Partial Differential Equations
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Lecture – 4.11
IBVP for Wave Equation Separation of Variables Method

Welcome to this lecture on initial boundary value problems for wave equation. We have already discussed this problem earlier. In this lecture we are going to see a method called separation of variables method for the solution of IBVP's.

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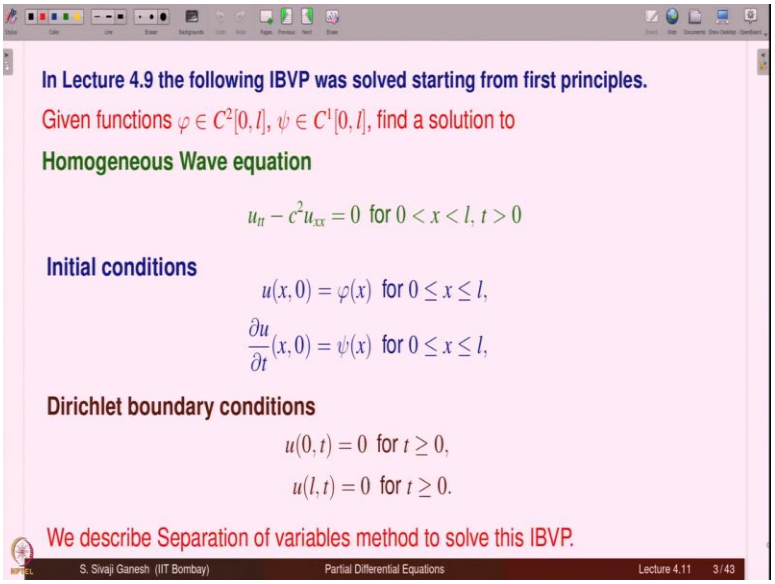

In lecture 4.9 the following IBVP was solved starting from first principle's which is the homogeneous wave equation and phi and psi is the Cauchy data on 0 Dirichlet boundary conditions. We describe separation of variable methods to solve this IBVP in this lecture.

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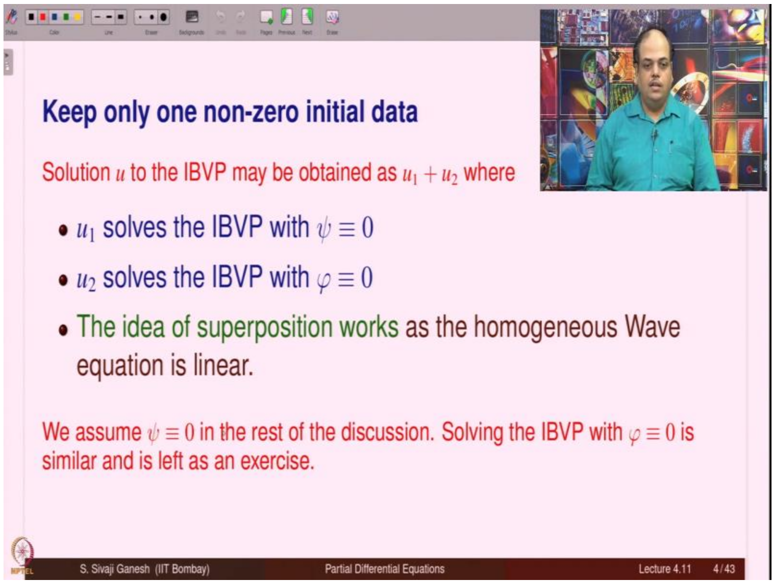

So, first thing is keep only one nonzero initial data that is keep either phi or psi that means keep phi make psi 0 or make phi 0 and keep psi. So, in this lecture we are going to consider with psi = 0, for obvious reasons because the equation is linear. Therefore if you solve this IBVP with psi = 0 and then one more IBVP phi = 0 and add these 2 solutions you will get solution to this IBVP. So, solution u to the IBVP may be obtained as u 1 + u 2 where u 1 solves the IBVP with psi = 0 and u 2 solves the IBVP with phi identically equal to 0.

The idea superposition works because the equation is linear and homogeneous. Therefore we assume that psi is identically equals 0 and the rest of this discussion solving the IBVP with phi identically equal to 0 is completely similar and is left as an exercise. Of course in the computation at we are going to do we can keep the psi as it is and see where it is going to play a role where the psi will appear in our computation. I will point out that part, it is only to keep the calculations to minimum and theorems to you know reasonable length that I am considering psi to be identically equal to 0.

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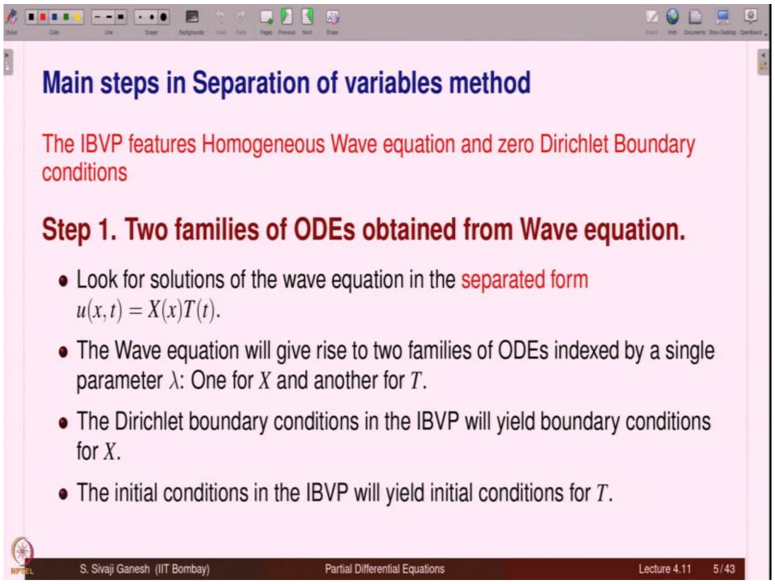

So, what are the main steps involved in separation of variables method, the IBVP features homogeneous wave equation and zero Dirichlet boundary conditions there is to be kept in mind. Step 1 is we derived 2 families of ODEs from wave equation. How do we get that look for solutions or the wave equation in these separated forms that is u of x t is not any arbitrary mix of x and t but it is a product of a function of x and a function of t.

Such solutions we are going to look for they are known as solutions in this separated form. So, the wave equation will give rise to 2 families of ODEs indexed by a single parameter lambda we are going to see that one 4 X one for T. Now from solving PDEs we are reduced to solving for ODEs which are considerably easier than the PDEs. So, the Dirichlet boundary conditions in the IBVP will yield boundary conditions for X. And the initial conditions in the IBVP will yield initial condition for T.

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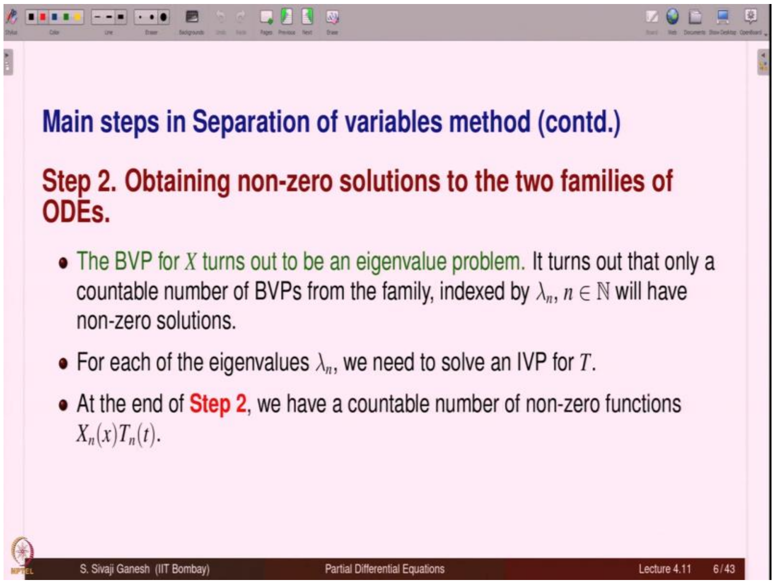

We will see exactly later, so step 2 is obtaining nonzero solutions to the 2 families of ODEs. So, the BVP for X turns out to be an eigenvalue problem, recall an eigenvalue problem from your ODE courses. It turns out that only a countable number of BVPs from the family indexed by lambda n will have nonzero solutions we will see that. And for each of the eigenvalues we need to solve an initial value problem for T. So that means first we solve the problem for X and then solve for T. At the end of step 2 we have a countable number of nonzero functions X n x into T n t.

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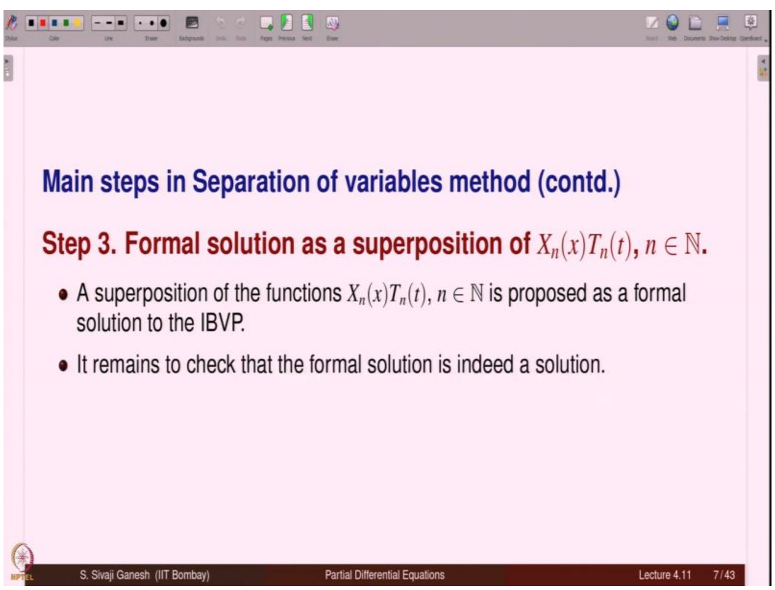

Then the step 3 is to propose a formal solution to the initial boundary value problem as a superposition of the functions that we have obtained namely X n x into T n t. So, a superposition of this function is proposed with a formal solution. It remains to check that the formal solution is indeed a solution. So, far these are methods that we add that is been described to obtain a solution. And once you think you have a solution you must check there is a solution.

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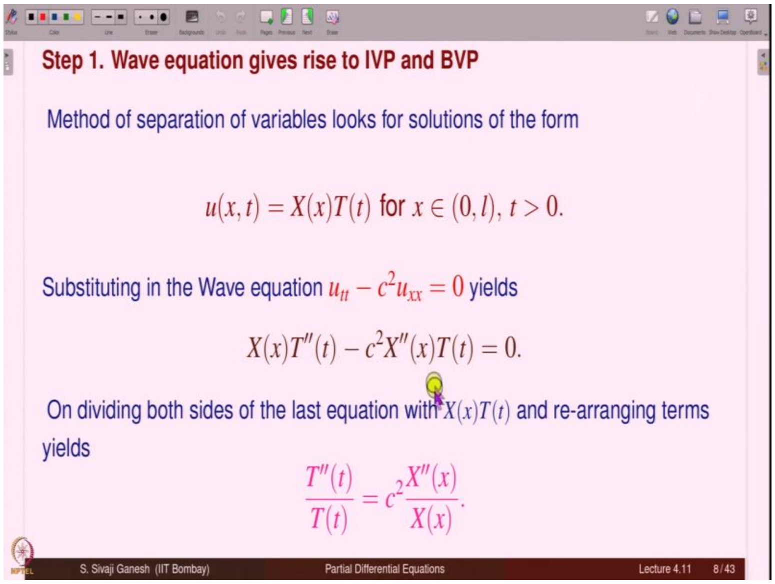

So, step 1 let us implement wave equation giving rise to an IVP and BVP how it is going to come let us see. So, separation of variables looks for solutions of this form as we said u x t is a function of x into a function of t we take this and substitute in the wave equation u tt becomes x T double dash and similarly u xx will become X double dash into T. Therefore the wave equation becomes this, now we divide this equation with xx into tt and then re-arrange terms we get this, we get T double dash / T = c square X double dash / X.

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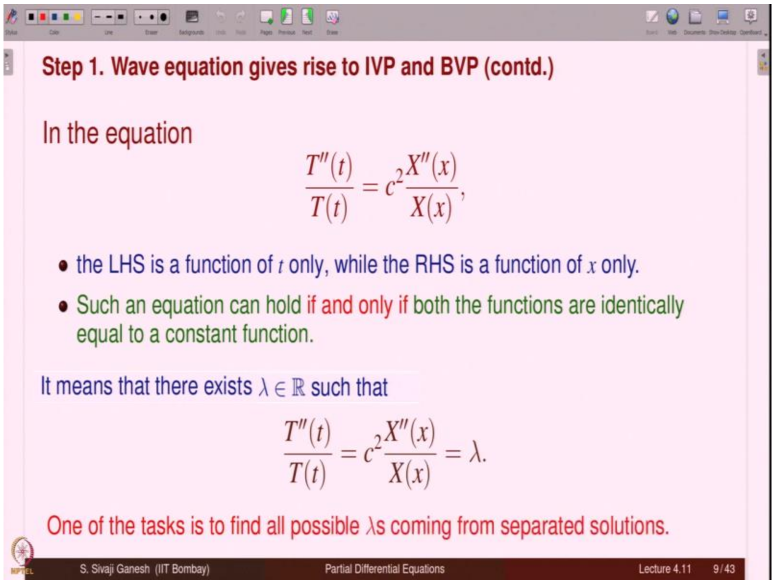

Now if you observe in this equation the left hand side is a function of t only because it depends only on T, T of t, T double dash of t so it is a function of t only. And RHS is a function of x only a function of t only equal to a function of x only. So, the only possibility is that there are constant functions. So, both functions are equal to a constant function, it means that there is a lambda real numbers such that this equal to lambda on this equal to lambda which is written as this. These are 2 equations T double dash / T = lambda and c square X double dash / X = lambda, so one of the tasks is to determine the lambda which arise out of the separated solutions.

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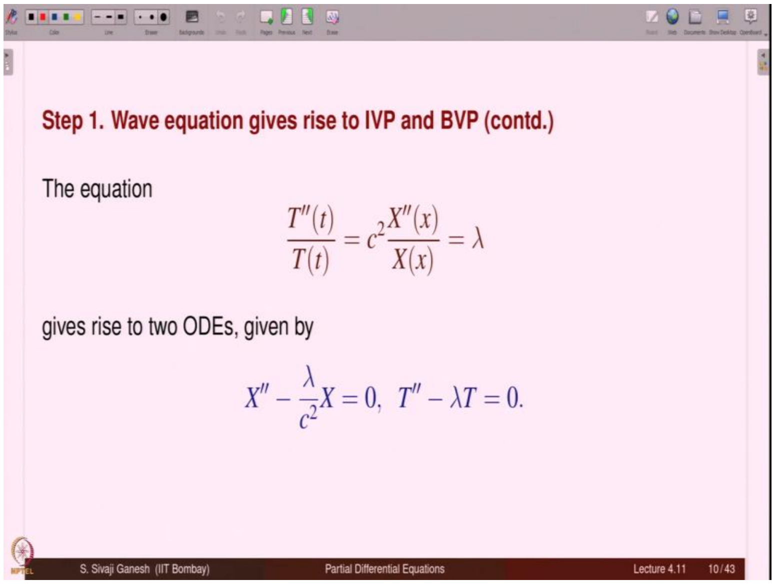

So, thus we get 2 ODEs which is X double dash - lambda / c square X = 0 and T double dash - lambda T = 0.

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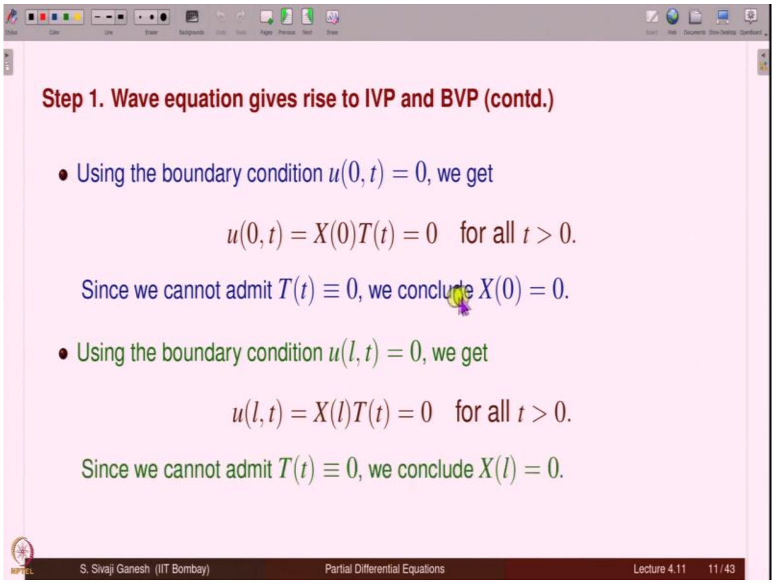

Now we have boundary condition u of 0 t = 0 from that boundary condition we get X 0 into T t = 0 for all t positive in fact t greater than equal to 0. So that means X of 0 must be zero if X of 0 is not 0 T of t is identically equal to 0 function once t identically is equal to 0 function then xx into tt will be identically equal to 0 function. So, we are not doing any great job in finding a 0 solution, so we are not interested in that.

So, we do not want to admit T to be identically equal to 0 therefore we conclude X of 0 is 0. Similarly we have one more boundary condition u of l t = 0 from which we get Xl into T t = 0 therefore Xl for the same reasons Xl = 0. So, we have got boundary conditions for X what are those X of 0 = 0 and X of l = 0.

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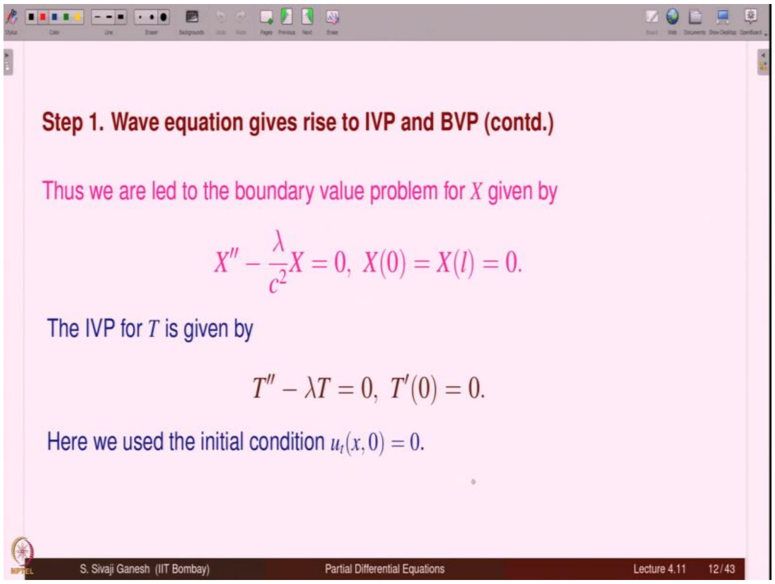

Therefore we have the ODE that we have already obtained. Now we have got the boundary value problem these are the boundary conditions and the IVP for T is given by T dash, T double dash - lambda T = 0 and T prime 0 = 0. Of course there is one more initial condition that I cannot apply. That is namely u of x 0 = phi X, we have the second one which is u, t of X 0 = 0. So, in that if we substitute u = xx into tt we get T prime 0 = 0. So, this one initial condition for T, so, if we had the function psi we cannot even write T dash of 0 = 0.

If psi was present we cannot get this initial condition in which case we work with only this ODE. We do not consider any initial value problem because we do not get one solve them and that features 2 constants being a second order ODE it we have 2 arbitrary constants and then we consider the product solutions. And in step 3 we propose your formal solution as superposition of this solutions and then determine the constants that we have in coming from this ODE using the 2 initial conditions that we have namely u x 0 = phi X, u t x 0 = psi X. Since we are assuming psi X = 0, in this lecture we get this condition T prime 0 = 0 by using this condition.

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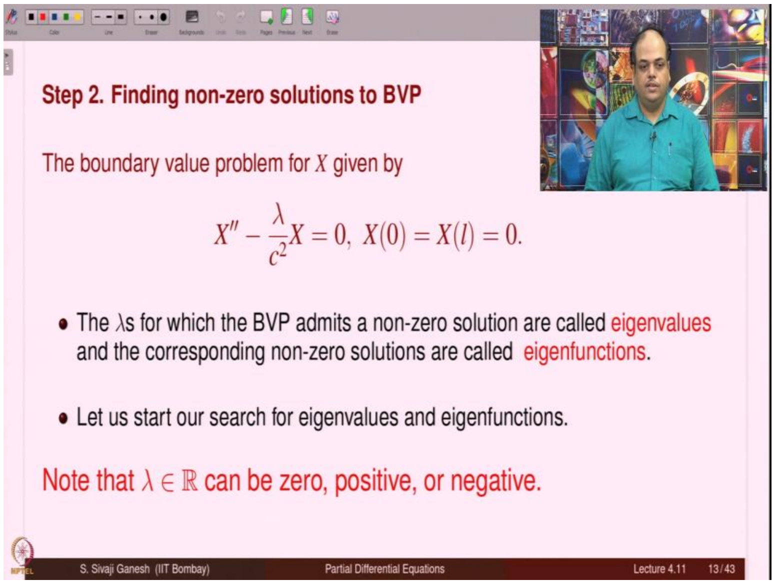

So, this is the boundary value problem that we have for X, X double dash - lambda / c square X = 0 X of 0 = X of L = 0. Now the lambdas for which the BVP admits a nonzero solution remember we are interested in only nonzero solutions. Such lambdas are called eigenvalues and the corresponding nonzero solutions are called eigenfunctions. So, let us start our search for eigenvalues and eigenfunctions.

Note that a real number lambda can be 0 positive or negative. So, we are going to make our divide our search into 3 parts, we will ask whether it is possible to get zero eigenvalue positive eigenvalues, negative eigenvalues. That is because the equation solution is like that it is X double dash plus some constant times X = 0. So, depending on that constant is 0 positive or negative the form of the solution changes. That is the reason for making these separate studies in the 3 cases.

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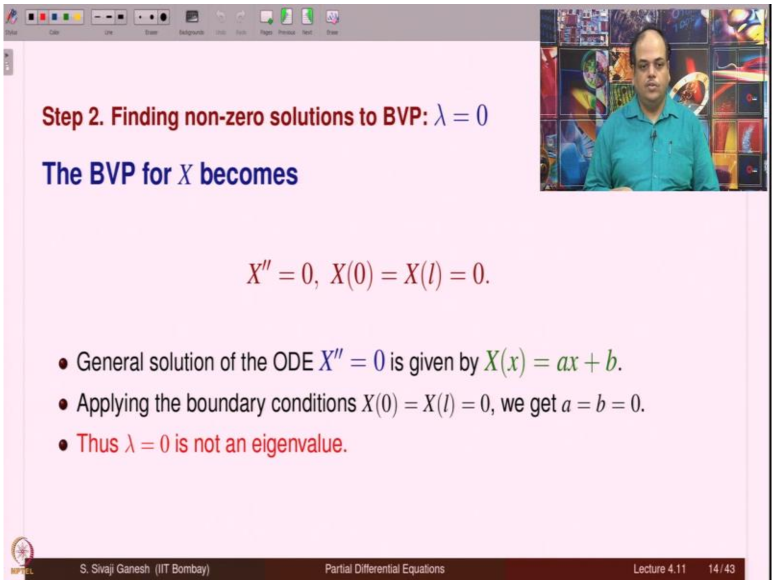

Let us look at lambda = 0, so the BVP is X double dash = 0 X of 0 = X of l = 0. General solution of the ODE X double dash = 0 is ax + b where a and b are constants. Now we are going to use these boundary conditions and determine a and b. So, applying the boundary conditions we get a and b to be 0. That is obvious because this is a straight line and these conditions are demanding that it should pass through 0 0 and l 0. That means the graph must be parallel to x axis which is the case only when it is 0 function. So, lambda = 0 is not an eigenvalue.

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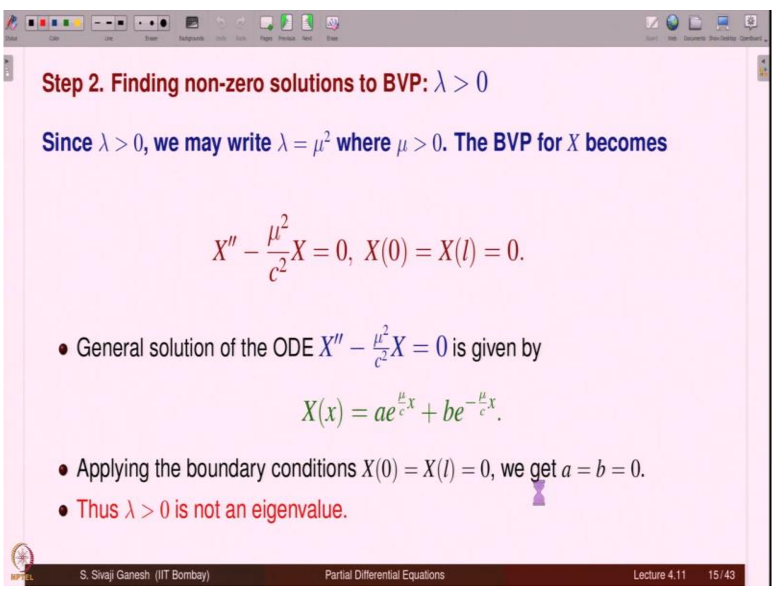

So, we do not bother about that it is not interesting, let us see whether it is possible to find eigenvalues which are positive lambda positive. So, once lambda is positive we can write it as lambda = mu square where mu is positive and the BVP becomes X double dash - mu square / c square X = 0 and boundary conditions are X of 0 and X of l both of them are 0. Now the general solution of this ODE is given by ae power mu / c x + be power -mu / c x.

If you are not written lambda = mu square we would have here a square root of lambda / c into x - square root of lambda / c into x. Just to reduce the notation clutter we have supposed that lambda looks like mu square where mu is positive. Supplying the boundary conditions we get a = b = 0. So, please go through these computations and conclude by yourself that this is indeed the case that a and b both of them are 0. Thus lambda positive is not an eigenvalue it cannot be eigenvalue that means no eigenvalues are positive.

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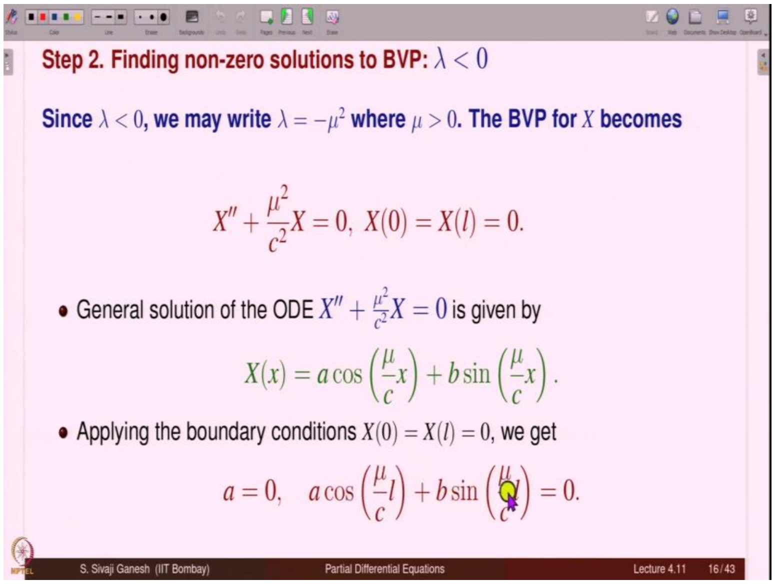

Now let us search for negative eigenvalues whether such eigenvalues are there or not. So, since lambda is negative we may write lambda = -mu square and mu is positive where mu is positive. The BVP for X becomes X double dash + mu square / c square X = 0 X of 0 and X of l both of them are 0. Now general solution of this ODE is in terms of sin and cosine. So, it is a cos mu / c x + b sin mu / c x.

Now we have to see whether there are a and b's for which these boundary conditions are satisfied. So, nonzero a and b's, so applying the boundary conditions what we get a = 0 because X of 0 if you put this is 0 sin 0 is 0 therefore what you get is a times cos 0 which is 1 equal to 0.

And when you put X = l this equation you get. Since a is already equal to 0 this relation is essentially b sin mu / c l = 0.

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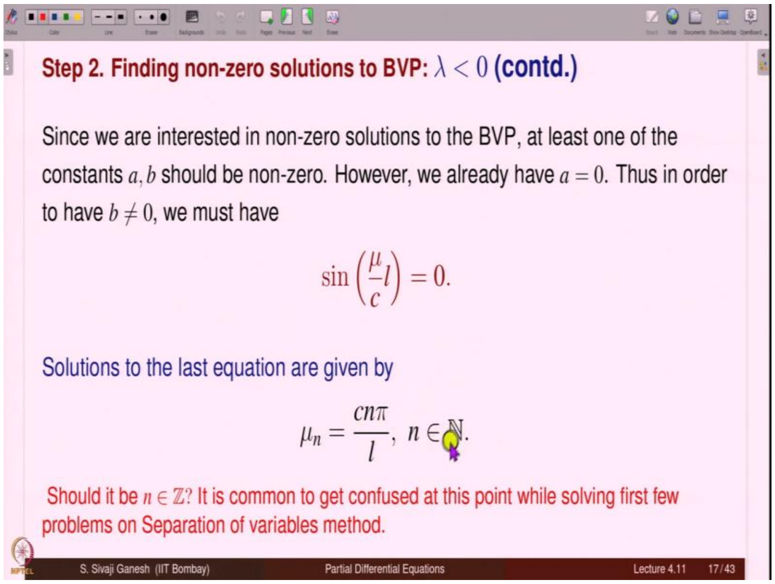

So, we want at least one of the constants a, b to be nonzero but a is already 0. Therefore to have been on 0 we must have sin of mu / c l = 0. So, sin of something is 0 means that gives a multiple of pi. So, mu n = cn pi / l, n belongs to N. So, these are the solutions of this should n belongs to Z is this is a mistake n belongs N and n belongs to Z because solutions of sin of something equals 0 means that has to be a multiple of pi it can be an integral multiple of pi.

Therefore should it be n belongs to Z, so it is a common thing to get confused at this point. It happens to everybody when they are solving first few problems involving separation of variables method, no it is n belongs to N not n belongs to Z this is not correct, because if n belongs to Z that means a negative number you take mu n becomes negative but you see when we are looking for lambda less than 0 we suppose that lambda = -mu square and mu is positive, therefore mu n is cannot be negative therefore n belongs to N is correct keep this in mind be very careful at such points.

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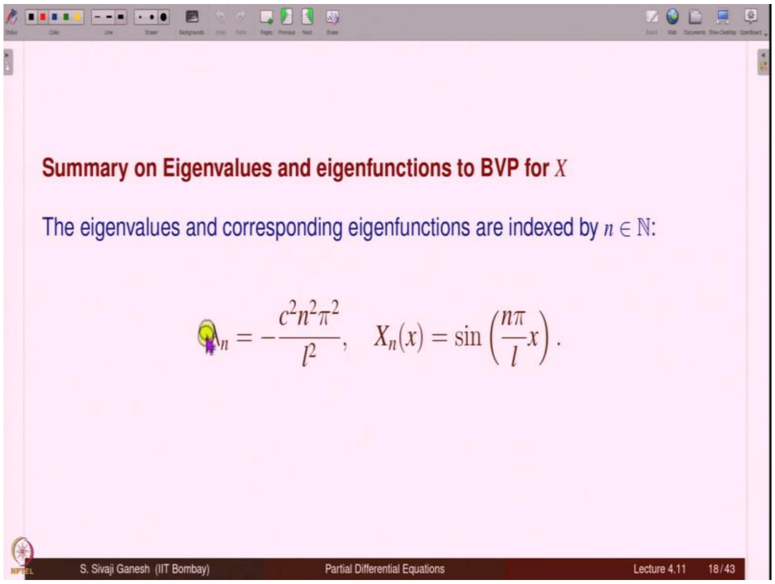

So, let us see what we have got so far about the boundary value problem for X. The eigenvalues and corresponding eigenfunctions are indexed by natural numbers they are actually countable. So, any countable set can be in exponential numbers, so lambda n = -c square n square pi square l square and X n of x is sin n pi / l x. So, we have got now some numbers lambda n now for each of such n we have to solve the initial value problem for T.

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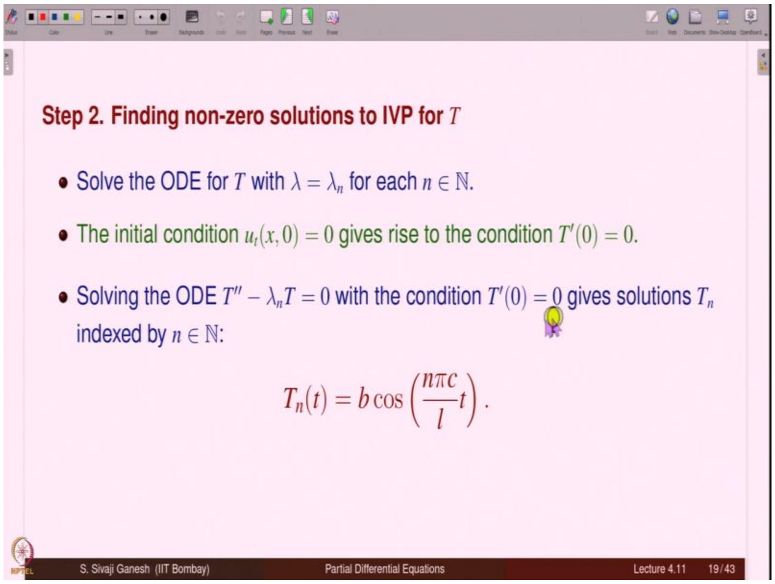

So, solve this ODE for T with lambda = lambda n for each n the initial condition u t of x 0 as we already discussed gives us the condition T prime 0 = 0. Therefore solution of T double dash - lambda n T = 0 with T prime 0 = 0 gives us T n = b cosine of n pi c / l into t.

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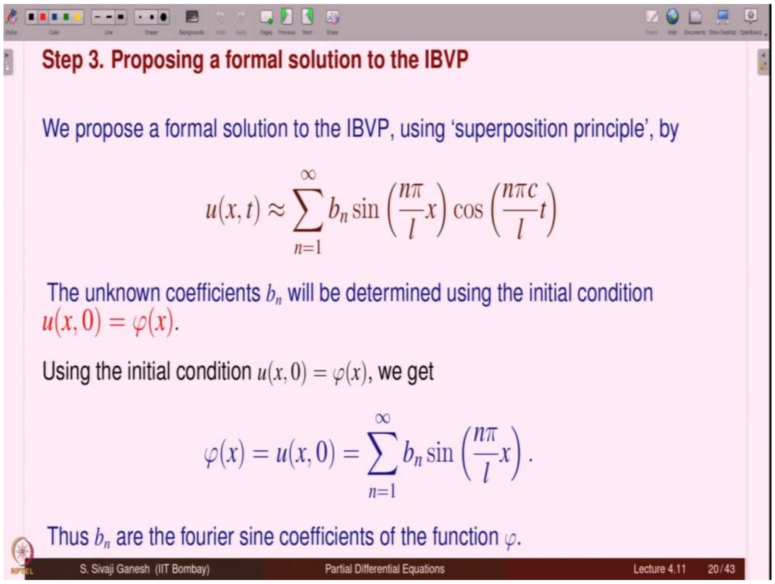

So, now we are in a position to propose a formal series solution to the IBVP using superposition principle, so that is summation n = 1 to infinity b n sin n pi / l x cos n pi c / l into t. So, this is precisely x n x into t n t, so we are considering superposition of x n x and t n t with coefficient b n summation n = 1 to infinity. Now we do not use equality away here that is because we do not know whether the series converges or not.

So, the unknown coefficient what are these b n's that also we should have an idea and they will be determined using the initial condition u x 0 = phi x, recall that we have considered psi = 0 if you are not considered psi = 0 what we will have here is sin n pi / l x into b n cosine of n pi c / l t + cn sin n pi / c l into t then we have determined both b n and c n using the 2 initial conditions one is this u x 0 is phi x and the second one is u t x 0 = psi x.

Because we assumed psi = 0 we only have one term here we do not have an extra term, that information was already used in that namely that psi = 0 was already used in obtaining solution for t. Now using the initial condition u x 0 = phi x what we have is put t = 0, cosine 0 is 1. So, phi x is b n sin n pi / l x. So, this gives us an idea of what should be the coefficient b n is. Later on we have to prove everything rigorously.

These only for guessing what should the b n being. So, b n's are the Fourier sin coefficient are the function phi that means phi when we express has a Fourier series it should have only sin terms that means cosine terms must be missing and b n will be the coefficients of sin terms, cosine terms must be missing tells us that function should be kind of odd function we will see that.

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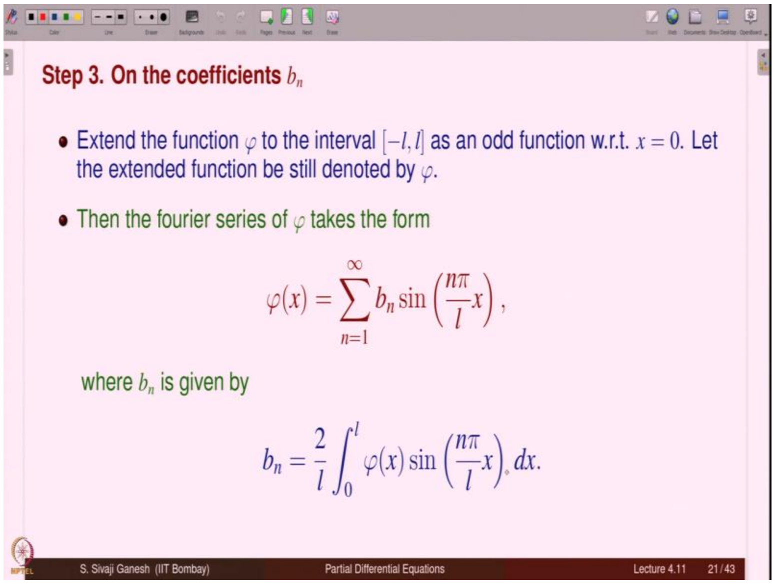

So, extend the function phi to the interval, phi is given only on 0 l. So, externally to minus l, l as an odd function with respect to x = 0. Let the extend function be still denoted by phi then the Fourier series of phi takes this form phi x = summation n = 1 to infinity b n sin n pi / l x note here there are no cosign terms here that is because phi is an odd function. The coefficients correspond to cosine terms will be 0. And what is b n? It is given by this expression. Formally if you want to get you can easily get multiply both sides with the sine k / l x integrate between 0 to l, you will get such an expression for b k.

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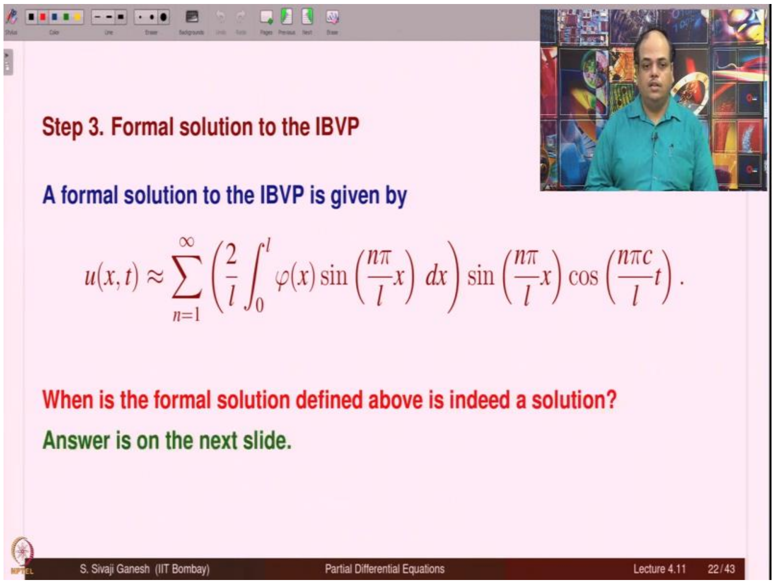

Now formal solution to IBVP is now given by this so I have just saw substituted values of b n. So, this is still a proposed candidate solution to the IBVP. That is why we still use this approximation symbol, just to make sure that we have not checked things. So, when this is formal solution which is defined above is indeed a solution.

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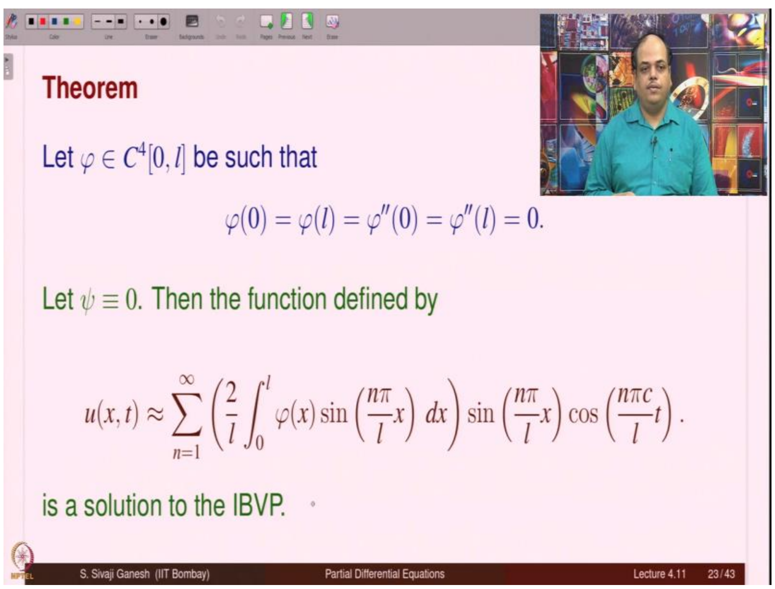

We have an answer on the next slide that is a theorem like phi be C 4, 4 times continuously differentiable and be such that phi 0, phi l, phi double dash 0, phi double dash l all of them are 0. Let us psi be identically equal to 0. So, what we want is phi smooth these are the compatibility conditions that we already encountered when we solve the same problem using first principles.

And we here we are assuming psi = 0 then the function defined by this formal infinite series and that is a solution to the IBVP.

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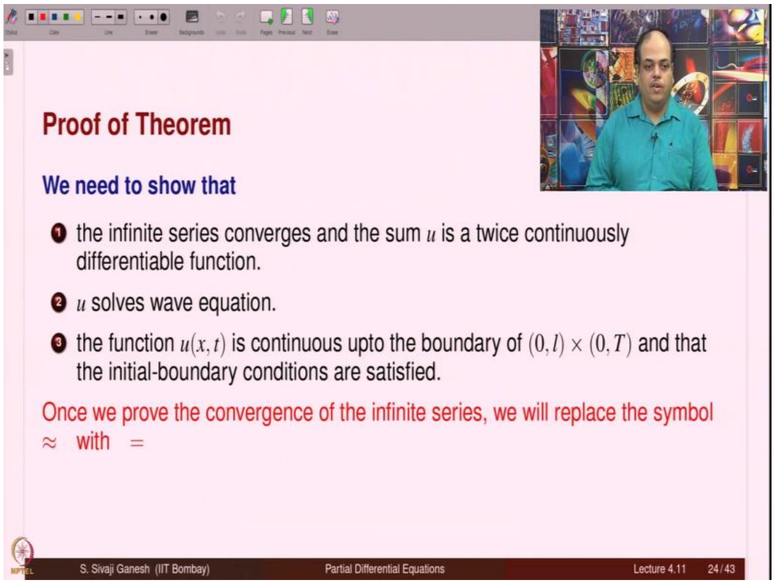

Let us prove this theorem, we need to show that the infinite series converges on the resultant function that we get is twice continuously differentiable function let us call it u itself. So, infinite series converges defines a function u of x t and that function is C 2 to function and then it solves a wave equation and it is continuous up to the boundary of 0 l cross 0 t. So, that the boundary conditions are meaningful namely u of 0 t and u of l t.

And initial conditions are meaningful that is u of x 0 is phi x we also need u t of x 0 = 0 that means the derivative of u should also be continuous to 0 l cross closed 0 T. So, once we prove the convergence of the infinite series we will replace the symbol that approximate symbol with equality.

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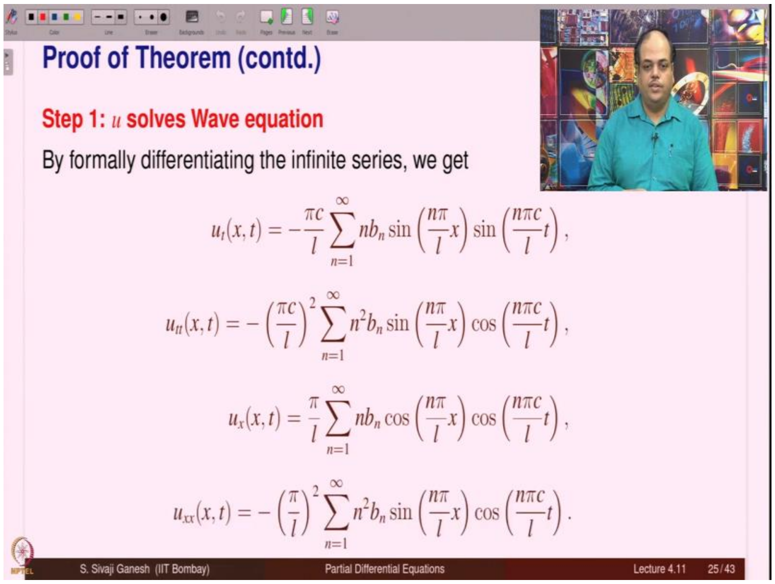

So, let us check by formally differentiating the infinite series we get u t equal to this you can see just by formal differentiation, so that means we are not saying that the infinite series converges or it can be differentiated term by term all such questions we are not answering we formally differentiate that means we pretend everything is alright and differentiate. So, I use u t because u denoted that infinite expansion infinite series.

So, u t will denote this similarly u tt, u x and u xx. If you notice whenever you differentiate once you pick up n here, in the expression for you there was no n it would just b n. Whenever you differentiate one derivative you get n here also n, if you are different 2 times you get n square and n square. If you look at these terms here sin terms are bounded by one. So, essentially if you want to prove the convergence by dominating this series with a convergent series you need that summation mod n b n converges. Here we need mod n square b n converges. So that in fact gives rise to our assumption why we have assumed that phi is C 4.

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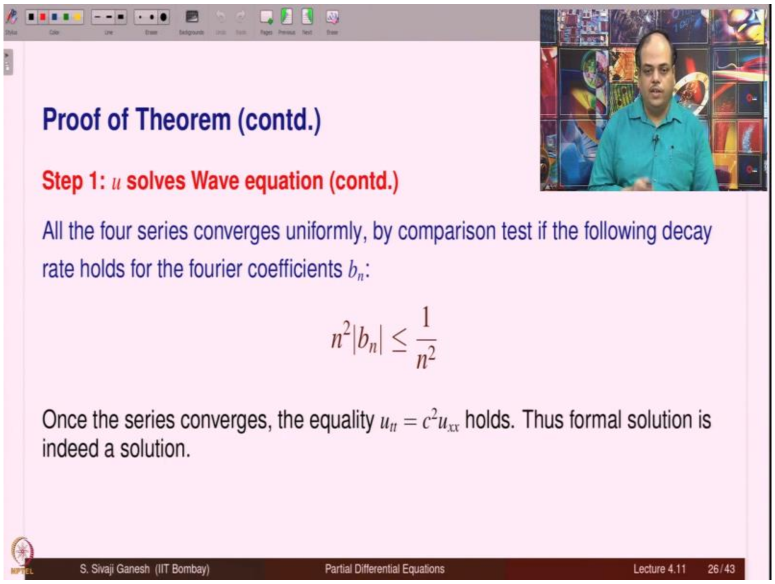

We will see those computations, so all the 4 series converges uniformly by comparison test that I have just mentioned, if we have the following estimate for b n n square b n summation n square b n modulus b n less than or equal summation 1 / n square and I know summation 1 / n square is finite it converges. So, this is one sufficient condition, so once the series converges there is no doubt that u tt = c square u xx holds we can see from the series that we have computed on earlier slides. So, therefore formal solution is indeed a solution of the wave equation.

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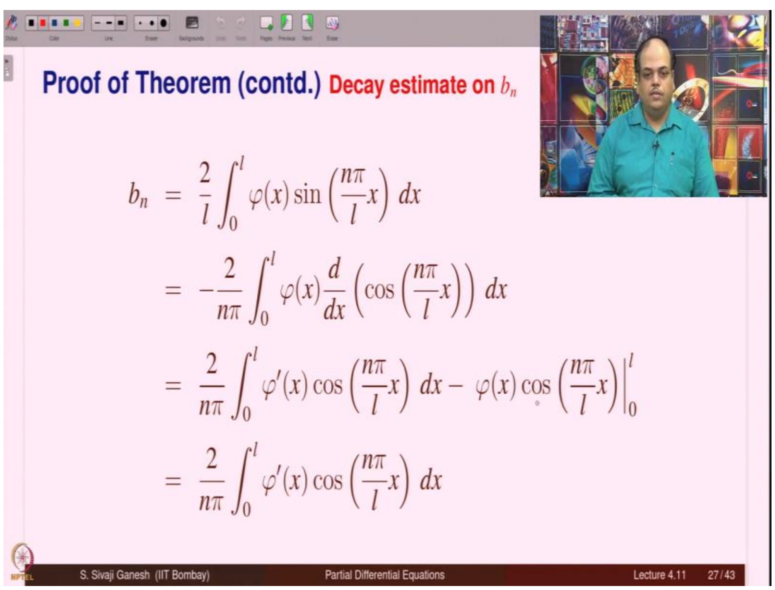

Now how do we get this DK estimate on b n, b n recall this is a formula for b n, now I write the sin term as d / dx of cos, so I get minus 2 / n pi 0 to l phi x d / dx of these cos. So, this means I am planning to use integration by parts, so d / dx will be shifted to phi I get phi dash x here and cosine term here that gave me a n you see one n came on differentiating ones and they will be bounded terms which is this, it is 0 because phi of 0 and phi of l are assumed to be 0. Now let us do this integration by parts once more namely we write cos as once again d / dx of sign.

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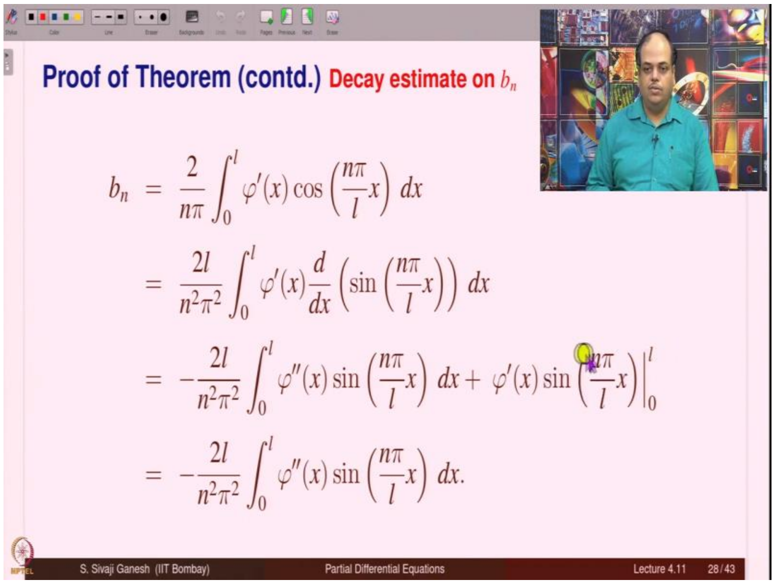

And that we pick up an n square and again integration by parts gives you this boundary terms once again are 0, why the boundary term is 0? That is exercises please see that earlier we got it from the compatibility conditions. Now we do not get it from compatibility conditions but we get it from this sin term when x = l or 0 this is going to be a multiple of pi and sin is 0 at those points that is why this is 0.

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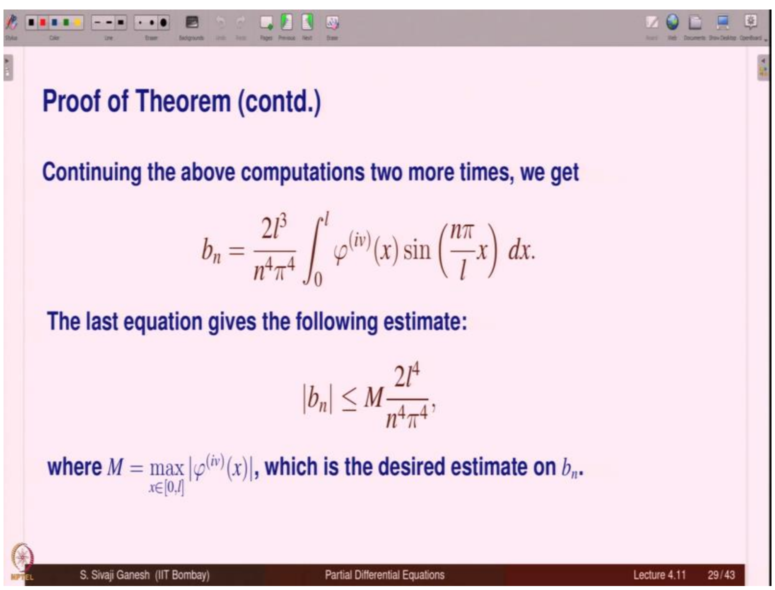

Now we have got this repeat this 2 more times you get this n power 4. So, therefore n square mod b n will be less than or equal to constant by n square. Therefore we get the desired decay estimate for b n. So, the smoother the function is the faster the decay is for the coefficients, so C 4 give us 1 / n power 4.

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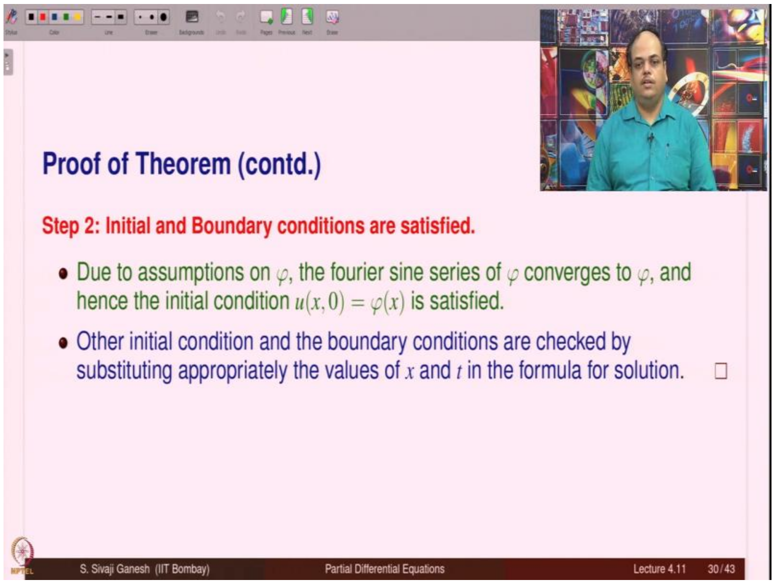

Now initial and boundary conditions, due to assumptions on phi the Fourier sin series or phi converges to phi and hence initial condition is satisfied and other initial condition can also be checked. So, this point I would like to mention that the proof of the theorem we are given is quite sketchy. The intention was to show that the formal solution that we have time using separation of variables method or classical solutions. Only if we assume sufficiently smooth Cauchy data, the actual theorems you can find in books on Fourier series. For example there is a book by GB Folland on Fourier analysis on applications there you will find such theorems.

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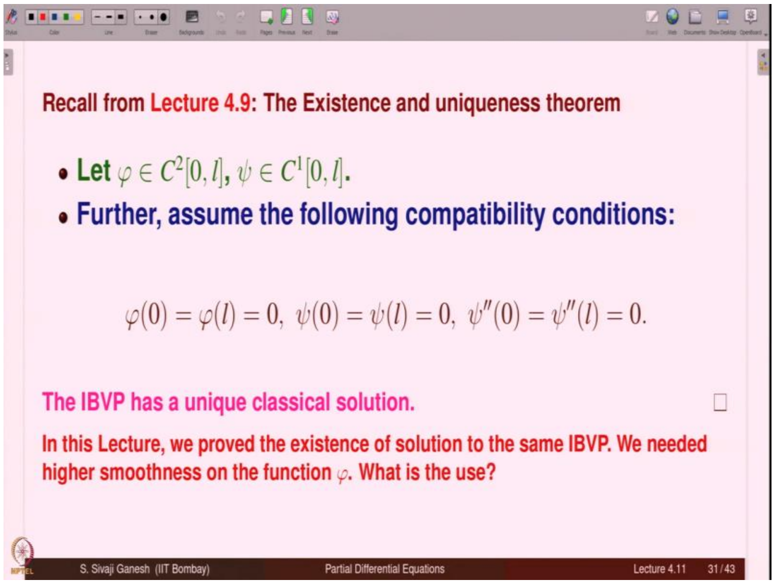

Let us recall from lecture 4.9 we have proved the following existence and uniqueness theorem following first principles for the same problem. Same IBVP of course we considered even psi term there u t x 0 was not assumed to be 0. In this lecture we assumed psi = 0. Now if you psi C 2 in the theorem in this lecture we proved the same theorem of existence but we needed a higher smoothness, so what is the use?.

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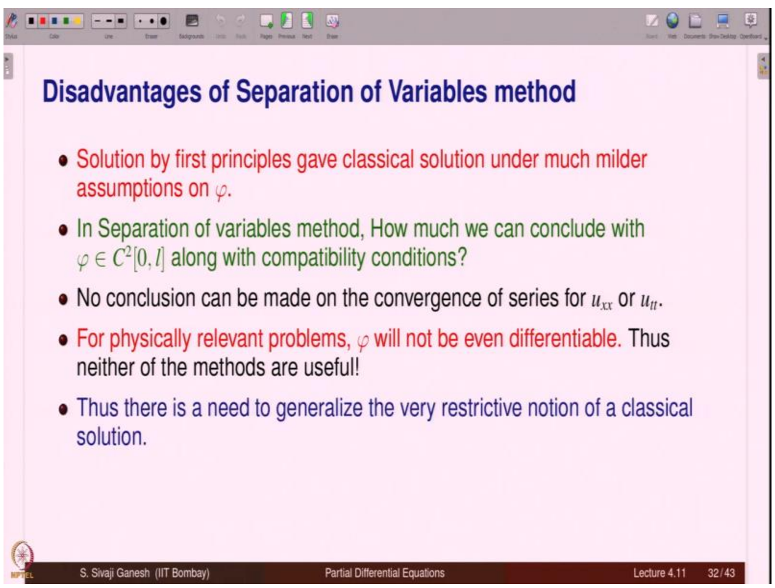

Let us see a few disadvantages of separation of variables method, solution by first principles gave classical solution under much milder assumptions on phi, in separation of variables method how much can we really conclude with those assumptions phi is in C 2, as you have seen the proof we needed n square mod b n to be less than or equal 1 / n square. So, that the series for second order derivatives converges.

Now phi is only C 2 b n are only like mod b n less than or equal to constant by n square. That means the series for the second order that we that we have obtained formally differentiating the formal solution may not converge. So, we cannot make any conclusion about the convergence of the series that we obtained for u xx or u tt because the Fourier coefficients there were n square b n. If phi is C 2 we do not know if summation n square mod b n is a convergent series or not

.

So, for physically relevant problems phi will not be even differentiable, phi will typically be a piecewise linear function which is continuous. So, therefore none of these 2 results are really applicable so we do not have to really worry that separation of variables method required much more smoothness. Finally if you see, both of them are useless as far as classical solution is concerned. Thus there is a need to generalize the very restrictive notion of the classical solution. We describe some ways to define notions of generalized solutions, for completeness sake it is out of the scope of the current course we will do this later on.

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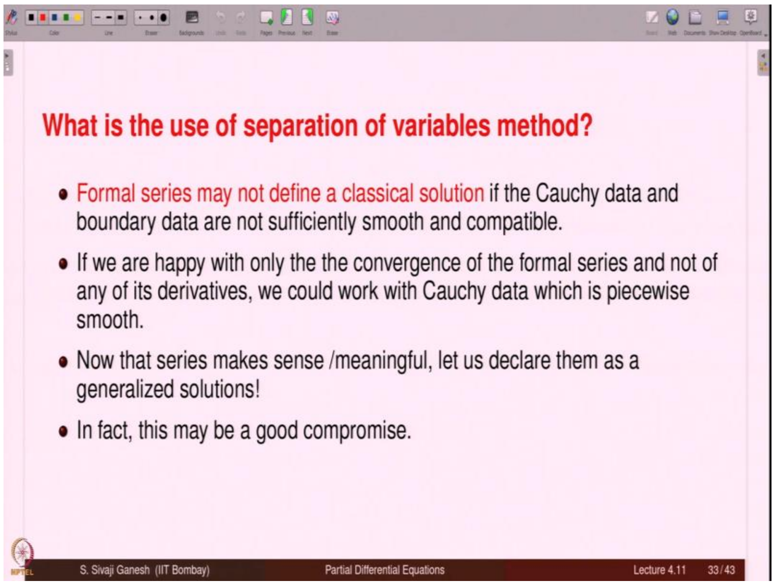

So, formal series may not define a classical solution if the Cauchy data and boundary data are not sufficiently smooth and compatible with each other. If you are happy with only the convergence or the formal series that means u x t equal to the infinite series in the series converges we are very happy with it we do not ask questions about the derivatives. Then we could work with Cauchy data which is piecewise smooth. Now that series makes sense or meaningful let us declare them as generalized solutions. These are the general ideas which go behind in generalizing notions of solutions. In fact this may be a good compromise.

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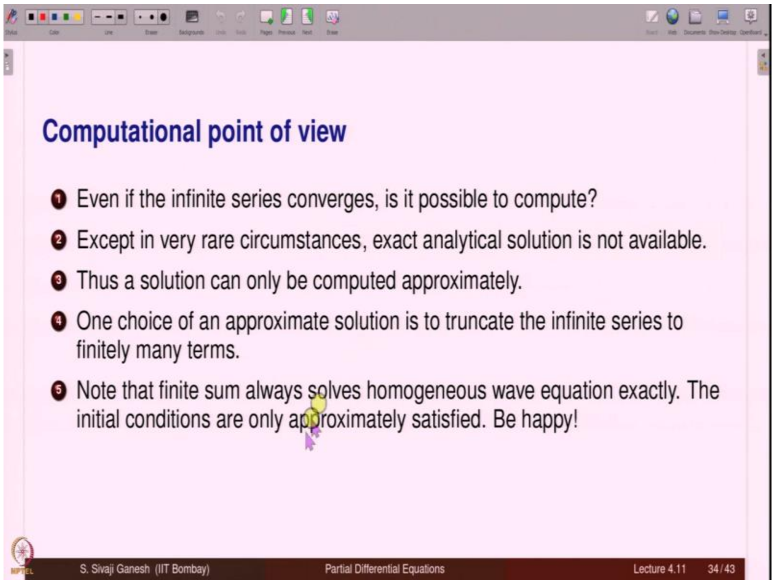

Let us see from computational point of view where is as separation of variables methods stand, imagine the infinite series that we obtained as a formal solution that we propose the, converges. Is it possible to compute? Except in very rare circumstances exact analytical solution is not available which can be easily computed, here we have an infinite series there is a solution can only be computed approximately anywhere.

Whatever may be the expression that you have for the solution except in rarest of the rarest cases you cannot compute exactly; therefore one choice of an approximate solution is to truncate the infinite series to finitely many times. Note that finite sum always solves homogeneous wave equation exactly on boundary conditions are also satisfied. Only initial conditions are satisfied in an approximate sense, be happy with that.

So, therefore some people take this point of view that yes I know that I have an infinite series, I know it does not converge also as long as it converges for the function may not be for the derivatives just the function it converges. Infinite series converges then I want to compute a solution on the of course notion of solution I have to generalize but I know a function which can be computed which is a candidate for solution. And by truncating this infinite sum I am computing it approximately that is one point of view.

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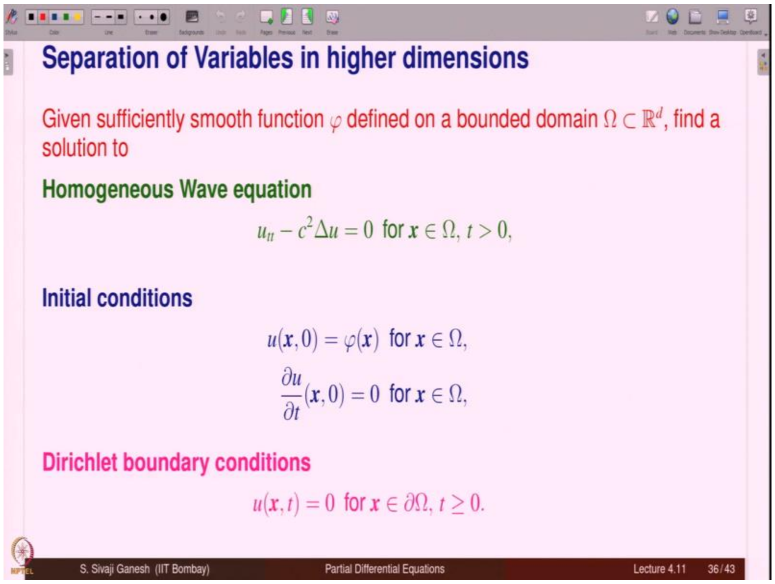

Of course one has to do an error analysis in that case, now separation of variables mattered in higher dimensions what are the new complications. So, problem as before the same given a sufficiently smooth function defined on a bounded domain in R d, find a solution to still homogeneous wave equation. Initial conditions where I have taken psi = 0 already, on Dirichlet boundary conditions for x belongs to boundary of omega you described u of x t to be 0

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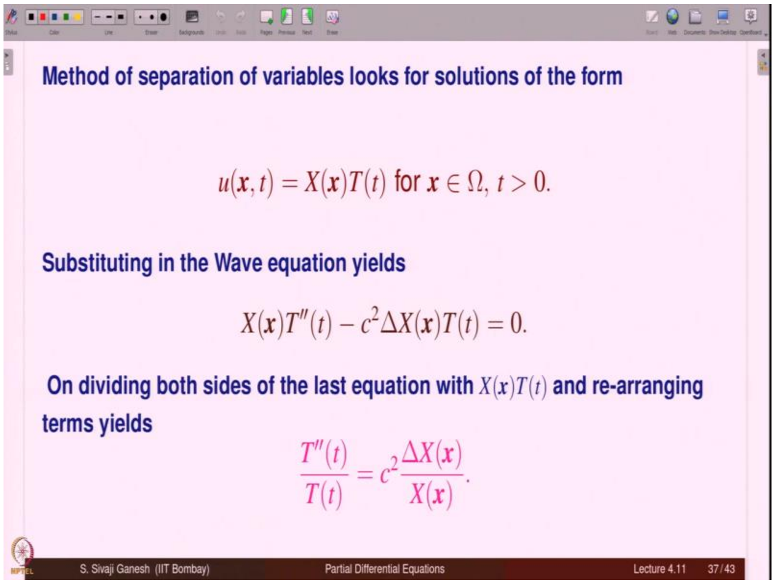

Homogeneous boundary conditions, perfect setting for separation of variables method the way we are seeing it. So, try for the separated solution substitute in the wave equation you get this and then divide both divide with Xx Tt rearrange you get this.

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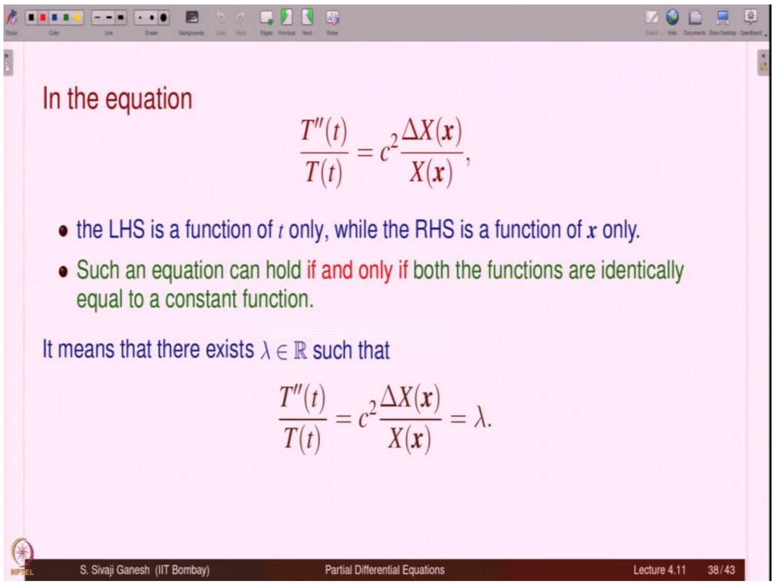

Now we get left hand side is a function of T here, right hand side is a function of X. Therefore there must be constant functions, thus giving rise to an ODE and PDE.

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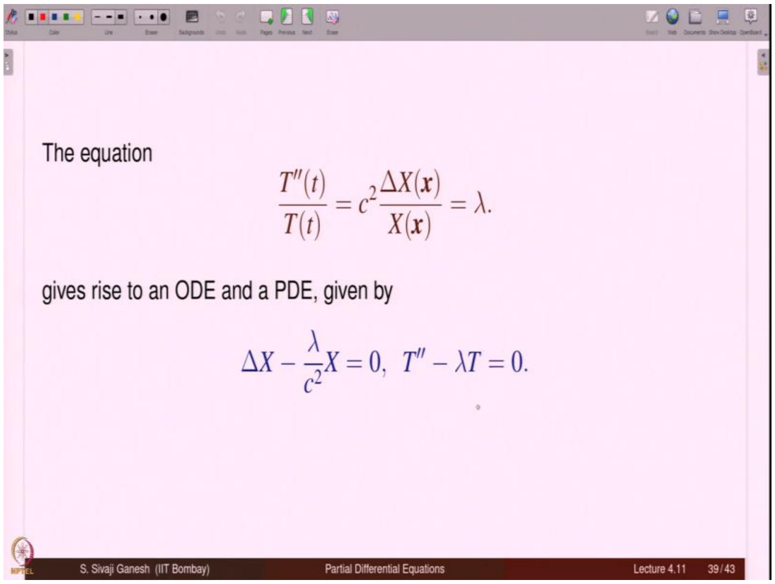

So, PDE for X and ODE for X and we have a boundary condition which is u of x t = 0 for all x in boundary of omega. Therefore we get here a boundary condition for X, X = 0 on the boundary of omega and here we get an initial condition for T as before.

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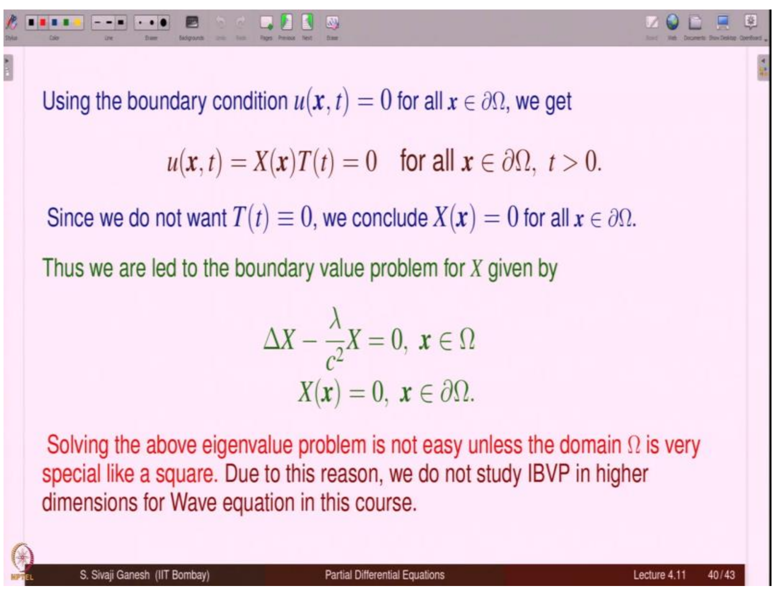

So, this is the boundary value problem we have to solve. Earlier the wave equation gave rise to 2 ODEs when we are considering one space dimension and ODEs is we could easily solve. Now here it is a linear equation once again but solving this is not easy we do not know. So, unless the domain is very special like a square imagine omega is a square. Once again we apply a separation of variables method and reduce that 2 ODEs and do something. So, it can be done but we are not going to discuss how to solve such problems. In this course just wanted to show how separation of variable method looks like in higher dimensions.

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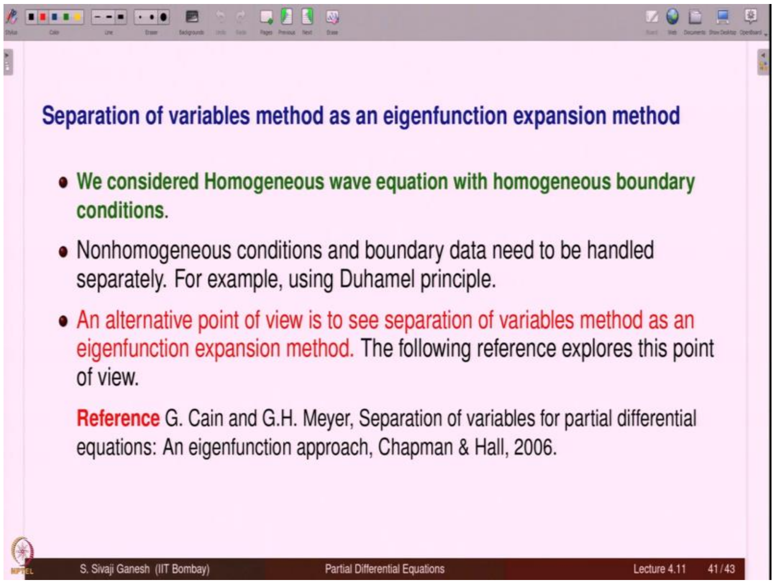

So, separation of variables method has an eigenfunction expansion with there, remember we considered homogeneous wave equation with homogeneous boundary conditions, non homogeneous conditions and boundary data need to be handled separately. Of course for example we have some tools like Duhamel principle. An alternative point of view is to see separation of variables method as an eigenfunction expansion method.

And the following reference of a book explores this point of view. Of course this is out of scope of the current course. The difference is a book by Cain and Meyer separation of variables for partial differential equations and eigenfunction approach.

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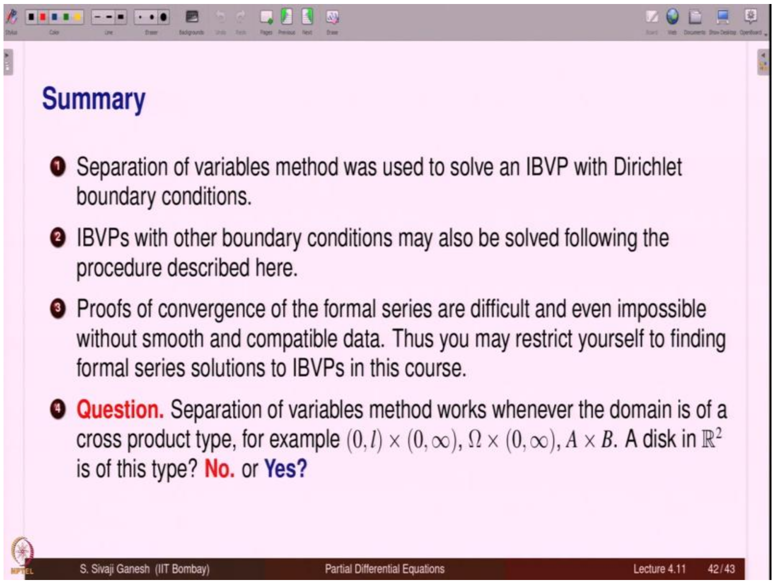

So, let us summarize separation of variables method was used to solve an IBVP with Dirichlet boundary conditions. IBVP with other boundary conditions may also be solved following the procedure described here; we will take it up in the tutorial. Proofs of convergence of a formal series are difficult and even impossible without smooth and compatible data that is you restrict yourself to finding formal series solutions to IBVPs in this course.

Question separation of variables method works whenever the domain is of a cross product type. We have seen 0, l cross 0 infinity X of x T of t, x belongs to 0 l, t belongs to 0 infinity, x belongs to omega t belongs to 0 infinity. A cross B x belongs to A, t belongs to B or x belongs to A, y belongs to B. A disk in R 2 is it of this type? Imagine you have a unit disc even. Does it look like a cross product of domains? Obviously not but in some other coordinate system it does look like, find out what is that coordinate system in which it does look like a cross product type perhaps after removing one point, Thank you.