Partial Differential Equations
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Lecture - 4.12
Tutorial on Separations of Variables Method for Wave Equations

Welcome to a tutorial on separation of variables method for wave equation. In this tutorial we are going to solve 3 problems which are carefully chosen I will comment about them at the end.

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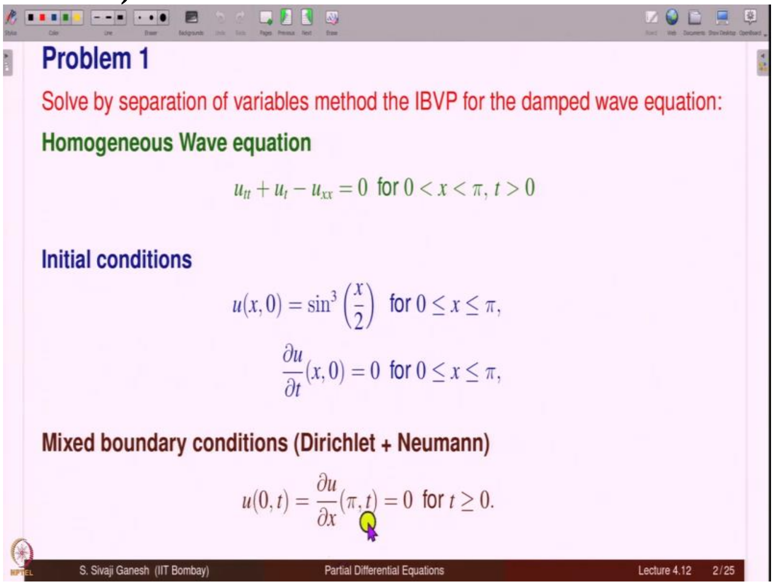

So, problem 1 is solved by separation of variables method IBVP for a damped wave equation this term u t because of the presence of this it is called a damped wave equation and the initial conditions are given where u of x, 0 is given to be sin cube x / 2 u t of x, 0 is given to be 0 then we deal with mixed boundary conditions u of 0, t is 0 and dou u / dou x at pi t is 0 for t greater than or = 0. So, let us go ahead and try to solve this by separation of variables method.

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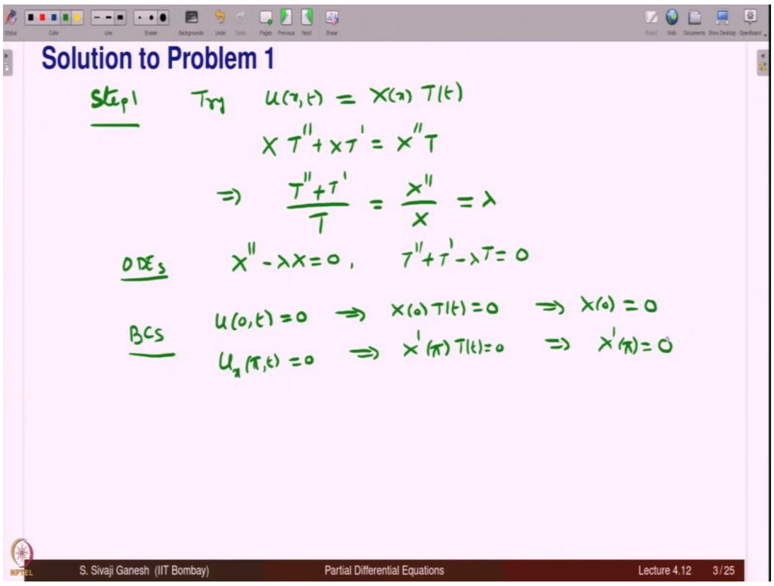

So, what is the step 1 in separation of variables method it is try u of x, t solutions to the given equation in the separated form. So, on substituting this expression in the equation what we get is X, T double dash + X, T dash = X double dash T and that further gives us T double dash + T dash / T = X double dash / X. Now if you notice here the left hand side is a function of T only on the right hand side is a function of X only.

Therefore as you know we set it equal to constant lambda. So, therefore we have got to 2 ODEs what are those? X double dash - lambda X = 0 and T double dash + T dash - lambda T = 0. Now from the given boundary conditions and initial condition we will try to get conditions for X and T. So, first boundary condition that we have is at 0 that is u of 0, t is 0 that implies that X of 0 T of t = 0.

And that implies X of 0 = 0 because we do not want T to be an identically equal to 0 functions. Now u, x pi of t is also given to be 0 that means X dash at pi into T of t is 0 and that will give us X dash at pi = 0. So, these are the boundary conditions that we obtain.

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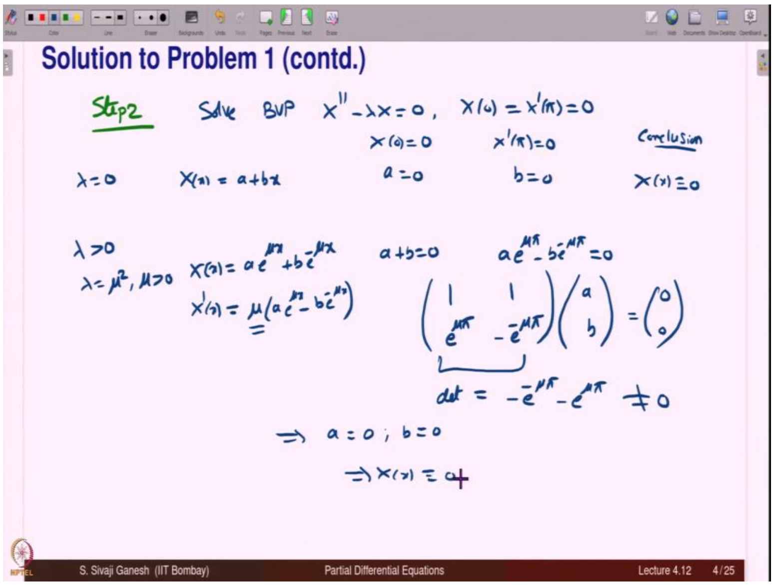

Now we move on to step 2 is to solve the boundary value problem X double dash - lambda X = 0 with the boundary conditions X of 0 and X prime of pi = 0 we need to solve for non 0 solutions of this of course X identically equals 0 is a solution that is useless because if X is identically equal to 0 u of x, t will be 0 we are not interested. So, we want to find non 0 solutions of this boundary value problem it would not exist for every real number lambda.

And those real number lambdas for which this boundary value problem admits non 0 solutions are called Eigen values and the corresponding non 0 solutions are called Eigen functions. In other words we are looking for Eigen functions of X double dash operator. So, first is lambda = 0 if lambda is 0 what is the solution equation is X double dash = 0 so therefore X of x = a + bx.

And when I apply this boundary condition X of 0 = 0 I get a = 0 if I apply X dash of pi = 0 I get b = 0. So, what is the conclusion? X is identically = 0 so lambda = 0 is not an Eigenvalue then we try for lambda which is positive we always write lambda = mu square and mu positive to avoid square roots in the expression of the solutions. Now when lambda = mu square what is the equation X double dash - mu square X = 0.

So, solution is a combination of exponentials which is a e power mu x + b e power - mu x. Now X of 0 = 0 when I apply what I get is a + b = 0 and I want to apply X dash pi therefore I should know what is X dash of x is a e power mu x - b e power - mu x. So, when I put x = pi what I get is a e power mu pi - b e power - mu pi = 0 I have not included this mu because mu is non 0 it gets cancelled in this equation.

So, A and B are satisfying a system of 2 linear equations it is always a good idea to write them in the matrix form. We are looking for a non trivial solution for this system that means at least one a or b is non 0 that will happen if and only determinant is 0 but what is the determinant? Determinant = - e power - mu pi - e power mu pi this is not equal to 0 therefore the only solution is the 0 solution a = 0 b = 0 that implies X of x is identically equal to 0 so no positive Eigen values in this problem.

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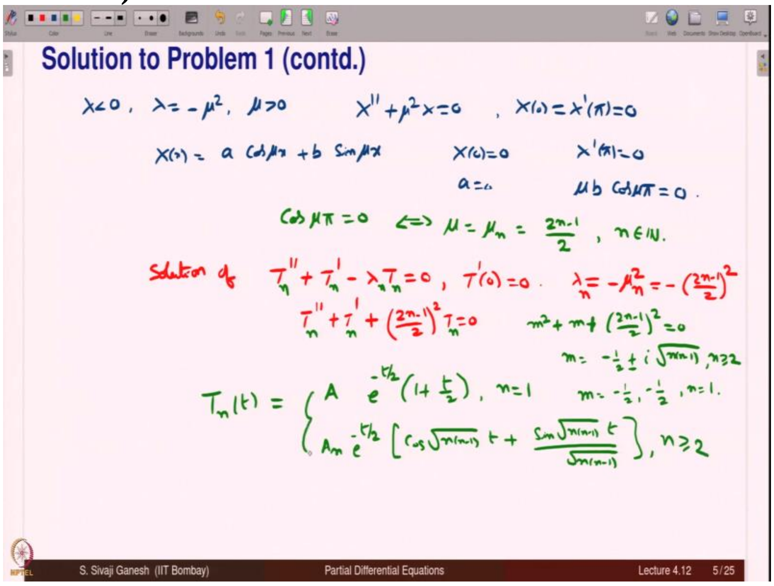

Now we look at negative Eigen values so suppose lambda is less than 0 then we as usual we put lambda - mu square mu is positive the equation is now X double dash + mu square X = 0 and the boundary conditions let me reiterate here on this solutions will be a combination of sin and cos here. So, x of x = a cos mu x + b sin mu x when we put X of 0 = 0 what we get is a = 0 when we put X dash of pi = 0.

We have to compute what X dash x is after doing that and substituting pi there X dash of pi = 0 this equation will give us mu b cos mu pi = 0 so cos mu pi = 0 if and only if mu = mu n = 2 n - 1 / 2 and n belongs to natural numbers having done this we are solved for x now we have to solve for t. So, the solution of T double dash + T dash - lambda T = 0 this equation needs to be supplemented with a condition that comes from u t x, 0 = 0 that will give us T prime of 0 is 0.

Now here what is lambda, lambda is minus mu square but we have a sequence so let us put lambda n - mu and square that is minus 2 n - 1 / 2 whole square now this is a second order

ODE. Now let us write the how the ODE looks now T dash + 2 n - 1 / 2 whole square into T = 0 the solution of this will go through via the auxiliary equations the m square + m + 2 n - 1 / 2 whole square = 0.

And m is minus 1 / 2 plus or minus i times root n into n - 1 and this is when n is greater than or equal to 2 and m = - half - half it is a repeated root when n is = 1. So, on solving this equation we need to now solve with the T n for T n so everywhere actually n now. The solution I write down the final solution please check for yourself is T n of t will turn out to be constant times e power - t / 2 into 1 + t / 2. If n is = 1 otherwise it is A n e power - t / 2 into cos root n into n - 1 into t + sin root n into n - 1 t / root n into n - 1 when n is greater than or equal to 2 so, we have got our X n's and T n's.

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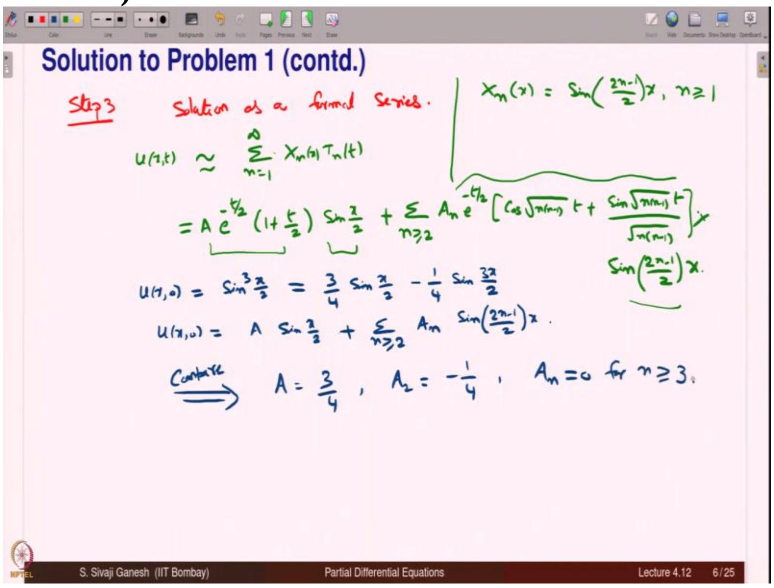

Now we are ready to move on to the step 3 is solution as a formal series. So, we write u of x, t summation n = 1 to infinity X n x T n t usually we put a constant here but the constant here we already put in T n. So, we will not write here so that that is nothing but A e power - t / 2 into 1 + t / 2 into sin x / 2 plus summation n greater than or equal to 2 A n e power minus t / 2 into cos root n into n - 1 t + sin root n into n - 1 t / root n into n - 1 into sin 2 n - 1 / 2 into x.

I forgot to mention when we determined the Eigen values we should have returned on the expression for X n's also X n of x is actually sin of 2 n - 1 / 2 into x this is n greater than equal to 1. So, this is T n into X n and this is T n and is x so we need to determine these coefficients A and A n's now this is where we will use the initial condition that we have that is u x, 0 = sin cube x / 2 which by trignometric identity is 3 / 4 sin x / 2 - 1 / 4 sin 3x / 2.

But what is u x, 0 from our series that A I am putting t = 0 so A sin x / 2 plus summation n greater than or equal to 2 A n sin 2 n - 1 / 2 into x. So, this implies compare the background there is some uniqueness result for a Fourier series that is used. So, what we get is A = 3 / 4, A 2 = - 1 / 4, A n = 0 for n bigger than or equal to 3.

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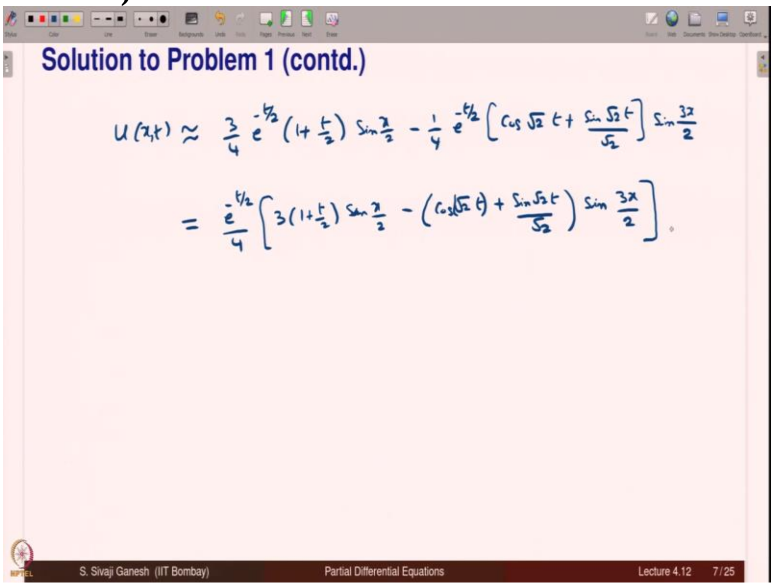

Therefore why did we get the solution as u of x, t expression is 3 / 4 e power - t / 2 1 + t / 2 sin x / 2 - 1 / 4 e power - t / 2 into cos root 2 t + sin root 2 t / root 2 into sin 3 x / 2 so this is a solution. So, if you want to simplify we can write e power - t / 2 / 4 what we get 3 into 1 + t / 2 sin x / 2 - cos root 2 t into t + sin root 2 t / root 2 sin 3 x / 2. So, please do these computations on your own that is when it will be clear whether it is cos root 2 into t or cos of root 2 t so thus we solve this problem 1.

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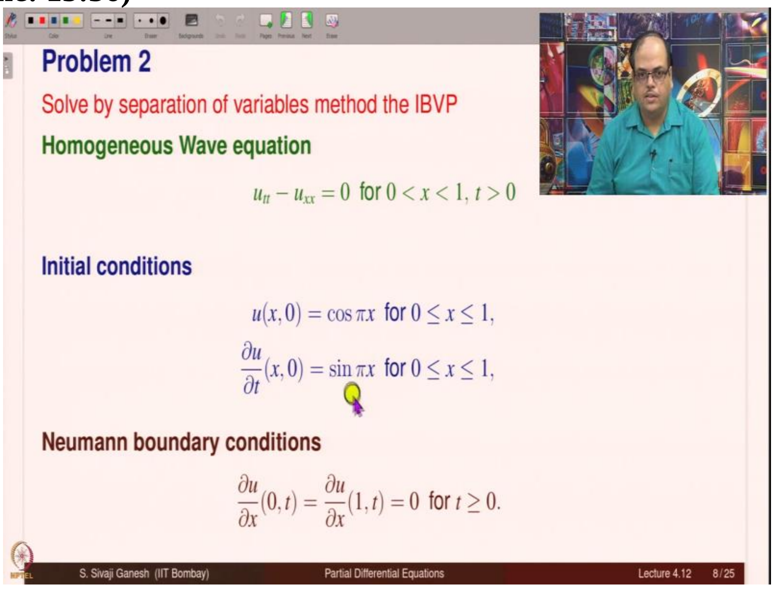

Let us move on to the problem 2 these again solution by separation of variables. Now we have the usual wave equation u tt - u xx = 0 on the interval 0, 1. Now the change here is more both initial data are non 0. If you remember in the lecture we have taken one of them to be 0 one of them to be non 0 but now we will see what the difficulties are by taking both of them. In fact there are no difficulties as such you will see that procedure is little more longer.

And now we have changed the boundary conditions to Neumann boundary conditions. In the lecture on separation of variables method we consider Dirichlet boundary conditions. Now we are considering different boundary conditions mixed ones we already considered in problem 1. So, after this tutorial you should be able to solve any mix of these boundary conditions.

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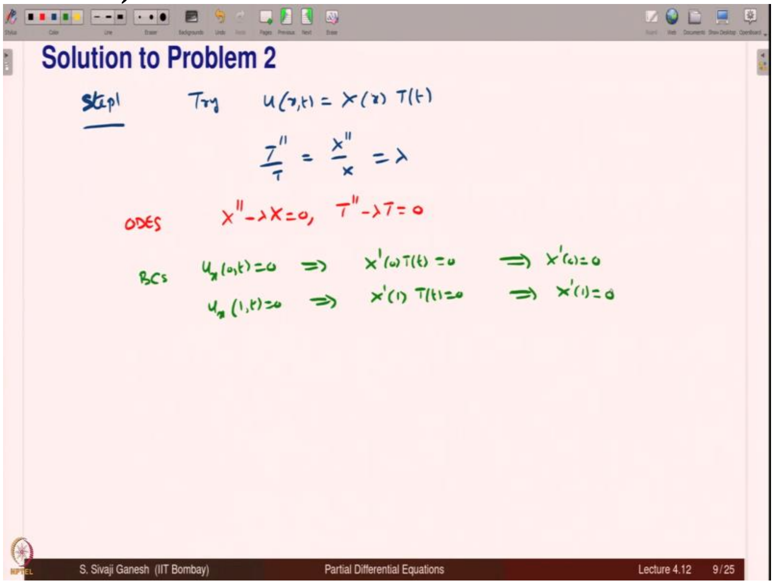

Let us turn to the solution to problem 2 step 1 as always is trying solutions which are in the separated form on substituting this in the wave equation we get T double dash / T = X double dash / X. And since left hand side is a function of T alone right hand side is a function of X alone it has to be a constant function. So, therefore this gives us as a 2 ODEs they are X double dash - lambda X = 0 T double dash - lambda T = 0.

And we also get boundary conditions from the given boundary conditions which are u x of 0, t = 0. That means that X dash at 0 into T of t is 0 we do not want T to be 0 functions therefore X dash of 0 is 0. The other boundary condition u x of 1, t is 0 but the same argument we get that X prime at 1 = 0.

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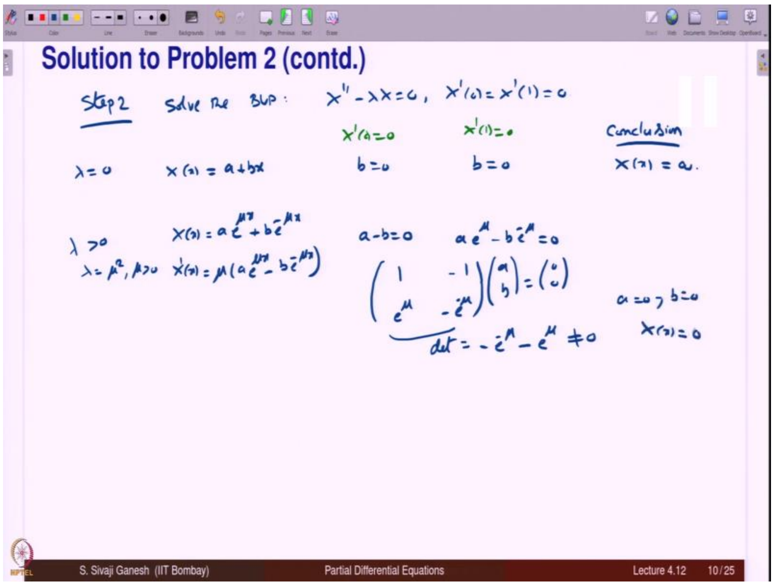

Step 2 is to solve the BVP or the eigenvalue problem X double dash - lambda X = 0 X prime of 0 = X prime of 1 = 0. So, we consider the cases lambda 0 lambda negative on lambda positive. So, when lambda is 0 solutions is X of x = a + bx and this boundary condition X prime of 0 = 0 will give us b 0 on the boundary condition X prime of 1 = 0 will also give us b = 0 which means the conclusion is that lambda = 0 is an eigenvalue what is a eiganfunction?

It is a function which is constant function a any constant function is a eiganfunction let us consider lambda to be positive lambda = mu square mu positive solution will be of exponential type X of x not exponential type combination of exponential this what we have and X dash of x is mu into a e power mu x - b e power - mu x. When I apply this boundary condition X prime of 0 is 0 we get a - b = 0.

When we apply X prime of 1 is 0 we get a e power mu - b e power - mu = 0 and we said we should always write this as a system because it is easy to check whether we have a non trivial solution or not depending on whether the determinant is 0 or not what is the determinant? Determinant is - e power mu there is a typo here - e power - mu - e power mu and that is not equal to 0. Therefore a and b are 0's and x is 0 so, negative real numbers are not eigenvalues.

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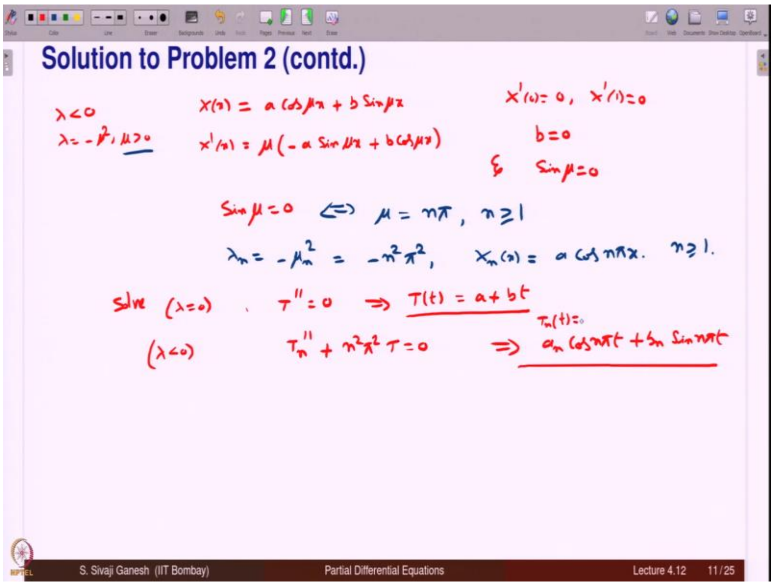

Now let us check for positive or not eigenvalues let us check for negative numbers lambda less than 0. So, lambda = - mu square mu positive now the solutions are combination of cos and sin a cos mu x + b sin mu x we would require x prime. So, let us compute that so the condition that we have X prime of 0 = 0 and X prime of 1 = 0 they give us that b is 0 and on substituting b = 0 we get sin mu = 0.

So, mu has to satisfy sin mu = 0 which is the case if and only if mu is a multiple of pi and we take natural numbers because mu is supposed to be positive. So, let us cut something and do something here mu = n pi n natural numbers because mu is supposed to be positive that is the assumption that is how we chose N. So, mu = N pi so what is the summary after this we have got lambda n which is equal to minus mu n square = - n square pi square.

And what are the eigenvectors X n of x = a cos mu that is n pi x n greater than or equal to 1 now we need to solve the ODE for T we have eigenvalues here lambda = 0 is an eigenvalue and lambda n = - n square pi square Eigen values. So, let us solve with lambda = 0 the equation is T double dash = 0 that implies T of t = a + b t we do not have any conditions on T because we are given non 0 Cauchy data.

Therefore we cannot do anything so, we keep it as this in the end we will determine these coefficients and when lambda is less than 0 we have a sequence of such problems T n double dash + n square pi square T = 0 and solutions to this are given by a n cos n pi t + b n sin n pi t. So, now we have solved for X n and T n. So, these are the things we have this is T n of t.

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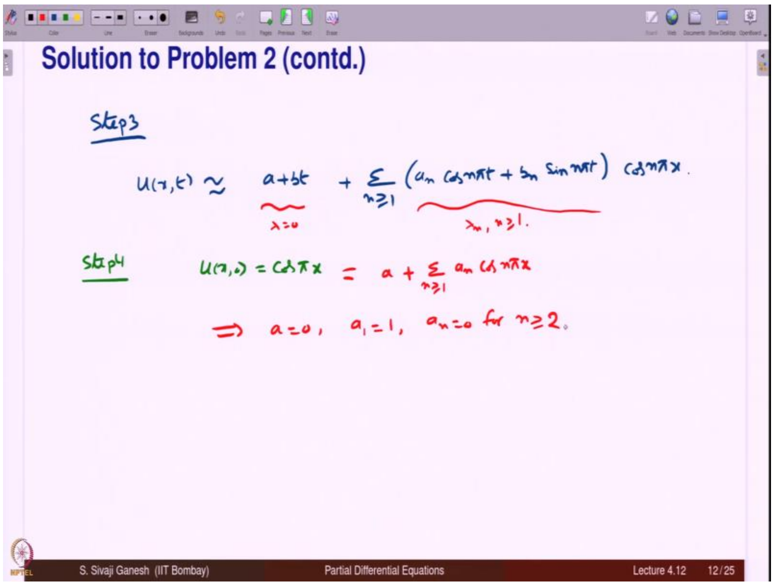

Now we are in a position to express the solution as a superposition of all these functions we have obtained. So, step 3 is u of x, t is a + b t plus summation n bigger than or equal to 1 a n cos n / t + b n sin n / t into cos n pi x. So, this corresponds to the eigenvalue lambda = 0 and this is t that we got X was constant. So that constants are there inside this a and b so we do not write explicitly a n.

These are the once which are corresponds to the eigenvalues lambda n n greater than or equal to 1. Now we move on to step 4 where we have to determine these constants. So, we use u of x, 0 = cos pi x this is what is given to us but from this series it will be equal to because t = 0 a plus summation n bigger than or equal to 1 a n cos n pi x. So that implies that a = 0 a 1 = 1 a n = 0 for n bigger than or equal to 2. So, we have determined a n's now we need to determine b n's for which we have another initial condition.

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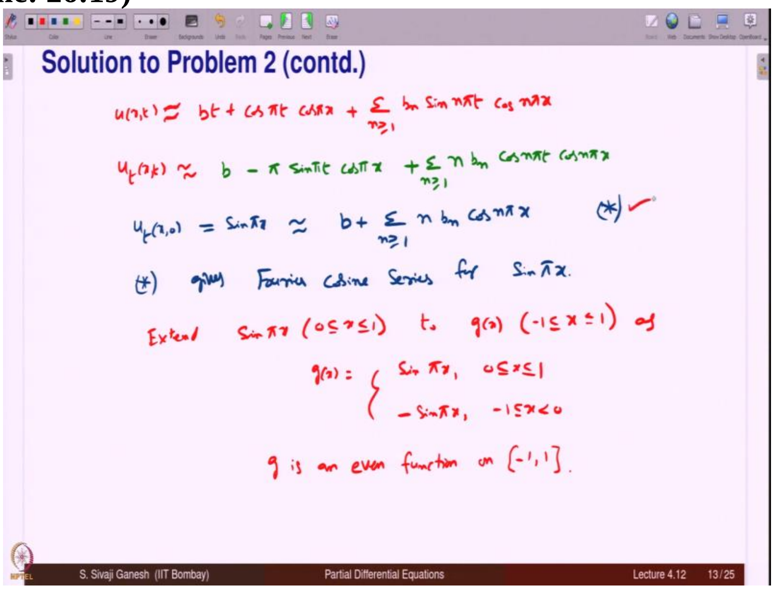

So, at the end of this previous step what we have is u of x, t equal to or this formal thing is b t + cos pi t cos pi x plus summation n bigger than or equal to 1 b n sin n pi t cos n pi x. So, therefore we have to see what is u t of x, t; u t of x, t is b - pi sin pi t cos pi x plus summation n bigger than or equal to 1 n b n cos n pi t cos n pi x. now we are to see u t of x, 0 because that is what is given to us u t of x, 0 is given to be equal to sin pi x is what we want.

And from the series when I put t = 0 what I get is b plus summation in bigger than or equal to 1 n b m cos n pi x is what we have. So, let us call this star so star says us that the series that we have on the RHS is a Fourier cosine series for sin pi x, star gives Fourier cosine series for this function sin pi x. So, which functions have only cosine series they are the even functions therefore what we need to do is to extend this function which function sin pi x it is given in this interval.

We have to extend this to a function let us denote it as g x which is now defined on the other side of 0 it extends this function sin pi x as even function so g of x = sin pi x in the interval 0, 1 - sin pi x in the interval minus 1 to 0 g is an even function on minus 1, 1 therefore when we write its full Fourier series it will only involve cosines and if the equality holds when you restrict the equality to the interval 0, 1 what you get is this star. So, using this idea we are going to determine these constants b and b n's.

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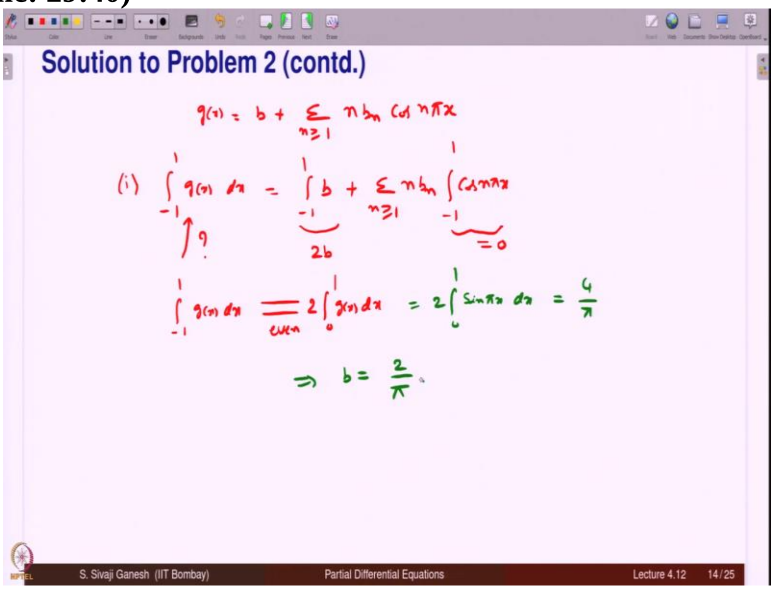

So, g of x = b plus summation n bigger than or equal to 1 n b n cos n pi x so first let us observe certain things integrate between minus 1 and 1 that is our interval g x d x that is minus 1 to 1 b so do not question whether this exchange of integral and summation is allowed when this allowed what happens that is what we are looking at the series we are going to everything we are doing formal computations this we have.

This is 0 and this is a 2b then what is this? Let us compute minus 1 to 1 g x d x because g is even this is 0 to 1 2 times g x d x even with respect to 0 x = 0 and that is nothing but 2 times 0 to 1 sin pi x d x that is 4 / pi. So, this gives us b = 2 / pi so we have got b now we need to get b n’s.

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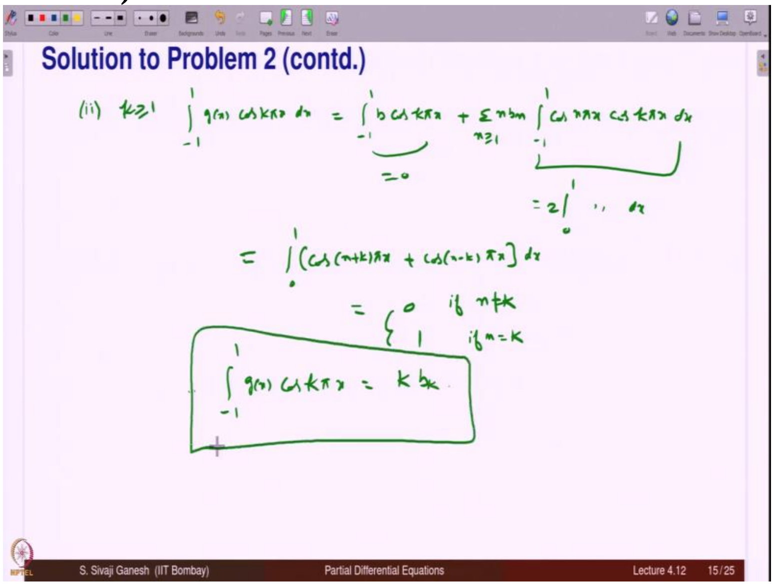

Second is this computation for k bigger than equal to 1, -1 to 1 g x cos k pi x d x does equal to -1 to 1 b cos k pi x plus summation nbn n - 1 to 1 cos n pi x cos k pi x dx. Now this integral is 0 because cosine minus 1 to 1 cosine k pi x so therefore this integral will be 0 and what is this integral this is an even function. So, therefore that is equal to 2 times 0 to 1 of the same integrand and that integrand is nothing but we can use cos a + b formulas.

So, what we get when use is so we get that is equal to 0 to 1 cos m + k pi x + cos n - k pi x dx. So that will be 0 if n is not equal to k and that will be equal to 1 if m = k therefore minus 1 to 1 g x cos k pi x = kb k.

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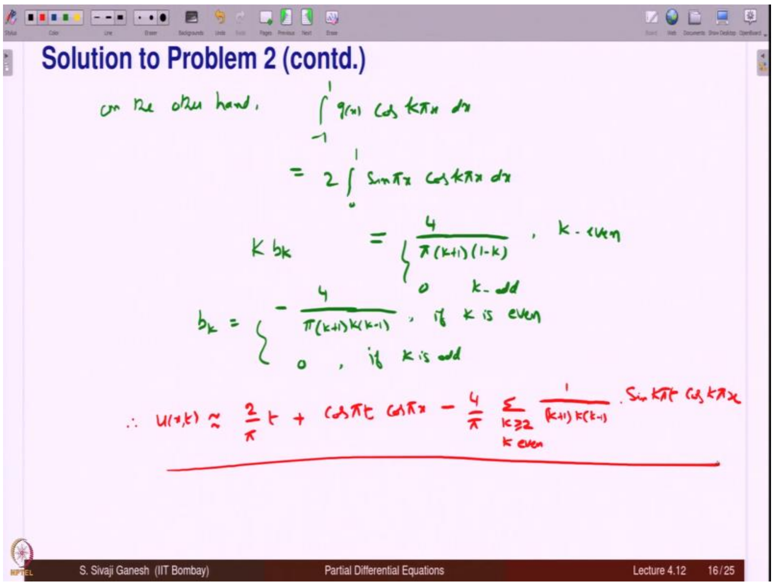

So, on the other hand -1 to 1 g x cos k pi x dx is given by 2 times 0 to 1 sin pi x cos k pi x dx this upon simplification will give us 4 / pi into k + 1 into 1 - k if k is even 0 if k is odd. So, the same thing is equal to k into b k that is what the expression we obtained by using the series. So, therefore this gives us the expression for b k so b k = - 4 / pi into k + 1 k, k - 1 if k is even and 0 if k is odd.

Therefore u of x, t is 2 / pi into t + cos pi t into cos pi x - 4 / pi summation k bigger than or equal 2 k even natural number 1 / k + 1 into k into k - 1 into sin k pi t cos k pi x. So, this is the solution that we obtained. In this example actually we ran into a Fourier cosine series and we have seen how to come to the coefficient and the solution.

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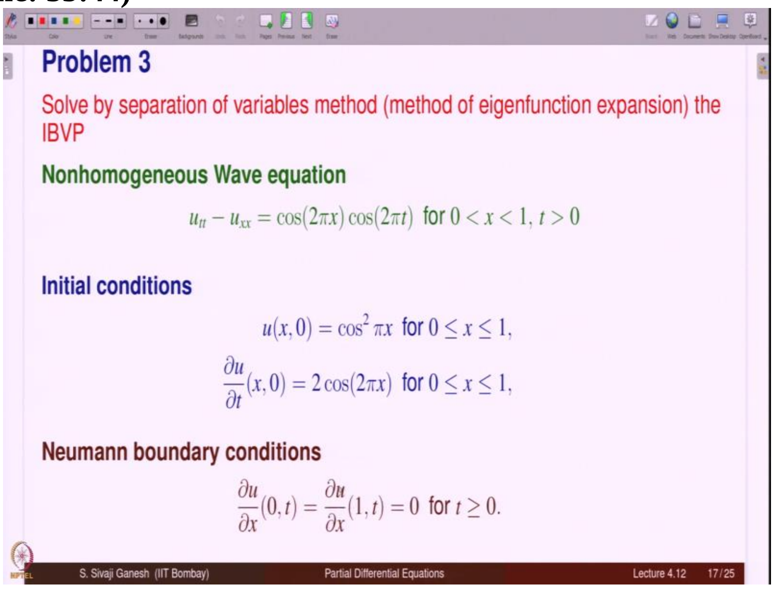

Let us move to problem 3, here it is a non homogeneous wave equation so as such we learned separation of variables method for homogeneous wave equation we will see how to modify it and get it for a non homogeneous wave equation. That is why it is also called method of eigenfunctions expansion. The idea is that keep the 0 get the eigenfunctions as we have computed and then propose a series in terms of them with the coefficients which are functions of t substitute the entire series in this equation and try to solve that idea. So, here once again we consider non 0 Cauchy data in both of them and we consider the Neumann boundary conditions.

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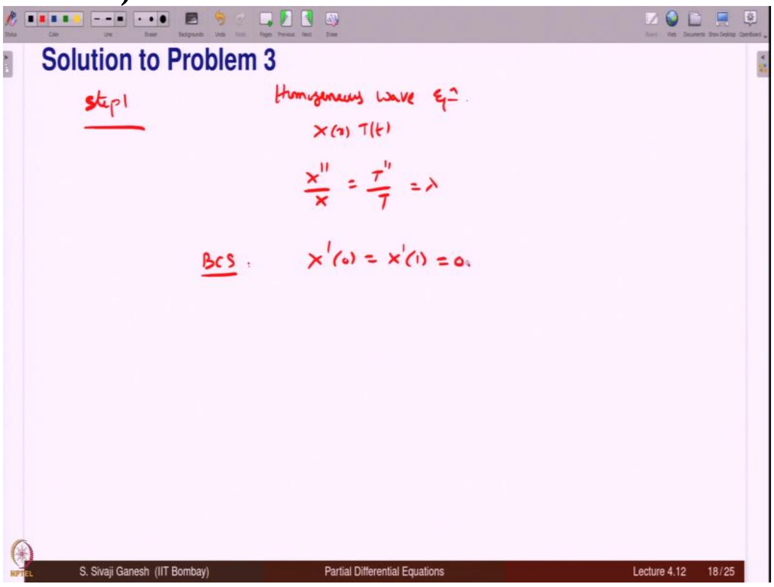

Step 1 is the same step 1 is take homogeneous wave equation and that is what I have indicated because we are now going to compute the eigenfunctions and tie separated solutions substitute in the wave equation that will give you X double dash / X = T double dash / T = lambda then we have to get boundary conditions and those are X prime of 0 = X prime of 1 = 0 we obtained this just in the last problem exactly the same computations.

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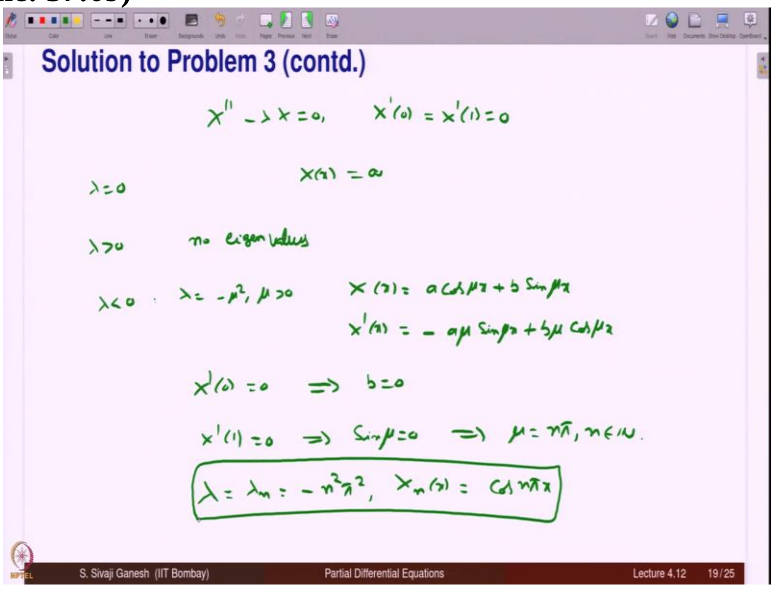

Now we need to solve the boundary value problem. So, X double dash - lambda x = 0 with X prime of 0 = X prime of 1 = 0. So, suppose you take lambda = 0 we saw that solution is constant earlier it is the same boundary conditions same problem and lambda positive is not no eigenvalues on lambda negative we set lambda = - mu square mu positive and that gives us that solutions are a cos mu x + b sin mu x and X prime of X = - a mu sin mu x + b mu cos x.

Using this boundary condition X prime of 0 = 0 that will give us b = 0 and X prime of 1 = 0 that will give us sin mu = 0 that implies our if and only is mu is n pi n belongs to n therefore lambda = lambda n = - n square pi square on X n of x is = cos n pi x. So, these are the eigenvalues and eigenfunctions coming from X.

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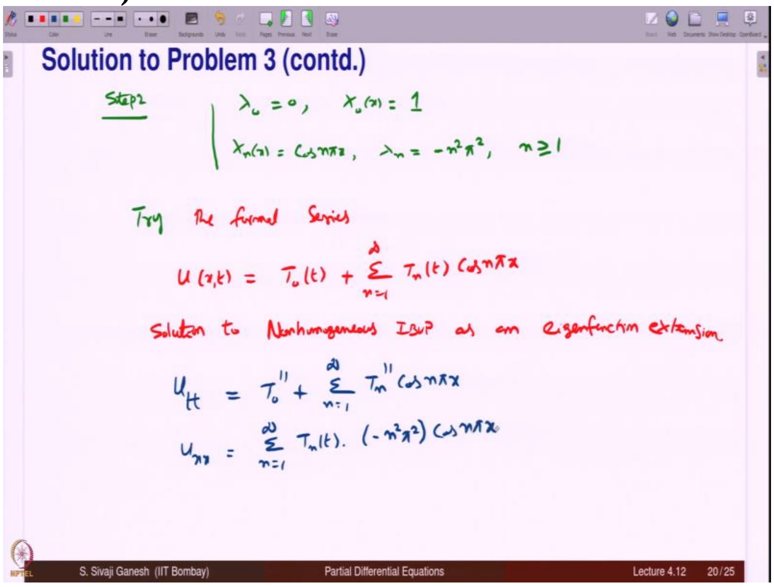

Now step 2 let us just recall what we have lambda 0 = 0, 0 is an eigenvalue X 0, x is any constant let me take it as 1 and X n of x is cosine n / x lambda n = - n square pi square and n greater than or equal to 1 these are the 2 sets of eigenvalues we got. This is with 0 Eigen value these for the negative eigenvalues and there is no positive eigenvalues here. So, now idea is try the formula series expansion as a solution what is that?

Try the formal series so we take series let us say infinity combination of the X 0 X n with coefficient which are function are t u of x t = T 0 of t into 1 so just T 0 of t summation n = 1 to infinity of T n of T into X n which is cos n pi x. So, what we are doing is solution to non homogenous equation or non homogenous problem as an eigenfunctions expansion this is the meaning of eigenfunctions expansion.

So, when we substitute into the equation of the given equation what we get is we need to compute u tt and u xx. So, what is u tt and u xx? Let us see that u tt is nothing but T 0 double dash plus summation n = 1 to infinity T n double dash cos n pi x what is u xx? That is summation n = 1 to n infinity T n of t into minus n square pi square cos n pi x is what we get?

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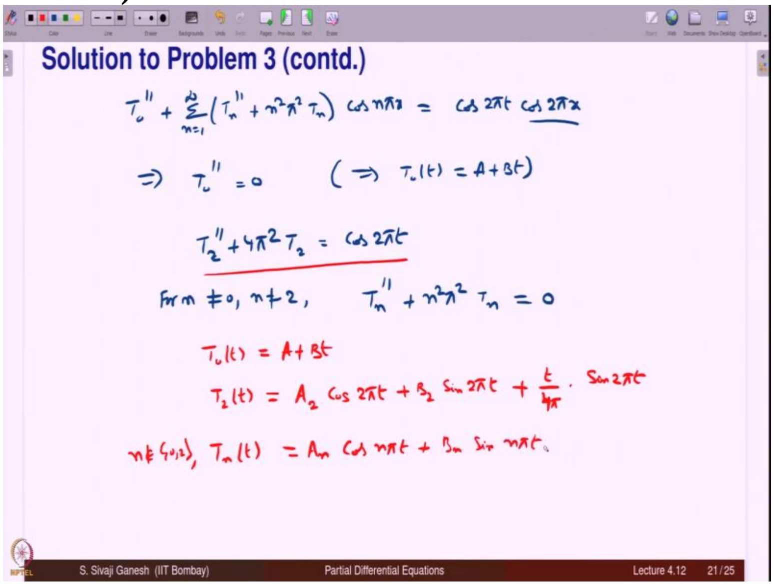

T 0 double dash summation n = 1 to n infinity T n double dash + n square pi square T n into cos n pi x = cos 2 pi t cos 2 pi x this is what we get? After we; substitute the expression given in the non homogenous wave equation. So, from here comparing T 0 double dash = 0 that is of course gives that T 0 of t is A + B t then T 2 double dash + 4 pi square T 2 = cos 2 pi t is is something like comparison of 2 vectors in a vector space.

Which are already expresses in kind of bases something like that component must be equal. So, cos 2 pi x coefficient is cos 2 pi t and here it is this T 2 double dash + 4 pi square T 2 and other things are 0. So, for n not equal to 0 and n not equal to 2 the equation will be T n double dash + n square pi square T n = 0. Now we need to solve these equations so T 0 we already solved let us write separately now T 0 of t is A + B t.

What is T 2 of t? We have to solve this non homogenous equation ODE that is A 2 cos 2 pi t + B 2 sin 2 pi t + t / 4 pi into sin 2 pi t. And T n of t for n not equal to 0 and 2 T n of t = A n cos n pi t + B n sin n pi t.

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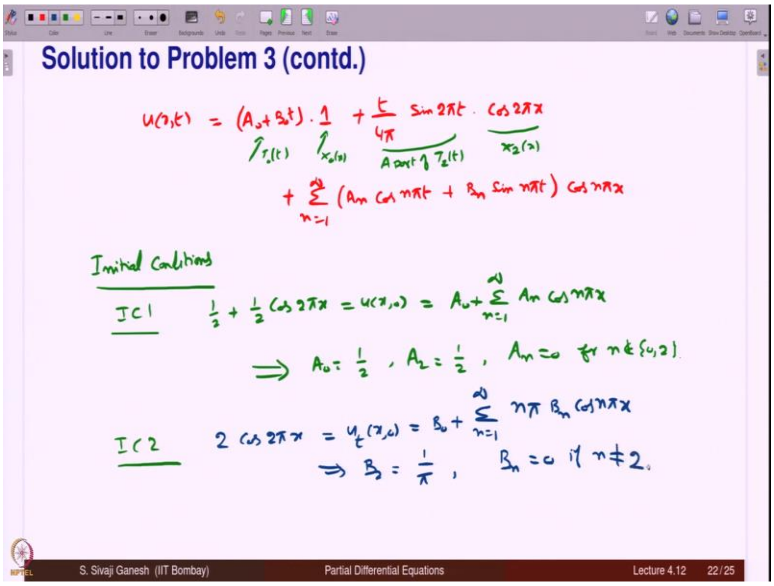

Therefore what is u x t? u x, t = = A 0 + B 0 t into 1 + t / 4 pi sin 2 pi t cos 2 pi x plus summation n = 1 infinity A n cos n pi t + B n sin n pi t into cos n pi x what are this terms? This is T 0 of t this X 0 of x this is a part of T 2 of t remaining part is clubbed in this expression this X 2 of x. So, this is expression for u x t now we have to apply initial conditions that we have.

So, what are the initial conditions? So, IC 1 is half + half cos 2 pi x this what is u x 0 you can check that that is equal to A 0 plus summation n = 1 to n infinity A n cos n pi x. Now for the same comparison A 0 = 1 / 2 A 2 is also equal to 1 / 2 and A n = 0 for n different form 0 and 2. Then IC 2 that is 2 cos 2 pi x = u t of x 0 and that is = B 0 plus summation n = 1 to n infinity n pi B n cos n pi x. That implies that B 2 = 1 / pi and B n = 0 if n is not equal to 2 so combining all this.

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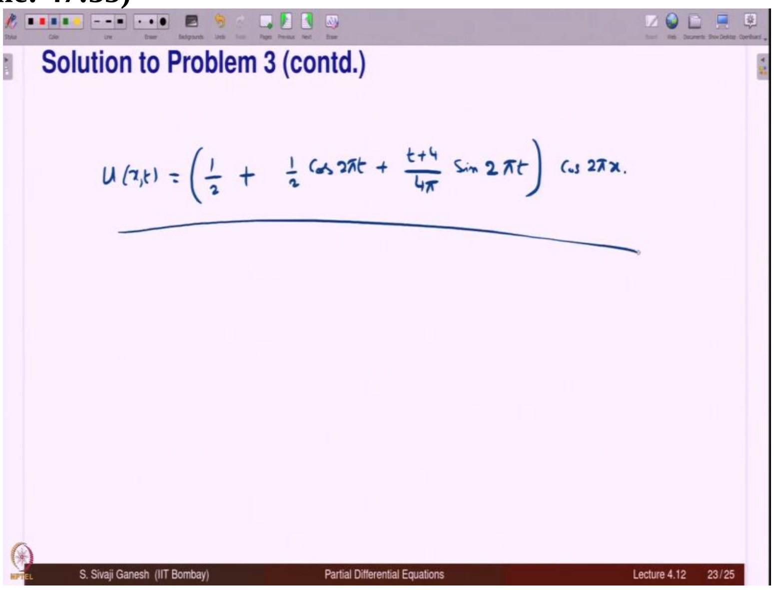

We get this expression for the solution u of x t = half + half cos 2 pi t + t + 4 / 4 pi sin 2 pi t into cos 2 pi x so, this is the solution.

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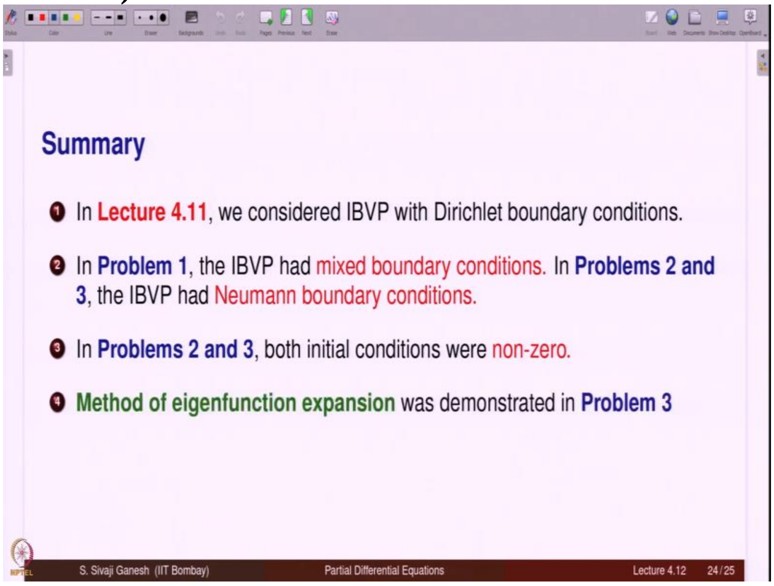

Let us see the summary of what we did in this tutorial in lecture 4.11 we considered IBVP with Dirichlet boundary conditions. In problem 1 we had mix boundary condition and problem 2 and 3 we had Neumann boundary conditions in problems 2 and 3 both initial conditions were non 0. So, method of eigenfunctions expansion was demonstrated in problem 3 to solve non homogenous equations, thank you.