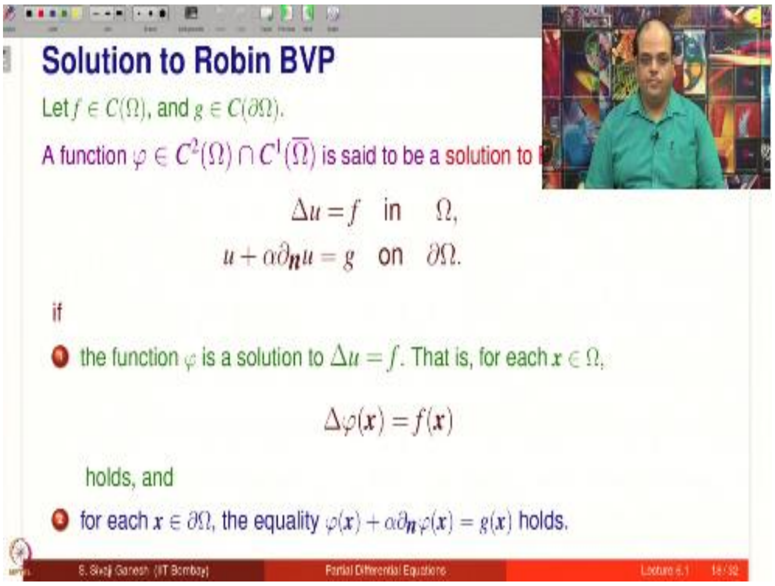

What is the definition of a solution to Robin boundary problem? Let f be a continuous function on omega and g be a continuous function on the boundary of omega. A function phi C 2 omega intersection C 1 of omega bar function phi in this space C 2 omega intersection C 1 of omega bar this is natural, because this is coming from the requirement of Laplacian phi = f and this will come because the requirement of the boundary condition. So, this function phi is said to be a solution to Robin boundary value problem.

If the function phi is a solution to a Laplacian u = f which means for every x in Omega Laplacian phi of x = f x, there is a reason why we assume phi C 2 of omega. And second condition is the boundary condition for every x on the boundary of omega, the equality phi x + alpha times dou n phi of x = g of x holds in order the that the left hand side is meaningful, we have assumed that phi is C 1 of omega bar.

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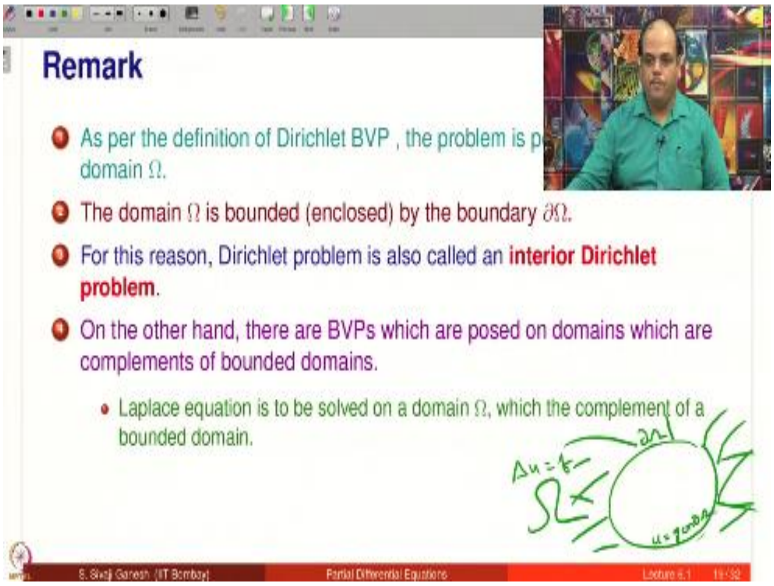

Remark as per the definition of Dirichlet boundary value problem, the problem is posed on a bounded domain omega. In fact, the other 2 boundary value problems are also posed on bounded domains by definition. The domain omega is bounded means what enclosed by the boundary of omega, for this reason Dirichlet boundary value problem is also called an interior Dirichlet problem. Similarly, the other notions we can say interior Neumann problem, interior Robin problem.

On the other hand, there are boundary value problems which are posed on domains which are complements of bounded domains. Laplace equation is to be solved on a domain omega which is the compliment of a bounded domain, for example, my omega is here. This is my omega it is a compliment of this set which is inside. And this is my boundary of omega, this curve is my boundary of omega, I am drawing the picture in the plane obviously.

So, omega is outside this outside region which is here, this is boundary of omega. So, we need to solve Laplacian u, let us say equals to f outside and here we are prescribing u = g for example, on boundary of omega.

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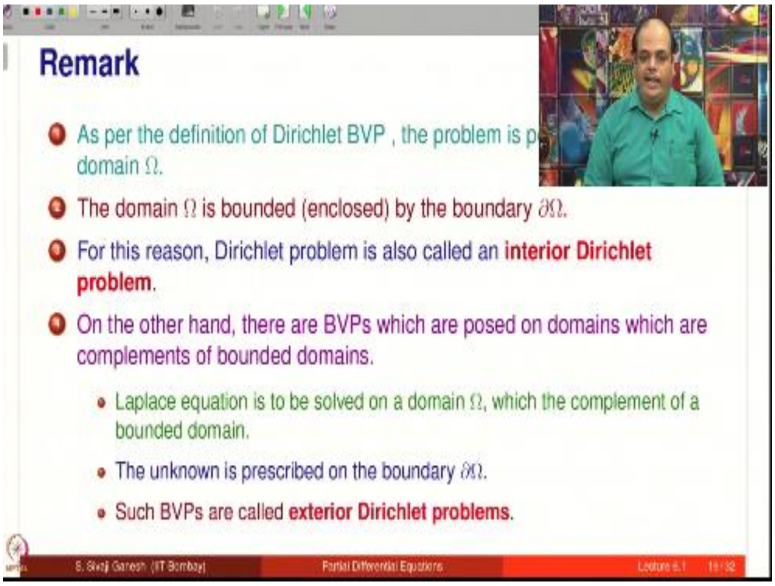

Unknown function is prescribed on the boundary for dou omega. Such boundary value problems are called exterior Dirichlet problems, similarly exterior Neumann problems, exterior Robin problems.

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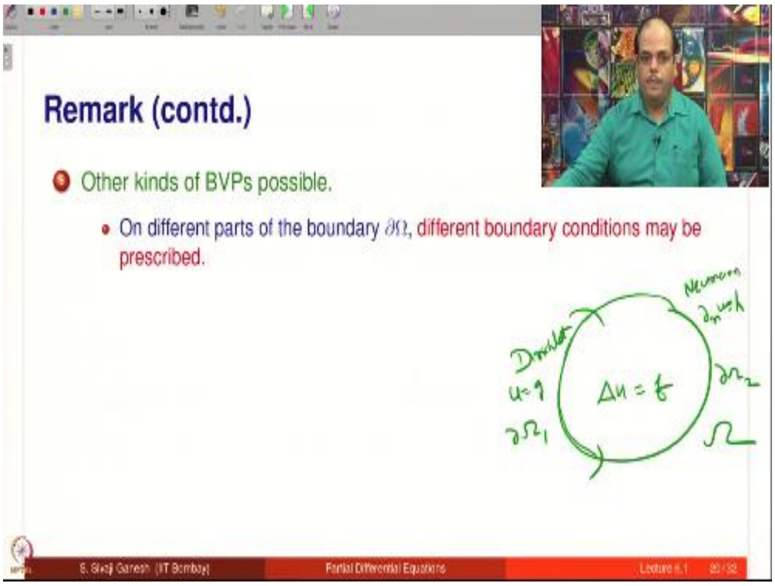

Other kinds of boundary value problems are also possible on different parts of the boundary, you prescribe different boundary conditions. For example, this is my omega then I consider this part, let us call this boundary of omega part 1, this is boundary of omega part 2. So, here I can ask u = g here I can ask dou n u equal to some function h. I want to solve Laplacian equation in omega = f in omega. And I asked that on this piece, u must be equal to g that is the Dirichlet boundary condition. On this piece, dou n u = h that is a Neumann condition.

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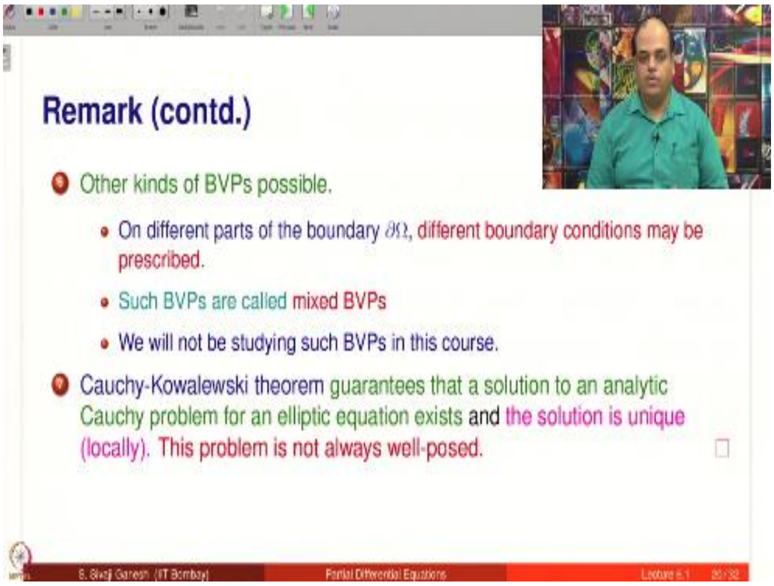

Such boundary value problems are called mixed boundary value problems, we will not be studying such problems in this course. Cauchy Kowalewski theorem these a comment about the initial value problem, Cauchy Kowalewski theorem guarantees that a solution to an analytic Cauchy problem for an elliptic equation in particular Laplace equation exists and the solution is unique locally. This problem is not always well posed, we are going to see towards the end of our discussion on Laplace equation, an example of well posed Cauchy problem for Laplace equation.

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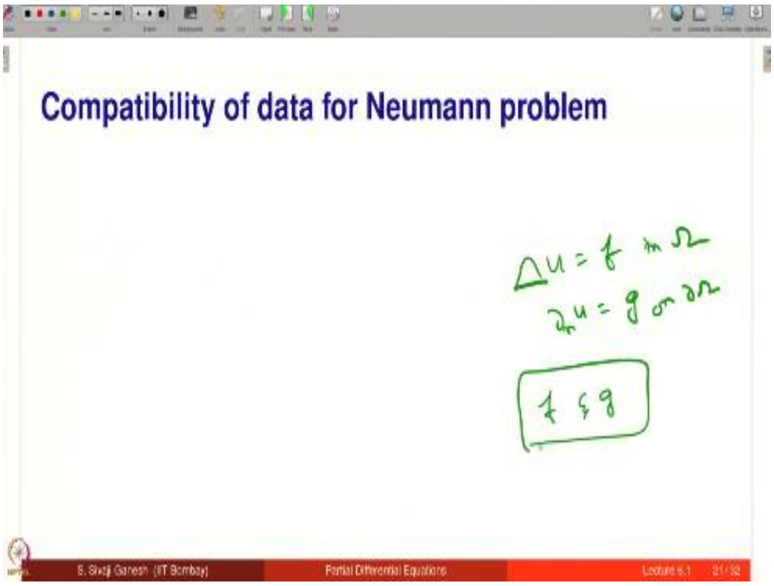

Now, the Neumann problem that we have namely Laplacian u = f and dou n u = g, this is in omega, this is on boundary of omega. If this problem has a solution, f and g are tied up with some relation, so you cannot have arbitrary f and g for which the Neumann boundary value problem has a solution. In other words, if the Neumann boundary value problem has a solution f and g must be compatible with each other in the sense that we are going to soon describe.

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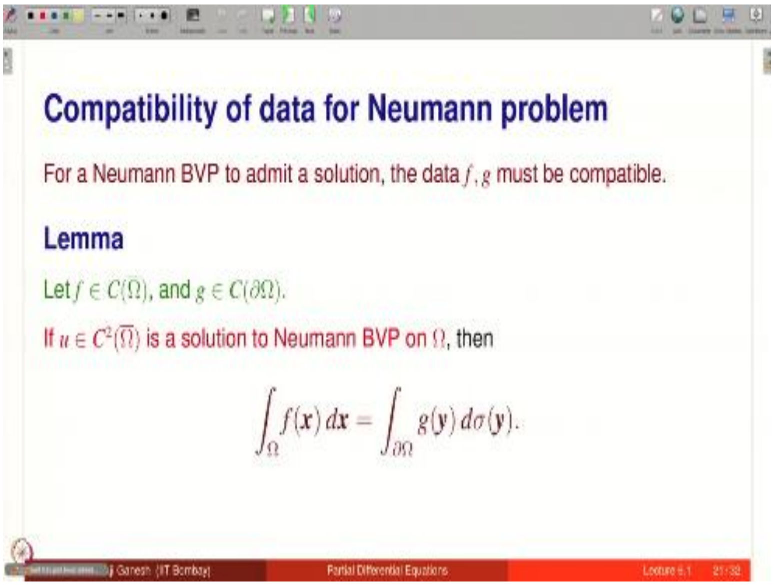

For a Neumann boundary value problem to admit a solution, the data f and g must be compatible. That is written in the form of a Lemma, it does not mean it is a big result. It is a simple observation, just to remember the observation, so, that we can recall whenever we want, we record them as a Lemma or as a theorem in this case, it is a Lemma. Let f be C of omega bar, we have to assume something more so, for in the definition of the problem we need a C of omega only.

But here we are asking for C of omega bar, because the tools that we are going to use will involve the continuity up to boundary. In fact, we are going to have certain integrals for them to make sense we need this assumption. And g in C of boundary of omega, if u belonging to C 2 of omega bar is a solution to Neumann boundary value problem and Omega, then integral over omega of f x = integral over boundary of g, g is defined on the boundary. So, integrate on the boundary, f is defined in Omega so, integrate on omega both of them must be equal.

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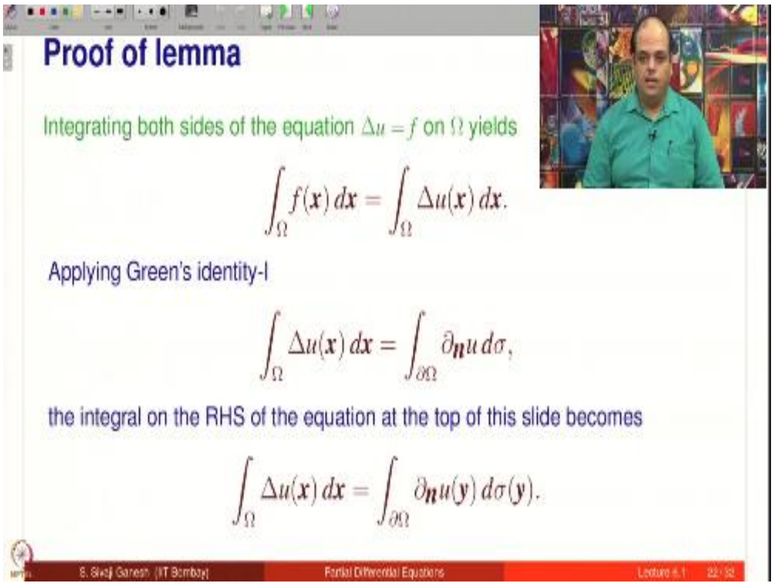

So, integrate both sides of this equation Laplacian u = f on omega what we get is this I have exchanged the sides. So, integral of f over omega = integral of Laplacian n u over omega fine. Now, applying Green's identity this right hand side integral can be written as this integral over boundary of omega dou n u d sigma. The integral on the right hand side does become this same, but now I know what is dou n u

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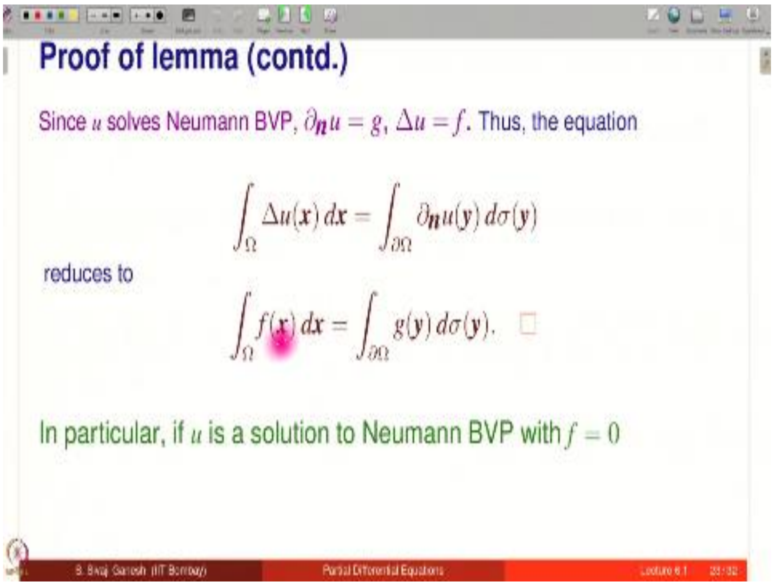

Since u solves the Neumann problem, dou n u is g and Laplacian u is f. Therefore, we have this equation reducing to an equation where f is here and g is here. So, thus we have proved the Lemma. In particular, if f is 0 that is you are looking at the homogeneous Laplace equation Laplacian u equal to 0 that means f is 0, then integral of g over boundary must be 0.

Now, let us discuss some uniqueness properties of solutions to the 3 boundary value problems that we have just introduced.

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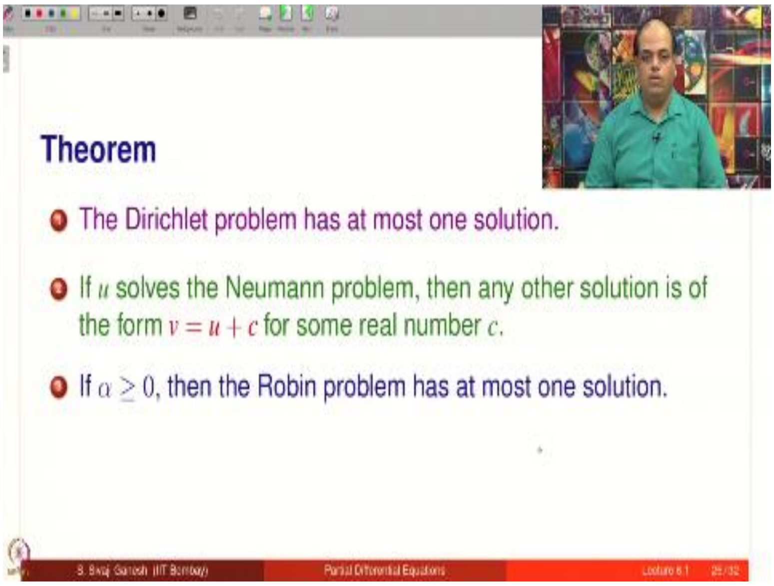

The Dirichlet boundary value problem has at most 1 solution, remember here we are not saying about existence at all, we are just saying it has at most 1 solution. That means, if it has a solution, then it has exactly 1 solution. That is the correct conclusion from here. And, if you solve some Neumann problem, then any other solution looks like u + c where c is a constant. In other words, difference of any 2 solutions to Neumann problem is a constant.

And for the Robin problem, we have to assume something an alpha. If alpha is greater than or equal to 0, then the Robin problem has at most 1 solution. Recall the Robin problem reduces to Dirichlet problem in alpha = 0. So, proving 3 is enough to prove 1. So, what we do is we just prove 3 and then we prove 2.

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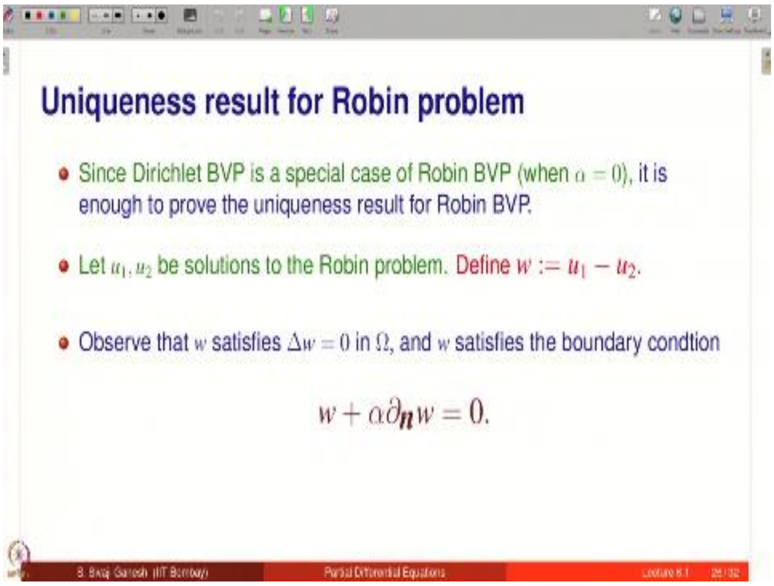

As I have already mentioned Dirichlet boundary value problem is a special case of Robin boundary value problem, it is enough to prove the uniqueness result for Robin boundary value problem. Please note I am calling here uniqueness result by this what I mean is that this problem admits at most one solution. So, I assume that u 1 and u 2 are solutions to the Robin problem. The uniqueness proofs always proceed like this. You take the difference of u 1 and u 2 and show that, that is 0.

So, therefore, you define a w which is u 1 - u 2, we want to show that w is 0. Observed that w satisfies Laplacian w = 0 because Laplacian u 1 = f, Laplacian u 2 = f therefore, Laplacian w is a difference of Laplacian u 1 and Laplacian u 2 both of them are f therefore, Laplacian w is 0, in other words w is a harmonic function in omega. And w satisfies the boundary condition w + alpha dou n w = 0 because u 1 satisfies the boundary condition with the same g and u 2 also satisfied with the same g therefore, the difference will satisfy 0, because this is linear in w. So, this is the boundary condition satisfied by w.

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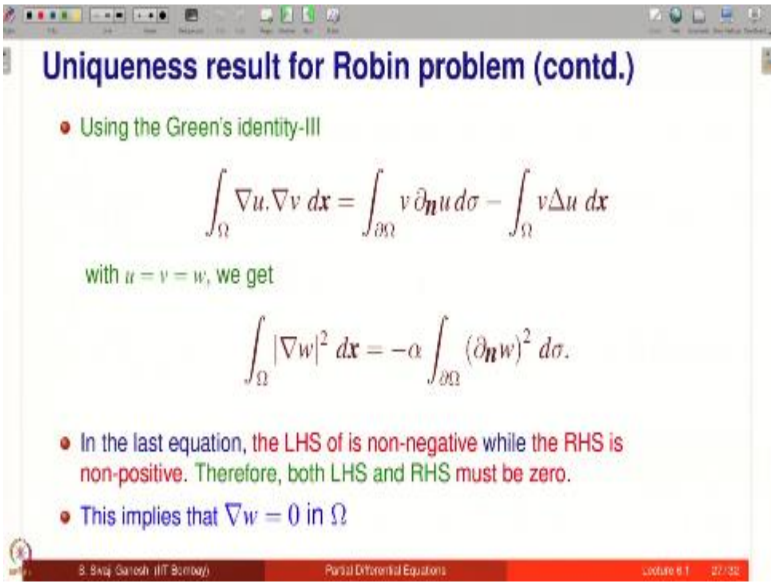

So, using the Green's identities 3 with this is the Green's identities 3, we are going to use this with u = v = w. So, the left hand side will be integral omega grad w dot grad w that is mod grad w square, right hand side Laplacian w is 0, so, this term is not there. So, what we have is this term? That is w into dou n w d sigma. Using the boundary condition, we get integral over boundary for omega w dou n w equal to this quantity please check that.

In the last equation, the left hand side is non negative because integrand is non negative, right hand side the integral is non negative, alpha is non negative therefore, the product is non negative but there is a minus sign. So, it is non positive. So, we have a non negative quantity equal to a non positive quantity which is possible if and only if both sides are 0. That means what? Grad w = 0 on omega, dou n w = 0 on boundary of omega.

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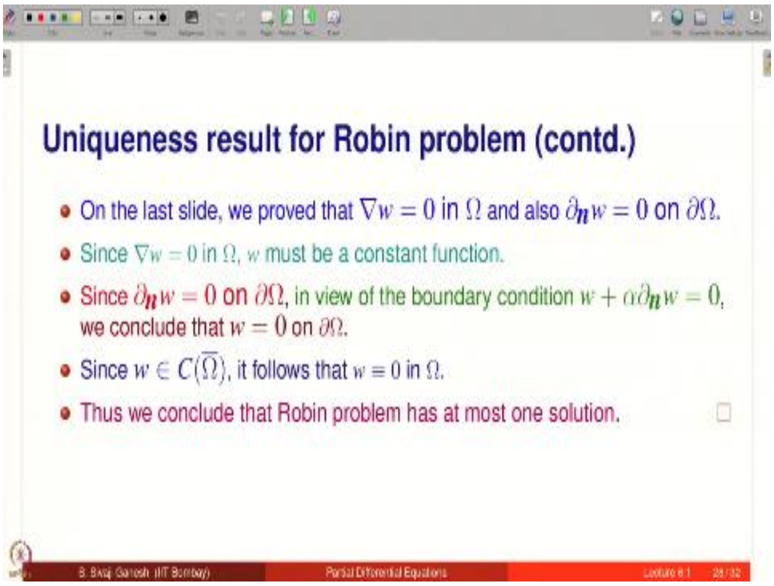

On the last slide we have proved that grad w = 0 in omega and also the normal derivative of w = 0 on the boundary of omega. Since gradient of w = 0 in omega w must be a constant function. Now, since the normal derivative is 0 on boundary of omega, if we use the boundary condition, we get w = 0 on the boundary of omega, but what is w? w is continuous up to boundary, w belongs to c of omega bar, w is a constant and it is 0 on the boundary. Therefore w must be 0 in omega also. In other words, we have Robin problem has at most one solution.

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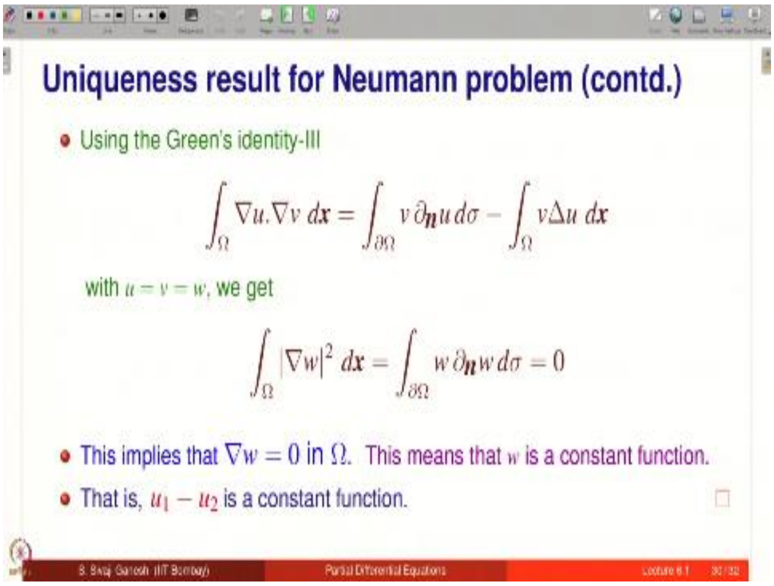

Now let us look at Neumann problem. Let u 1 u 2 be solutions to Neumann problem. Consider the difference u 1 - u 2, call it w, look at the problems satisfied by w. Laplacian w will be equal to 0 and normal derivative w = 0 on the boundary of omega. Now using once again, Green's identities 3 with u = v = w, we have integral over omega mod grad w square that is coming from the left hand side. On the right hand side as before Laplacian w is 0. So, second term drops out, what you have is this term?

Which I have written here, w dou n w d sigma, but dou n w is 0, therefore, this integral is 0. This means grad w = 0 in omega, that is all we have information nothing more. If grad w is 0 w must be a constant function. That means u 1 - u 2 is a constant function therefore u 1 = u 2 + constant.

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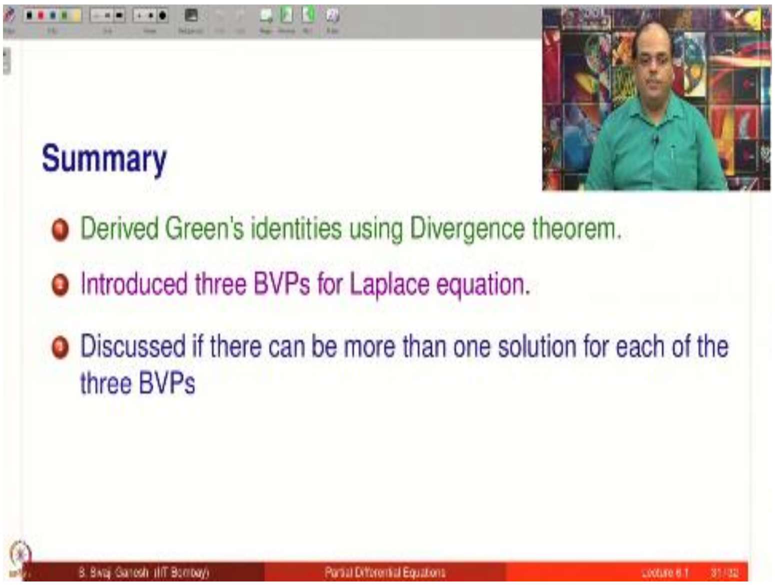

Let us summarize what we did in this lecture, we have derived Green's identities using divergence theorem, introduced 3 boundary value problems for Laplace equation, discussed if there can be more than 1 solution for each of the 3 BVPs. Thank you.