Partial Differential Equations
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Lecture – 5.7
Qualitative Analysis of Wave Equation
Propagation of Waves

Welcome. In this lecture, we are going to discuss about the propagation studies for the wave equation and with this lecture; we are going to end up discussion on wave equation.

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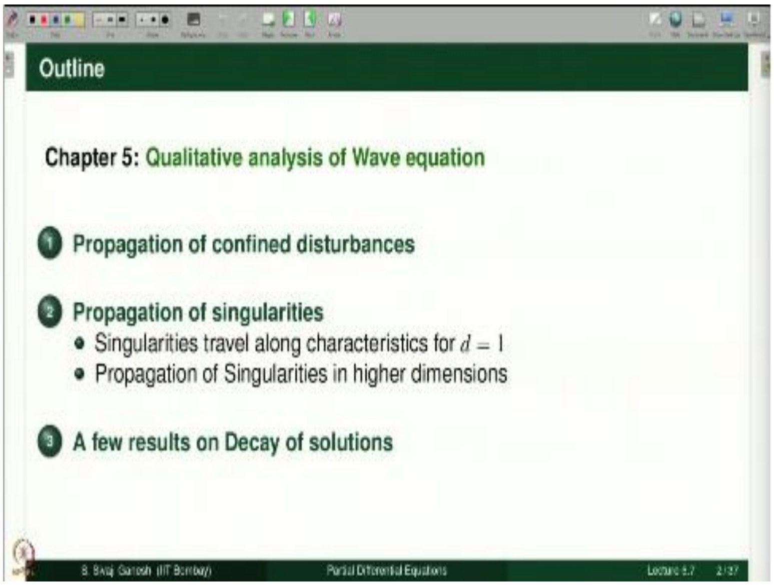

Outline for this lecture is as follows. First, we discuss about propagation of confined disturbances. We will mention what they are later and then propagation of singularities and we give a few results on decay of solutions with the proof.

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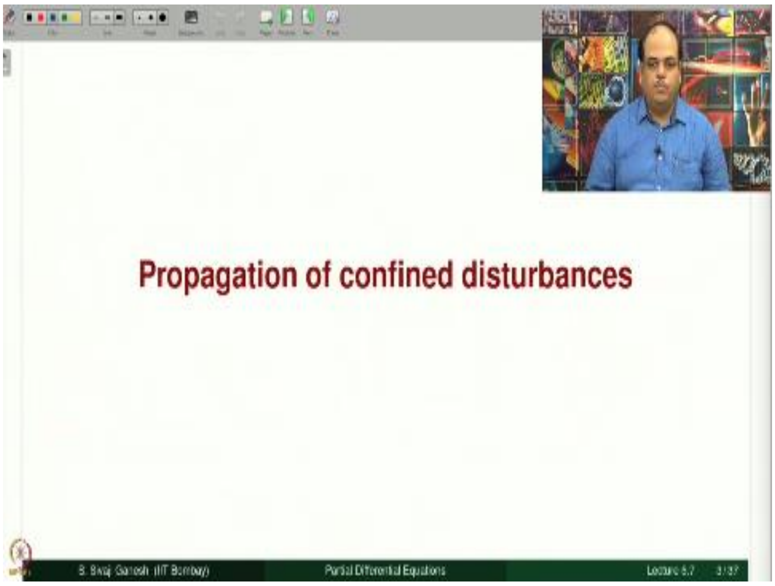

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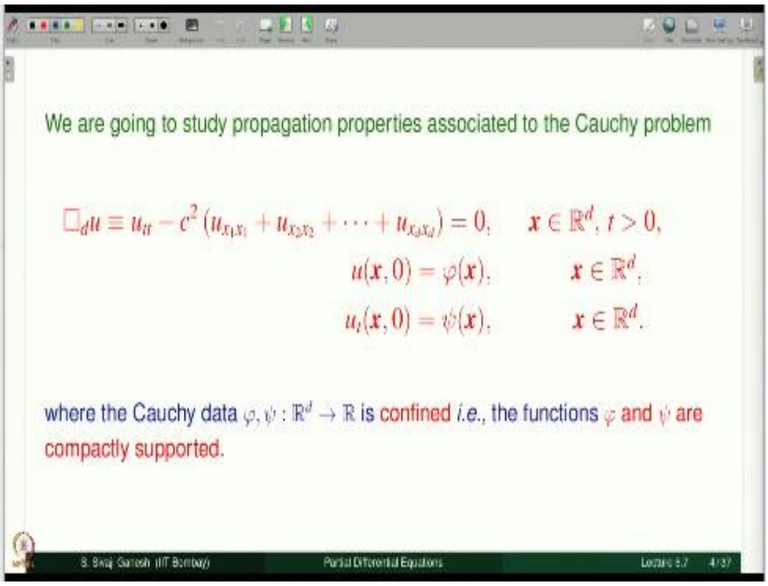

Propagation of confined disturbances: So, we are going to study propagation properties associated with the Cauchy problem where the wave equation is homogeneous and we have the Cauchy data phi and psi which is confined. It means, the functions phi and psi are compactly supported.

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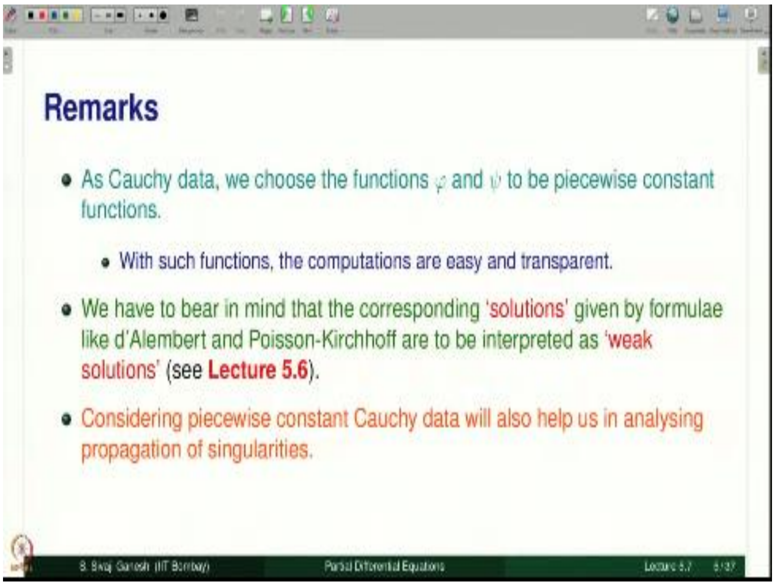

As Cauchy data, we choose functions phi and psi to be piecewise constant functions with such functions, the computations are easy and transparent. We have to bear in mind that the corresponding solutions given by the formulae like d'Alembert, Poisson-Kirchhoff, are to be interpreted as weak solutions or weaker solutions that is why I put them in quotes. Of course, we have introduced the notion of weak solutions in the lecture 5.6.

Considering piecewise constant Cauchy data will also help us in analysing propagation of singularities in the Cauchy data.

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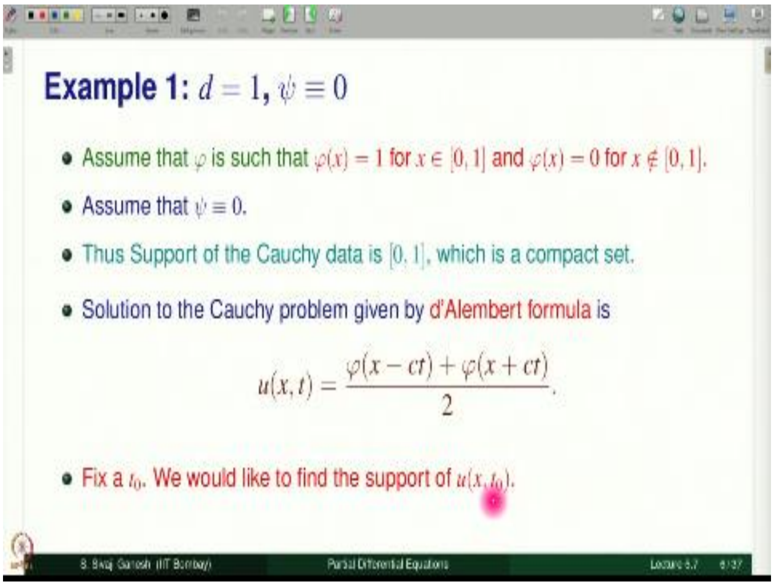

Let us see an example dimension 1; 1 dimensional wave equation, psi is identity equal to 0 and phi is such that it is equal to 1 on the interval 0, 1; 0 otherwise, outside the interval 0, 1 function is 0. So, it is a discontinuous function but piecewise constant function 0 up to 0, 1, from 0 to 1 once again 0 from 1 to infinity that is the function phi. The support of phi is precisely 0, 1. So, we are assuming that psi identically equal to 0.

Therefore, as we see the support of the Cauchy data is the interval 0, 1 which is a compact set. Support of phi is 0, 1; psi is identically equal to 0. Now, a solution to the Cauchy problem given by d'Alembert formula is this. Fix a t 0, we would like to find the support of u of x, t 0. So, we want to study the support of the solution to the wave equation at a time t = t 0.

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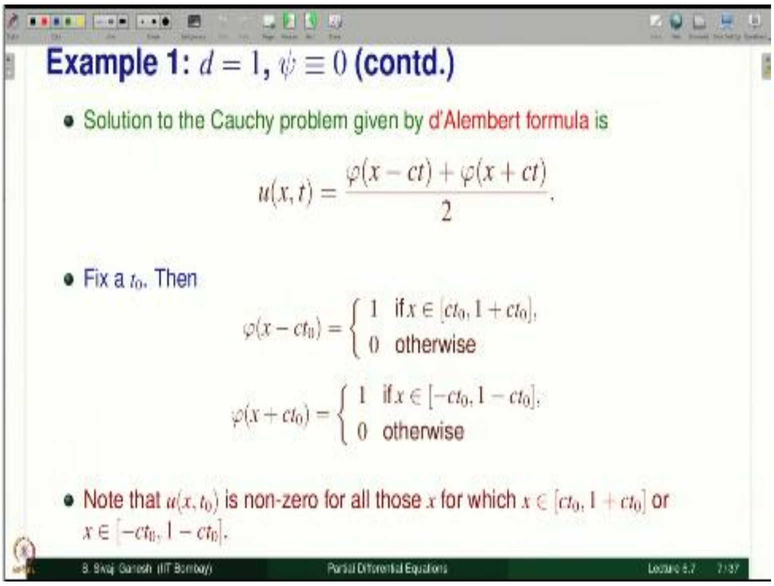

Fix a t 0, then what we need is u of x, t 0. So, it depends on x – c t 0 and x + c t 0. Therefore, we will find out what is phi of x – ct 0 and phi of x + c t 0. We can easily see that these formulae hold because phi is equal to 1 whenever the argument is an interval 0, 1. The argument is interval 0, 1 means x is in the interval. Similarly, here this argument is an interval 0, 1 means x is in the interval. So, note that u of x t 0 is nonzero for all those x for which x is in this interval or this interval.

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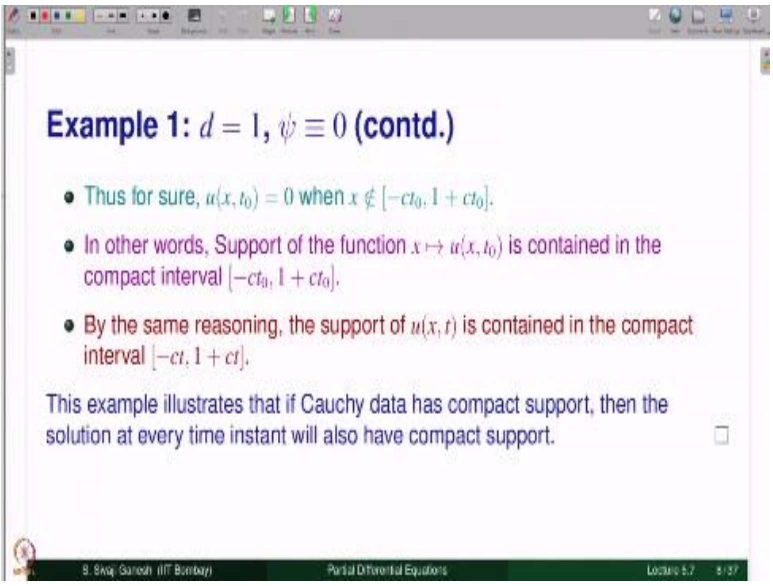

Thus, for sure u of x, t 0 is 0 when x is not in the interval – ct 0, 1 + ct 0. In other words, support as a function x mapping to u of x, t 0 is contained in the compact interval – ct 0, 1 + ct 0. By the same reasoning, the support of u of x t that is x going to u of x t is contained in the compact interval – ct, 1 + ct. This example illustrates that if Cauchy data has compact support, then the solution at every time instant will also have compact support.

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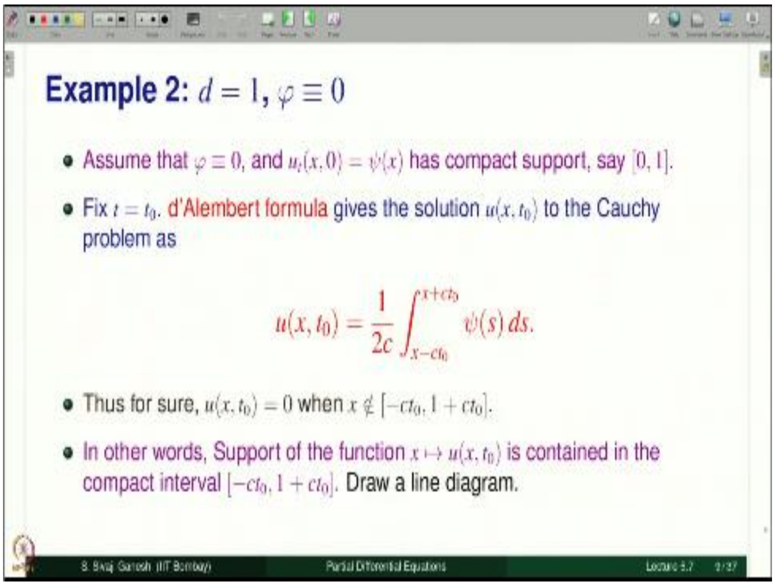

Now, let us like another example where phi is 0 but psi will be there. In the example 1, psi is 0, In example 2, phi will be 0. Psi has compact support say 0, 1. Now, fix t in t 0, then d'Alembert formula gives you this as a formula for the solution. Thus, for sure, u of x t 0 = 0 when x is not in this interval. This is – ct 0; this is 1 + ct 0. So, when x is not in this interval, let us say x is here, what happens?

x is bigger than 1 + ct 0 that means x – ct 0 is bigger than 1. What does this mean? If I take the interval 0, 1, x – ct 0 is here. So, where will be x + ct 0 this side. What does this mean? Psi is supported in interval 0, 1 and x – ct 0, x – ct 0 which is the domain of integration is not intersecting 0, 1. Therefore, the integral will be 0. Similarly, if x is here because if x is not in this interval means x is either to the right side of 1 + ct 0 like here or to the left side of 1 – ct 0.

So, if x is like this, what does it mean? x + ct 0 is less than 0. What does that mean? x = ct 0 is here. Where will be x – ct 0? It will be here. Once again this interval on which we are integrating side does not intersect with 0, 1 therefore, this we have this u of x t 0 is 0. In other words, the support of this function x going into u of x t 0 is contained in this compact interval. That is what we have drawn the diagram and shown.

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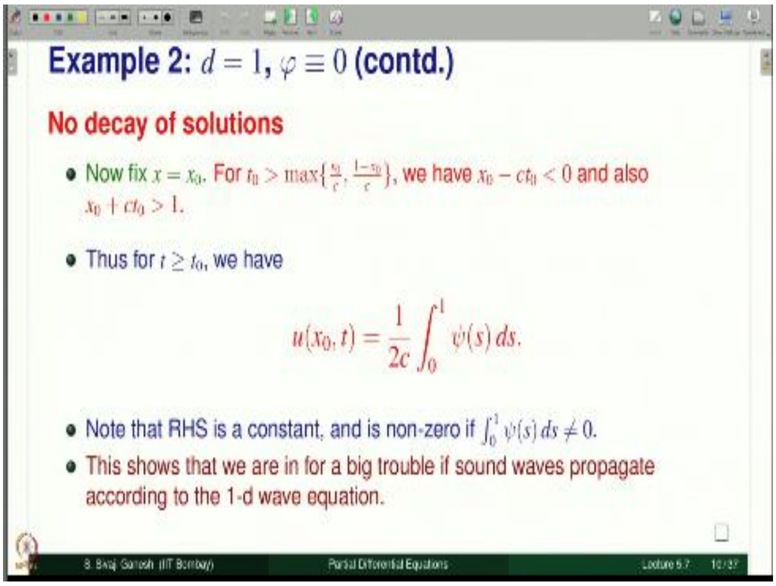

No decay of solutions. This is another property that there is no decay when we are dealing with 1 dimensional wave equation. Fix x equal to x 0, what is the meaning of decay? You stand at a point x 0 and look at u of x 0, t as t varies as t goes to infinity. So, for t 0 bigger than this quantity, we have x 0 – ct 0 to be less than 0 and x 0 + ct 0 bigger than 1. Therefore, what will happen is that 0 and 1 are here; x 0 – ct 0 and x 0 + ct 0 will be here.

If you recall the d'Alembert formula that will be an integral on this interval x 0 – c t 0, x 0 + ct 0. So, if t is bigger than this t 0, x 0 – ct 0, x 0 + ct 0, this interval will always contain the interval 0, 1 on which psi is supported. Therefore, the solution is actually 0 to 1 psi s ds that the integral part. So, 1 by 2 c will be there. So, we have this. This will be the solution. Let us see that. So, for t bigger than t 0, this is a formula that comes from the d'Alembert formula.

Now, RHS is a constant because as I told you this integral is from x 0 – ct to x 0 + ct but x 0 – ct and x 0 + ct always contains 0, 1, the moment t is bigger than or equal to t 0 where t 0 is equal to this or bigger than this whatever equal to we can set. So, and it is nonzero, this is not 0. It all depends on what the integral of psi on the interval 0, 1. If it is not equal to 0, this is nonzero constant for all t.

It means, the solution does not becomes 0 or does not decay at it stays constant. So, we are in a big trouble if sound waves propagate according to 1D wave equation, because of this reason, luckily, they do not.

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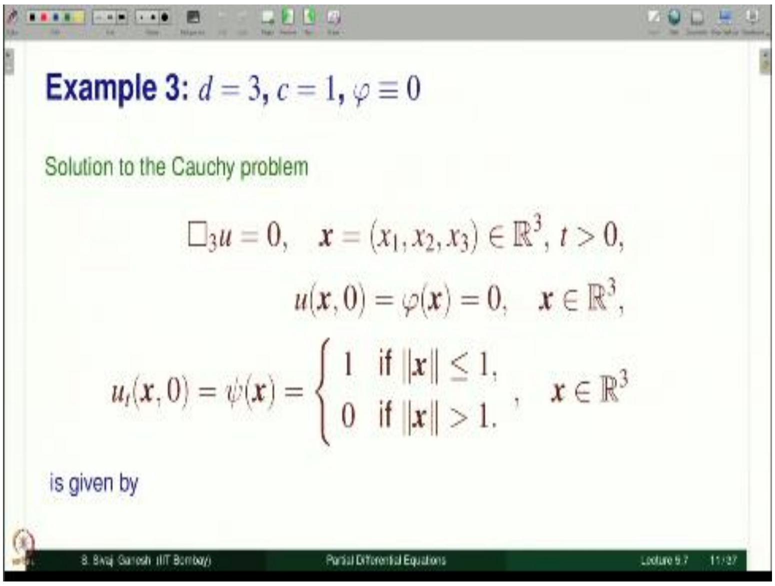

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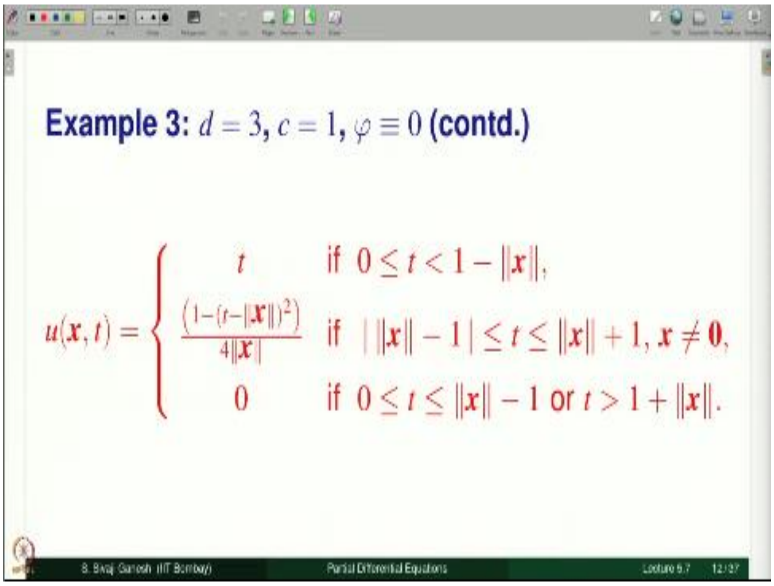

So, d = 3. Straight away we do the example for dimension 3. I assume speed is 1, c = 1 and phi = 0. So, solution to the Cauchy problem, this, we have worked out in an earlier lecture to be this.

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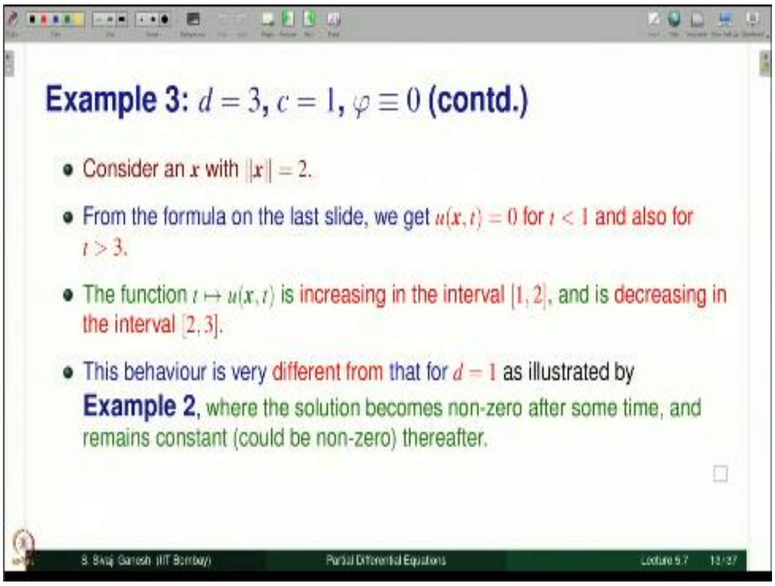

Now, consider an x with norm x = 2 and analyse what is it? It says here, I fixed an x with normal x = 2 and I want to study u of x t as t goes to infinity, what is the behaviour? From the formula on the last slide, we get u of x t = 0 for t less than 1 and for t bigger than 3. The function t going to u of x t is increasing the interval 1, 2; decreasing in 2, 3 and then becomes 0 after t greater than 3.

This is what we have already learned in earlier lectures that in 3 dimension solution for the wave equation, there is a time upto which information has not reached that is up to 1, after that image information has reached and after this information goes away, 0. So, this is the leading edge and trailing edge that is what we have seen even the pictures when we discussed Huygens principle, we have discussed.

So, this behaviour is very different from that for d = 1, we have just seen as illustrated by example 2. So, where the solution become nonzero after some time and remains constant, that could be nonzero if integral of 0 to 1 of psi is not 0.

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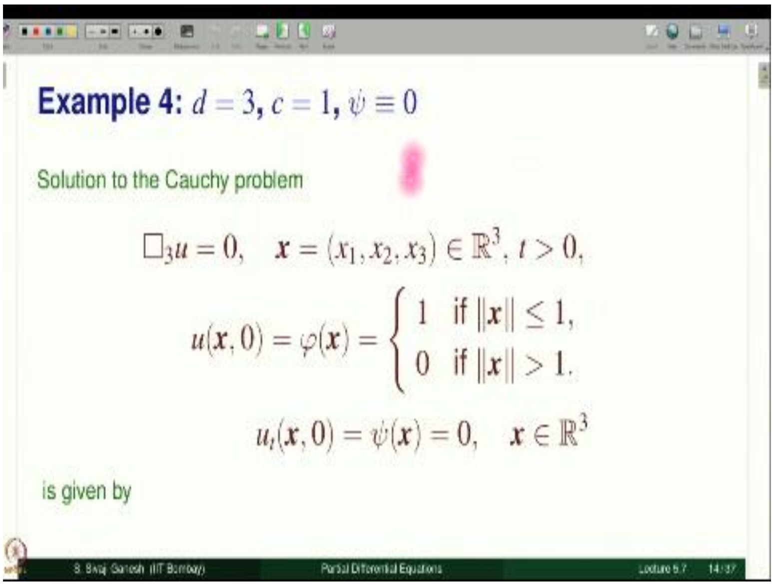

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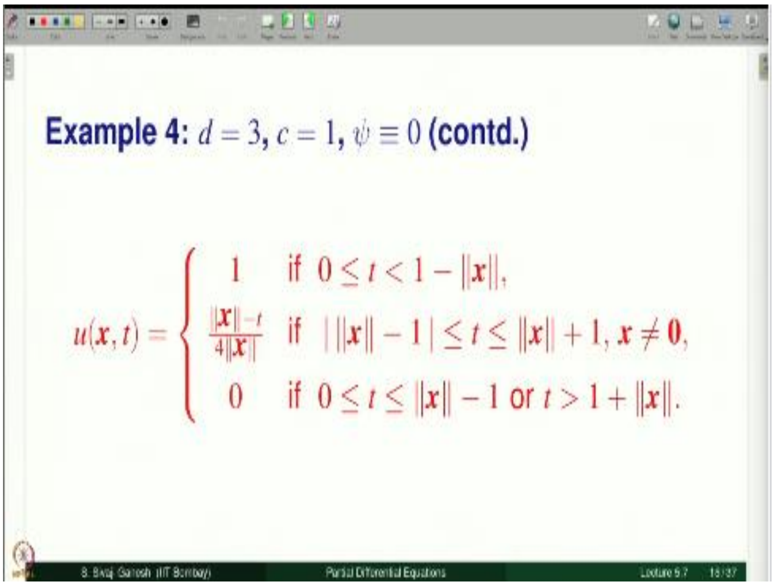

Now, d = 3, c = 1, psi = 0, this Cauchy problem we have solved in course, maybe a weaker notion of solution; we got u to be like this.

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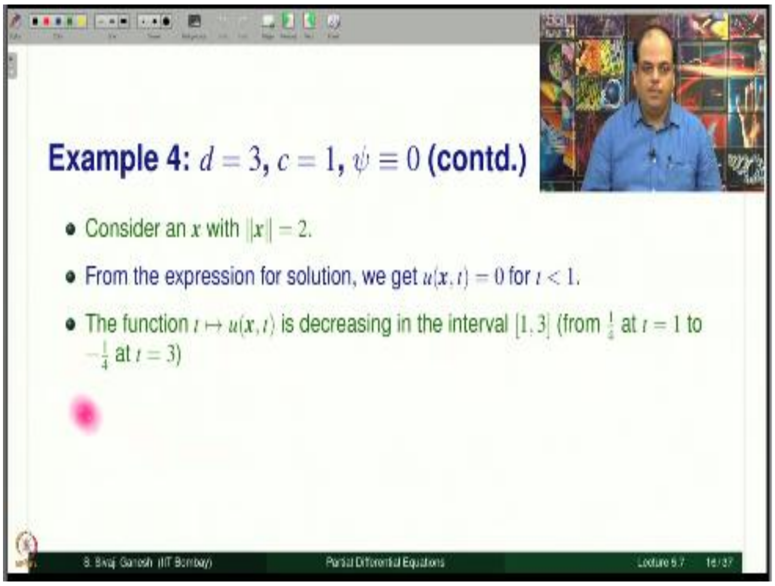

Once again consider an x with norm x = 2, from the expression for solution, we get u is 0 of a t less than 1. It is function is decreasing in 1, 3; in fact, this, please verify all this assertions by yourself by looking at the formula and then becomes 0 for t greater than 3.

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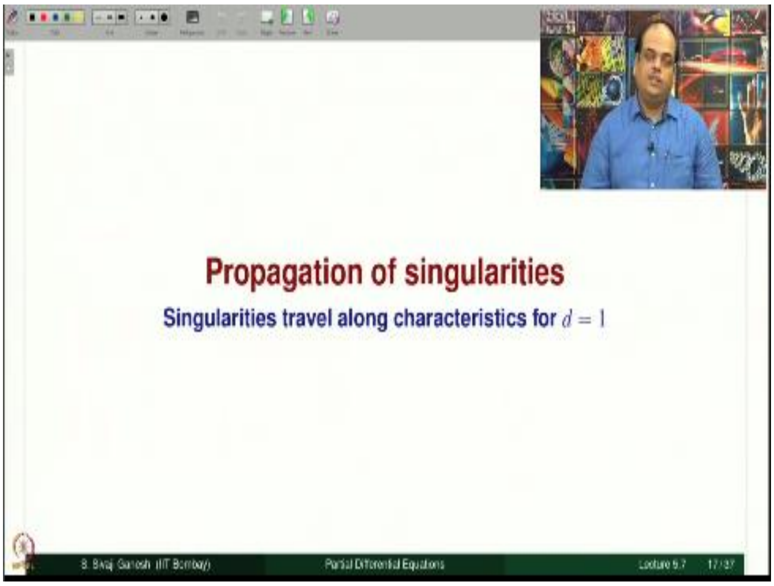

Now, let us discuss propagation of singularities. Singularities travel along characteristics for d = 1.

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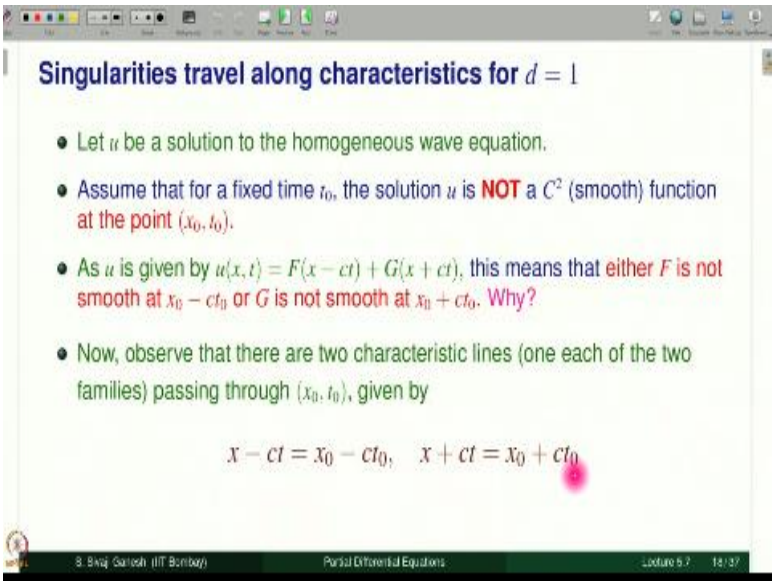

Let u be a solution to the homogeneous wave equation. Assume that for a fixed at time t 0 that means that at a fixed time t 0, the solution u is not a C 2, sometimes we just call smooth function at the point x 0 t 0 that means that formula for u has a trouble at x 0 t 0. u is given by this expression F of x – ct + G of x + ct. u has a trouble at x 0 t 0 means F should have a trouble at x 0 – ct 0. Or else, G should have a trouble at x 0 + ct 0 or both.

So, this means either F is not smooth at x 0 – ct 0 or G is not smooth at x 0 + ct 0. Why? Please justify this. If they are smooth, then you can conclude that u is smooth. Smooth, if you think it is continuous, it is continuous; if it is differentiable, it can be differentiable. If you think it is C 2, it is C 2. Now, observe that there are 2 characteristic lines which pass to the point x 0 t 0.

Recall that there are 2 families of characteristic lines for the wave equation. One member each passes through this point x 0 t 0 namely x – ct = x 0 – ct 0, x + ct = x 0 + ct 0. These are 2 characteristic lines which pass through the point x 0 t 0.

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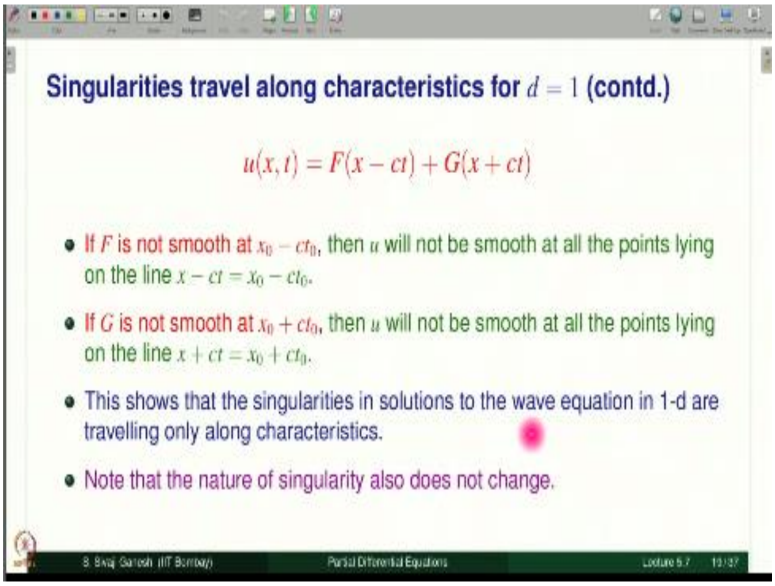

Now, in view of this formula for u, if F is not smooth at x 0 – ct 0, then F will not be smooth at all those x and t such that x – ct is equal to x 0 – ct 0. After all, it depends on whether F as a function of one variable, what happens to at a particular location x 0 – ct 0? So, therefore, whenever x – ct is equal to x 0 – ct 0, you have the same problem that is why it is not smooth. So, if G is not smooth at x 0 + ct 0, u will not be smooth at all points on this line x + ct = x 0 + ct 0 for the same reason.

This shows that the singularities in solutions to wave equation are travelling only along characteristics. Note that the nature of singularity also does not change. That is what I said, if F is not continuous, u will not be continuous. If F is not differentiable, u will not be differentiable. If F is not C 1, u will not be C 1 and so on. So, the nature of singularity does not change.

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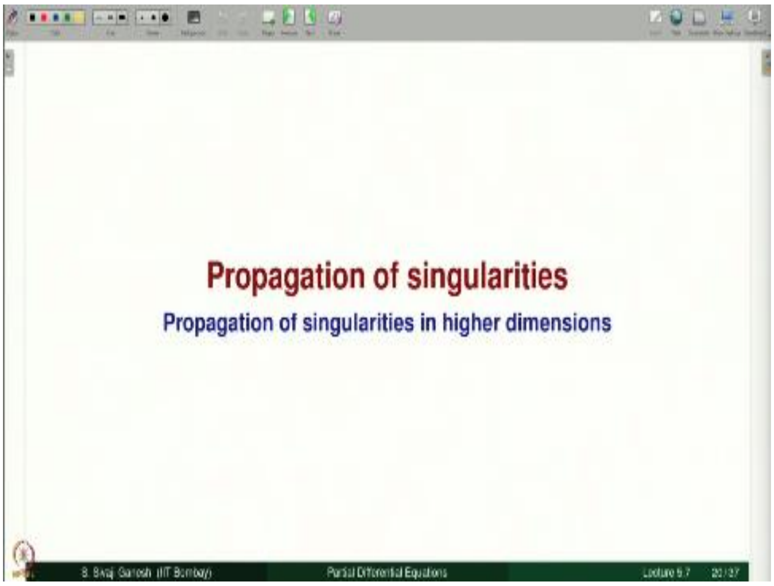

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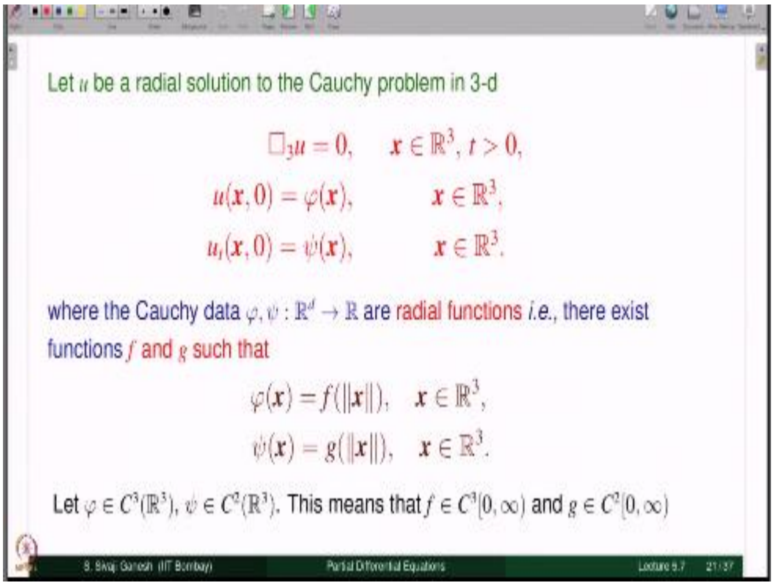

Let us look at what happens in higher dimensions. Let u be a radial solution to the Cauchy problem in 3D where functions phi and psi are also radial. What do you mean by radial? It means that the function depend only on the distance to the origin. So, phi of x will be a function of norm x that there exist f and g such that phi x = f of norm x, psi x = g of norm x. In other words, phi and psi are constant at all points of any sphere with centre at origin. Since, phi and psi are functions of norm x only.

In other words, it depends on the distance of x to the origin. These functions are called radial functions. Because if you consider a sphere with radius norm x, norm x is the radius. That is why these are called radial functions. Let phi belongs to C 3 of R 3 and psi belongs to C 2 of R 3. These are the assumptions that we need. So, that this problem will have a classical solution given by Poisson-Kirchhoff formula and that in turn means that f and g, they are C 3 and C 2 on this interval 0, infinity.

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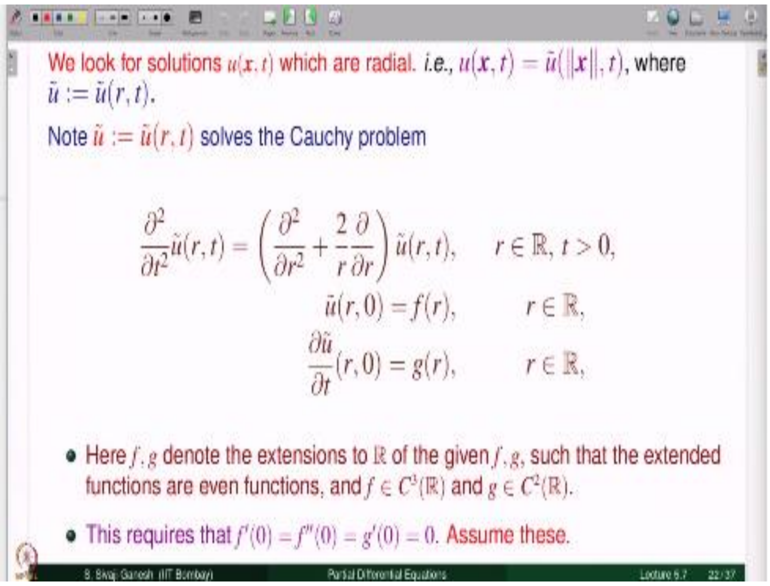

So, we look for solutions. u of x t which are radial, that is, u of x t is looking like u tilde of norm x, t where u tilde is a function of R, t; R is in the interval 0, infinity and t is in 0, infinity once again. So, note that u tilde solves the Cauchy problem, which is given here. It is very easy to derive. In fact, we have already done this before, when we are trying to get solutions to the Cauchy problem.

So, this is in fact what is called as radial Laplacian. Here, f and g denote the extensions to R of the given f and g because the given f and g are defined only on the closed interval 0, open infinity, whereas here, we need r belongs to R, because we are trying to pose a problem for r belongs to R and hence, we extend the given f, g to f, g. So, we still use the same notations f g, such that the extended functions are even functions.

And of course, f is in C 3 and g is in C 2. This requires that f dash of 0 and f double dash of 0 is 0. Similarly, g dash f of 0 equal to 0. If f and g satisfy these conditions, then we can do this extension as mentioned here in this point; assume these conditions.

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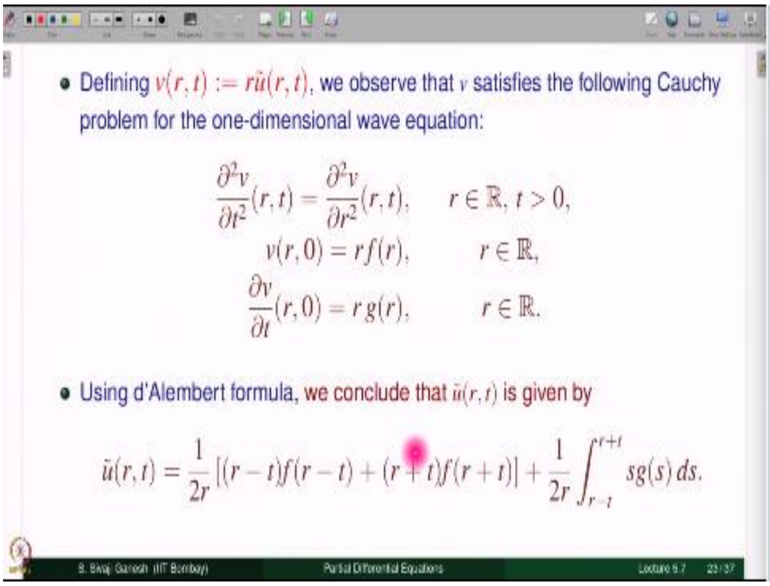

Now, defining v of r, t = r into u tilde of r, t, we see that v satisfies the Cauchy problem for the 1D wave equation because under this change of the dependent variable, the radial Laplacian will simply become dou 2 v by dou r square. So, this is exactly 1 dimensional wave equation for v and these are the Cauchy data, v of r, 0 is r f r; dou v by dou t r, 0 is r times g of r.

Now, using d'Alembert formula, we conclude that u tilde r, t is given by this formula. So, of course, the d'Alembert formula gives you v of r t, but once you know v of r t, you know what is u tilde r t is divide with r. Therefore, I divide with r, I get this formula for u tilde r t.

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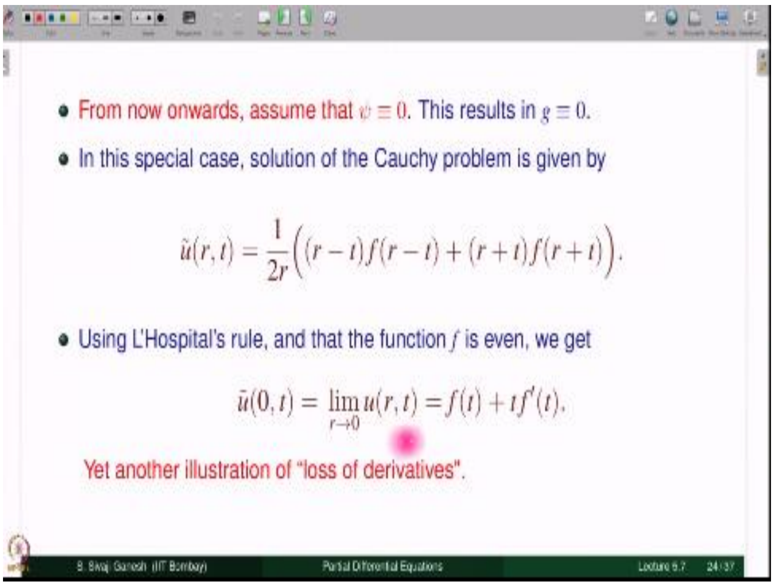

From now onwards, assume that psi is identical equal to 0. This means that there is no g. g identical equal to 0. So, the formula simplifies in this special case to this formula, u tilde r t =

1 by 2 r into r – t into f of r – t + r + t into f of r + t. Using L'Hospital's rule and that the function f is even we get u tilde of 0, t equal to limit of this quantity as r goes to 0 which using the L'Hospital's rule turns out to be f of t + t times f prime of t.

This is yet another illustration of loss of derivatives, because we have assumed f is C 3, but now you have f dash so, that means it is just C 2. So, u tilde will only be C 2. u tilde of 0 t if it exists, it will be only a C 2 function. So, we have lost the derivatives.

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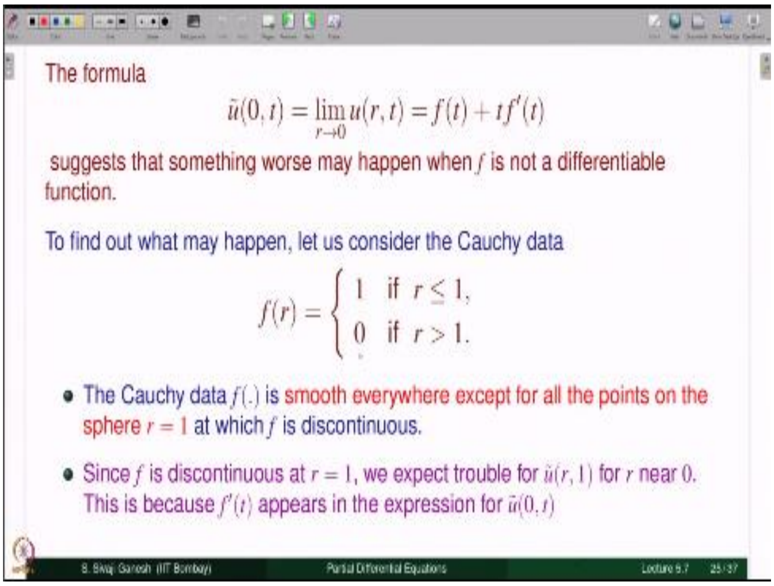

So, the formula u tilde of 0 t equal to f of t + t times f prime of t suggests that something worse may happen when f is not a differentiable function. To find out what may happen, let us consider the Cauchy data f of r given by this formula f of r = 1 if r is less than or equal to 1, 0 if r is bigger than 1. In other words, this Cauchy data f takes the value 1 on the closed unit ball with centre at 0 and outside the closed unit ball, it is 0.

The Cauchy data f of r is a smooth function everywhere except when r = 1 at which the function is discontinuous. Since f is discontinuous at r = 1, we expect trouble for u tilde r, 1 for r near 0. This is because f prime of t appears in the expression for u tilde of 0 t and f prime of 1 is not meaningful because function itself is discontinuous.

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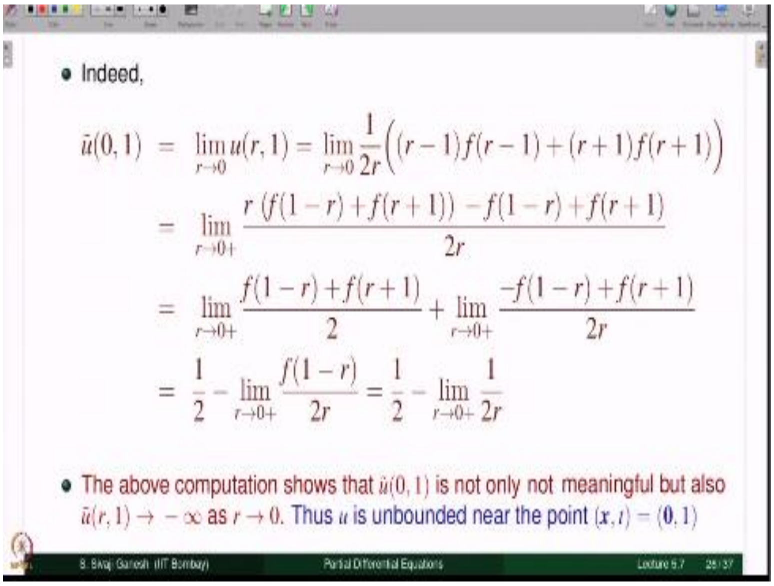

Indeed, u tilde of 0, 1, let us compute; it is limit r goes to 0 of u of r 1. We do not know whether limit exists or not, let us compute. So, what is u of r 1? I use a formula on the previous slides there is a u of r, 1. Now, let us simplify this expression after the limit. So, I take r common, I get f of r – 1 + f of r + 1. I have written r – 1 as 1 – r because f is an even function. So, f of r – 1 equal to f of 1 – r.

Now, what is remaining is – f of r – 1 which once again write as – f of 1 – r and + f of r + 1 by 2 r. I have not done anything, I have just rearranged the terms in this expression to be this. Now, there is a r here; these is r here, I can cancel these 2 r, so I will separate these into 2 terms. This limit can be computed as limit of this plus limit of this provided these 2 limits exists. Now, we see what is the limit of this and what is the limit of this separately.

The first term is just 1 by 2 because when I am coming to r from the right side of 0, 1 + r is always bigger than 1 and f is 0, if the argument is bigger than 1, so, this term is not there. What I have only this term and that term f of 1 – r; 1 – r is always less than 1 and hence, this is always 1. So, 1 by 2. So, in fact, this quantity does not depend on r by the nature of the definition of the function f that we are considering.

Then we have once again f of 1 + r is 0. So, I dropped that term and I take this minus sign here, limit r goes to 0 + f of 1 – r by 2 r; f of 1 – r is 1. I have 1. Now, I know this 1 by 2 r has no limit, which is a real number, but legally speaking, this is going to infinity and half minus infinity is like minus infinity. So, there is no limit here. So, after observing this limit equal to this minus this tells you that this limit exists if only if this limit exists.

And we now just check this limit which is actually this limit that does not exist. So, u tilde of 0, 1 is not only not meaningful, but also u tilde of r, 1 goes to minus infinity as r goes to 0. Thus, u is unbounded near the point x t = 0, 1; x = 0, t = 1.

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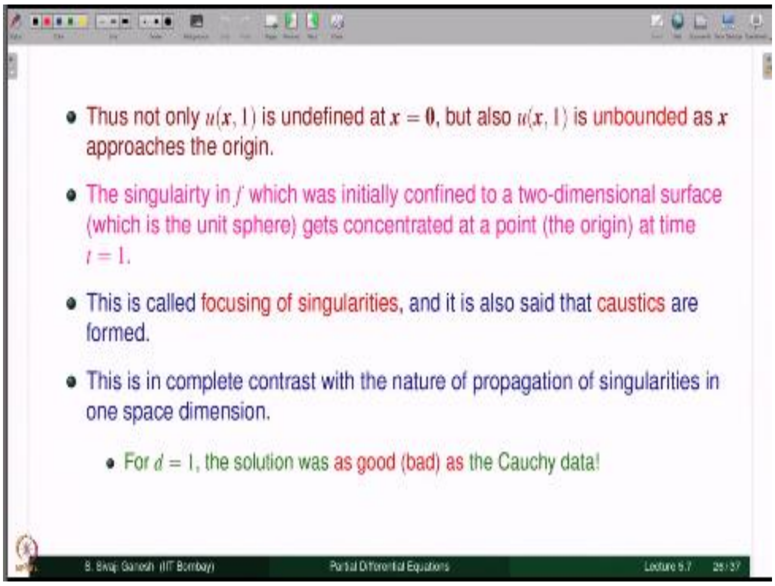

Thus, not only u of x 1 is undefined at x = 0, but also u of x 1 is unbounded as x approaches the origin. The singularity in f which was initially confined to a 2 dimensional surface, which is the unit sphere gets concentrated at a point which is now the origin as time goes to 1 that is at the time t = 1. This is called focusing of singularities and one also says that caustics formula.

This is in complete contrast with the nature of propagation of singularities in one space dimension. In one space dimension, the solution was as good or as bad as the Cauchy data.

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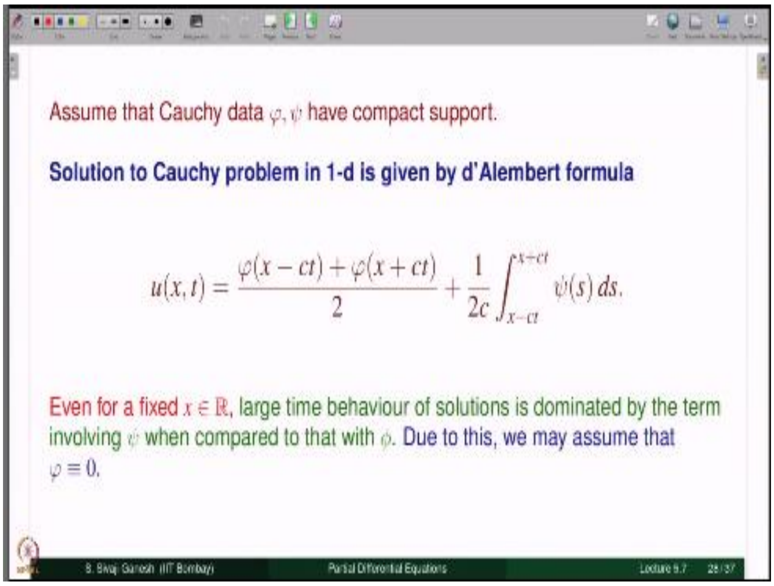

Now, let us look at decay of solutions. Assume that the Cauchy data phi and psi have compact support. So, throughout our discussion on decay of solutions, we assume that the Cauchy data is having compact support. Otherwise, we do not expect such results. Solution to Cauchy problem in 1D is given by d'Alembert formula which is given by this formula u of x t equal to phi of x – ct + phi of x + ct by 2 + 1 by 2 c integral x – ct to the x + ct psi s ds.

Even for a fixed x in R, large time behaviour of solutions, solutions given by this formula, is dominated by the term involving psi. We saw this in one of the tutorials, where we saw the point wise u of x t converges to a certain constant in that example. Phi never played a role there. So, due to this, let us assume phi = 0.

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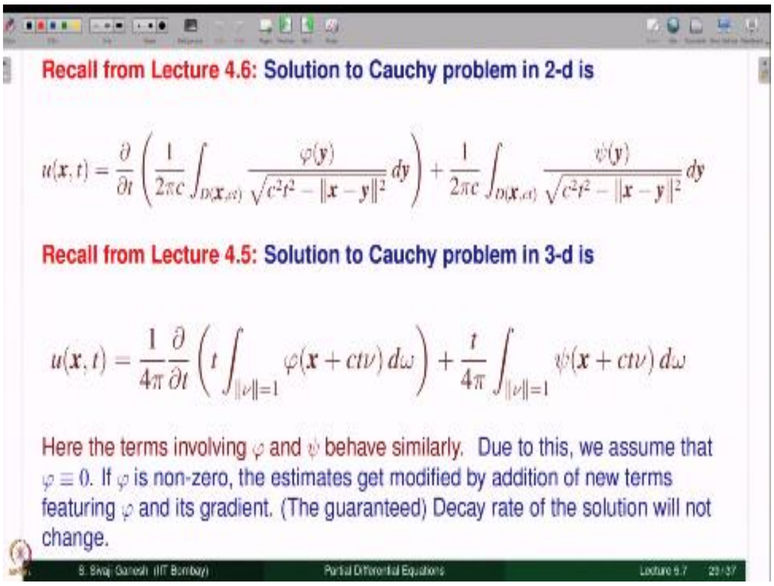

Now, recall from lecture 4.6 and 4.5, the Poisson-Kirchhoff formulae for the Cauchy problem in 2D and 3D which are given by this formula. Here, the terms involving phi and psi behave similarly. Due to this, we assume that phi is identically equal to 0. If it is nonzero, the estimates get modified by addition of new terms featuring phi and its gradient. The guarantee decay rate of a solution will not change.

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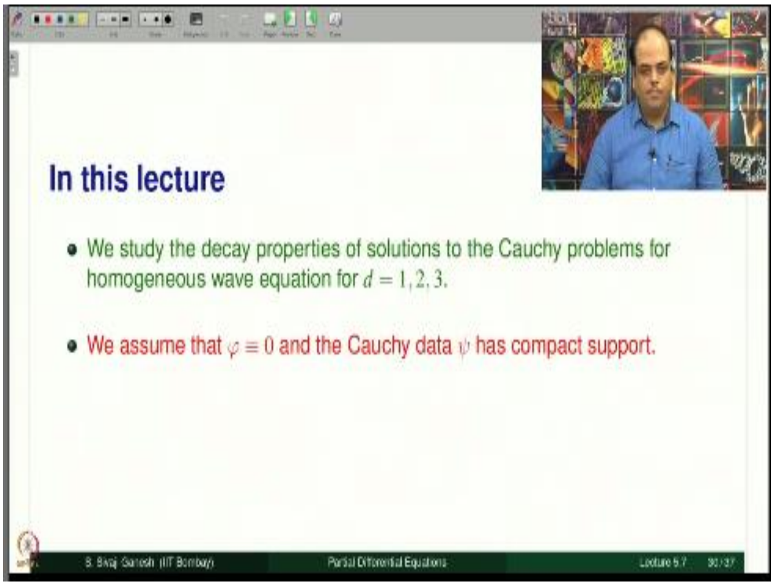

In this lecture, we are going to study decay properties of solutions to Cauchy problems for homogeneous wave equation across all the 3 dimensions 1, 2 and 3. We assume phi 0 and Cauchy data has compact support. So, we are going to assume Cauchy data has compact support.

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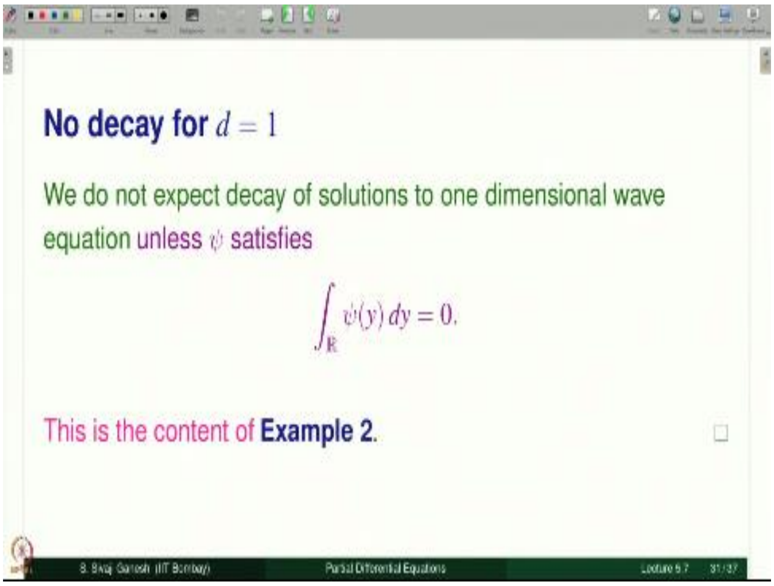

No decay for d = 1. We already saw this in example 2. We do not expect decay of solutions to one dimensional wave equation unless psi satisfies the integral over R equal to 0, in fact, 0 to 1 in example 2, because we assumed psi supported in 0, 1. That is the content of example 2.

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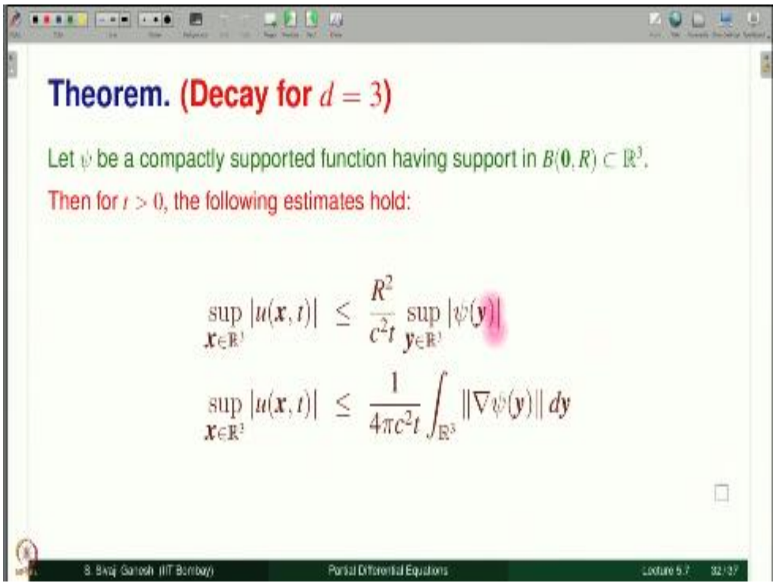

So, theorem for decay for d = 3, let psi be a compactly supported function with support in the ball of radius r with centre origin. Then for t positive, we have the following 2 estimates. If you notice here, this is a uniform estimate because this will go to 0 as t goes to infinity, as t goes to infinity, RHS goes to 0. So, this goes to 0 and the way, it goes to 0 does not depend on x at all because this is valid for all x.

The right hand side does not depend on x that is why this is called a uniform decay estimate. The only difference between these 2 estimates is that here we have supremum of psi; here, we have integral of norm psi. So, whenever these 2 things are meaningful, we are assuming, it has compact support. So, both are meaningful. So, we have these estimates. Of course, we do know that u of x t is actually 0 after some time for every fixed x, but these are uniform decay estimates.

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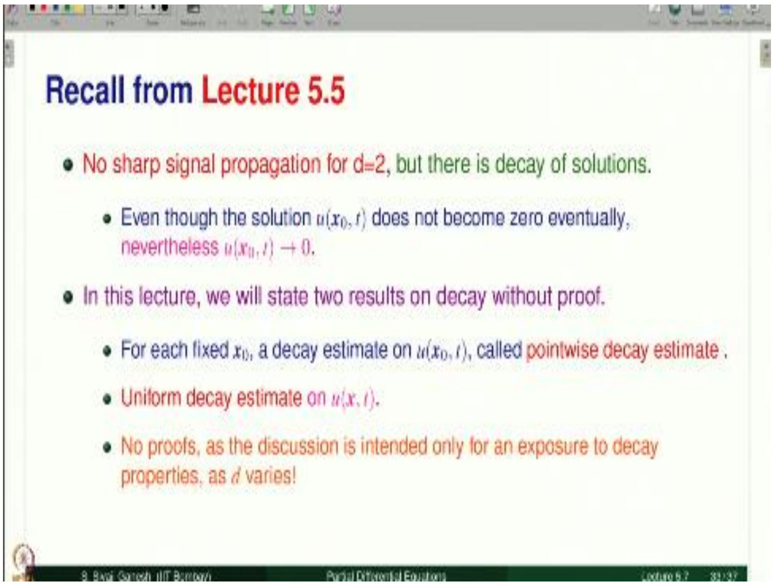

So, recall from lecture 5.5 that there were no sharp signal propagation for d = 2, but there is a decay of solution, this what we mentioned. Even though solution u of x 0, t it does not become 0 eventually that means there is a time after which u of x 0 t is 0 for d = 3 that does not happen here. Something becoming 0 eventually means, after some time, it is equal to 0 that is the meaning of saying eventually 0 that does not happen.

Nevertheless, u of x 0, t goes to 0 as t goes to infinity. In this lecture, we will stick to results on decay with the proof, of course, even for d = 3, we are not given proof. For each fixed x 0, a decay estimate on u of x 0, t that is one decay estimate. It is called point wise decay estimate because x 0 is fixed and we are talking about the decay of u of x 0 t. Second one is uniform decay estimate on u of x t. No proofs as a discussion is intended only for an exposure to decay properties as d varies.

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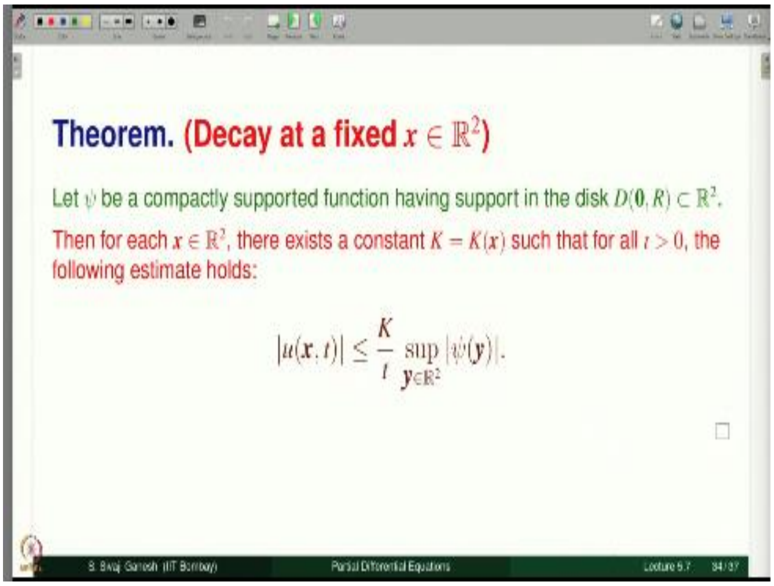

So, decay at a fixed x that is the point wise decay. As before, psi will be compactly supported function. Now, in a disk of radius r centred at the origin in R 2, then for each fixed x, there is a constant K which appears in this inequality that depends on x such that for all t, this estimate holds. So, as t goes to infinity, right hand side goes to 0 and therefore, left hand side also. But this estimate depends on K and K depends on x. That is why it is called decay at a fixed x or point wise decay estimate.

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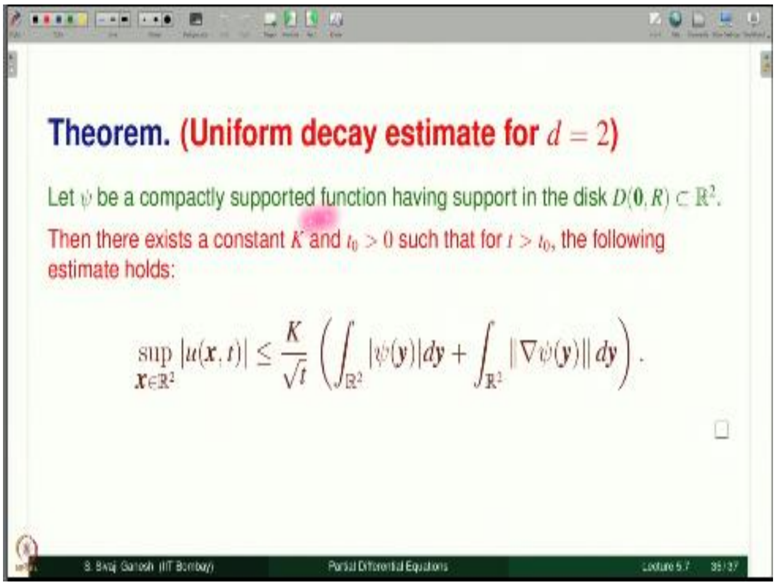

Uniform decay estimate: same assumption as before on psi, then there exists a constant under t 0 such that whenever t is bigger than t 0, this estimate happens. If you are interested as t goes to infinity, it is okay, this goes to 0. t goes to infinity means t will become bigger than t 0 after some time. So, estimate is valid. So, this goes to 0. Therefore, this goes to 0, so, uniform decay, but notice, it is a root here in the point was decay, it was 1 k by t.

Now, it has become root t because of the uniform decay. So, you are having a stronger statement here. So, you have a weaker estimate. That is what is true in general.

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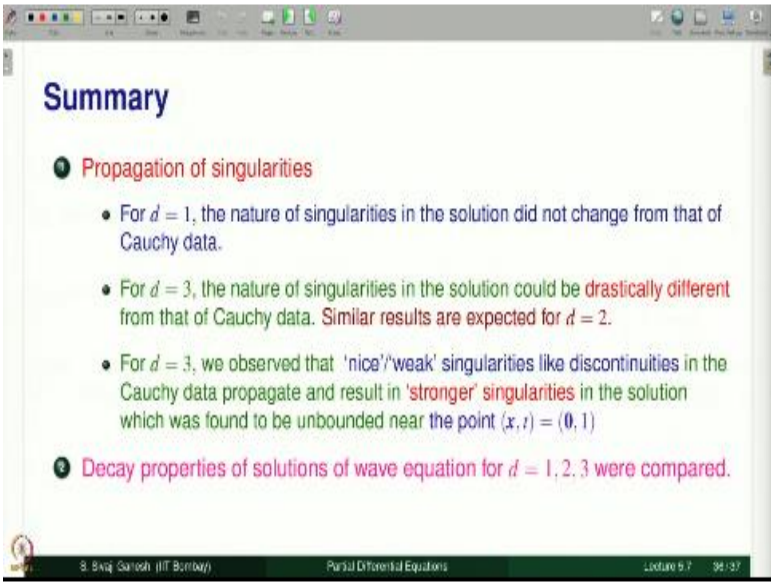

So, let us summarize. Propagation of singularities for d = 1, the nature of singularities in the solution did not change from the Cauchy data at different timings. For d = 3, the nature of singularities in the solution could be drastically different from that of Cauchy data. This, we would be expected even from the Poisson-Kirchhoff formula where even if phi is a C 3 and psi is C 2, eventually the u, you get is only C 2.

Therefore, if you fix time t and look at u of x, t 0 for example, this is not C 3. It is only C 2. So, there is a loss of derivatives there. So, therefore, that is the reason behind this. Similar results are expected for d = 2. For d = 3, we observed that nice or weak singularities like discontinuities in the Cauchy data propagate and result in stronger singularities in a solution which was found to be unbounded near the point x t = 0, 1 in that example that we saw.

Decay properties are solutions of wave equation for d = 1, 2, 3, we have compared. d = 1, no decay; d = 3, there is a uniform decay of 1 by t type and for d = 2, we had a point wise decay and a uniform decay, one was like 1 by t, one was like 1 by root t. Thank you.

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Thank you