Partial differential Equations
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Lecture – 5.5
Qualitative Analysis of Wave Equation
Huygens Principle

Continuing our qualitative analysis of the wave equation, today, we are going to see what is called Huygens principle.

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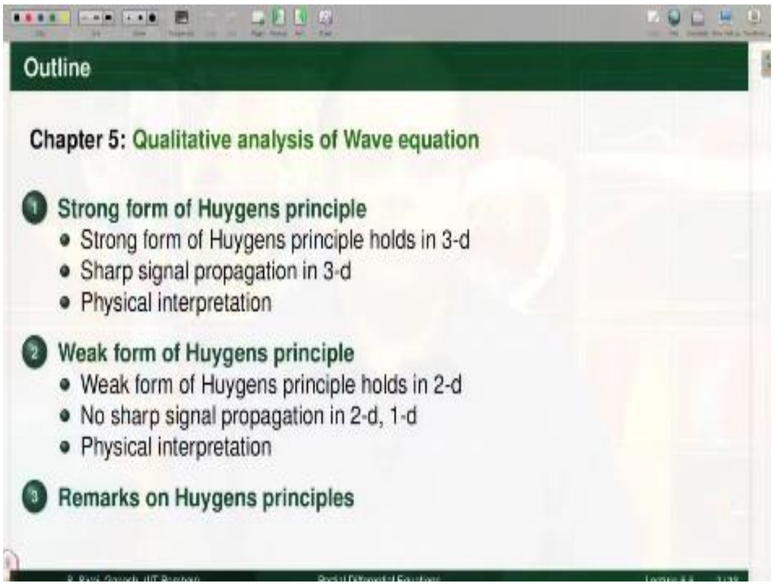

Outline for today's lecture is: first we introduce what is called a strong form of Huygens principle and we show that it holds in 3D, consequence sharp signal propagation in 3D and we give a small physical interpretation of the strong form of Huygens principle. Then we go on to introduce what is called weak form of Huygens principle. And we show that it is satisfied in 2D and no sharp signal propagation in 2D and also in 1D and physical interpretation of the same and then we give a couple of remarks on Huygens principles.

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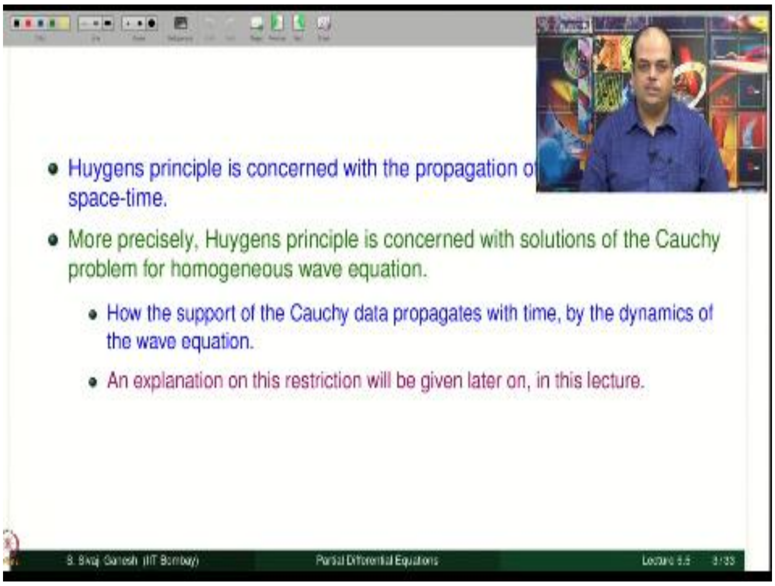

So, Huygens principle is concerned with the propagation of information in space time. More precisely, Huygens principle is concerned with solutions of the Cauchy problem for homogeneous wave equation. How the support of the Cauchy data propagates with time by the dynamics of the wave equation? An explanation on this restriction will be given later on in this lecture, which is towards the end of this lecture.

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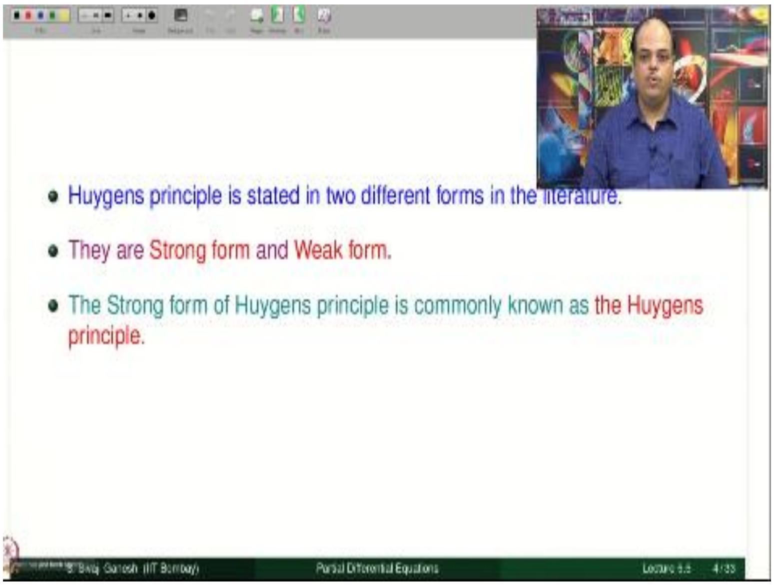

Huygens principle is stated in 2 different forms in the literature. They are strong form and weak form. The strong form of Huygens principle is commonly known as the Huygens principle. So, when somebody is saying Huygens principle, it means, it is a strong form of the Huygens principle.

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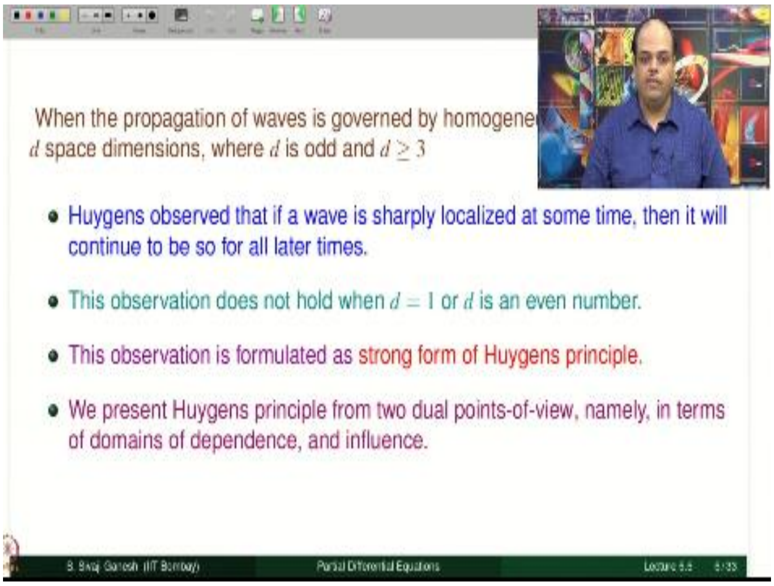

So, when the propagation of waves is governed by homogeneous wave equation in d space dimensions, where d is odd and d greater than or equal to 3, Huygens observed that if a wave is sharply localized at some time, then it will continue to be so for all later times. This observation does not hold when d = 1 or, d is an even number. This observation is formulated as strong form of Huygens principle. We present Huygens principle from 2 dual points of view namely in terms of domains of dependence and influence.

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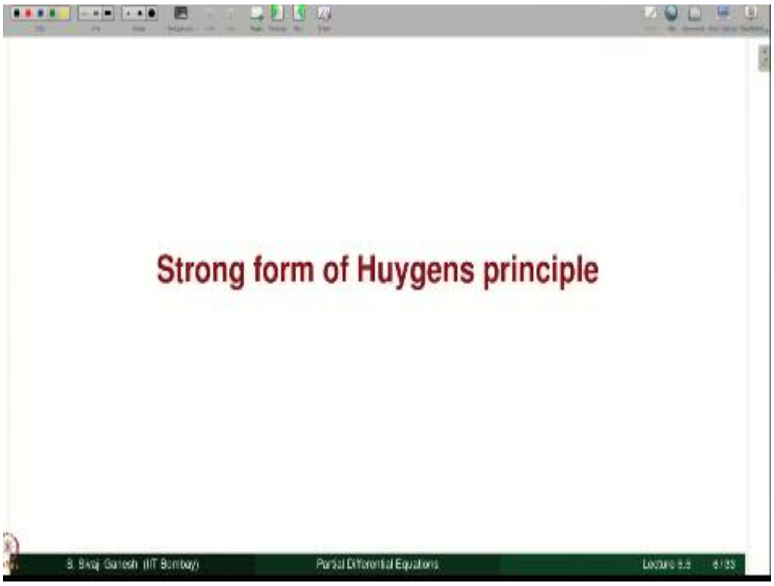

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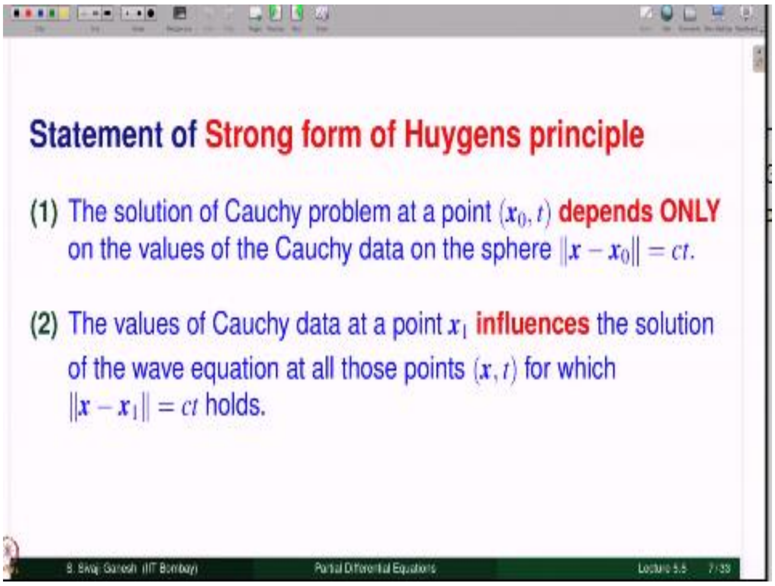

Strong form of Huygens principle: The statement point 1: the solution of Cauchy problem at a point x 0, t depends only on the values of the Cauchy data on the sphere norm x – x 0 = ct. The values of Cauchy data at a point x 1 influences the solution of the wave equation at all those points x t for which norm x – x 1 = ct holds.

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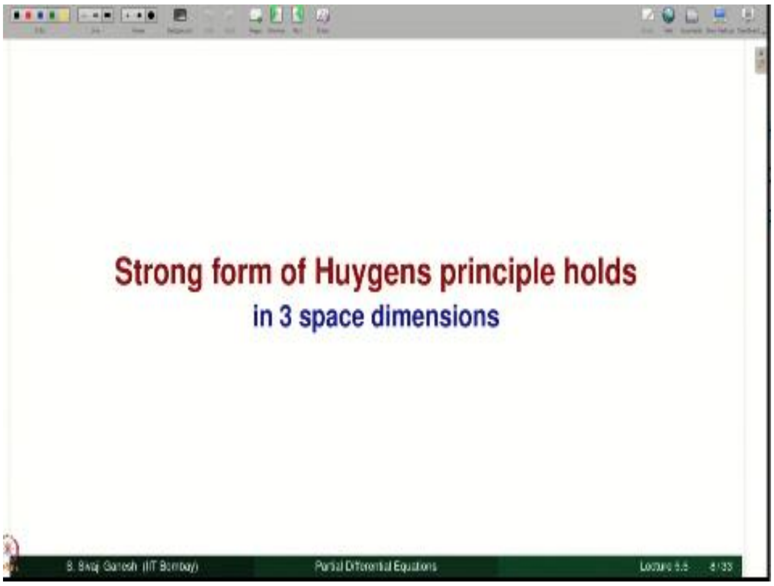

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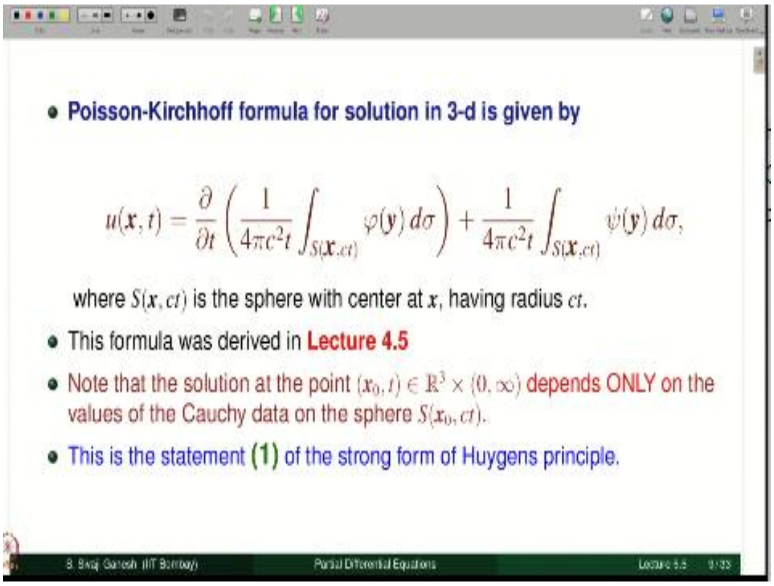

Strong form of Huygens principle holds in 3 space dimensions. Recall the Poisson-Kirchhoff formula for the solution in 3D which is given by this formula where S of x, ct that is the domain of integration is a sphere with centre at x having radius ct. This formula, we have derived in lecture 4.5. Note that the solution at a point x 0, t that is put x = x 0, it depends on the values of phi and psi only on S of x 0, ct that means those y which are a distance of ct from the point x 0 when we are looking at u of x 0, t. This is a statement one of the strong form of Huygens principle.

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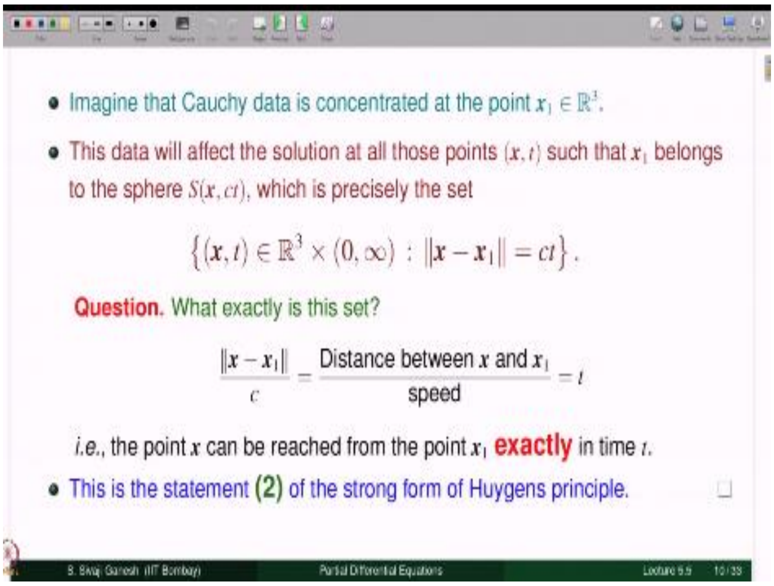

Imagine that Cauchy data is concentrated at the point x 1 in R 3. This data will affect the solution at all those points x t such that x 1 belongs to the sphere S of x, ct, which is a consequence of Poisson-Kirchhoff formula which is precisely the set, set of all x t in R 3 cross 0 infinity such that norm x – x 1 = ct. What exactly is this set? Let us look at this quantity, norm x – x 1 by c. In this norm x – x 1 is what?

Distance between x and x 1 and c is the speed. So, distance by speed is time. So, that is equal to t. So, that is what happens for the point in this set. So, this set consists of those points x, t such that x can be reached from the x 1 in time t with speed c that is the point x can be reached for the point x 1 exactly in time t exactly means sharp as we normally use, sharp in time t, nothing less, nothing more. This is the statement 2 of the strong form of Huygens principle.

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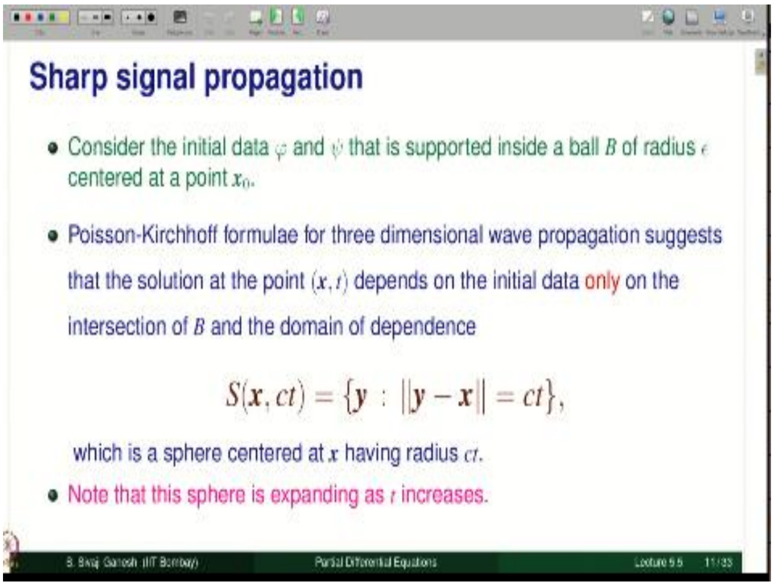

Let us turn our attention to sharp signal propagation. What does this mean? Consider the initial data phi and psi that is supported inside a ball of radius epsilon centred at a point x 0 a small ball. Poisson-Kirchhoff formula for 3D wave propagation suggests that the solution at the point x t depends on the initial data only on the intersection of this B where the outside which Cauchy data is 0 on this sphere because the solution u of x t is given in terms of integrals on the sphere.

So, therefore, the sphere must intersect the support of phi and psi which is contained in B of 0 epsilon. Otherwise, solution will be 0. Note that this sphere is expanding as t increases. As t increases, sphere is expanding.

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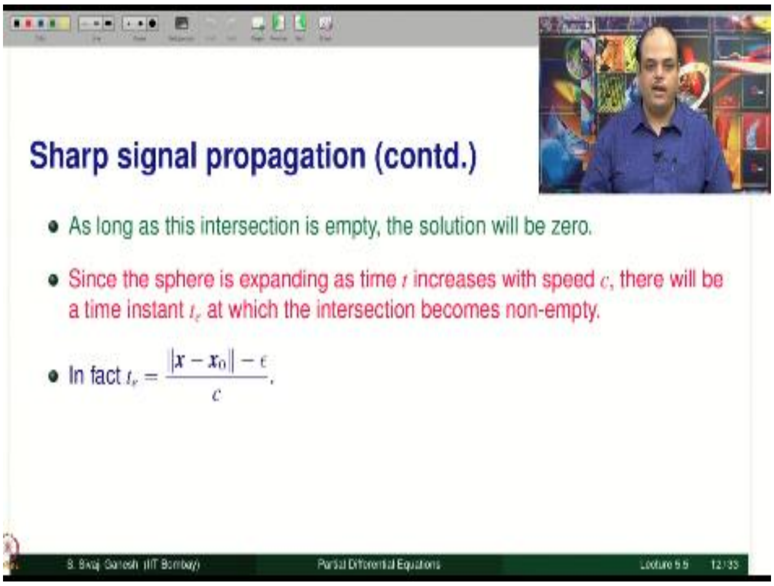

As long as this intersection is empty, solution will be 0. Since, the sphere is expanding as time t increases with speed c, there will be a time instant t e at which the intersection becomes non-empty. In fact, t can be given expression norm x – x 0 – epsilon by c.

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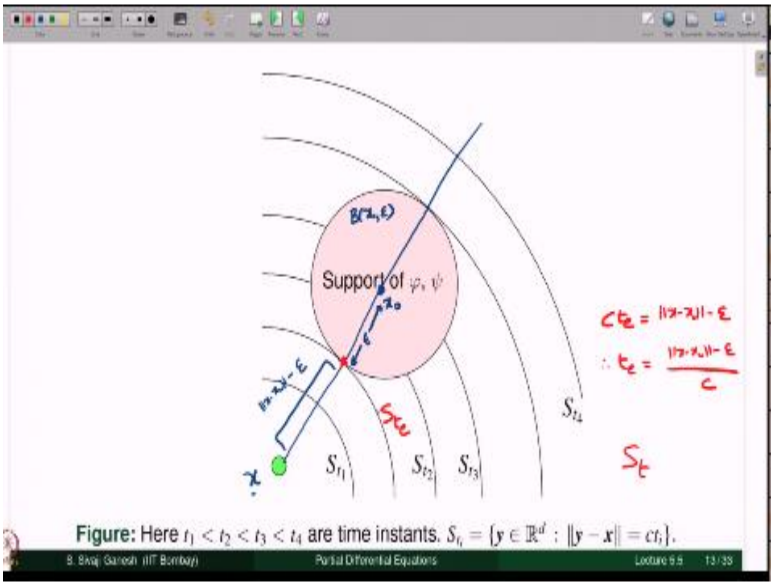

So, this is the ball where the support of the Cauchy data lies in B of x 0, epsilon. So, let me draw this line. This is the centre x 0. Let us call this point as x and this distance is epsilon. Therefore, this distance is norm x – x 0 – epsilon; this point lies on S t e. What is t e? Let us compute. c t e is equal to norm x – x 0 – epsilon. Therefore, t e is equal to norm x – x 0 – epsilon by c.

Recall S t e is the sphere with centre at x and the radius as c t e and it touches the support of phi, psi which is in the inside is ball x 0, epsilon at this point. So, therefore, this point lies on

S t e. So, as you observe these spheres S t which are centred at the point x and having radius ct as t increases t = t e is the precise time at which the sphere intersects B of x 0, epsilon for the first time.

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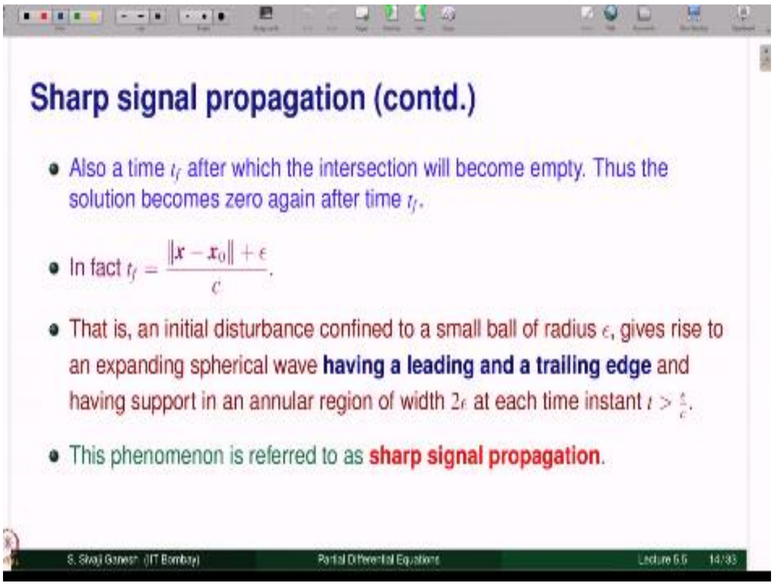

Also, a time t f after which the; intersection will become empty. When does it become empty? In this picture, after this so, after this let us call t x f f of final after this t f if you draw any other sphere for which t is bigger than t f, it is going to be like that. So, it does not intersect this bond. Thus, the solution becomes 0 again after time t f. In fact, t f = norm x – x 0 + epsilon by c that is an initial disturbance confined to a small ball of radius epsilon gives rise to an expanding spherical wave having a leading and a trailing edge and having support in an annular region of width 2 epsilon.

Two, epsilon is precisely the diameter of the ball in which the Cauchy data is supported at each time instant t bigger than epsilon by c. In other words, ct is bigger than epsilon for such times, this happens. This phenomenon is referred to as short signal propagation.

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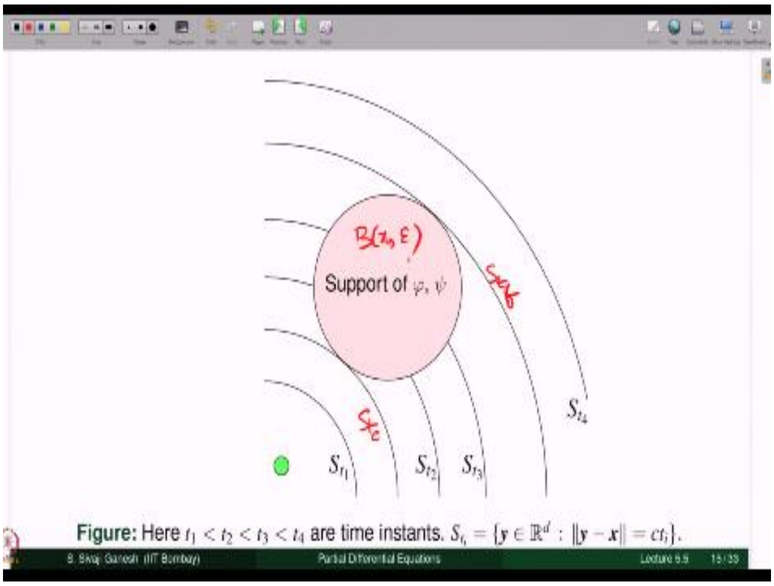

So, this is S t e and this is the S t f. For t less than t e, S t does not intersect the support B of x 0, epsilon and for t bigger than t f, once again S t does not intersect B of x 0, epsilon.

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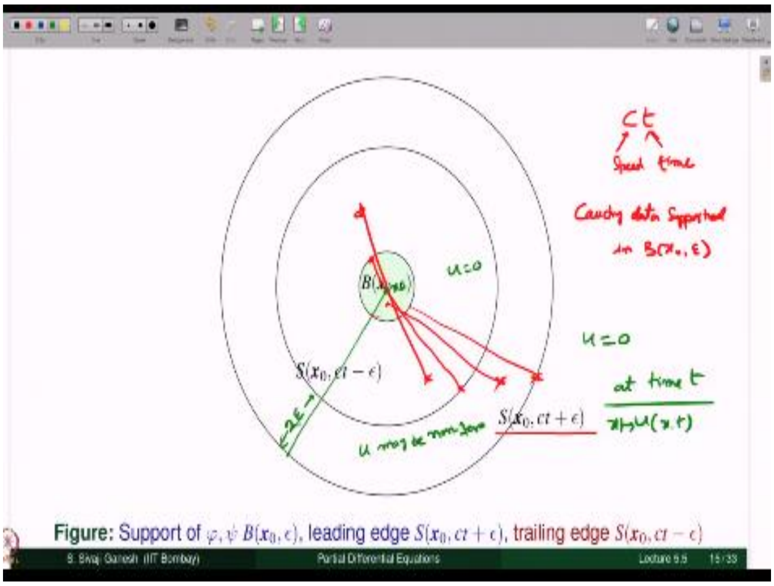

First, to understand what the ct means. c is the speed; t is the time. So, if you are moving at speed c, you will travel a distance of ct in time t. So, ct is the distance travelled in time t when you are moving with speed c. So, recall that the Cauchy data is supported in B of x 0, epsilon. So, what does this sphere S of x 0, ct plus epsilon represent? It consists of those points which can be reached from this ball in time t, exactly a time t, which can just touch here.

Of course, if they do all the points inside this even these points will also do. In time t, they will travel maybe at this point, it will reach. So, this is the point which will reach precisely the other end point. So, if you take a point here in time t, it would go away, it will go out. So, therefore, solution will be 0 here. Let us call this as the centre x 0, this region for draw a radial like this, this width is actually 2 epsilon.

So, only if you are inside this angular region, u may be nonzero. We would like to say, it is nonzero but it could be 0, but definitely outside, this u is 0. If you are here, u is 0. If you are here, u is 0 at time t. That is support of x going to u of x, t at a time t is inside the angular region of let us say, width 2 epsilon.

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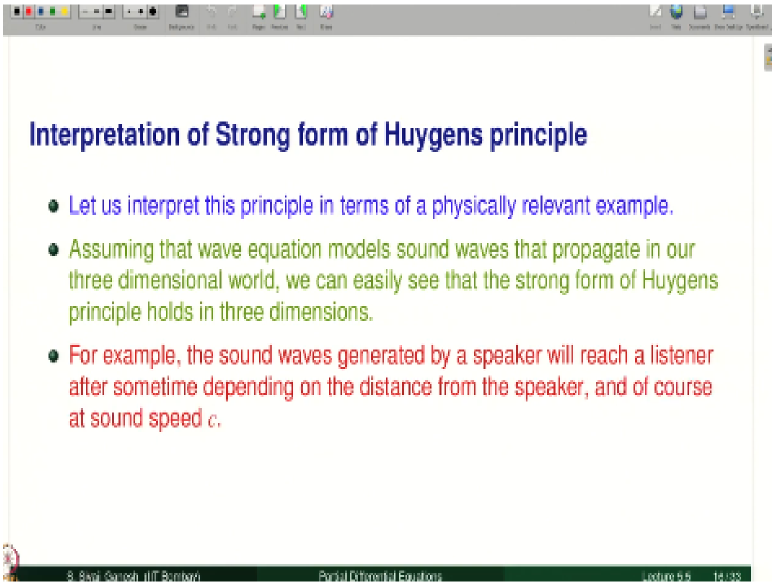

So, interpretation of strong form of Huygens principle: let us interpret this principle in terms of a physically relevant example. Assume that the wave equation models on waves that propagate in our 3 dimensional world. We can easily see that the strong form of Huygens principle holds in 3 dimensions. For example, the sound waves generated by a speaker will reach the listener after some time depending on the distance from the speaker and of course, at the speed, which is the sound speed, speed of sound in that medium.

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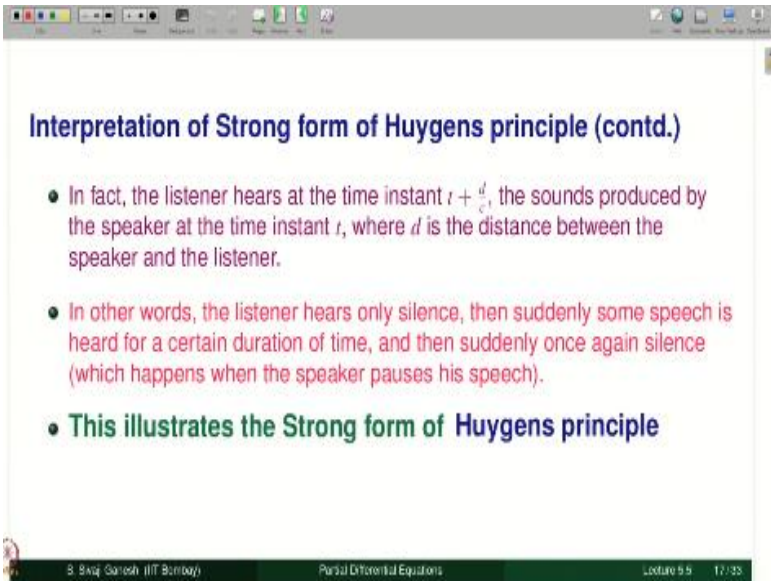

In fact, the listener hears at the time instant t + d by c, the sounds produced by the speaker at the time instant to t and d is the distance between the speaker and the listener. In other words, the listener hears only silence, then suddenly some speech you heard for a certain duration of time, the time for which the speaker is producing sound. And then suddenly once again silence which happens when the speaker pauses his speech or stops his speech. This illustrates a strong form of Huygens principle.

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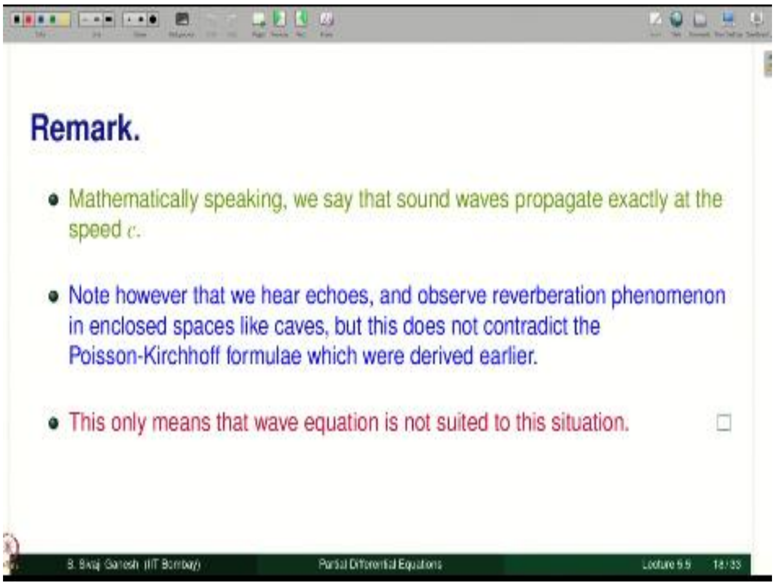

Mathematically speaking, we say that sound waves propagate exactly at the speed c. Note however, that we hear echoes and observe reverberation phenomena in enclosed spaces like caves, but this does not candidate the Poisson-Kirchhoff formulae which were derived earlier. They were derived for wave equation in the full space, not with boundaries. This only means that wave equation not suited to this situation we have to model differently, not model it as a variable position on R d or R 3.

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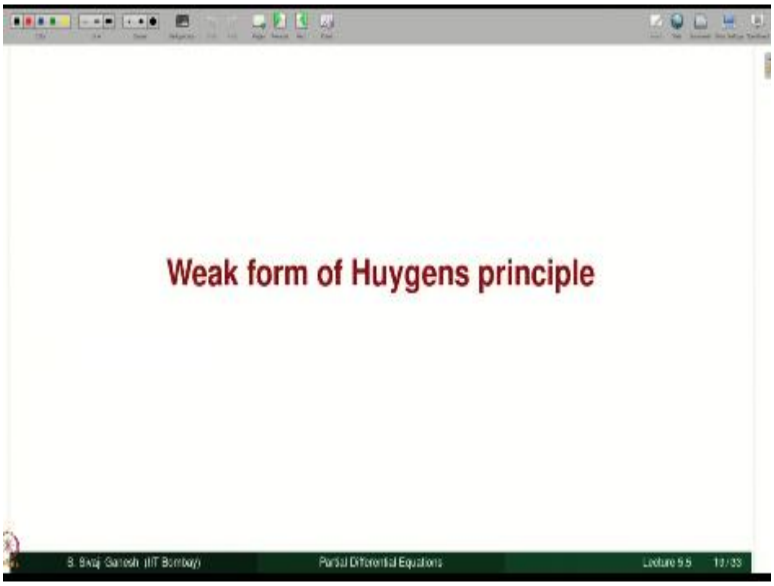

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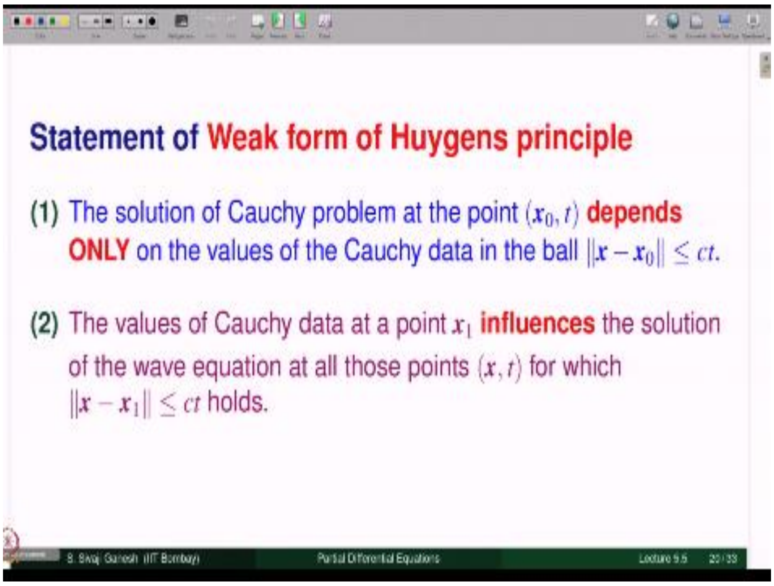

Let us look at the weak form of Huygens principle. The solution of the Cauchy problem at the point x 0 t depends only on the values of the Cauchy data up to this, it is common with the strong form also in the ball. In the strong form, it was the sphere. In a weak form, we allow the ball; norm x – x 0 is Less than or equal to ct. That is a difference. The values of Cauchy data at a point x 1 influences the solution of the wave equation at all those points x t for which norm x – x 1 less than or equal to ct holds.

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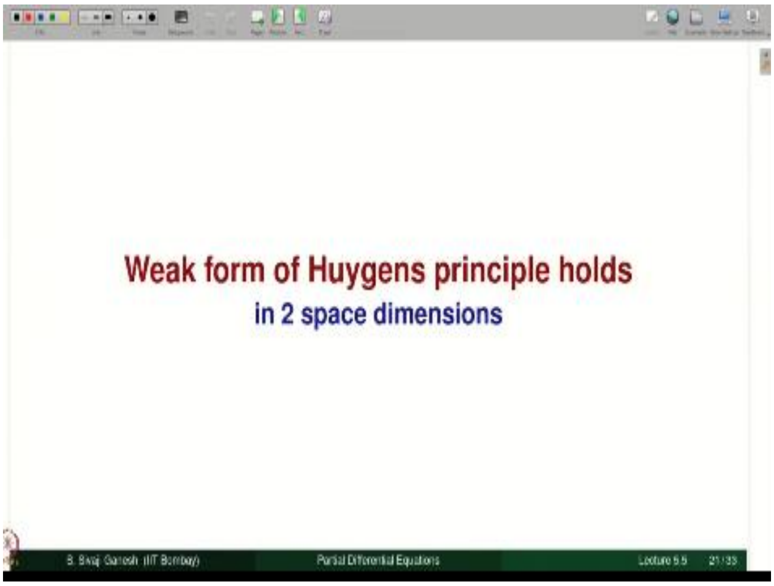

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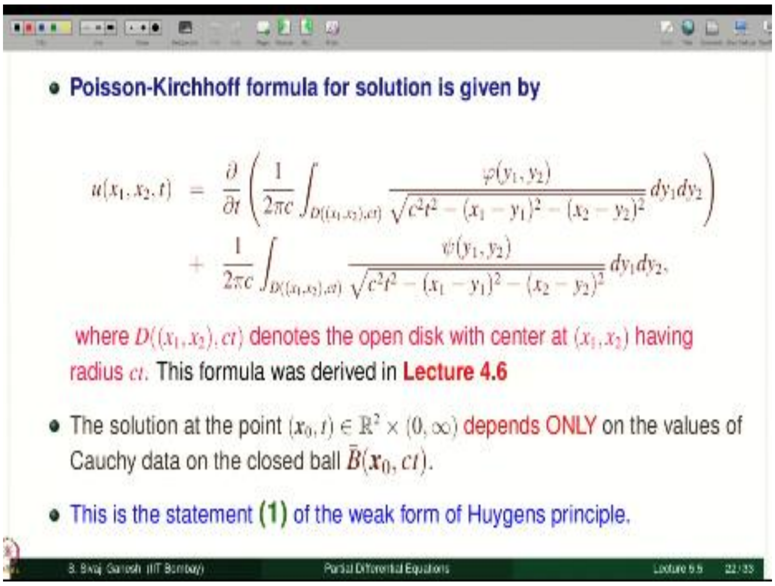

So, weak form of Huygens principle holds in 2 space dimensions. These are Poisson-Kirchhoff formula in 2D which, if you want to compute the value of u at a point x, t, x = x 1 x 2 here, what you need is the values of phi and psi on this disk for all the y which belong to this that means what distance between y and x is less than or equal to ct, less than ct but nothing wrong in saying less than or equal to ct also.

So, because integrals, it does not matter what happens on the boundary? So, this formula was derived in lecture 4.6. And as I already pointed out, if you want to find a solution at a point x 0, t that is x 1 x 2 is now called x 0. It depends only on the values of Cauchy data on this d of x 0, ct. I should have used the d because this is 2 dimension. In any general dimension we use of course, ball nothing wrong in using ball but it might confuse.

Do not get confused in 2D ball means disk. So, this is a closed disk centre x 0 radius ct. This is the statement one or the weak form of Huygens principle.

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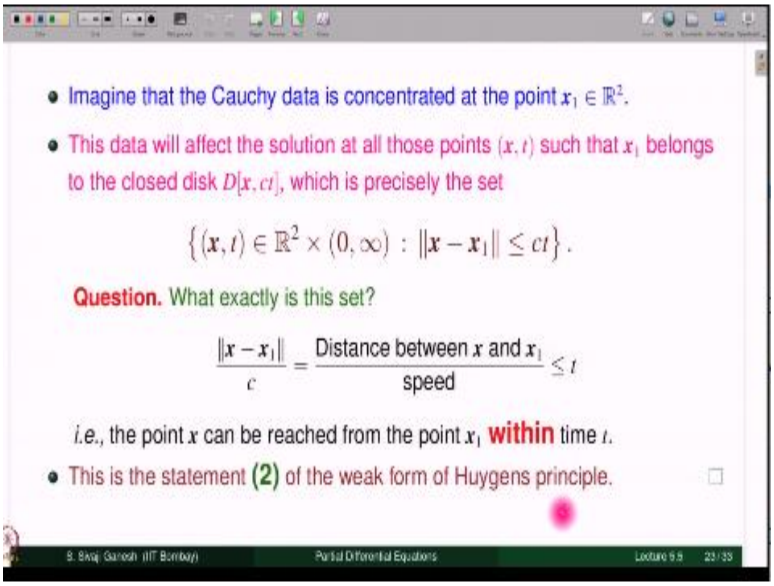

Imagine that the Cauchy data is concentrated at a point x 1 in R 2. This data will affect the solution at all those points x t such that x 1 belongs to the closed disk D x, ct centre x radius ct, which is precisely the given by the set. This is coming from Poisson-Kirchhoff formula once again. What exactly is the set? Let us look at norm x – x 1 by c; norm x – x 1 is the distance between x and x 1, c is the speed.

Therefore, what you get is time and time is less than or equal to t in this case, it means this set consists of those x t such that x can be reached from x 1 in a time which is less than or equal to t. Compare this with a situation in 3 dimensions where this time was exactly in time t. Now, within time t that is a difference that is the point x can be reached from the point x 1 within time t and this is the statement 2 of the weak form of Huygens principle.

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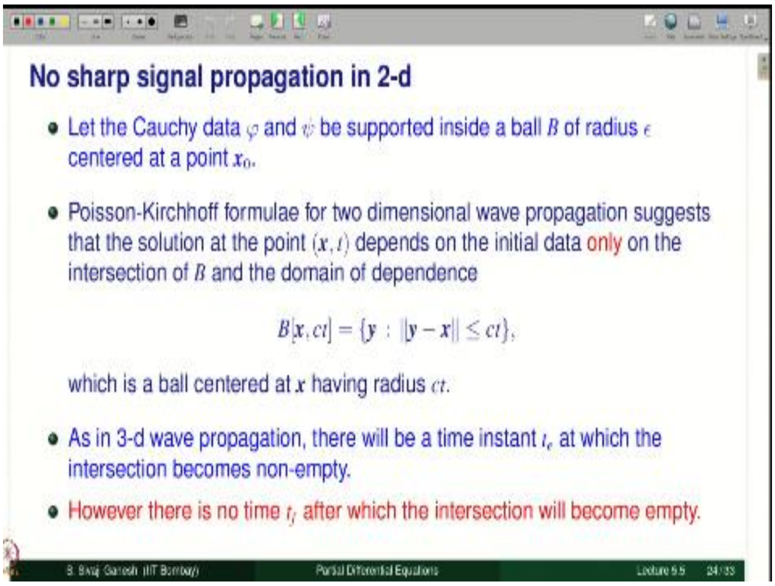

No sharp signal propagation in 2D. Let the Cauchy data be supported inside a ball of radius epsilon centred x at the point x 0 as before. Poisson-Kirchhoff formula for 2D wave propagation suggests that the solution at a point x t depends on the initial data only on the intersection of B this ball and this ball which is the domain of dependence for u of x t. As in 3D wave propagation, there will be a time instant at which the intersection becomes non-empty. However, there is no time after which the intersection will become empty.

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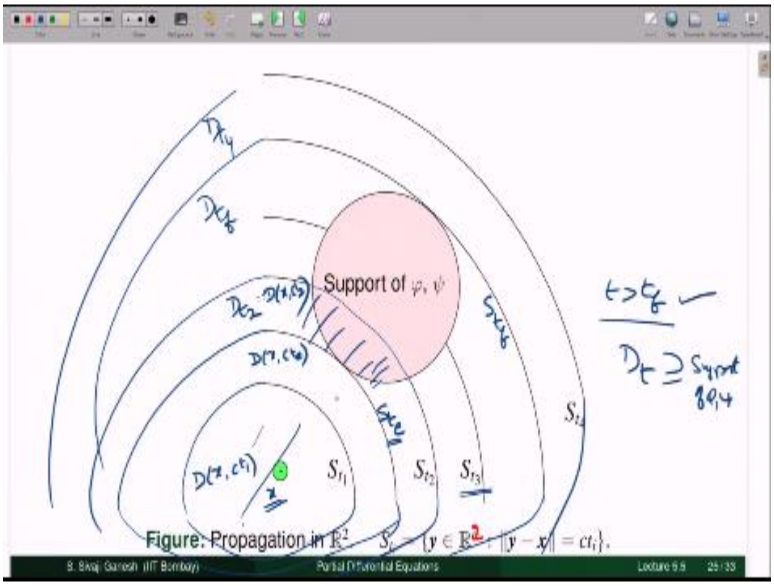

So, let us illustrate in a picture I have demonstrated only S t 1. So, let us draw the disk here that is what it comes in Poisson-Kirchhoff formula. Disk with centre x, let us call this point as x; x, radius is ct 1, if you are writing this picture for S t 1, the disk is the inside part. This disk clearly does not intersect the support. As t increases, the disk size is increasing. It is expanding this. So, let us look at this disk.

I am not able to draw the complete disk here because obviously some picture is not fitting. Now, this is let us call the S t e because this disk now of radius ct e actually starts intersecting the support and hence, it can start to pick up nonzero signals from the initial data. Now, if you look at S t 2 corresponding thing will be let us call D t 2. For simple notation D t 2 is nothing but disk of radius ct 2 centred at x.

So, that intercepts because in this portion, it considerable portion, it intersects. So, therefore, you expect nonzero solution and same is the case with the D t 3 which corresponds to the boundaries S t 3. Till this time, this is the let us call it S t f. Of course, notation is bad notation here I will explain why that is so. These are disk; D t f goes but then if t is bigger than t f which is depicted in the picture with the t 4, the disk is going like that D t 4; even D t 4 intersect is support which is not a situation in 3 dimensions.

In 3 dimensions, S t f is what that mattered and the S t f is the last time t f at which it intersects the support. For t bigger than t f, S t never intersects the support, but in 2 dimensions, it is not S t what is important. It is a D t which plays the role and D t continues to intersect the support. In fact, if you take t bigger than t f, the corresponding D t will always contain the support from this picture, support of phi and psi.

So, therefore, there is a time up to which the signal from the initial data is not received at this point x namely, t after which he starts receiving after that, it may continue to receive forever. Therefore, there is a leading edge in this case, but there is no trailing edge. What we can expect is a kind of decaying trailing edge if you want. We will discuss this later.

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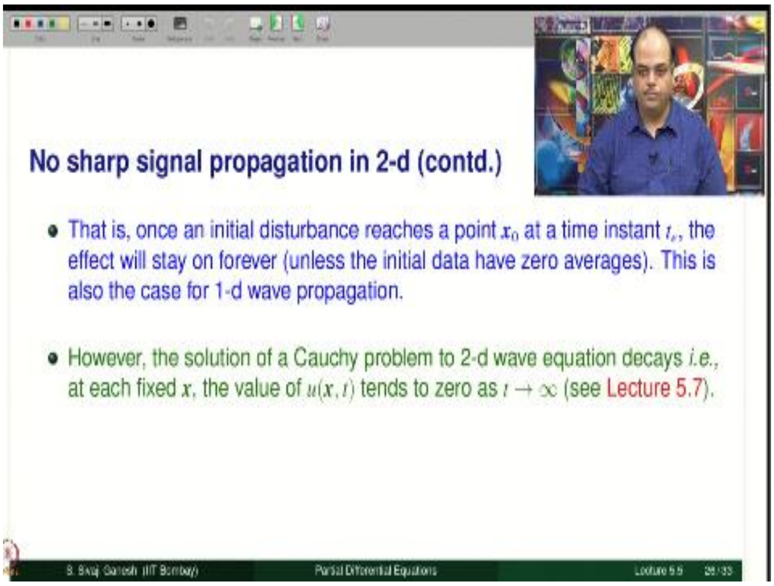

So, that is once an initial disturbance reaches at a point x 0 a time instant t e, it remains. The effect will stay on forever, unless the initial data have 0 averages; this, you ignore for the moment. This is also the case for 1D wave propagation, we have already seen that. However, the solution of a Cauchy problem to 2D wave equation, decays; it is true that it becomes nonzero but the effect slowly decays that means it goes to 0 slowly that is at each fixed x, the value of u x t tends to 0 as t goes to infinity. This we will see in lecture 5.7.

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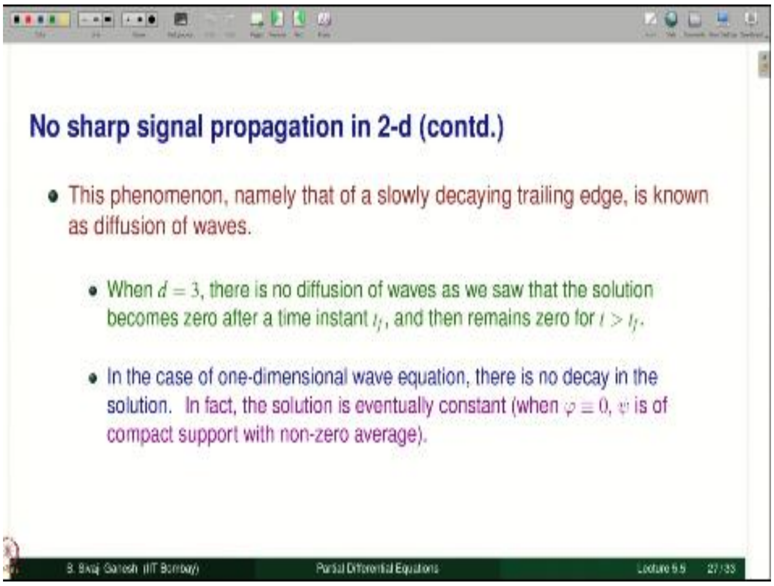

In 3D, it was 0 after some time; in 2d, it is not going to become 0 after some time, but it is going to decay as t goes to infinity which will be seen in lecture 5.7. So, this phenomenon namely that of a slowly decaying trailing edge is known as diffusion of waves. When d = 3, there is no diffusion as we saw that the solution becomes 0 after a time instant t f and then remains 0 for all t bigger than t f.

In the case of 1D wave equation, we saw this that there is no decay in the solution. In fact, the solution is eventually constant when phi is identical equals 0 on size of compact support with a nonzero average.

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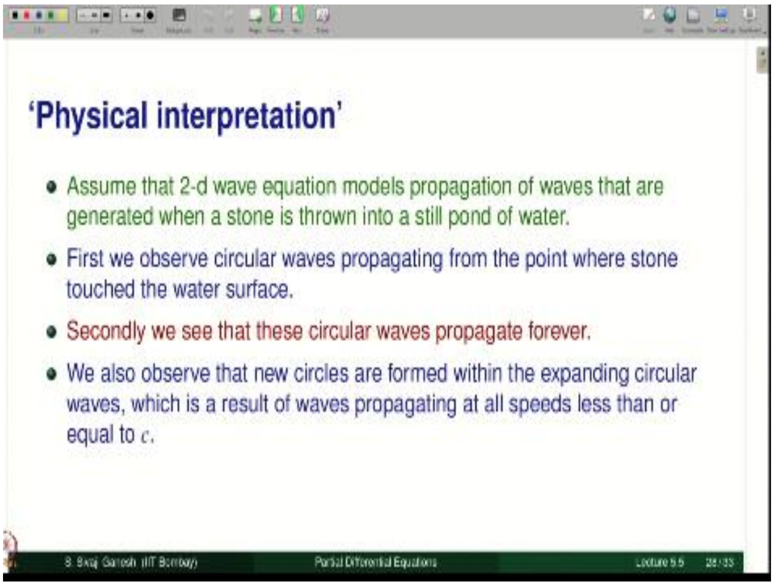

Let us do the physical interpretation of a weak form. Assume that the 2D wave equation models propagation of waves that are generated when a stone is thrown into a still pond of water. First, observed circular waves propagating from the point where stone touches the water surface. Secondly, we see that these circular waves propagate forever. We also observed that new circles are formed within the expanding circular waves, which is a result of waves propagating at all speeds less than or equal to c.

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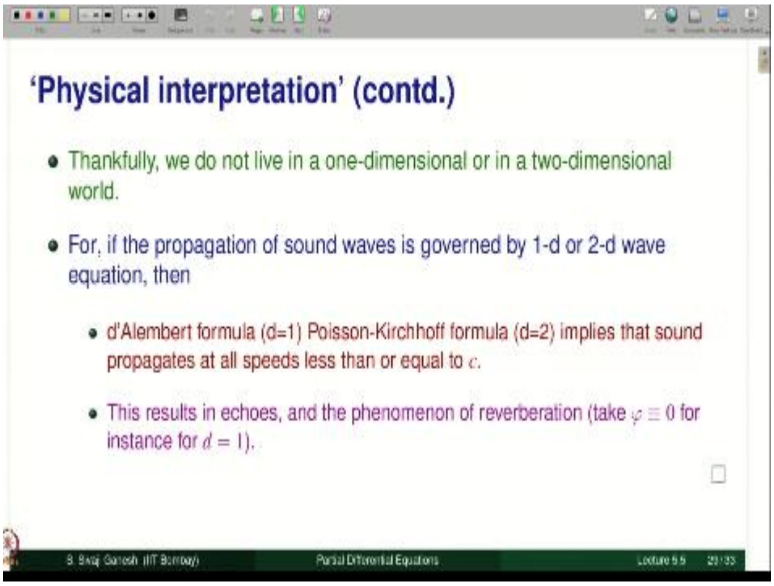

Thankfully, we do not live in a 1 dimensional or 2 dimensional world because the propagation of sound waves is governed by 1D or 2D wave equation. What will happen? D'Alembert formula, Poisson-Kirchhoff formula imply that sound propagates at all speeds less than or equal to c that results in echoes and phenomena of reverberation. So, we should be happy that sound waves do not follow 1D, 2D wave propagation.

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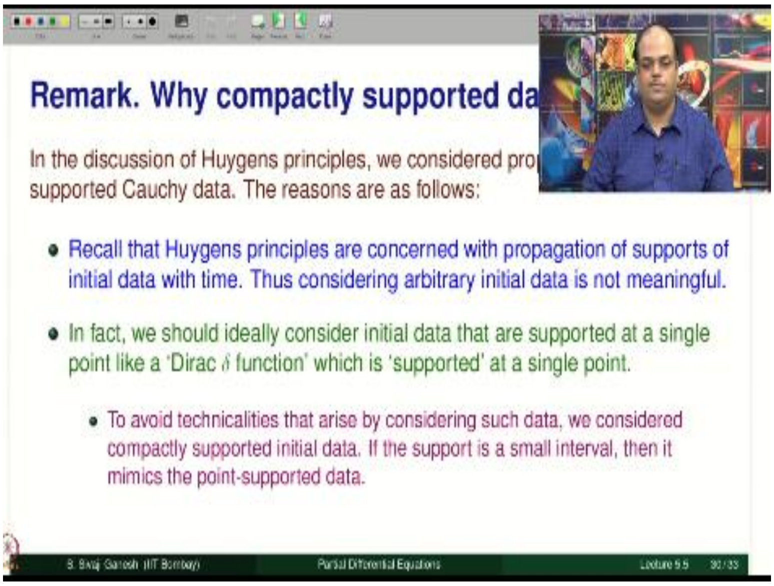

Remark: why compactly supported data? In the discussion of Huygens principles, we considered propagation of compactly supported Cauchy data. The reasons are as follows. What are the reasons? Recall the Huygens principle is concerned with propagation of supports of initial data with time, thus, considering arbitrary initial data is not meaningful. In fact, we should ideally consider initial data that is supported at a single point like what is called a derived delta function. It is just at one point.

To avoid technicalities that arise by considering such data, we look at a good approximation of it namely that data is supported in a small ball, in other words, a compact set. If the support is a small interval, then it mimics the point supported data or a small ball, then mimics point supported data in higher dimensions.

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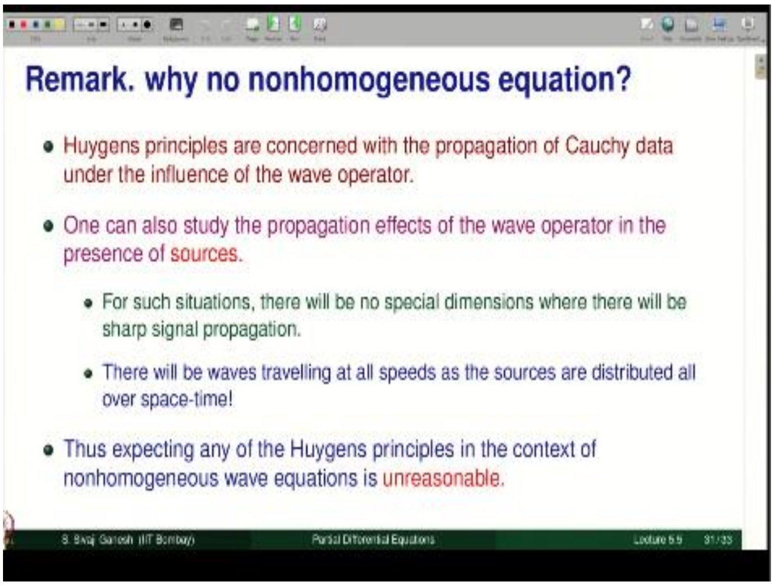

Now, why no non-homogeneous equation? We considered only homogeneous equations. Why? Huygens principles are concerned with propagation of Cauchy data under the influence of the wave operator. One can also study the propagation effects or the wave operator in the presence of sources that is when the non-homogeneous wave equation is considered, one can study.

For such situations, there will be no special dimensions where there will be sharp signal propagation. Why? There will be waves traveling at all speeds as the sources are distributed all over the space time f of x t, thus expecting any of the forms are the Huygens principles to hold in the context of non-homogeneous wave equations is unreasonable. So, that is why we do not consider non-homogeneous equations.

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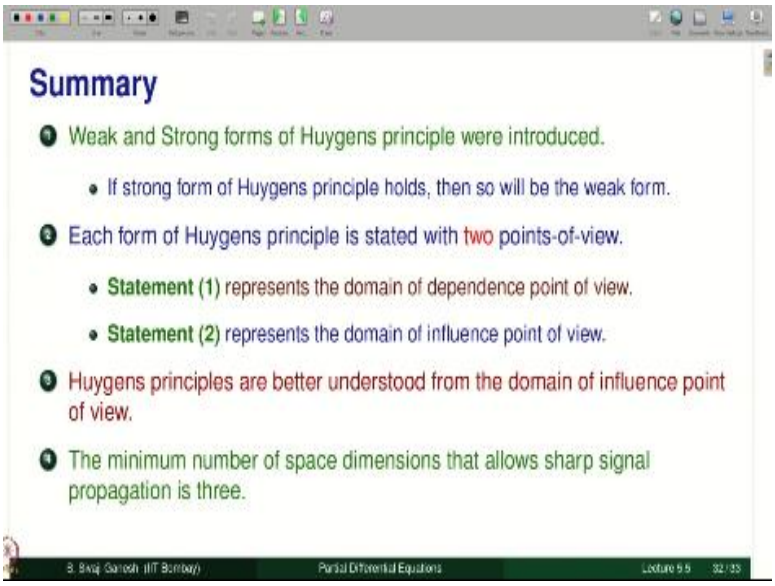

Let us summarize what we did in today's lecture. Weak and strong forms of Huygens principle were introduced. If strong form of Huygens principle holds, then so will be the weak form. Each form of Huygens principle is stated with 2 points of view. Statement one represents the domain of dependence point of view. Statement 2 represents the domain of influence point of view.

Huygens principles are better understood from the domain of influence point of view. The minimum number of space dimension that allows sharp signal propagation is 3. We have observed this. Thank you.

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