Partial Differential Equations
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Lecture – 2.3
First Order Partial Differential Equations
General Solutions to Linear and Semilinear Equations

In this lecture and the next lecture, we are going to discuss how to solve first order partial differential equations, in sense that of how to obtain general solutions. So, in today's lecture, we will discuss about general solutions to linear and semilinear equations, because it is easier. And we see how to extend these ideas to the case of Quasilinear equation in the next lecture.

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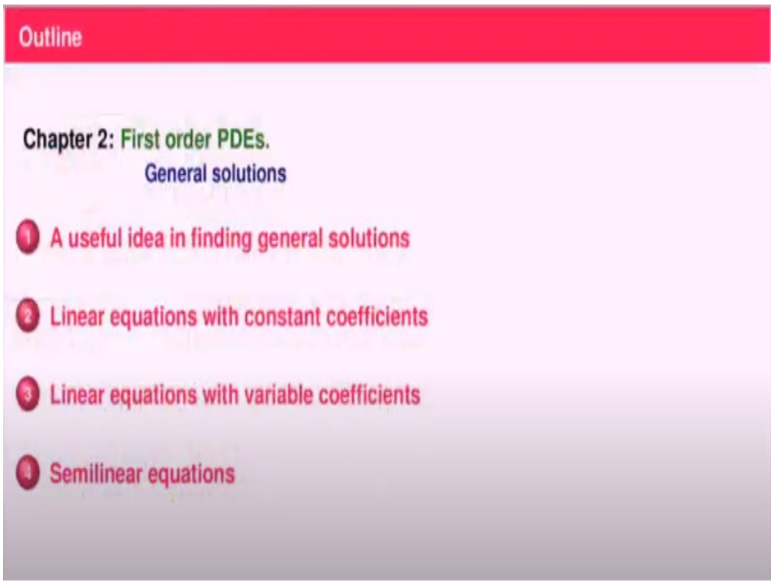

So, the outline of today's lecture is, first we discuss a useful idea in finding general solutions. And then we move on to linear equations with constant coefficients, because that is a simplest case. And from where we get an idea and try to solve linear equations with variable coefficients. Semilinear equations follows very similarly to linear equations, because the part where the first order partial derivatives appear in both linear and semilinear equation they look alike.

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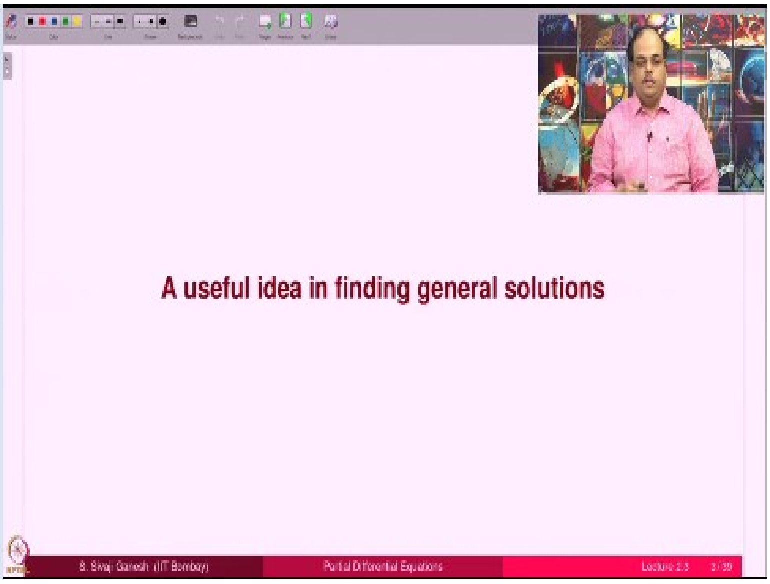

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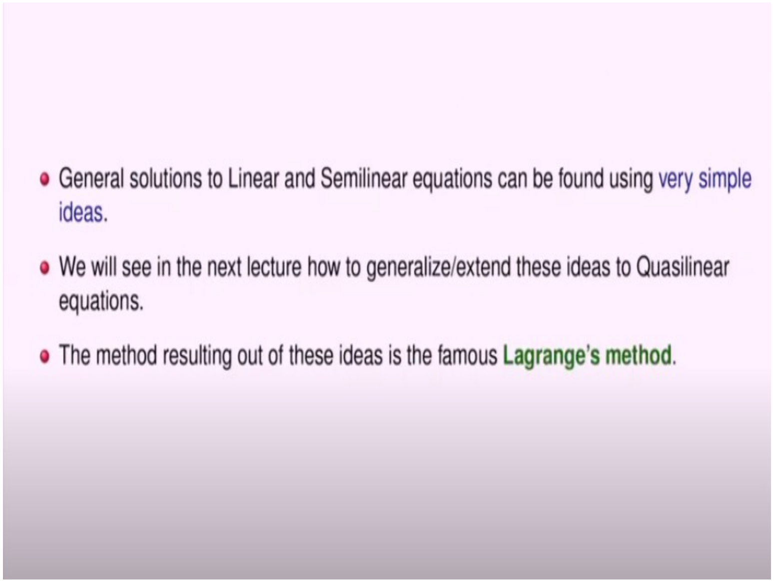

So, a useful idea in finding general solutions: So, General solutions to linear and semilinear equations can be found using very simple ideas. We will see in the next lecture, how to generalise or extend these ideas to the case of Quasilinear equations. The method resulting out of these ideas is a famous Lagrange's method for finding general solutions of Quasilinear equations.

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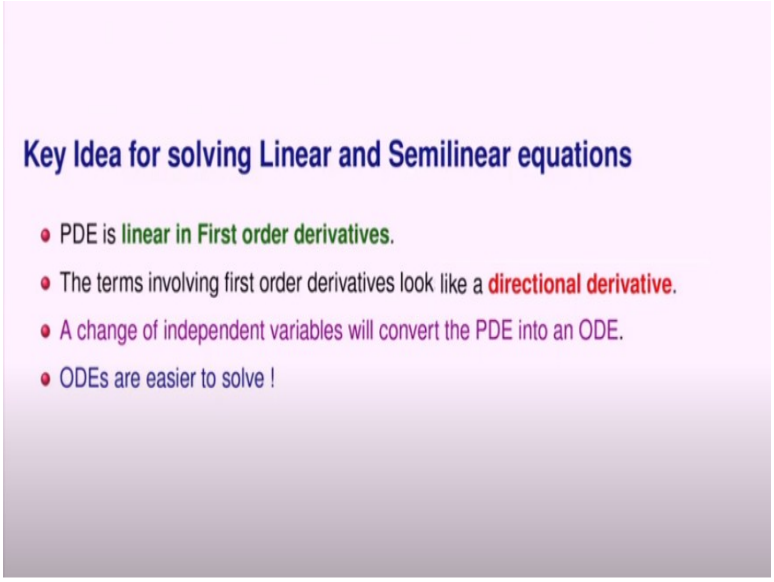

So, let us discuss the linear and semilinear equations now and what is the key idea behind this method. So, first of all observe that the partial differential equation is linear in first order derivatives. The terms involving first order derivatives look like a directional derivative. We will see more on this in the next slides. Now, what happens is change of independent variables will then convert our PDE into an ODE.

And we are experts in solving ODEs, thus ODEs are easier to solve. So, that is why this strategy we will follow.

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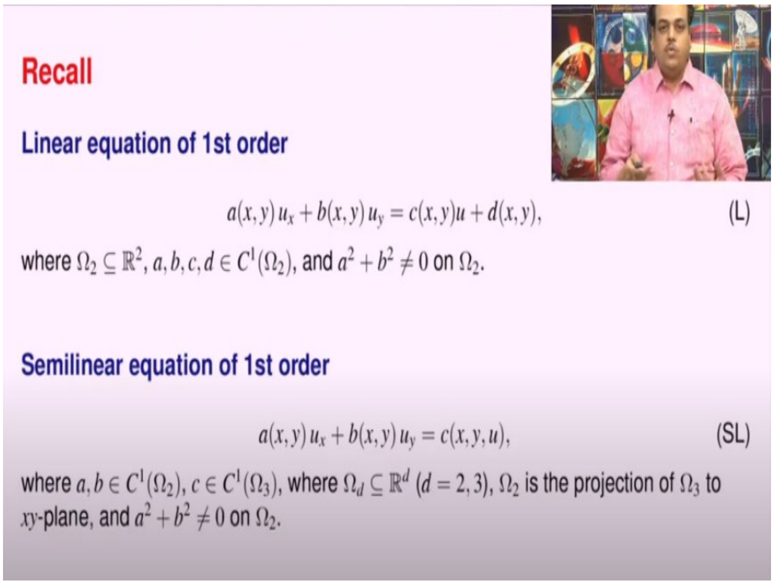

Recall linear equation of first order is f this form $a(x,y)u_x + b(x,y)u_y$ = c (x, y)u + d (x, y).

This we denote by L and the coefficients a b c d are $\mathbb{C}^1$ functions of omega 2, omega 2 is a subset of $\mathbb{R}^2$. Of course, there is another way to think of this as a special case of Quasilinear equations in which case a (x y z)will appear, but a (x y z) is independent of z. So, asking it that a (x y z) is $\mathbb{C}^1$ of omega 3 is same as asking a (x y z) is $\mathbb{C}^1$ of omega 2, because a does not depend on z.

And we do not want both these coefficients of the partial derivatives to vanish at the same time, in a sense at the same point in omega 2. So, at every point in omega 2, at least one of them should be non-zero. And then semilinear equation of first order looks like this $a(x,y)u_x$ + $b(x,y)u_y$ the left hand side is exactly same as that of L. Whereas, the right hand side, c (x y u) that is a semilinear equation.

And the assumptions natural are a b or $\mathbb{C}^1$omega 2 and c is $\mathbb{C}^1$ of omega 3. Of course, omega d s are subsets of $\mathbb{R}^d$; d equal to 1, 3. Omega 2 is a projection of omega 3 to the x y plane. And a squared plus b squared is not equal to 0 on the domain omega 2.

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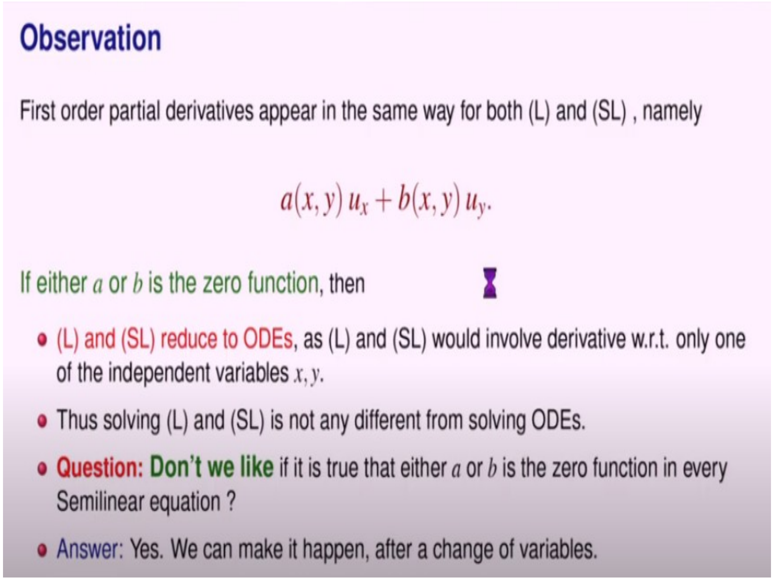

Let us see an observation that first order partial derivatives appear in the same way for both L and SL, we have already observed this. Namely, this is the part $a(x,y)u_x$ + $b(x,y)u_y$; this is the left hand side in both L and SL. If either a or b is a 0 function, what does it mean?

Imagine a 0, then this above expression may simply $b(x, y)u_y$ and the equation would look like $b(x, y)u_y$ equal to c (x ,y) u plus d in the case of L and v (x, y) u_x equal to c of (x ,y)u in the case of SL.

That means there is no derivative with respect to x will appear if a is 0. In other words, it is ODE, it reduced ODE. Therefore, solving L and SL in such a case is same as solving ODEs; is no different from solving ODEs. Now, we ask a question. Do not you like if it is true that either a or b is 0 function in every semilinear equation? Answer is yes, of course we like it. But we know that we cannot expect that, but we can make it happen. We can make it happen after a change of variables.

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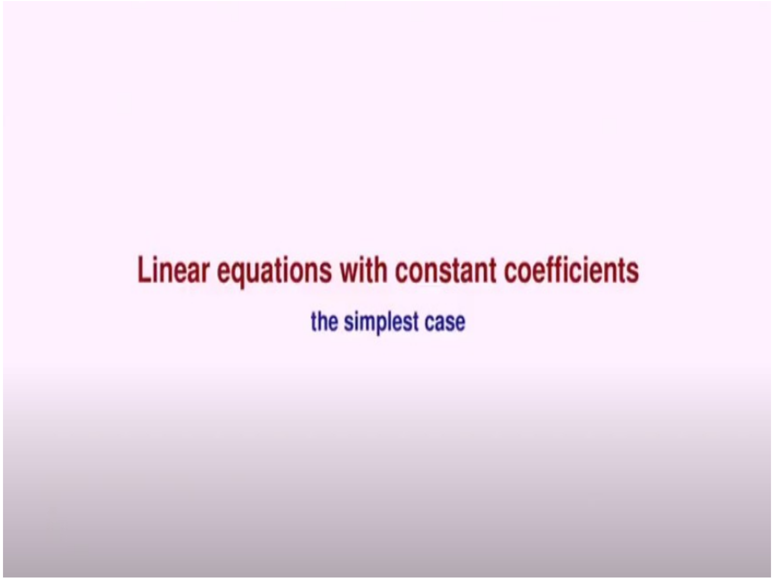

We will, we are going to do that now. First, let us understand the simplest case. It is a linear equation with consta coefficients.

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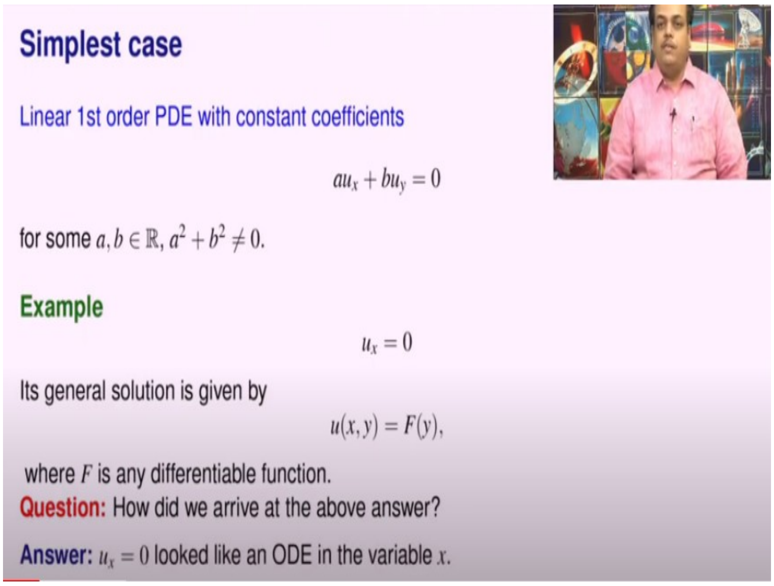

So, let us look at this equation here $au_x + bu_y$ = 0, coefficients are constant. So, a and b are real numbers. At least one of them is nonzero real numbers. If one of them is 0, it is a ODE. So, we do not want to discuss. So, you can as well assume both a b are nonzero. Of course, it is not needed for the purposes of our lecture. But you can imagine that both are nonzero numbers that is when it is not clear what you wanted, what you can do immediately. If any one of them is 0, it was ODE already solved.

So, we placed this assumption that one of these a b must be nonzero real number. Let us look at an example, of course, in this example, I made it, b = 0 and a =e 1. This is just an example. This will guide us in further proceedings. So, let us look at this example u_x equal to 0, its general solution is given by u(x, y) equal to F(y), where F is any differentiable function, because you need to differentiate this and check that derivative is 0.

So, derivative should exist that is reasonable to assume F is a differentiable function. Though, it does not matter here, even if F is just a function, not even continuous does not matter. Because when you write the partial derivatives expression with respect to x, all different coefficients are 0. So, how did we arrive at the above answer? It looked like an ODE in the variable x.

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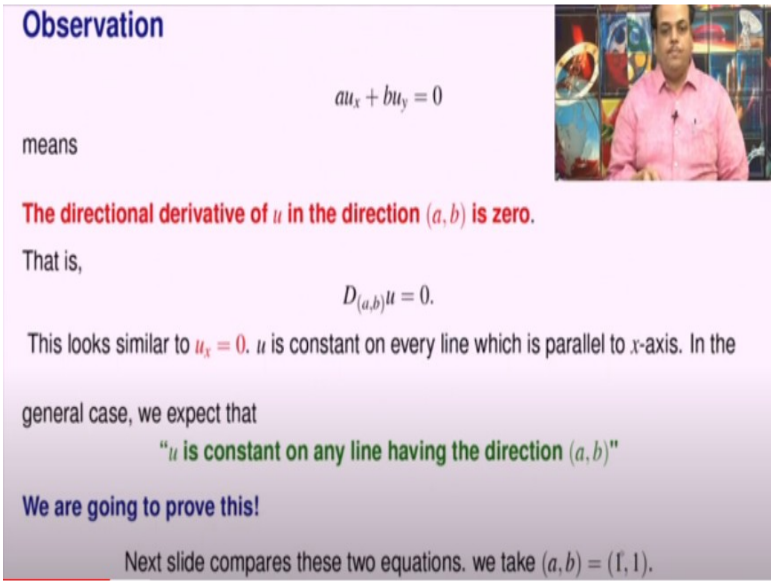

So, $au_x + bu_y$ =0 means the directional derivative of u in the direction a b is 0. In terms of the notations, it is like this $D_{(a,b)}u$ = 0. This looks similar to u_x = 0, u_x =0 is nothing but d is 1,0; the vector a b is 1,0. So, it is a direction derivative of u in the direction of 1,0 that is a partial derivative with respect to x. So, of course, $D_{(a,b)}$u = 0 is more general than u_x = 0.

But important thing is that it is a directional derivative, we know the direction derivative is like one variable derivative. It looks like one of the other variables is not there in the equation; exactly like u_x = 0, y is absent. So, in this case, we have to find out what are the variables? One of them is there in this equation, but, another one is not there. We need to identify. In this case, it is very simple; x is there, because the partial derivative of u with respect to x is there, y is not in the equation that is very obvious here.

Now, let us understand u_x = 0 more closely. u_x = 0, it means that u is constant on every line, which is parallel to x axis. Parallel to x axis means y equal to constant of the equation. So, solution we saw u (x, y) =F (y). So, if y is constant, u x y is constant. So, in general, we expect by analogy in this case, u is constant on every line is parallel to x axis. Now, in this case, I expect u is constant on any line having the direction a b.

In the case of u_x ,the direction is 1,0 right; parallel to x axis has the direction 1,0. So, therefore, I expect from this experience that solutions of $D(a,b)$u = 0 are constant on any line having the direction a b. We are going to prove this. So, in the next slide, we compare these 2 equations side by side. First, let us look at u_x = 0 and a, b we need to make some special numbers. So, a, b we take 1, 1.

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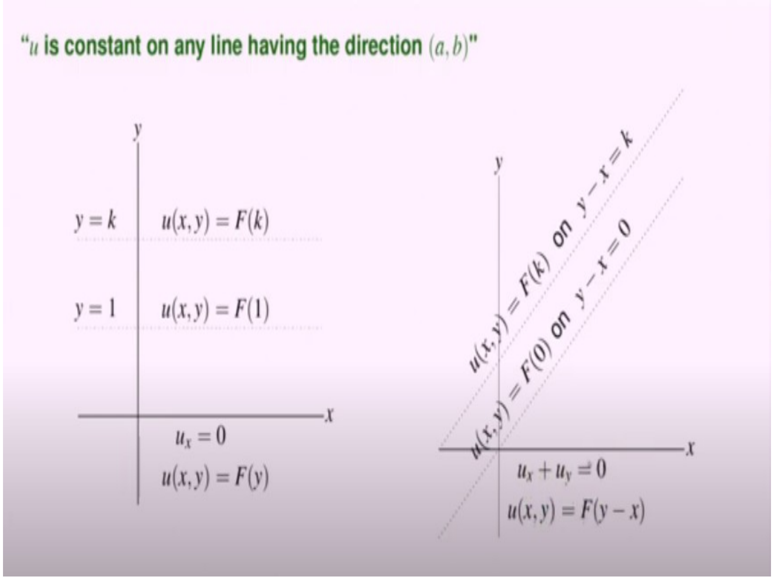

So, now u_x = 0, solution is F (y). So, whenever y = 1 that is a line parallel to x axis, it has a direction 1,0 and u is F of 1; u is constant. Now, if you take any other line, parallel to x axis that means any line having the direction 1,0 will look like y = k for some k. And u on that is F of k. So, it is constant on each line parallel to x axis. Now, in this case, $u_x + u_y$ = 0, this is having the direction 1,1; 1,1 is this direction, the line y = x has the direction 1,1.

So, u should be constant along each of these lines parallel to the line y equal to x. Any line parallel to y = x looks like y – x = k for some k. And u = F (y – x) . You can check by substituting in this equation that this u is indeed a solution to this equation. So, we have obtained a general solution in the case of au_x + bu_y =0. So, we need to find what that is. We got from a b = 1,1. If it is generally a b, what will these b? We will see that formula.

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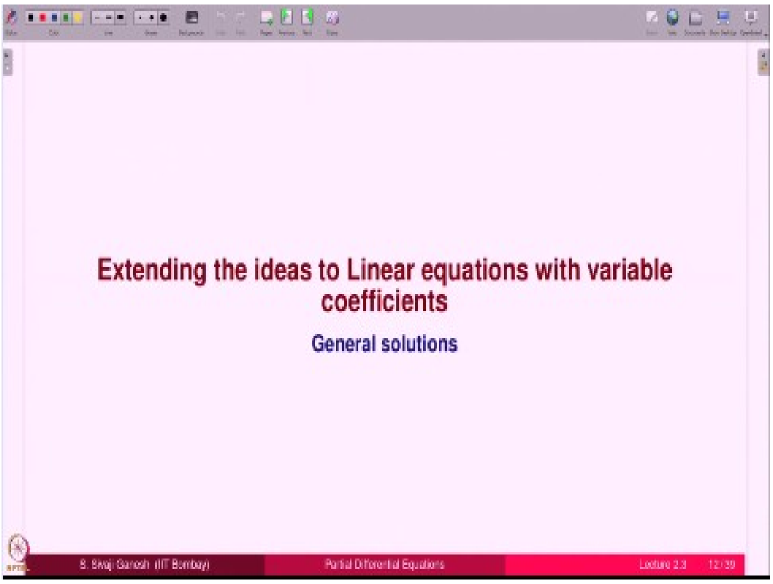

Now, how do I extend these ideas to linear equations, but with variable coefficient? How will we do that?

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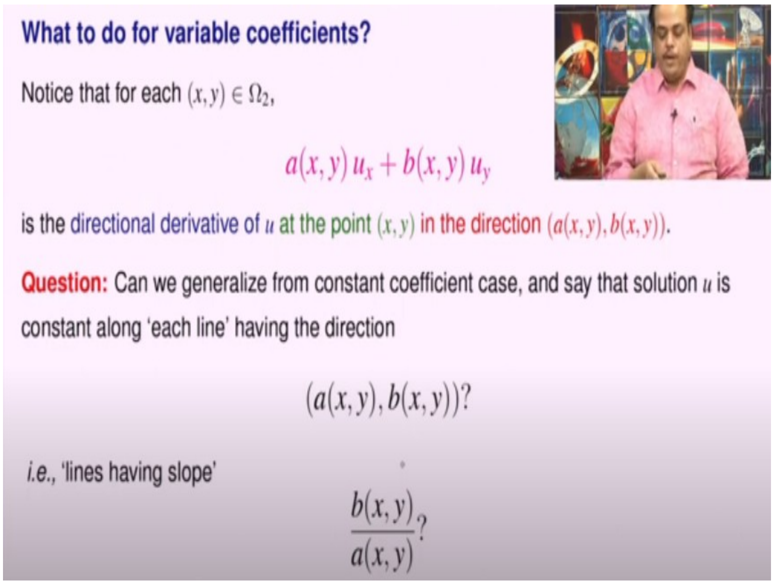

So, what to do? That is the first question. So, notice that for every point in omega 2, this $a(x,y)u_x + b(x,y)u_y$ is also a directional derivative of u at the point x y. But, the catch is that the direction depends on the point x y. It is not constant a comma b as in the constant coefficient case. Here, the direction is (a(x,y),b(x,y)) it varies from point to point. This is the difference. Otherwise, it is also directional divert.

Now, can we generalise from constant coefficient case? And say that u is constant along each line having this direction; we would like to say that. Of course, what it means that a line having this direction is questionable, because this changes from point to point right. In the constant coefficients case, it was just a comma b, it never depended on x y, but now, it depends that is why I put in quotes.

Can we expect such a result? Having this direction means this is the slope y coordinate x coordinate, b by a that is the slope of the line. What line? x y is varying; b by a will vary from point to point.

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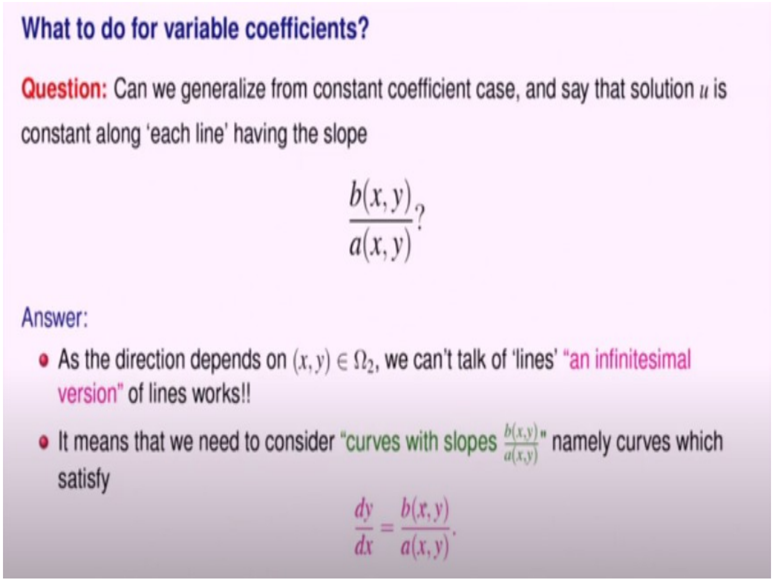

So, as the direction depends on the point answer to the above question is, as the direction depends on the point x y in omega 2, we cannot talk of lines. But an infinitesimal version of lines works. What is that? We are going to present that soon. It means that we need to consider curves with slopes b by a, we cannot say lines all the time. Okay, considered curves lines are also. After all curves having a constant slope that is why they are lines.

But here, we have to admit now more general things than lines. So, we admit curves with slopes b by a that is okay. Of course, one condition is that we have a in the denominator, so we must assume that a is not 0 in the domain. Now, let us assume that we will assume that. And we will also show that does not make much difference. If you have not assumed that a is not equal to 0 throughout. It is enough at a point that a is nonzero.

We know that every point x 0 y 0 in omega 2, one of them is nonzero a or b. So, we assume that a is nonzero and we can do the things. If a is zero, we will work with b. So, that can be done. So, this means what? Curves with slopes means you are look at solutions are this ODE.

This is the slope of the curve at a point x y and that I am saying should be b by a. So, if b and a are constants, the solutions are straight lines. So, now these are curves. So, $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$.

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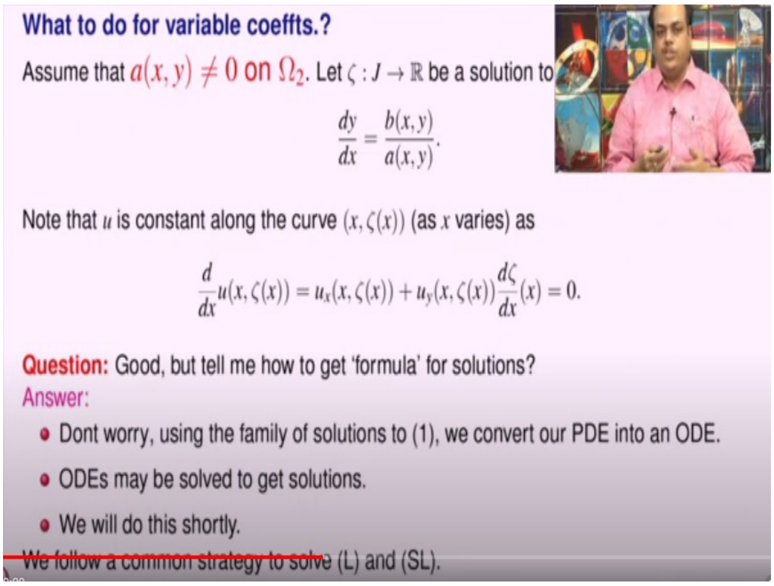

So, we have got hold of analogous things, 2 straight lines which are curves with slopes b by a. So, we assume that a is not equal to 0 on omega 2, we will come back to this discussion after we finish doing this analysis where we say that does not matter if you do not have to assume that it is nonzero throughout omega 2, because of certain things which will come into the analysis soon.

There is nothing that you gain by assuming the a is not 0 throughout omega 2. We will come back to that. So, let zeta be a solution to this. Define as some interval J in x, whenever this solution, one observation is that u is constant along this curve x , , x , ζ (x) as x varies describes a curve and u (x) , ζ (x) is constant whenever ζ is a solution to this ODE.

We can verify that so, $\frac{d}{dx}$ u (x, ζ (x)), we want to compute this. Now, x dependency is there in both the coordinates here as well as here both the components. So, first we will differentiate u with respect to x and derivative of x with respect to x is 1, then differentiate u with respect to y and differentiate ζ (x) with respect to x that is what the chain rule says. So, that will give us a zeta dash of x, d zeta by d x through this dependence, we get this.

So, as I told you chain rule will keep coming definitely throughout this lecture and the next one. But chain rule if you remove them, I think you can do PDE, we can we may also say that is one of the most fundamental results in differential calculus. It does not look like a big result to us, but it plays a role everywhere. So, question, good, but tell me how to get a formula for solutions.

You have observed that if you have a solution that will be constant along curves, which are solutions of this ODE fine. But from there, how do we get formula for the solution? That is a question. The answer is that do not worry using this family of solutions to 1 because it is a ODE. So, solutions will be a one parameter family. So, using that family, we convert our PDE into ODE, we convert our PDE into ODE.

And ODE may be solved to get solutions. So, now you are as good as your ability to solve ODE's. We will do this shortly. Of course, this assumes that you are able to solve this equation one get the family and then converting the PDE into ODE very simple and your ability to solve ODE. It hinges on that. But definitely there is an algorithm. So, we follow a common strategy to solve L and SL because as we observed the LHS in both the equations and L and SL is the same, both are the same.

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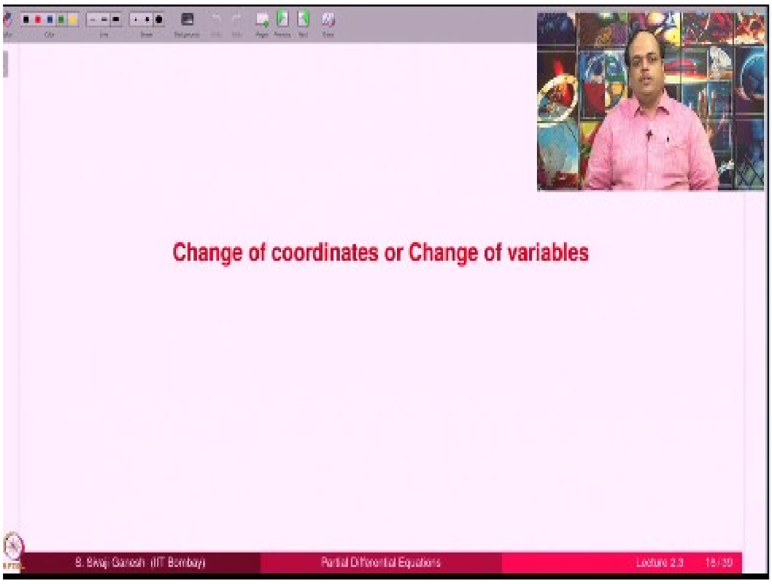

So, now we have to do what is called change of coordinates; sometimes called change of variables.

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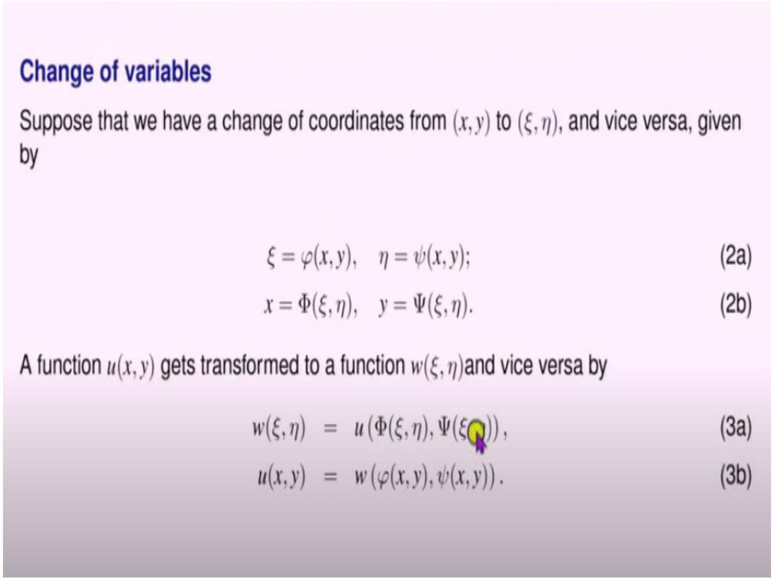

So, suppose that we have a change of coordinates from x y to xi eta and vice versa. Change of variables are always like you are going from x y description to xieta description and you should be able to come back only then it is useful. Otherwise, things are lost. So, change of variable always both sides and vice versa given by this. So, $\xi = \varphi$ (x y). $\eta = \psi$ (x ,y) and x = $\Phi(\xi, \eta)$; y = Ψ (ξ, η) .

It is very important to write this kind of setup whenever you do change the variables, so that you will not have confusion. This is another part. Chain rule will be used. Chain rule as such is easy, very easy, you can apply. But here, when you do change of coordinates that is where your real test of understand will come. And to be careful, better you always use these kind of notations.

Normally people write $\xi = \varphi$ (x y), $\eta = \psi$ (x ,y) avoid that that is the first message. So, this is the change of coordinates. Obviously, it means that a certain domain has this change of coordinates in R^2 . A function now, under this change of coordinates, u (x,y) gets transformed to a new function; call it new function new notation w (ξ ,η); do not call u (ξ , η)that will cause confusion; w (ξ ,η) and vice versa of course.

By this formula, u (x,y) but x = $\Phi(\xi, \eta)$; y = Ψ (ξ, η). So, you substitute you get a function of (ξ ,η); similarly, here. u (x , y) = equal to $w(\phi(x, y), \psi(x, y))$.

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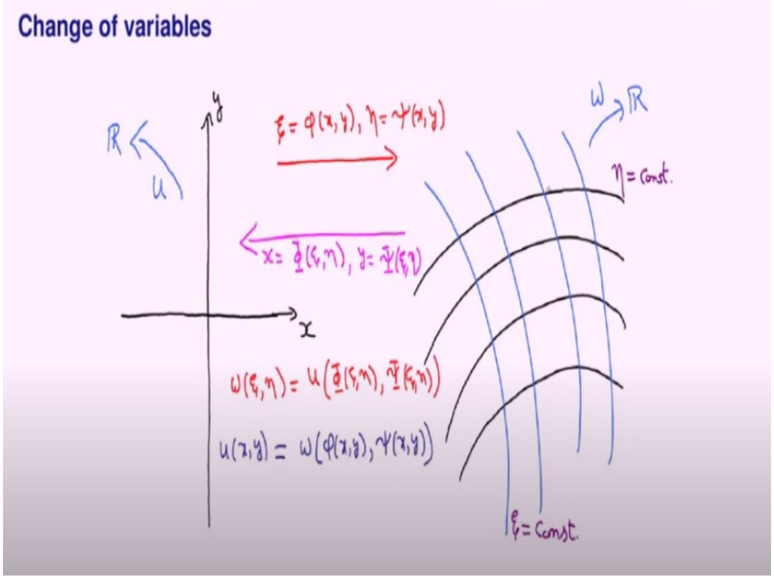

Now, we illustrate this in a picture. So, here we have the rectangular coordinates x y and we have gone from here to (ξ ,η) coordinate system through these functions φ ψ. ψ equal to constant are these blue curves; η equal to constant are this black curves. From here, we can come back where this transformation and if you have a function defined from here x y thing to R, you can define a function from (ξ ,η) description to R.

Basically, it is the address that you are changing with respect to the new coordinates. Earlier to describe a point in the plane you are giving x address and y address, x coordinate and y coordinate. Now for example, you are using a different coordinate system, if you want to describe this, this might be a η equal to some number $\mathbb{C}^1$, this may be ξequal to some number $\mathbb{C}^2$, so $\mathbb{C}^1$, $\mathbb{C}^2$will uniquely fix u.

For example, look at our globe, you can easily give coordinates of any location using the longitude and latitude numbers. Exactly like that. So, it is a different change, here it is a change of coordinates. So, you need to give address fully, full address.

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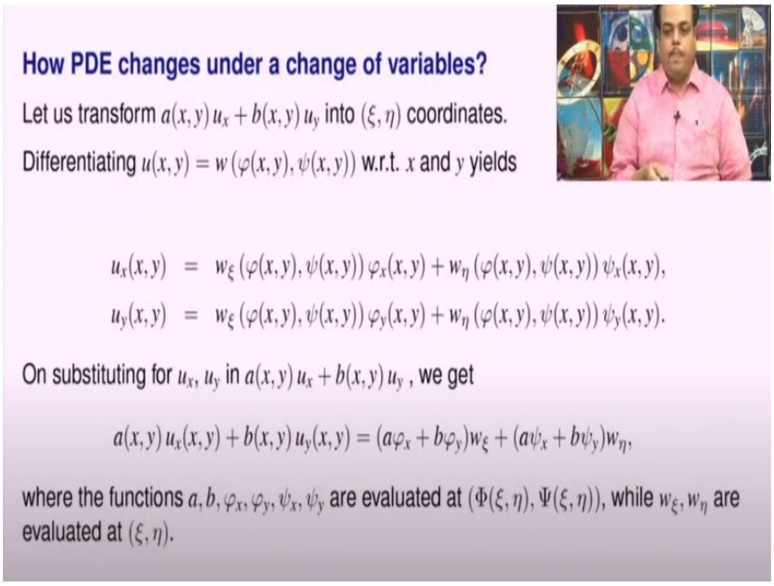

So, how PDE changes under change of variables? Very important, we need to do that. So, a$(x,y)u_x$ + b $(x,y)u_y$ is our LHS in both L and SL equations. Now, we will change it to (ξ ,η) coordinates, very easy. We have to look at the relations that we have between u and w and differentiate and find out what is u_x; what is u_y , substitute for u_x and u_y . Similarly, for x and y here, you substitute, then you will get the new expression.

So, $u_x(x,y)$ is derivative of w look x appears both in this location and this location. So differentiate w with respect the first variable which is ψat this point, $\varphi(x,y)$ $\psi(x,y)$, $\varphi(x,y)$ $\psi(x,y)$ and then differentiate ϕ with respect to x at the point x y. Then differentiate w with respect to the second coordinate that is w η at this point $\varphi(x,y)$ $\psi(x,y)$. This is chain rule and then differentiate ψ with respect to x at the point x y.

Similarly, you do for u_y. Now, substitute u_x u_y in this expression a$(x,y)u_x$ + b $(x,y)u_y$. So, you get this expression. What have you gained? Nothing, because earlier it was a$(x,y)u_x$ + b $(x,y)u_y$ now, you got something into w ψ or something else in w η . Now, the coefficients are all functions of (ξ ,η) , you have to convert them into a (ξ ,η) . Because ϕ x is a function of x y, a is also a function of x y, but you know, the expression of x y in terms of (ξ ,η) . substitute that you get this.

So, these are the functions evaluated these points; x equal to this and y equal to this. So, we have got one expression in x y coordinates equal to another expression. You may call this cap A (ξ ,η) w ψ + capital B of capital B chi eta into w eta. So, we have not gained anything, but these were we can play because (ξ, η), it is something that I want to make a choice. So, that it will be useful for me; convenient to me.

In fact, we wanted to convert into ODE. So, we would like to remove one of these derivatives. How do we remove one of the derivatives? Just ask that (ξ, η) should be such that this guy 0 or maybe this guy 0. Then it becomes a ODE in one of the variables that is the idea.

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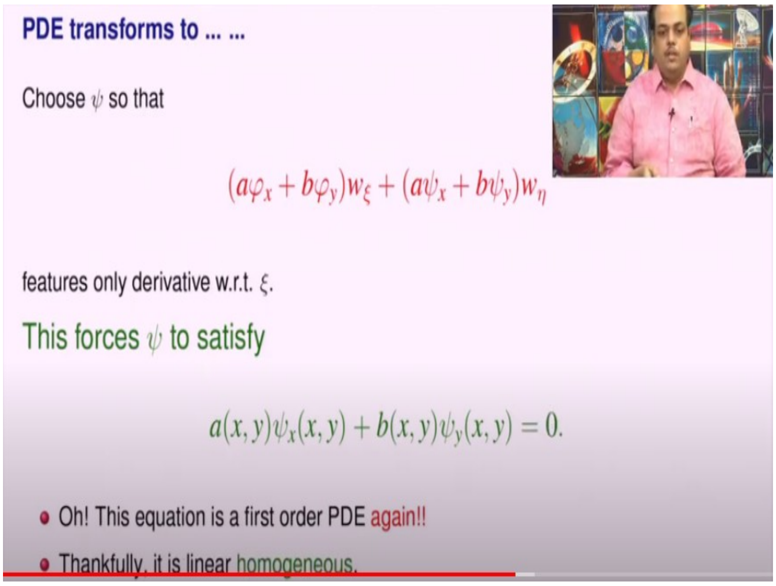

So, choose size so, that this features only derivative with respect to ψ. I decide to let only derivative with respect ψ to be there. So that means I do not want this term. Therefore, I demand these guys 0; choose size that is 0. This forces ϕ to satisfy this. Of course, if you said, there is, I do not want derivative for η; you would have set this equal to 0, but if you see, equation is the same write a (x,y)ϕ_x (x, y) + b (x,y) ψ_y (x,y). So, the equation is the same equation.

So, let us find a $\psi$with this property. Now, this is also first order PDE right. We are trying to solve one first order PDE. We ended up with another first order PDE. So, what have we gained? Thankfully, this equation is a homogeneous equation. So, that helps us; right hand side is 0 linear homogeneous. How does that help?

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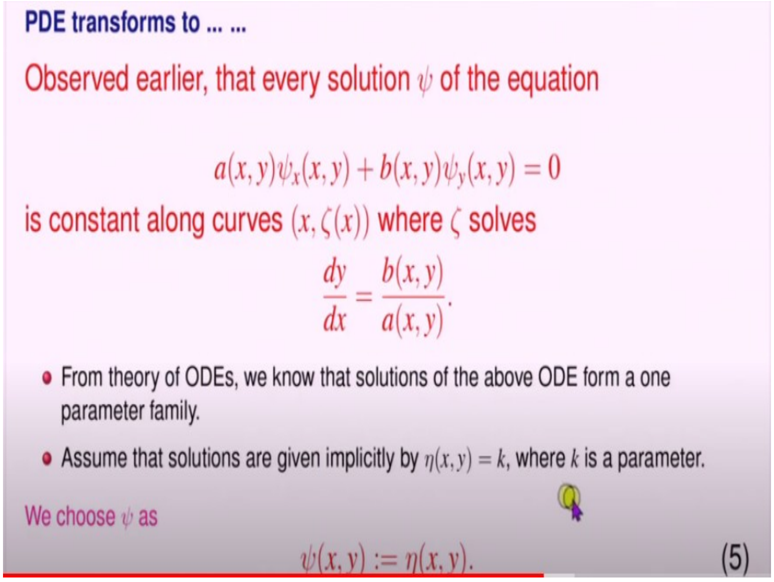

Earlier, we observed that every solution of this particular equation is constant along solutions of some ODEs which is this, we observed this. So, from theory of ODEs, we know that solutions of the above ODE, this ODE for a one parameter family. So, assume that solutions are given implicitly by $\eta(x,y)$ = k where k is a parameter. Why one parameter family? Because this is the first order ODE one initial condition you give and you get a solution that is a parameter.

Same thing, we can express a differently but that is why I am assuming that they are given in this form. Once you have this form, you set ψ = η. So, ψ (x,y) = $\eta(x,y)$. So, you know ψ of. Now, you still need to find fine. So, you have set ψ like this, it means your PDE that the transformed equation is good, because you do not have this time, but ϕ is still there. So, we need to find ϕ only then we will have a change of coordinates, we will do that.

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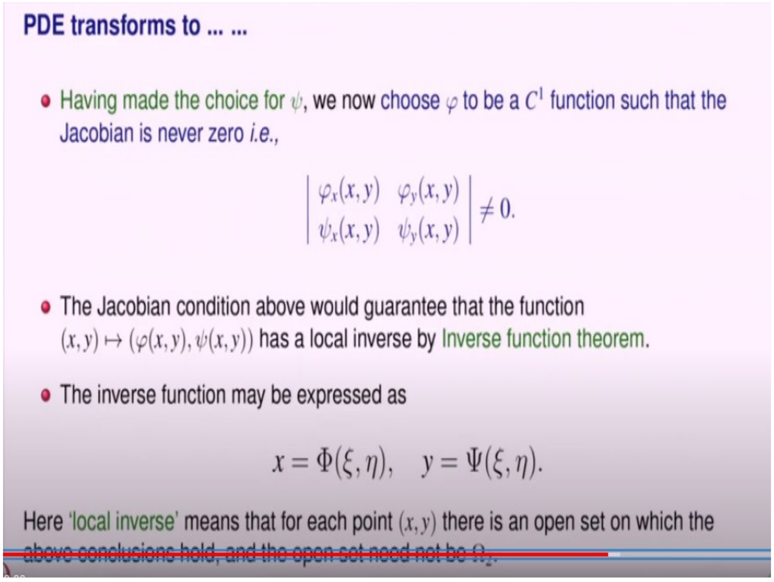

Now, having chosen ψchoose ϕ to be a $\mathbb{C}^1$ function such that this Jacobian is never 0, the Jacobian is never 0. The Jacobian condition above will guarantee that this function x y going ϕ (x y)and ψ (x ,y) has a local inverse and ϕ ψbeing $\mathbb{C}^1$ functions, the inverse will also be $\mathbb{C}^1$function that means we have a diffeomorphism. So, we have to apply what is called inverse function theorem.

Therefore, there is a second most important theorem in differential calculus of multi variables inverse function theorem which is equivalent to another theorem, which is called an implicit function theorem. You can prove one from each other. So, that is why both are equivalent and you must be thorough with application of these theorems, how to apply these theorems?

So, the inverse function may be expressed; now, that x $= \Phi$ (ξ ,η) and y $= \Psi$ (ξ ,η), you can do that. But if you look at the inverse function theorem, the assumption will be at some point x 0, y 0, Jacobian is nonzero, then neighbourhood of x 0, y 0, you have this as a diffeomorphism. So, inverse function theorem conclusion are what are called local.

Even, if you have the Jacobian everywhere nonzero, you cannot say that throughout the domain x y on which these nonzero; this defines a inverse function diffeomorphism, you cannot that is why a local inverse is always a conclusion from the standard inverse function theorem. What is local inverse? It means each point there is an open set on which the above conclusions hold. And the open set need not be omega 2.

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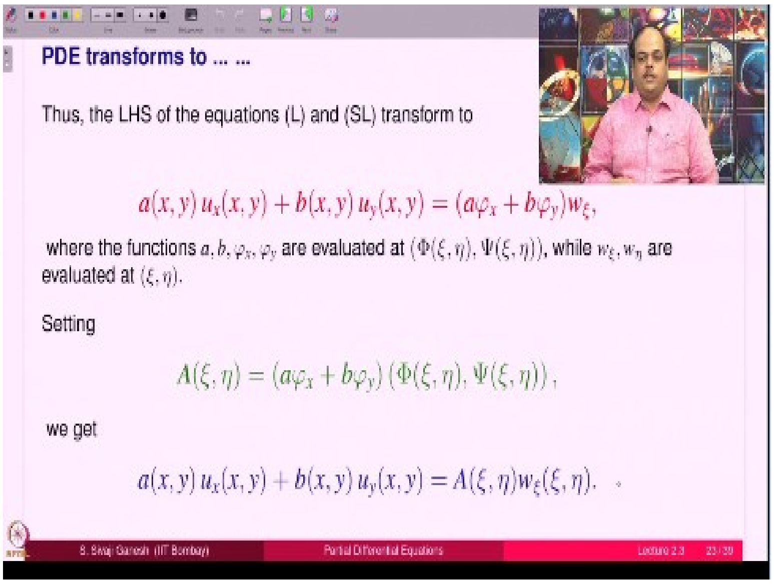

Thus, the LHS of both L and SL which is here a (x,y) u_x (x, y) + b (x,y) u_y (x,y)becomes this. We are just writing where these coefficients are evaluated that they are evaluated this w ξor w η . Right hand side is functional, (ξ ,η). So, we can give a name to that let us call it A (ξ ,η) as this quantity. Therefore, this LHS of L and SL becomes simply A (ξ ,η) w(w(ξ ,η)). So, no η derivative here. So, it is going to become ODE.

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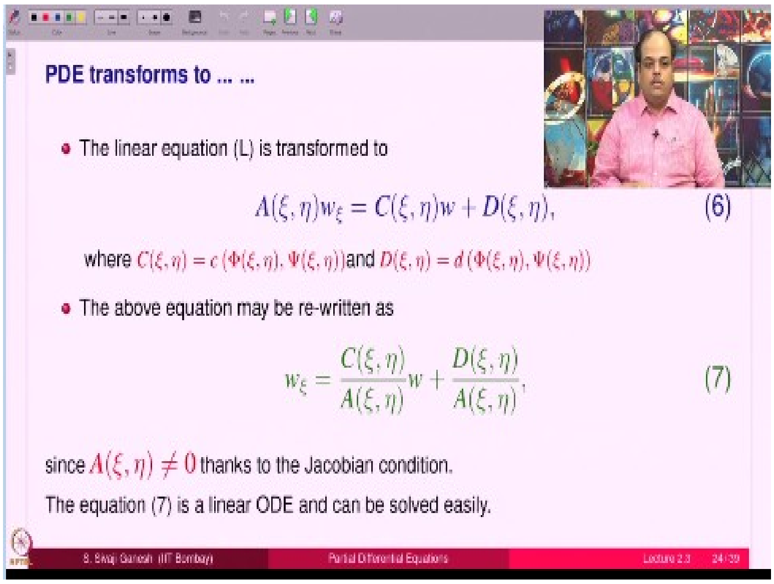

The linear equation now becomes this. These LHS; right hand side C of x y is expressed as (ξ ,η) by this change variables, u is w and D also we express using the change of variable, D of x y, Capital D(ξ ,η) that is the equation. Now, you may write it like this. It is a ODE. It is a linear ODE. Therefore, it can be solved.

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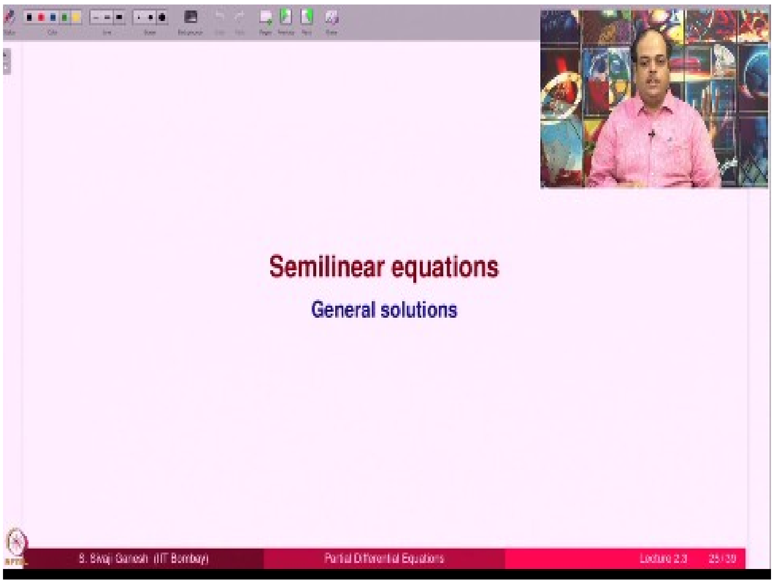

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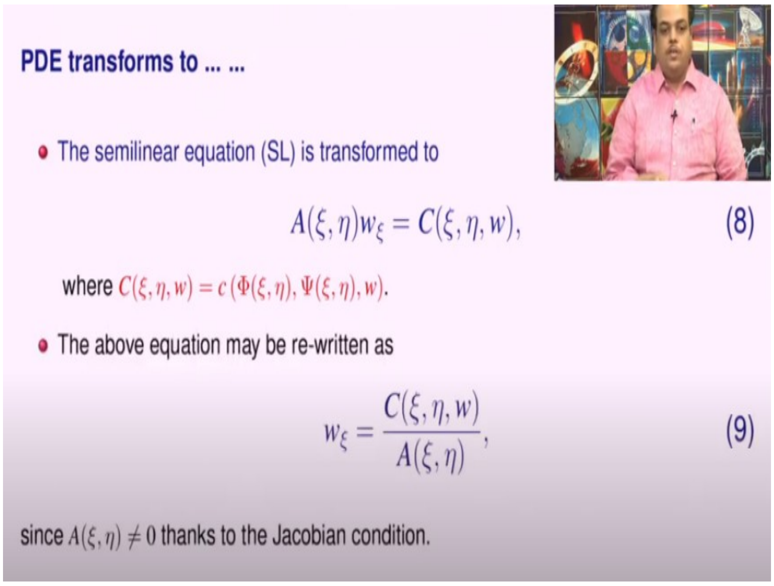

Now, let us see what happens to semilinear equations. The PDE will transform to this because the right hand side, there is nothing to do much. It is just c of small c of x y u earlier; now it becomes capital C (ξ, η, w).

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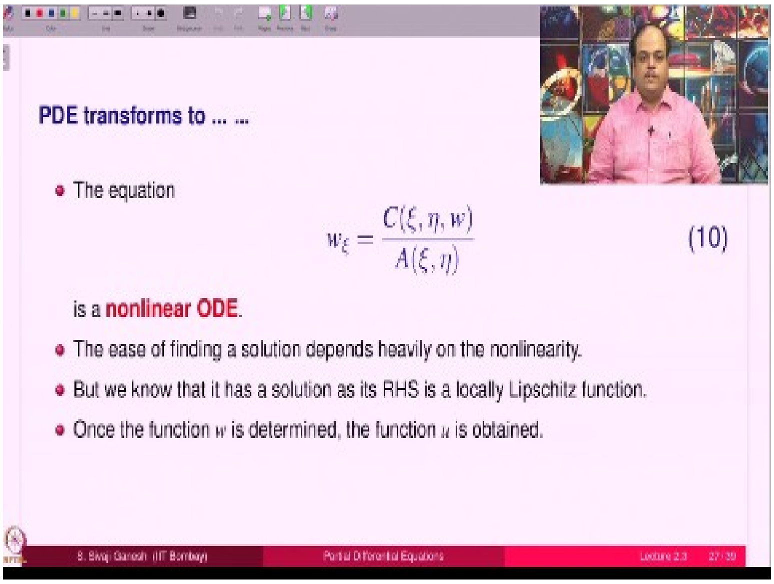

Now, you write like this and the equation is nonlinear ODE. The ease of finding a solution depends heavily on the nonlinearity the capital C and capital A. But we know that it has solution as the right hand side is a local ellipsis function, solutions are there. Once the function w is determined, function u is known now, we know, the correspondence between u and w.

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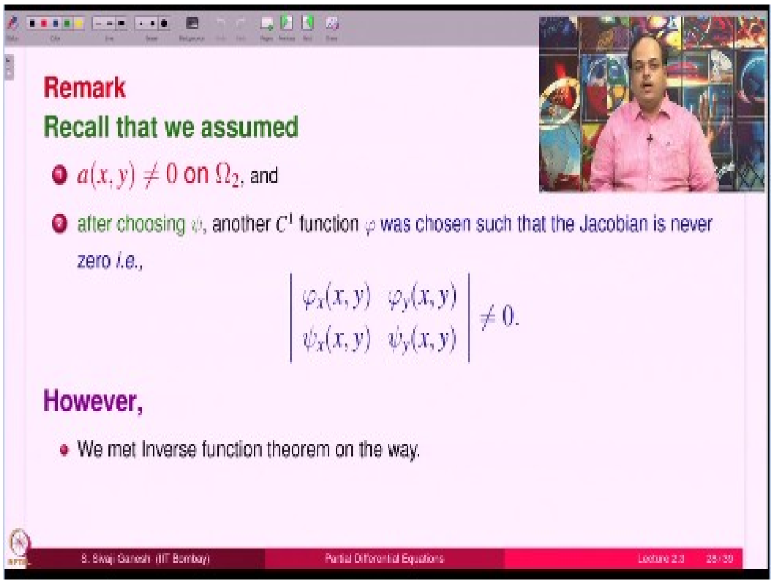

Now, a small remark on our assumption that we assumed this right a is not equal to 0 on omega 2. And one more thing we assumed on the way, after choosing ψ we said, choose another ϕ C 1 function says that the Jacobian is never 0 on omega 2. However, we might inverse function on the way that theorem even if you go with the global conditions like this, this is one, it will give you only local conclusions, there is a problem of the inverse function theorem.

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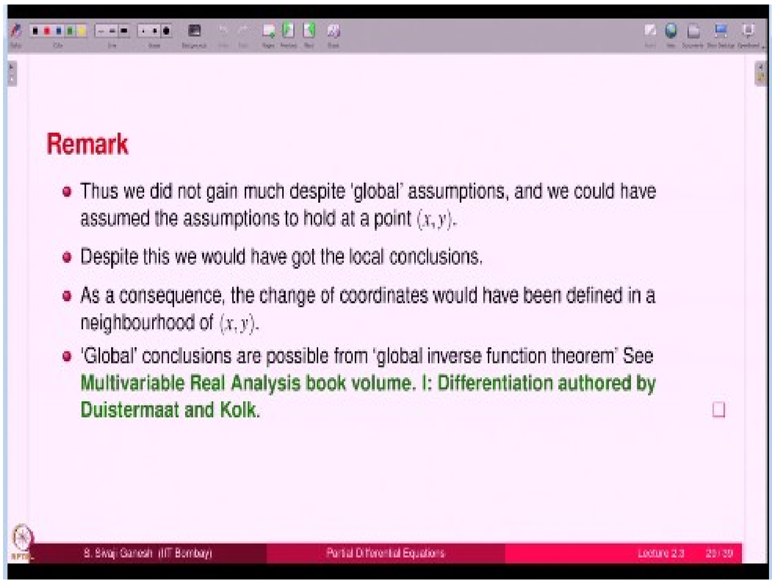

So, we did not gain much despite our global assumptions. That tells that hey, there is no need to assume globally. Assume that a is nonzero at some point x 0, y 0; the conclusion that you will get it will be around x 0, y 0. In any case, even if you assumed a is not equal to 0 on omega 2, when you went the inverse function theorem, it only gave you a conclusion around the point x 0, y 0.

So, as a consequence of change of coordinates, you will find in a neighbourhood of that point, whichever point you are fixed; here I have fixed x y. So, but there are also a global inverse function theorems, this is just for you to be aware of. Global conclusions are possible from global inverse function theorem. For this, I refer you to this book of multivariable real analysis, volume 1, it is called differentiation. It is authored by Duistermaat and Kolk.

They also have a second volume which is about integration. These are very slightly fat books, but they write very nicely.

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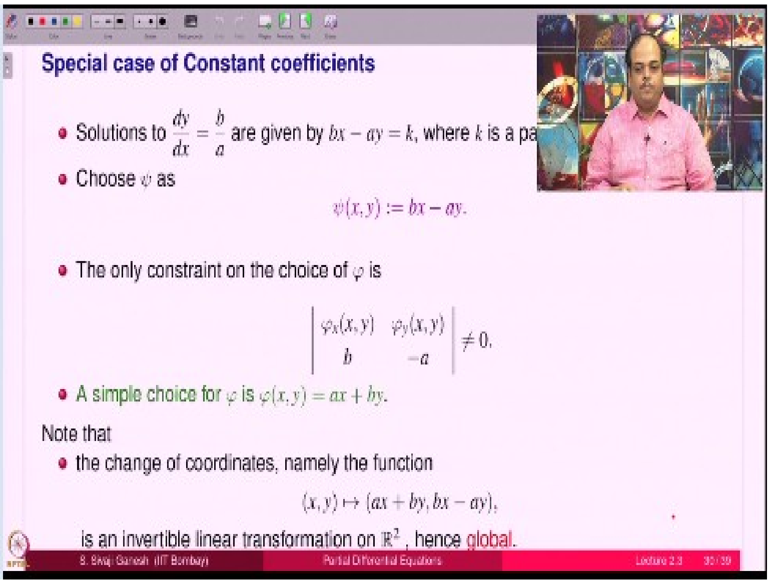

So, let us now specialise to constant coefficients. So, solutions to $\frac{dy}{dx}$ dy = $\frac{b}{a}$, because we have not yet solved a u_x + b u_y =0 so far. Now, we are going to solve. So, equal to $\frac{b}{a}$ are given by b x - a y = k, where k is a parameter. Choose φ as b x -a y, solutions. a dy =b dx. So, b after integrating you get b x - a y equal to constant that is how we get ψ .

Now, you have to choose φ. The only condition on φ, it should be a $\mathbb{C}^1$ function and the Jacobian non zero. So, this is the Jacobian. ψ, I know the derivatives, I put in here, this is ψc x and ψy. We want this to be nonzero. In this case, there is a easy choice for phi. It is a x plus b y. What is the relation between like a x + b y equal to constant and b x - a y equal to constant? They are perpendicular, they are like x axis and y axis right.

Note that this change of coordinates x y going to a x + b y, b x - a y, you may call this as ξ = ψ (x,y) , η = φ (x y) , . These are global. It is an invertible linear transformation, not to say invertible, global.

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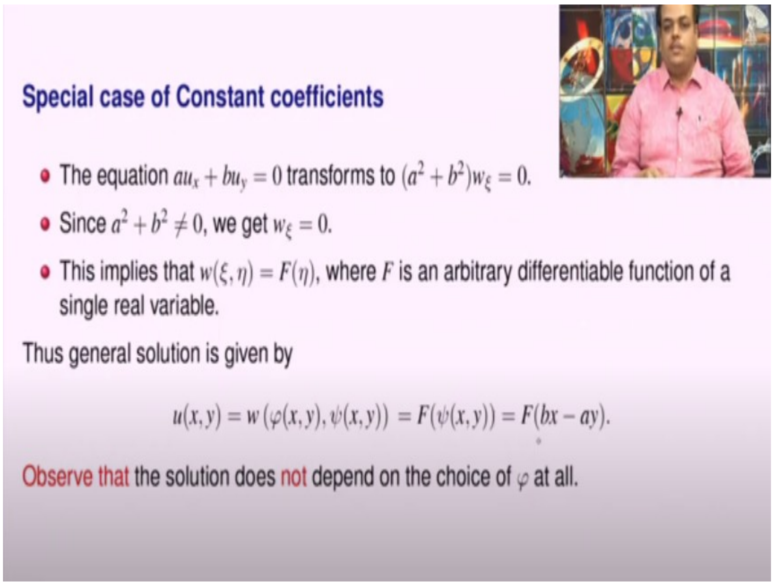

This is a linear. Linear if things are invertible, we will get global. But in the general situation, these are nonlinear equations. A special case of constant corrections: Let us continue the equation a u_x + b u_y , now becomes(a^2 + b^2) w_ξ= 0. Now, I need not even bother to change this into some new coordinates because this is nonzero .You can cancel and conclude w_ξ = 0

And solution it means it is a function of η alone, right = 0. It is a ODE, say. From a u_x + b u_y = 0, we have got a ODE w_ξ= 0, of course, after a change of variable. So, this implies that w is simply a function of η. Now, what is η? We can substitute back, get the u;
u (x ,y) = w (φ (x, y), ψ(x, y), but w, I know, is only arbitrary functional second coordinate that is F(ψ(x, y))and what is your ψ (x ,y), b x - a y.

So, just to connect it back to our original equation u_x= 0 that example, u_x = 0 equation means a is 1; b is 0. So, therefore, it is an arbitrary function of minus φ. Arbitrary function of minus φ is same as arbitrary function of y. So, we got back. So, observe that solutions do not depend on the choice of φat all. Now, you see b x - a y equal to constant, the function is constant on that line. So, these again means that u is constant on any line which is having the direction of a b.

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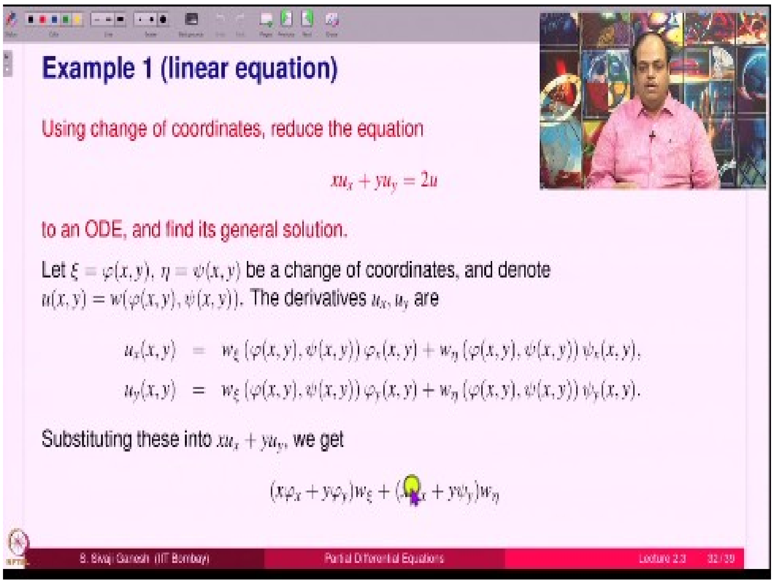

Now, let us look at an example x u_x + y u_y= 2 u, here I said using change of coordinates reduce this equation and find the general solution. So, this is the recall of the change of variables. We have to do this and then go on substitute for u_x , u_y expressions inside this get; the expression (x φ_x + yφ_y)w_ξ plus +(x ψ_x +y ψ_y) w_η. Now, set this equal to 0. Find solutions of this.

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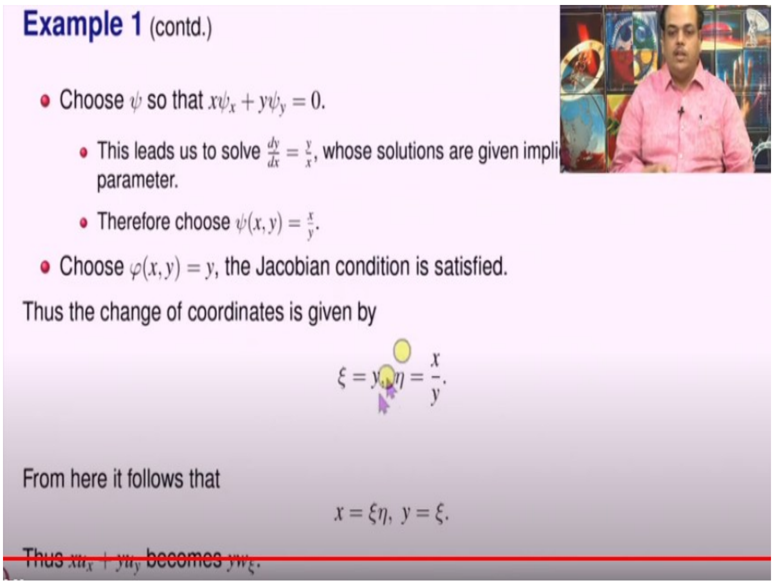

So, that mean,$\frac{dy}{dx}$ is $\frac{y}{x}$and solutions are given by $\frac{x}{y}$ equal to constant that is the family. Therefore ψ(x, y) is$\frac{x}{y}$. Now, we have to find φ. Choose φ = y and Jacobian conditionally is satisfied. So, we have got ξ and η , this is a function of p (x ,y) , η is a function of x y. ψ (x, y)= $\frac{x}{y}$ And then you get the relation between x and y and ξ and η a , then the LHS becomes y w_ξ because we made sure that $w_\eta$coefficient is 0. So, y w_ξ, but what is y? y is ξ . So, it is ξ w_ξ is LHS.

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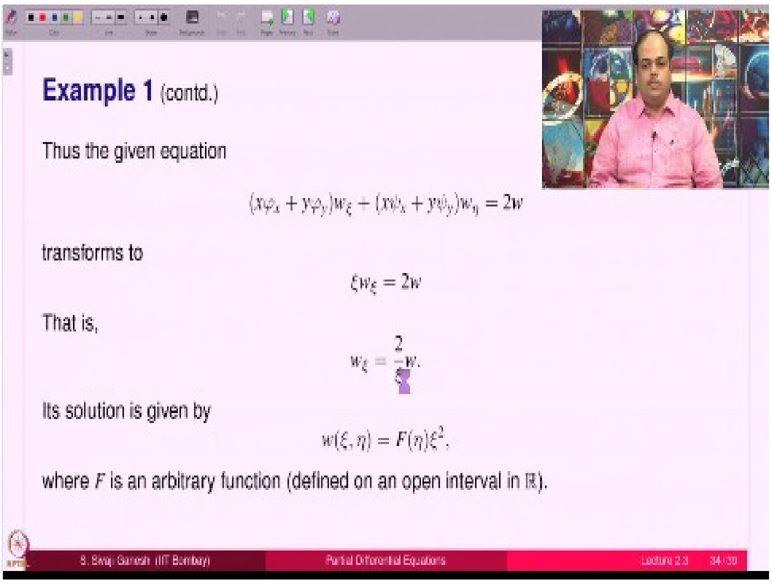

What is RHS? It is 2 u, so it is 2 w. So, ξ w_ξ = 2 w. So,w_ξ = $\frac{2w}{\xi}$, it suggests that we should avoid ξ = 0. So, solution will be this, w (ξ ,η) = equal to F (η)ξ^2 after solving this. So, these are the way to solve using change of variables.

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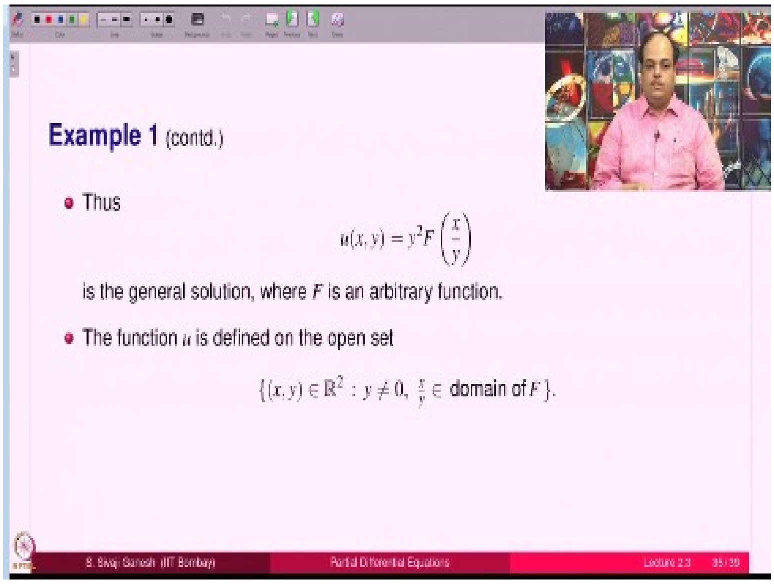

And then going back to x and y, we get u (x, y) =y^2 y F ($\frac{x}{y}$) . Now, it is clear that it is defined for all x on every y such that x by y belongs to domain of F. So, usually any arbitrary function that you like have one variable and x by y should make sense that means y should be nonzero and then it should be in the domain of F. So, you get so many solutions.

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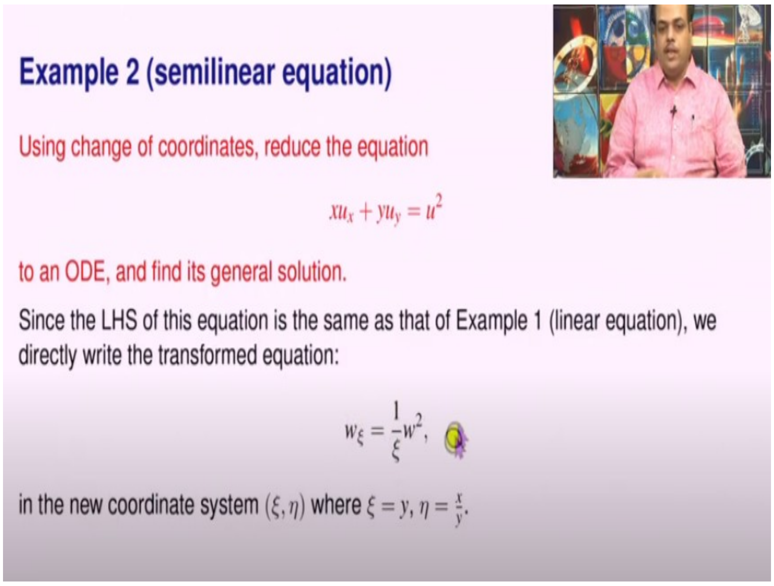

Now, this is another example where the right hand side is u^2 , nonlinear equation right, but as a PDE is a semilinear equation, LHS is the same. So, we can skip the computations. Finally, you end up with w_ξ w equal to $\frac{1}{\xi}$ w^2square. Actually, what you get is $\xi\, w_\xi = w^2$.

So, I take it this side.

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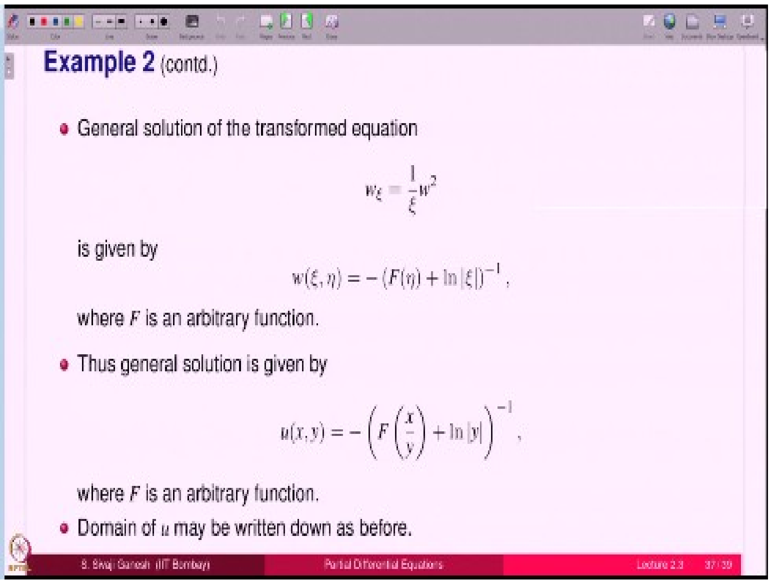

If you solve, you get this answer w (ξ ,η) = $\frac{-1}{F(\eta)}$ $+log\,|\xi|$, F is an arbitrary function. Going back to x y coordinates, we get this expression. So, this clearly tells y = 0 has to be avoided because a logarithm is there. And once y is nonzero, it is is fine. So, **(37:23)** belong to the domain of F. So, choose any F of one variable function of one variable with some domain. And then this solution is valid for all those x y such that y is non zero; $\frac{x}{y}$ belongs to domain of F.

And $log\,|y| +$ F ($\frac{x}{y}$) is nonzero. So, domain of u may be written down that is not important, one way of said.

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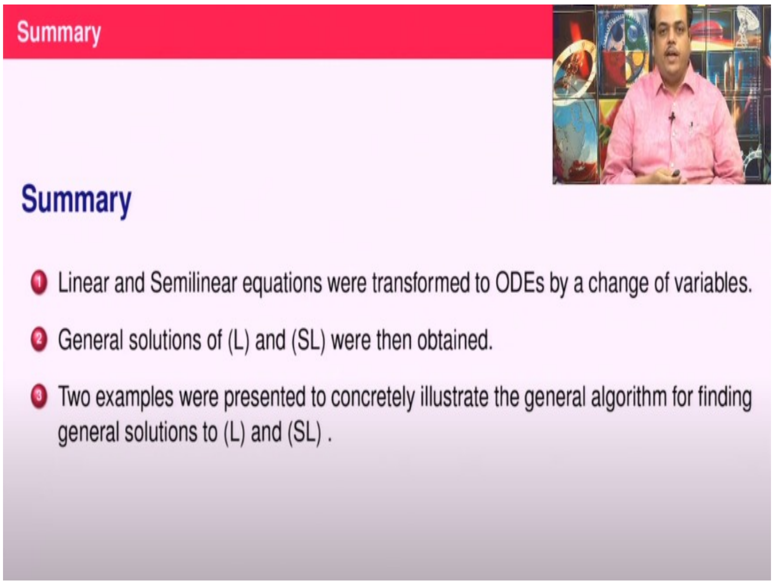

So, summarise what we did is that linear and semilinear equations were transformed to ODEs by a change of variables and general solutions were obtained. Two examples were presented to concretely illustrate the general algorithm that we have presented for finding general solutions to L and SL. So, in the next lecture, we are going to see how these ideas can be generalised are extended to the case of Quasilinear equations.

Quasilinear equations will pose a new troubles. We will see how to overcome them. Thank you.