Partial Differential Equations
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Module No # 06
Lecture No # 33
Wellposedness of Cauchy for Problem Wave Equation

Welcome to this lecture on Wellposedness of Cauchy problem for wave equation. So far we have solved the non-homogeneous Cauchy problem, for non-homogeneous wave equation. Now we are going to show that this problem is well-posed which we are going to define what is the meaning of Wellposedness and then probe?

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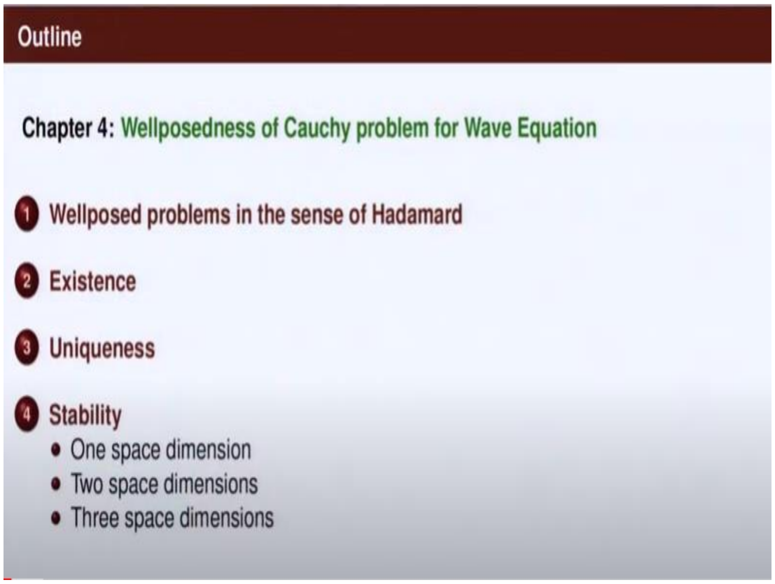

So the outline for this lecture is as follows first we introduce the concept as the notion of well-posedproblems in the sense of Hadamard. Then we discuss about existence of solutions, uniqueness of solution and stability of solutions. These 3 properties are what are required for a problem to well-posed in the sense of hadamard.

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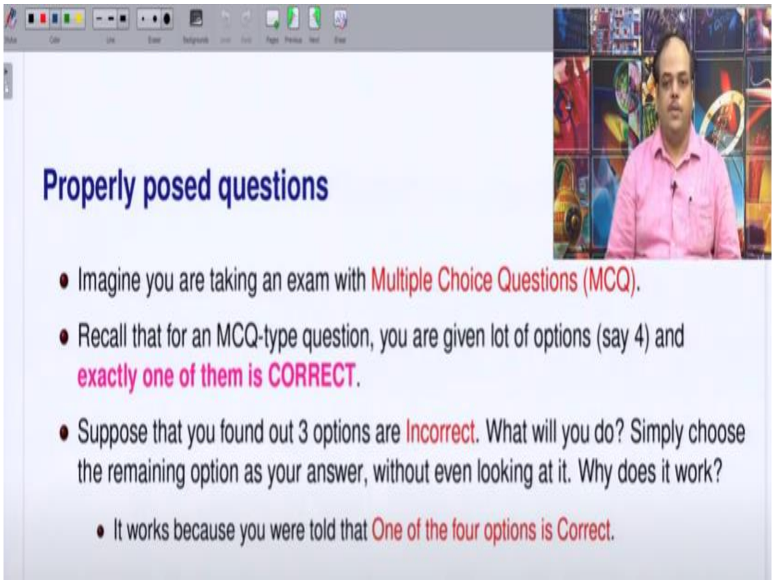

So let us discuss what is called a property pose questions? Imagine you are taking an exam with the multiple choice questions.Recall that for in MCQ type question you are given a lot of options usually 4, let us say 4 options are given. And exactly one of them is correct that is what is called multiple choice questions. Suppose that you found out 3 options are incorrect, what will you do? Simply choose the remaining option as your answerwithout even looking at it.

Why does it work? It works because you were told that one of the 4 options is correct. You are figured it out that 3 of them are incorrect therefore what is remaining must be correct. So existence of an answer is guaranteed.

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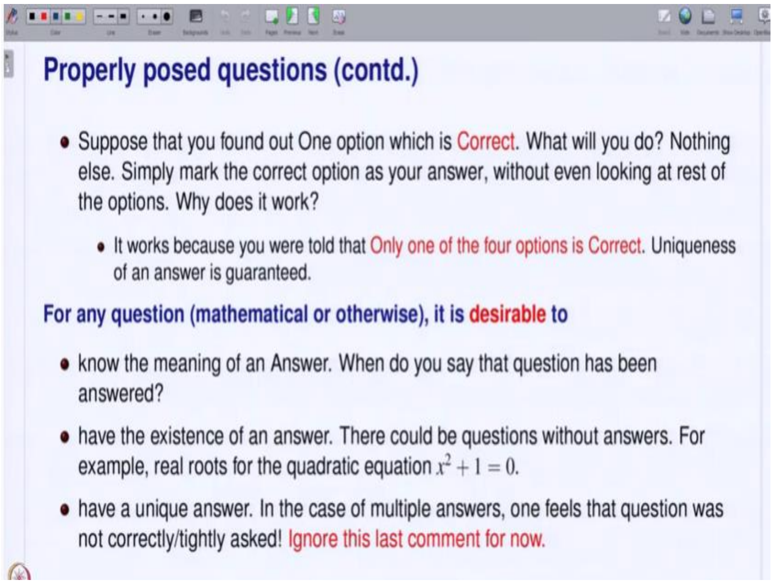

Suppose you found out one option which is correct what will you do? Nothing else simply, mark the correct option as you are answer without even looking at the rest of the options.Why does it work? It works because you were told that only 1 of the 4 options is correct. Uniqueness of an answer is guaranteed. So for any question mathematical or otherwise it is desirable to know the meaning of an answer that I tell you when do, you know that the question actually answered.

So when do you say the question has been answered? And desirable to have the existence of an answer having defined what is the meaning of an answer? It is desirable to have the existence of an answer. There could be questions without answers that means we understand what is meaning of an answer to a question, that question does not have an answer. For example the real roots for the quadratic equation x square + 1 = 0.

What do you mean by real root for this equation? It is real number alpha so is that alpha square + 1 = 0 very clear. But there is no real number which satisfy alpha square + 1 = 0 that means concept of an answer defined we understand what, is the meaning of an answer? But this question does not have an answerand also desirable to have a unique answer. So in the case of multiple answers you imagine you have more than one answer one feels that question was not you know tightly posed or currently posed.

So please ignore this last comment for now. Imagine that I have not mentioned this part. So these are the 3 conditions first thing is we should the meaning of answer it is desirable to have an existence of an answer and desirable to have a unique answer.

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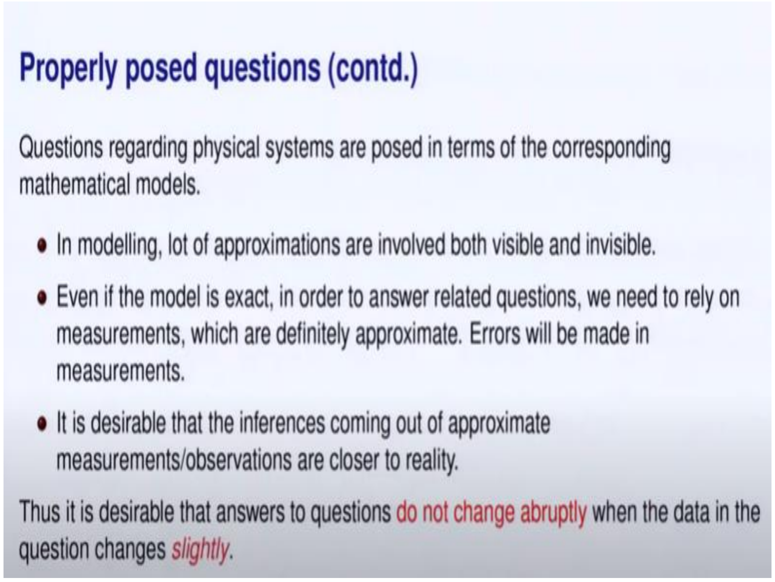

From here the hadamard concept of properly pose questions will be state. Now if you go back to questions regarding physical systems are posed in terms of the corresponding mathematical models. In modeling lot of approximations are involved some are visible some are non-visible. Even if the model is exact in order to answer related questions we need to rely on measurements somewhere we have to give input of the data and ask, what is the output of the system?

System is already model exactly using your equation mathematical models. But measurements are definitely approximate. Errors will be made in measurements. Therefore it is desirable that the inference coming out of such approximate measurements are observations are closer to the reality that is closer situation where such errors are not been made. Thus it is desirable that answer to the questions do not change abruptly when the data in the question changes slightly.

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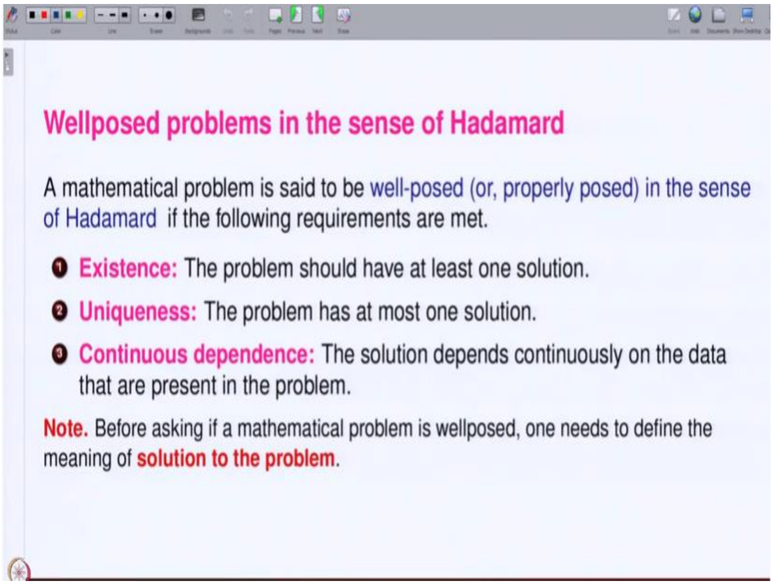

So well-posed problems in the sense of hadamard. A mathematical problem is set to be well-posed are properly posed in the sense of hadamard. So hadamard has given in this French translation some people translated as well-posed some people translated it as properly posed. If the following requirements are met what are the requirements? Existence the problem should have at least one solution.

Uniqueness the problem should have at most one solution. Continuous dependences the solution depends continuously on the data that are present in the problem. Note before asking if a mathematical problem is well-posed, one needs to define the meaning of solution to the problem.Only when it is defined we can ask whether a new solution is exists or not, unique are not etc.

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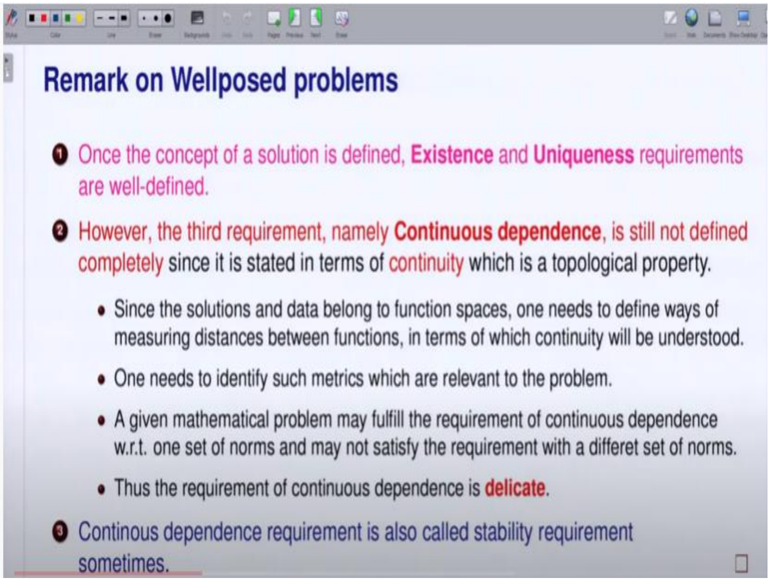

Remark on wellposed problems. Once the concept of solution is defined, existence and uniqueness requirement are well defined. However the third requirement namely continuous dependence is still not defined completely. Why? It is stated in terms of word continuity which is a topological property. Since the solutions and data belong to functions spaces, one need to define ways of measuring distances between functions, in terms of which continuitywill be understood.

One needs to identify such metrics which are relevant to the problem. A given mathematical problem may fulfill the requirement of continuous dependence with respect to one set of norms. and may not satisfy this requirement with different set of norms. Thus the requirement of continuous dependence is delicate. Continuous dependence requirement is also called stability requirement sometimes particularly in the problems were there is a time variable involved people call it stability requirement.

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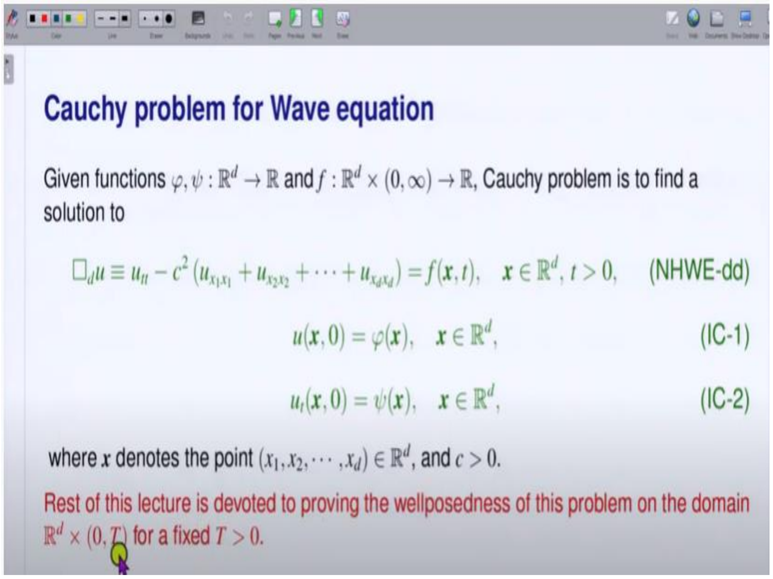

So let us recall the Cauchy problem for wave equation. It is given functions phi psi and f**.** defined on the appropriate domain. Cauchy the problem is to find a solution to the d'Alemberitan operator equal to f and satisfying the 2 initial conditions. So rest of this lecture is devoted to proving the Wellposedness of this problem on the domain R d cross 0, T. I will point out where this exactly required for a fixed T positive.

If you see here, it is R d cross 0, infinity but this result of Wellposedness will hold on R d cross 0, T only T cannot made infinity.

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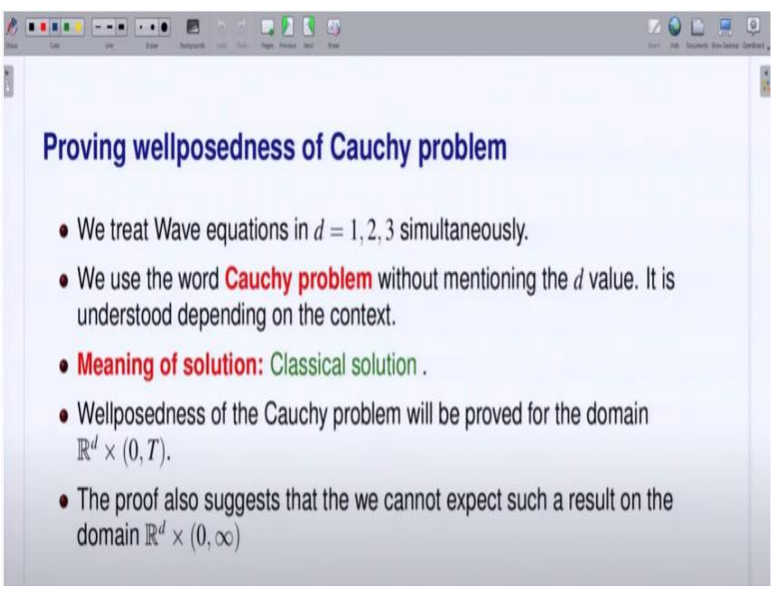

So proving Wellposedness of Cauchy problem we treat wave equation in 1, 2, 3 dimension simultaneously. That means we cover existence of all the 3 dimensions then we go to uniqueness and then we go to stability results. So we use the word Cauchy problem without mentioning the d value. It is understood depending on the context. What is the meaning of solution? Because that is the first thing we have to do define before saying the problem is well-posed in the sense of hadamard is to defined motion of the solution to the problem that is the classical solution.

The notion of classical solution we have driver the classical solution to the non-homogeneous Cauchy problem in the previous lectures. So Wellposedness of the Cauchy problem will be proved for the domain R d cross 0, T. The proof also suggests that we cannot expect such a result on the domain R d cross 0, infinity.

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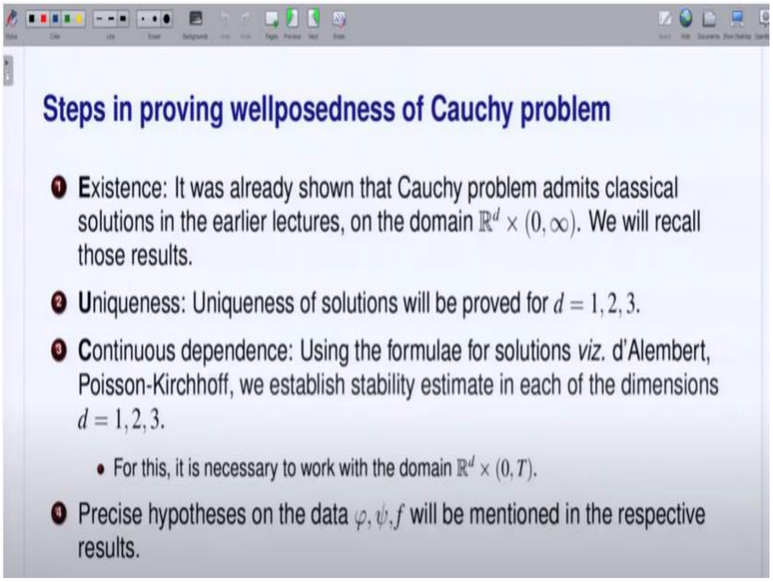

So what are the steps involved improving Wellposedness of Cauchy problem? First thing is existence it was already shown that Cauchy problem admits classical solution in the earlier lectures, on the domain R d cross 0, infinity. We will recall those results. Uniqueness, we have not proving uniqueness, we will prove that in this lecture. And then we go to continuous dependence using the formula for solutions like d'Alembert formula or Poisson-Kirchhoff formula and dimension 2 and 3 d'Alermbert in dimension 1.

We establish stability estimate in each of these dimensions d = 1, 2, 3 so for this it is necessary to work with the domain Rd cross 0, T. So it is for the stability estimate that we need to work with the finite time. Precise hypothesis on the data phi, psi f will be mention in the respective results.

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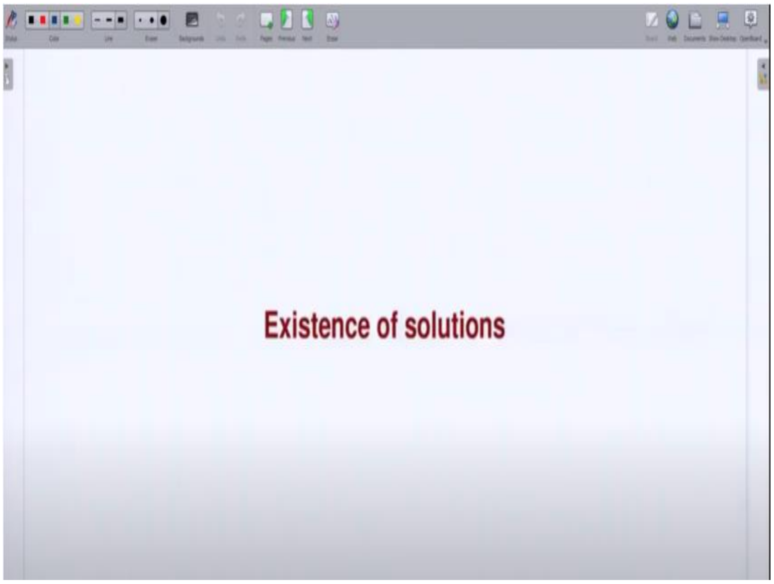

So let us go to existence of solutions. Here I am just going to recall what we have done in the earlier lectures

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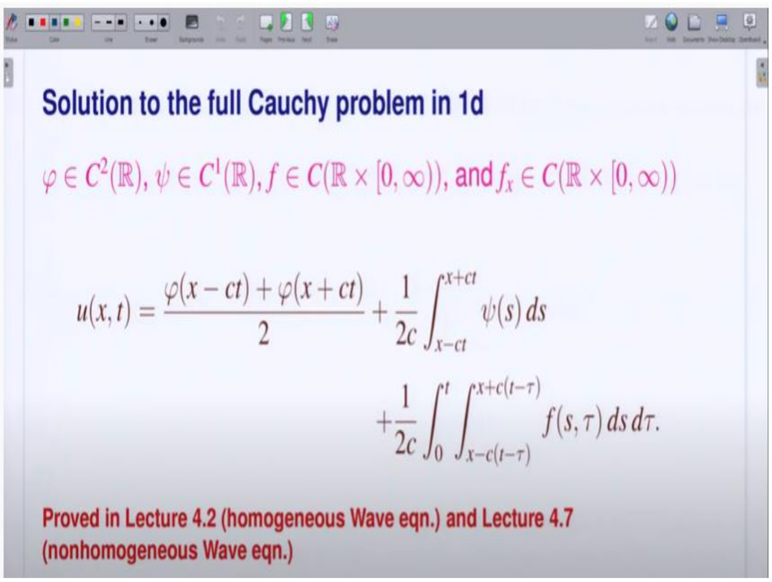

Solution to the full Cauchy problem,sometime I call it as non-homogenous Cauchy problem in 1 d. Phi should be C 2 of R, psi should be C 1 of R, f should be continue R cross 0, infinity and f x should be continuous R cross 0, infinity. These are the formula we obtain reference is lecture 4.2 and lecture 4.7

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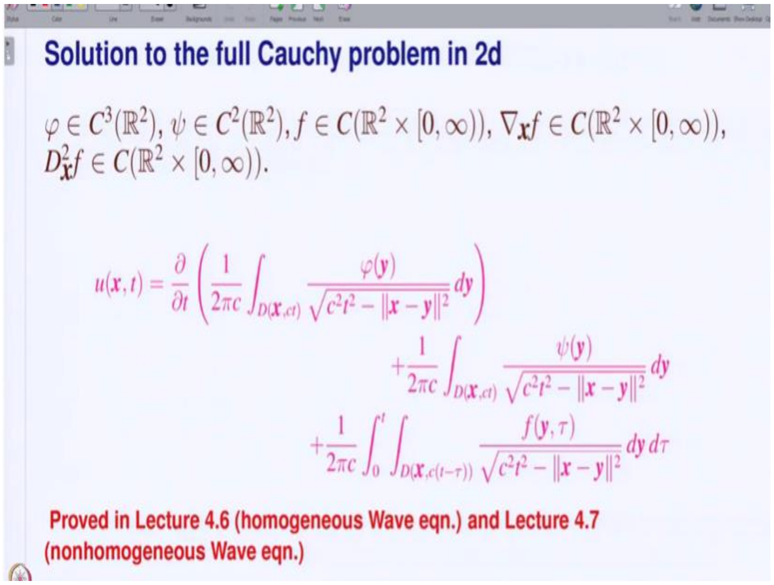

Now in 2 dimensions the hypothesis required is here, I will not read fully but let us say phi and psi should be C3 and C2 respectively and f gradient of second derivative should be continuous on r to cross 0, infinity. And this is the formula that we obtain, reference lecture 4.6 and lecture 4.7 so this represent a classical solution to the Cauchy problem.

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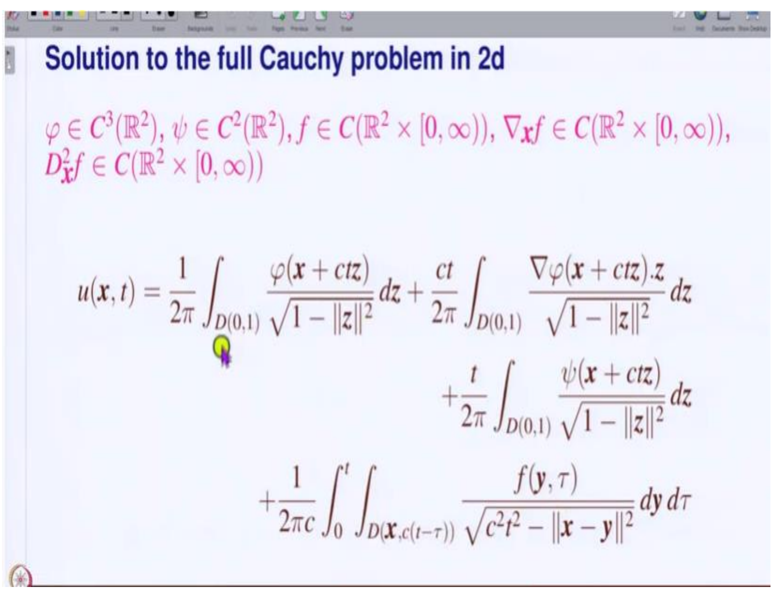

Now we may also write it on this domain as D 0, 1. Disk of radius 1 center at the origin because we going to use this formula improving the stability estimate that is why I have written down this particular formula.

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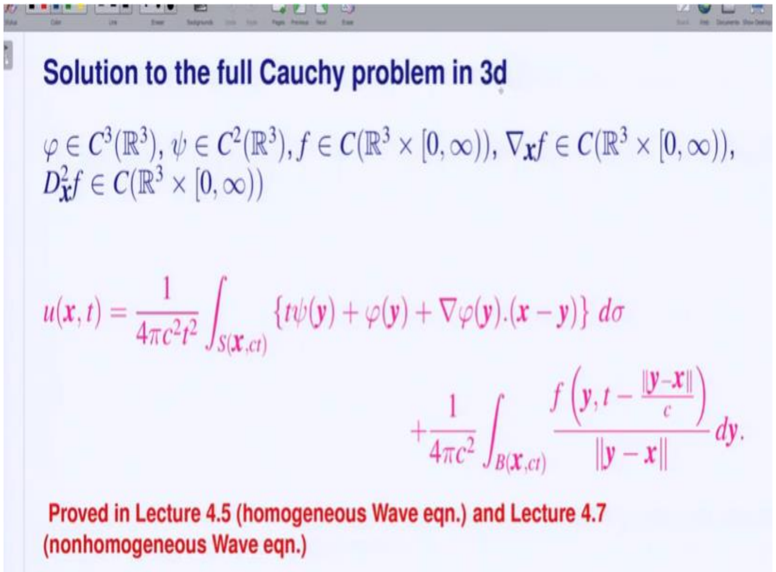

How about in 3d, similar hypothesis as in 2d and these is the formula. And reference lecture 4.5 and lecture 4.7 that is where we have derived on shown that this a classical solution to the non-homogenous Cauchy problem in 3 dimensions.

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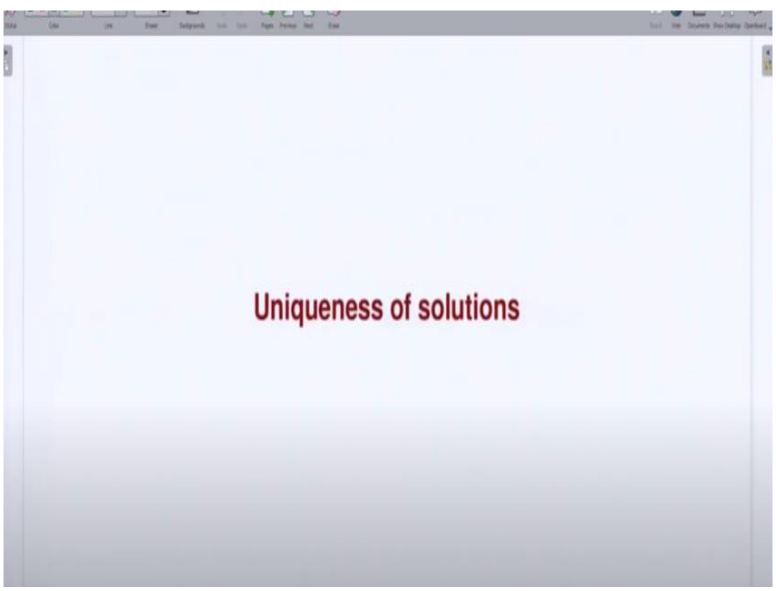

So let us discuss the uniqueness of solutions.

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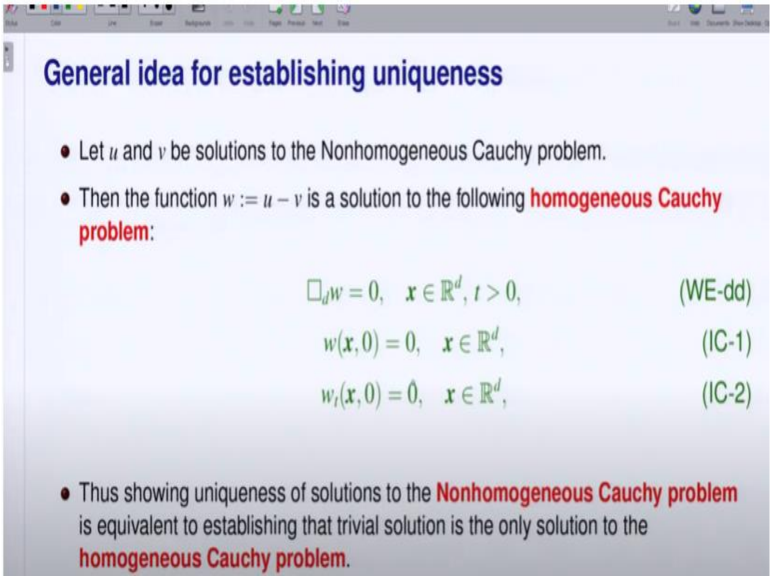

The general idea for establishing uniqueness always is like this to any problem if you want to show it has uniqueness. You just take solutions u and v consider the difference it works usually in the liner situations even in the nonlinear situations one consider as that and then try to show that is as 0. Let u and v be the solution to the non-homogeneous Cauchy problem. Then the difference let us call w u - v it is a solution to the following homogeneous Cauchy problem what is that?

Homogeneous wave equation and 0 initial data. Because both u and v satisfy initial data phi and psi for initial displacement and velocity respectively difference will be 0 because of the linearity. The problem is linear d'Almerbtian and value at a point, derivative value at a point these are all linear operations. Now showing uniqueness means what? I wanted to show that w is 0 so therefore showing uniqueness to the non-homogeneous Cauchy problem is same as showing that this homogeneous Cauchy problem and everything is homogeneous with 0 initial data has 0.

Of course 0is the solution very clear substitute to w is satisfies but we show is 0 is the only solution. That sometimes 0 solution is also called as trivial solution. 0, solution is the only solution to the homogeneous Cauchy problem which is here. Therefore we will concentrate and showing only this. That solution of such a homogeneous Cauchy problem with0 initial data the only solution is this 0 solution. There by establishing uniqueness of solution to the non-homogenous Cauchy problem.

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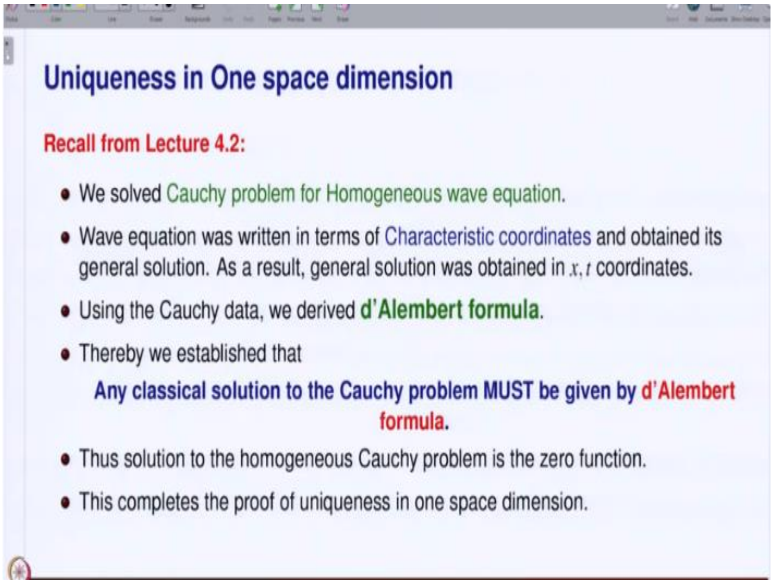

Let us move to the one dimension recall from lecture 4.2. We saw the homogenous Cauchy equation. First how do we do it was written in terms of characteristiccoordinates and obtained its general solution. As a result, General solution was obtained in xt coordinates. After that we use the Cauchy data and we drive the Alembert formula. Thereby we established that any classical solution to the Cauchy problem must be given by Alembert formula.

Thus solution to the homogeneous Cauchy problem is the 0 function, because for the Cauchy problem with homogeneous wave equation and 0 initial data. 0 is already known to be solution advantage of uniqueness that is the only solution function. This completes the proof of uniqueness in 1 space dimensions. In other words we made no compromises in deriving the d'Alembert formula.

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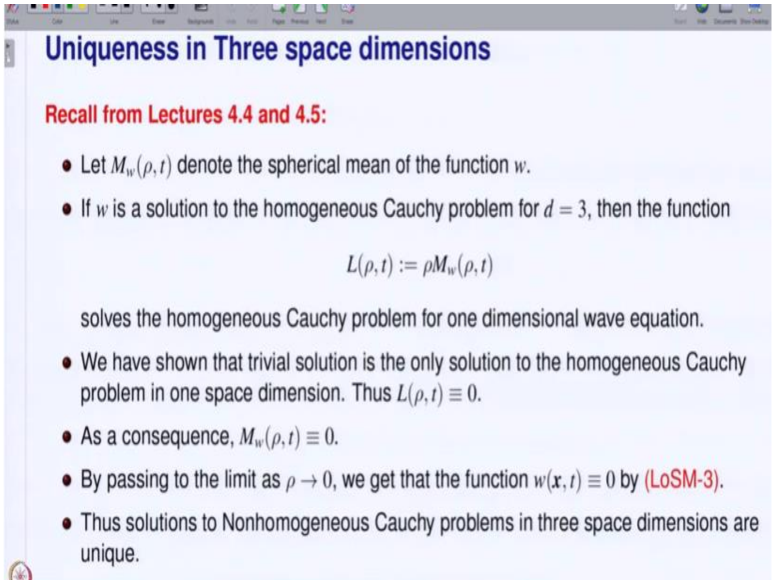

Now let us move on to uniqueness in 3 space dimensions. Remember these are the order in which we solve the Cauchy problem. First we solved dimension 1, then we solved dimension 3 then we used hadamard method descent to solve in 2 dimensions. So recall from lecture 4.4 and lecture 4.5. M w of rho t it is the spherical means associate to the function w. If w is a solution to the homogenous Cauchy problem in d = 3.

Then this L of rho t given defined by rho times Mw of row t this is what we did in lecture 4.5. This L of rho t satisfies the 1 dimension wave equation. And if w satisfies a homogeneous Cauchy problem then L also satisfy homogeneous Cauchy problem in one dimension.And there we have just shown on the previous slide the trivial solution is the only solution therefore L of rho t must be the 0 function.

Once L of rho t is 0 M w of rho t must be 0. Once Mw of rho t 0 it means all spherical average are 0 by L0SM Lemma on spherical means w itself must be 0, which means we have our uniqueness therefore solutions to non-homogeneous Cauchy problem in 3d are unique.

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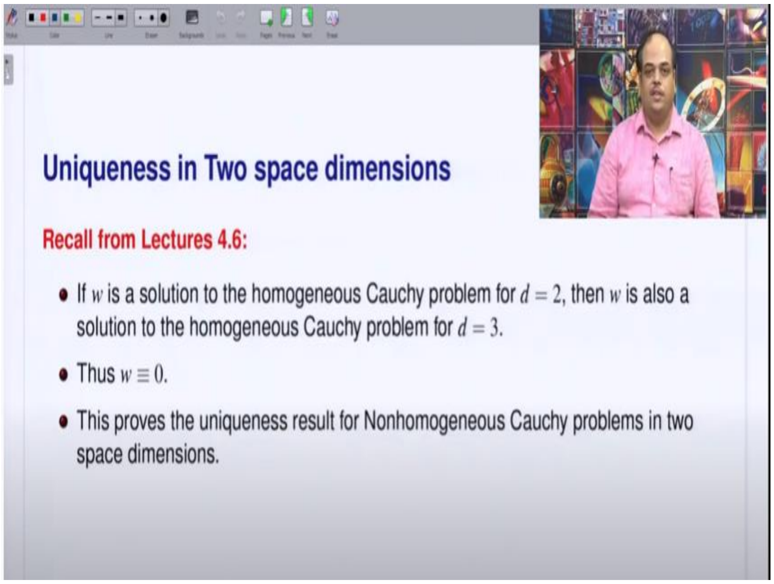

Now let us move to 2 space dimensions. Recall from lecture 4.6. If w is solution to the homogeneous Cauchy problem for d=2, then w is also a solution to the homogeneous Cauchy problem for d=3. We have just proved the uniqueness therefore w must be 0. This proves uniqueness result for non-homogeneous Cauchy problems in 2 space dimensions.

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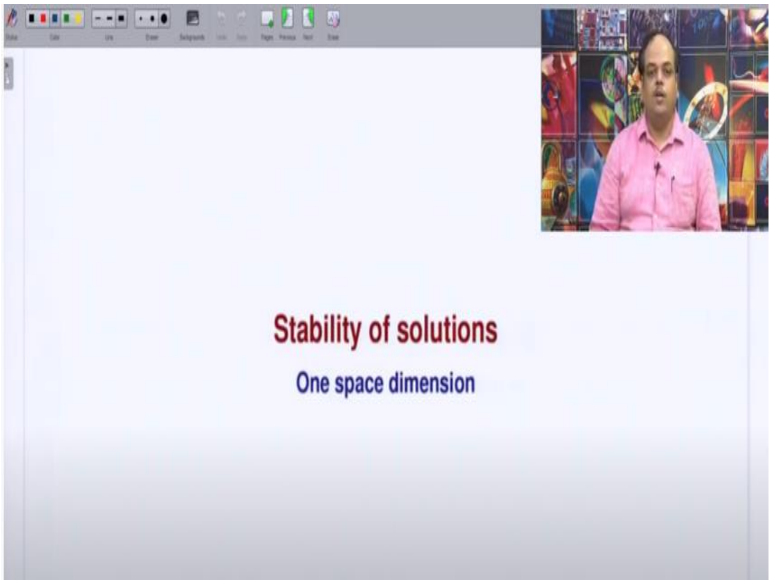

Now let us move on to stability of solutions. What do you mean by that? We are going to state in the form of result. Let us do 1space dimensions first.

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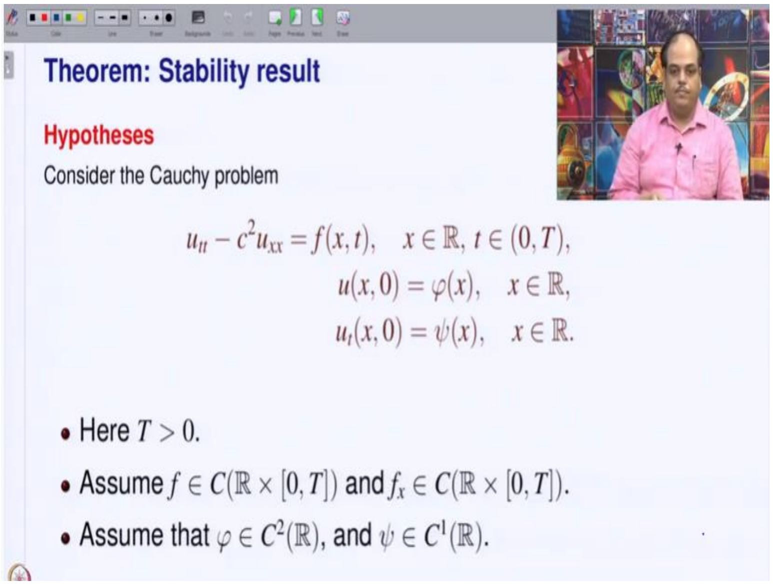

Stability result, Hypothesis is consider this Cauchy problem of course we have to write f phi psithe hypothesis on them so that. We have a classical solution to this problem that is what is going to be hypothesis.Of course we have introduced a T, so T should be positive. You will see this T will place a crucial role. The last step of the proof you will see exactly where and how the T play a role. So assume hypothesis on f phi psi which are required to have a classical solution to this problem.

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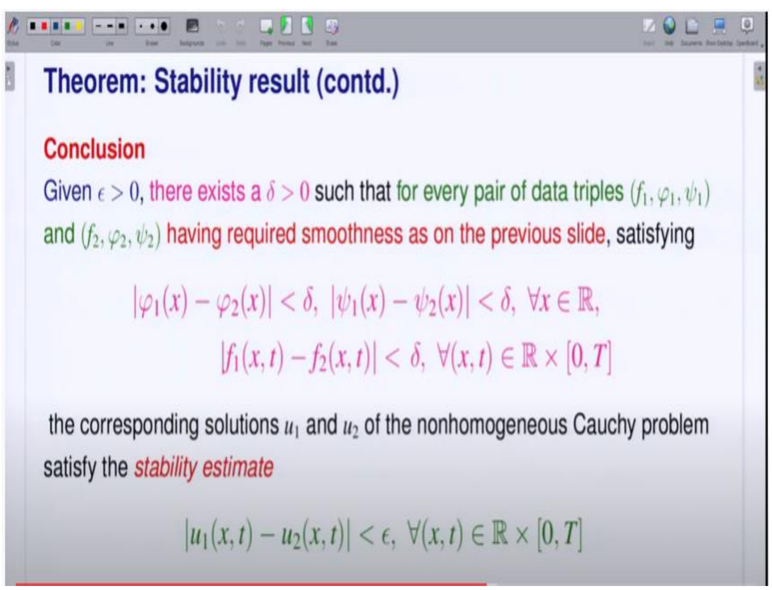

So conclusion is given epsilon positive. Now we are going to write about continuity right. So given epsilon positive there is a delta such that whenever 2 data that is f 1 phi 1 psi 1 these 1 data set, these another data set f 2 phi 2 psi 2.Whenever these 2 are in the distance of utmost delta, am using very loosely using the word distancemod phi x – phi 2 x is always is less than delta for every x similarly for mod psi even x – psi 2 x is less than delta for every x and mod f1 xt – f2 less than delta for every xt in r cross 0, t that happens.

Then corresponding solutions u 1denotes solutions with this initial data u 2 denotes with this not initial data f 2 is a source phi 2 psi 2 are the initial displacement and velocity respectively. Sou 1 and u 2, they satisfy what is call stability estimate. Mod u on st – u t x t is less than epsilon for every x t in R cross 0, t so this tells that if the 2 data set f 1 phi 1 psi 1 and f 2 phi 2 psi 2 are sufficient close then solutions will remain arbitral close, the close that you want. You prescribe this epsilon then such a delta exists. This is continuity kind of requirement.

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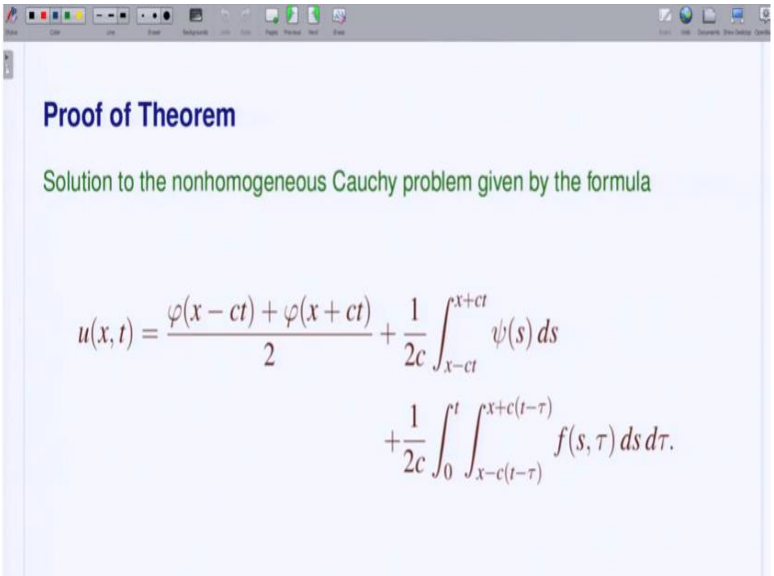

Solution to the non-homogeneous Cauchy problem is given by this formula we know this. And what is the stability estimate? It is in terms of u 1- u 2. So we write this formula for u 1 u 2 and subtract and get u 1 - u 2 that is the idea.

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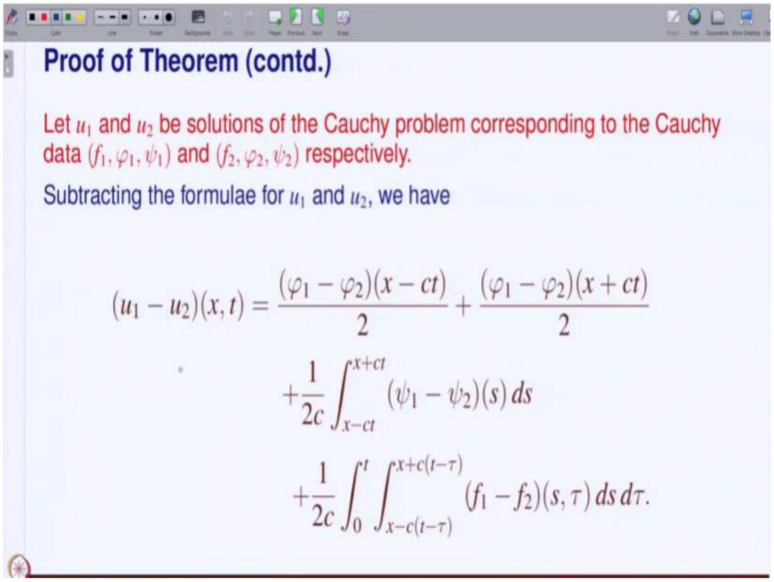

So let u 1 and u 2 be solution to the Cauchy problem with not really Cauchy data. This is the Cauchy data and this is the source term. Similarly phi 2 psi 2 is Cauchy data and f 2 is the source term.Subtracting the formula for u 1 and u 2, we get this. I just substitute and then subtracted. Now notice how the things look phi 1 - phi 2, psi 1 - psi 2, f 1 - f 2. What we have to show is there is a delta says that whenever these differences are at most delta in modulus this can be made as an epsilon that is what we want to show.

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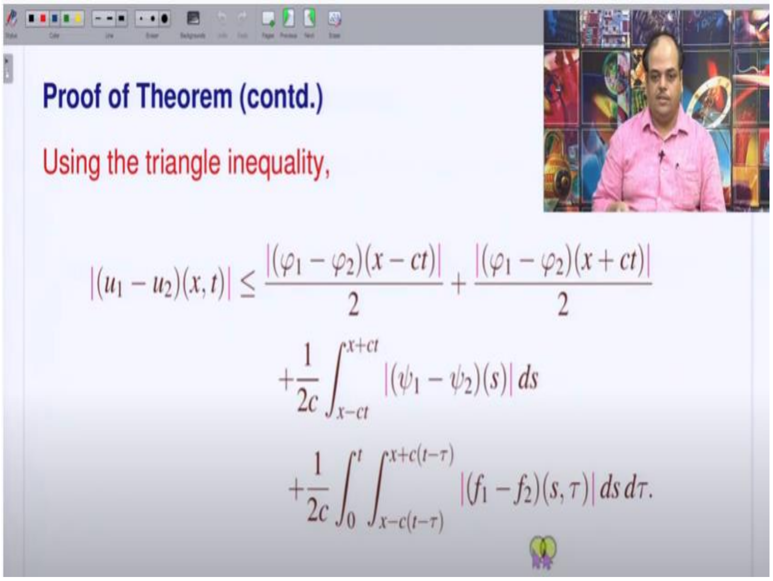

Then let us apply in the triangle inequality and we get this modulus of the LHS less than or equal to modulus. Modulus of some is less than able to some of the modulus so apply this. And modulus of the integral is less than able to integral of modulus using all that we get this letters from the previous day.

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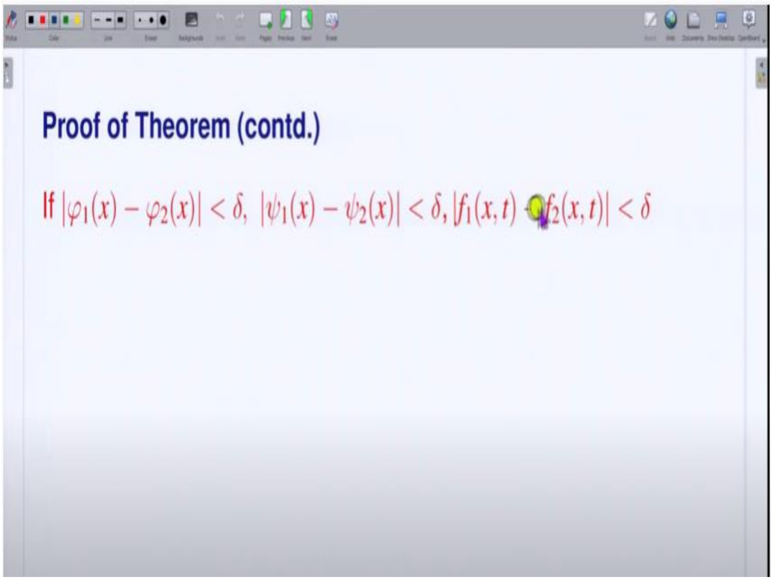

Now if there is a delta like that, we are going to find the Delta

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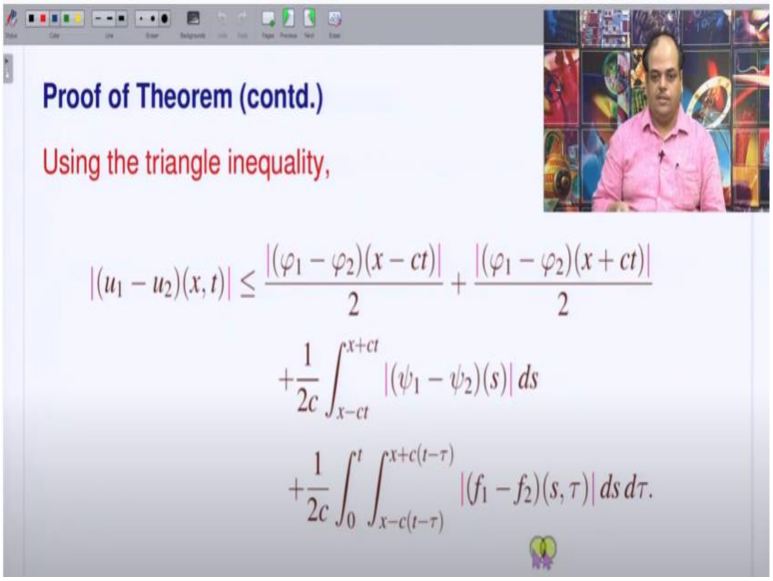

But if at all there is this thing then I will go and see what consequence it has to this estimate. So I get mod u 1 – u 2 less than equal to this is less than delta, so delta by 2. This is less than delta no matter what is argument so delta by 2 + delta by 2+ 1 by 2 c x - ct this is also less than delta.

Delta comes out side then you get length of this interval which 2 ct similarly here this less than delta it comes outside then as you know this is a triangle so that triangle area will come.

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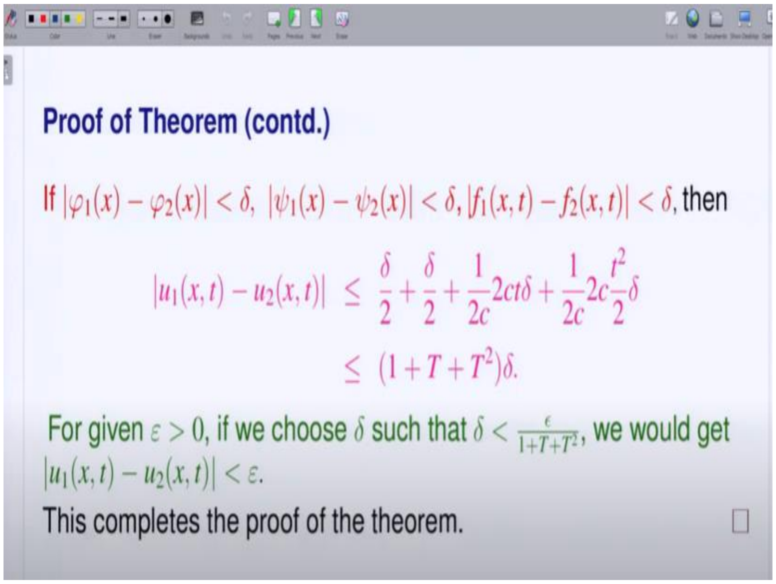

This is the first phi 1 - phi 2 at x -ct this is the second term phi 1 – phi 2 at x +ct this integral term with psi this other integral term with f. Please do this competition very simple. How simplifying we get this. Now what is our idea? We want to make this lesson epsilon. Therefore given epsilon I want to make this lesson epsilon can I choose delta so this is less than epsilon, yes choose delta to be less than epsilon 1+ T + T square.

So the above competition shows but mod x t - x 1 is less than epsilon. Now comes to comment if T is a higher and higher, this delta will be smaller and smaller. Imagine T is infinity loosely speaking is 0. That means you would not be able to choose this Delta that means it gives an idea that if you consider the Cauchy problem r cross 0, infinity we may not have stability estimate. That means there may be there will be or there may be the data sets of, f 1 phi1 psi1 which are close to of f 2 phi 2 psi 2 but solutions are not.

So that I leave it for you as an exercise to think about it how to get that given that we are dealing with the linear equation. You may consider one of the data set to be 0, 0, 0. So this completes the proof of theorem.

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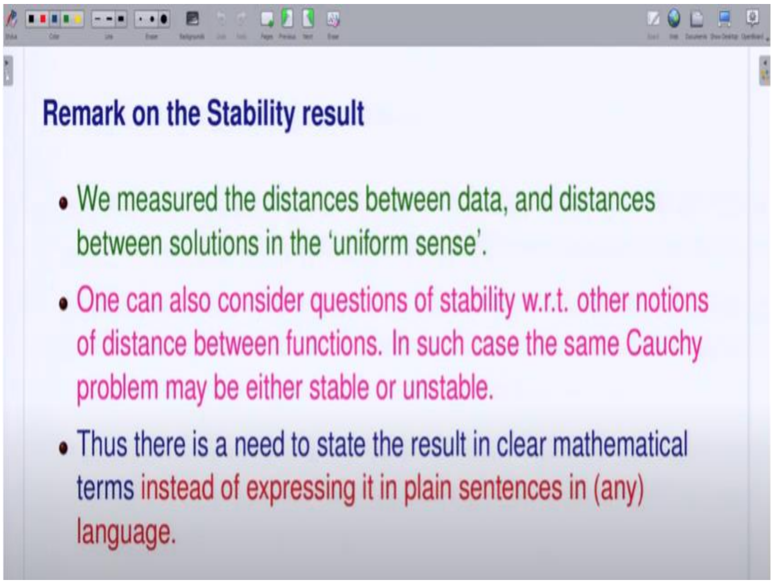

So we measure the distances between data and distance between solutions in the kind of uniforms sense. I have put this quotes because exactly not uniform sense, but some kind of uniform set. It not uniform metric or if you know it is not supremum, no. Because on the spaces we are considering there are like C of R cross 0, t. With respective to R you may not have supremum for any continuous function. Imagine f x = x. Of course, it is in Cf R cross 0 infinity. Supreme does not make any sense.

That is why I am not using that word here that why I have put quotes here. Uniform in the sense for every x distance between phi 1 and phi 2 is less than delta that is what I am mean. So one can also consider Cauchy of stability with respect to other notions of distances between functions in such case the same Cauchy problem may be stable and unstable anything can happen. Because if you are studied function norms you know that an infinite dimensional nominal spaces 2 norms may not be equivalent that is the problem.

So topologies could be different so thus there is a need to state the result in clear mathematical terms. This is always the case, if you think you proved some result, you should be able to state it very cleanly. May be very clearly it can run into any number of pages, but it should be very clearly stated that is one will know what we actually prove instead of expressing it in some plain sentences.

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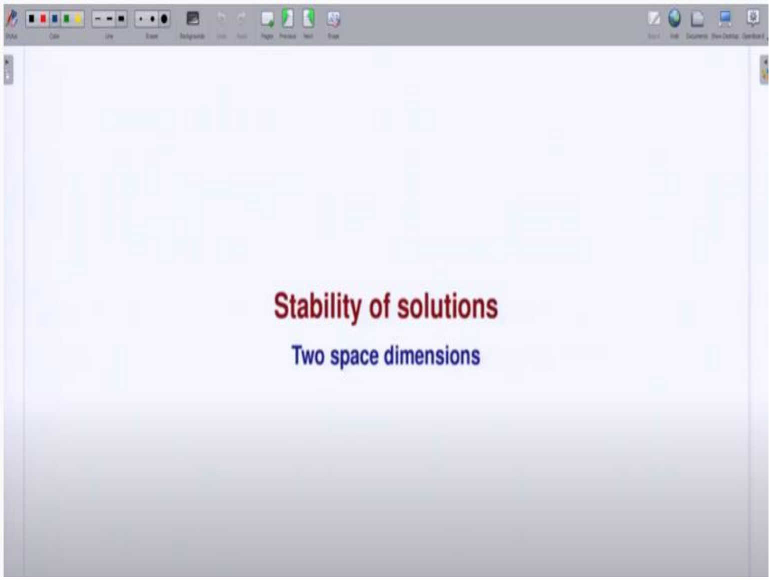

Let us move on to the 2 space dimensions stability of solutions.

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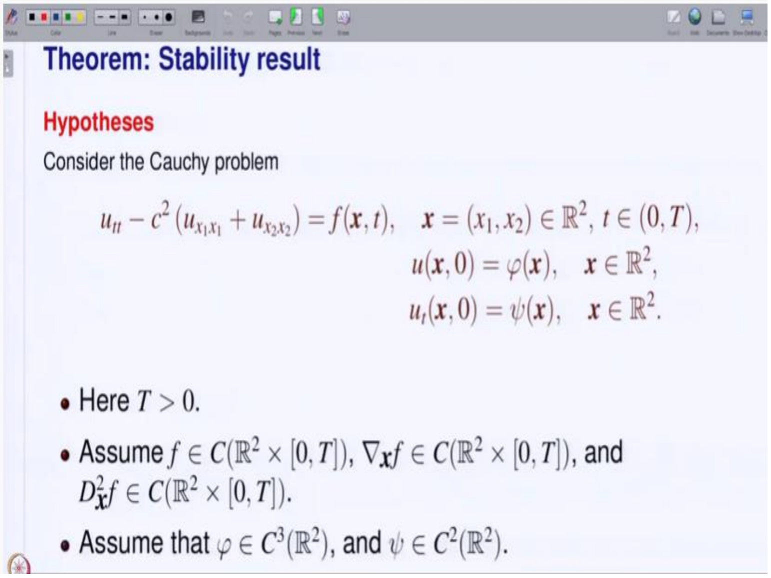

Result is going to look exactly as before instead of 1d wave equation you have 2d wave equation f phi and psi so assume f belongs to continuous grad of continuous and D x 2 is continuous assume phi is c 3 psi is c 2. So that we have a classical solution to this problem.

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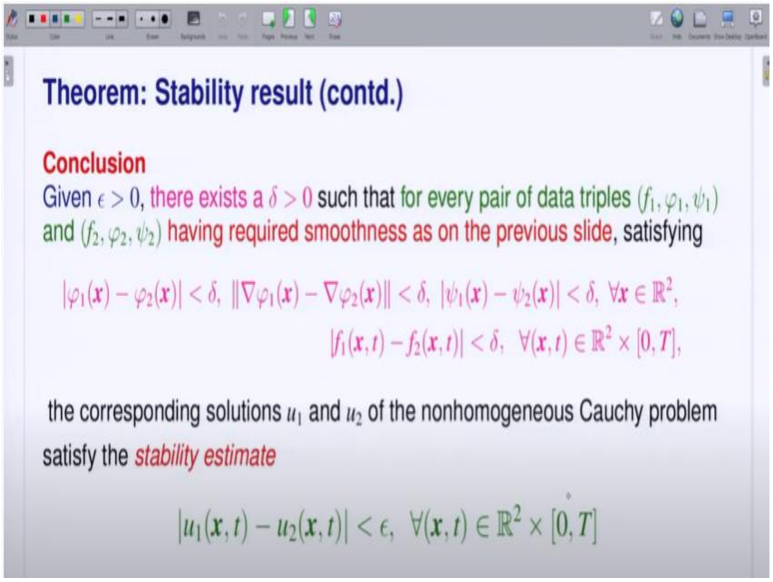

So the conclusion given epsilon there is a delta so this that when are the data triples are at most of the distance in some sense of delta. Uniformly with respect to the x or uniformly with respect to x t in this case. Then the corresponding solutions will satisfy the stability estimate. The idea proof is the exactly same write down the expression for u 1 and expression for u 2 to subtract apply triangle inequality and try to show this.

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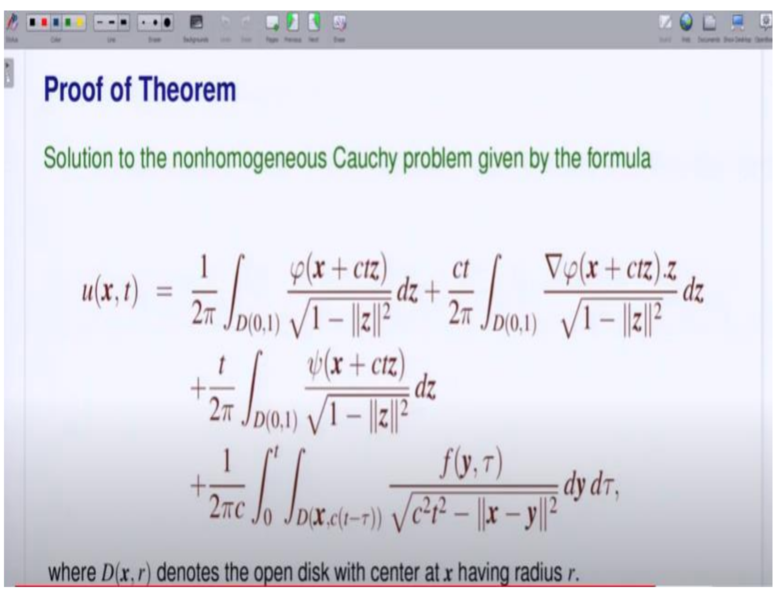

So this formula for u which we know, this is a convenient formula. That is why we are using this formula.

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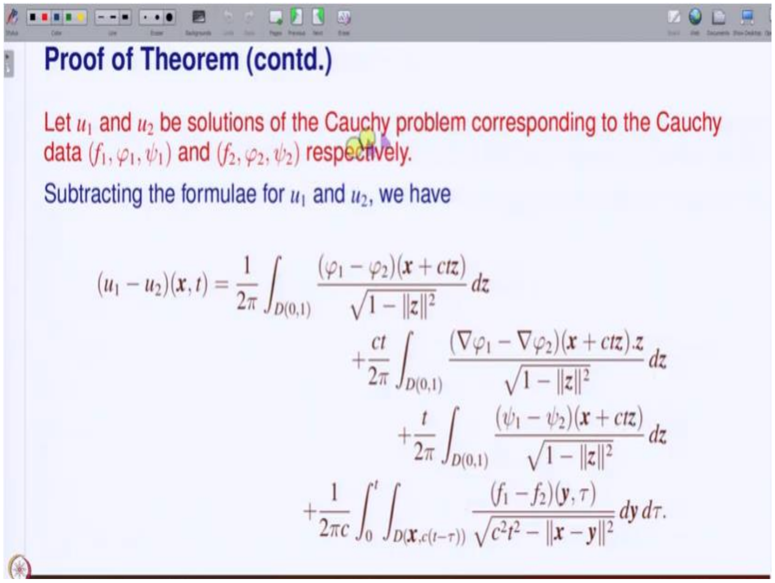

Let u 1 and u 2 be solutions for the data f 1, phi 1, psi 1. F 2, phi 2, psi 2 subtract formula we get this expression now take the modulus on both sides and apply triangle inequality here.

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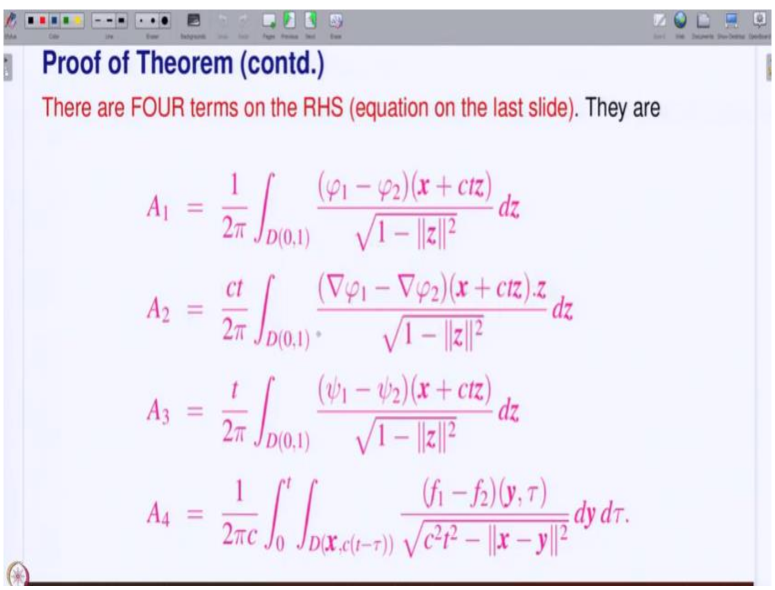

There are 1, 2, 3, 4 terms in the RHS just for convenient let us give some names as A1, A 2, A 3 and A 4so there for modulus of u 1 - u 2 of x t is less than or equal to mod A 1 + mod A 2 + mod A 3 +mod A 4. And what is mod A 1? Is less than or equal to the modulus insights and phi 1 - phi 2 is always less than delta that is our assumption that comes out delta by 2 pi and what is left is integral D 0 1 dz by root 1 - nom z square. Similarly here you can bring things out.

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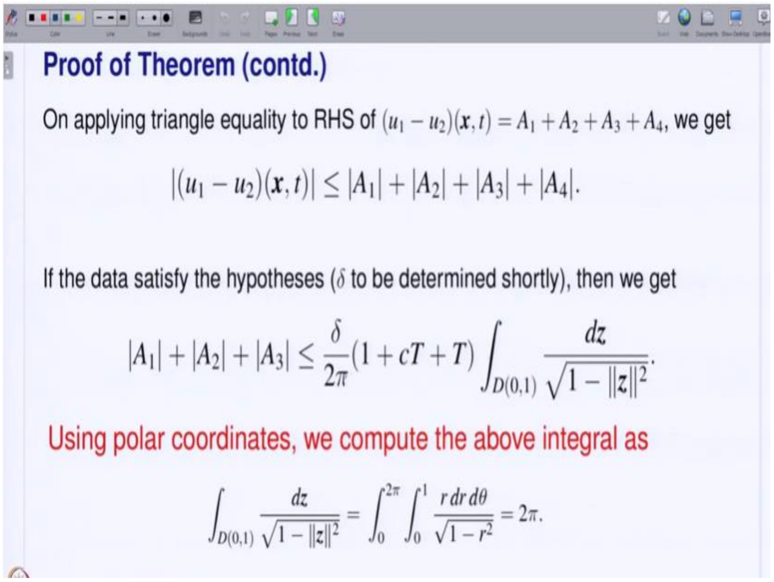

So what we get is mod A one plus A 2, first 3 times I am considering. First time will be delta by 2 phi, as we already saw into this integral. Second term will be less than equal to c T delta by 2 phi into this integral term.Third term mod A 3 less than or equal to delta by 2 pi into D into this integral that is what you will get. So we have to compute was integral is use polar coordinates. And then this evaluates 2 pi so therefore for first 3 terms are less than or equal to delta into 1 + c T + T that 2 pi is cancel with this 2 pi.

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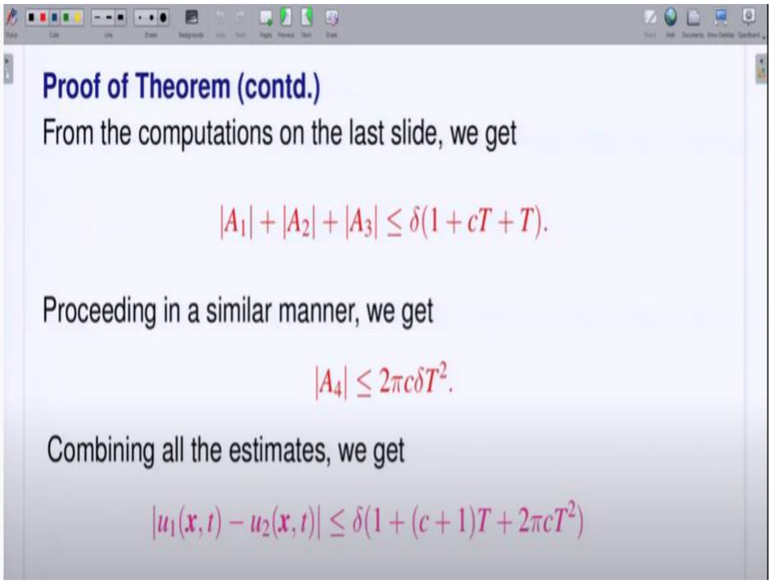

Now one more term is there Proceeding in a similar manner the fourth term will satisfy the system Mod A4 is less than or equal to 2 pi c delta T square. Combining all estimates we get this expression. Now can I make this less than epsilon? Yes, choose delta to be less than epsilon divided by this quantity

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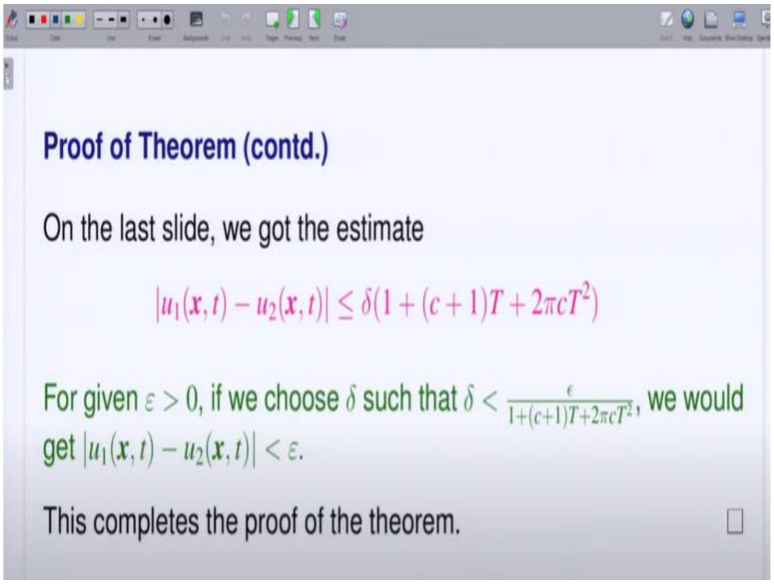

So we will get this. So this completes the proof of the theorem.

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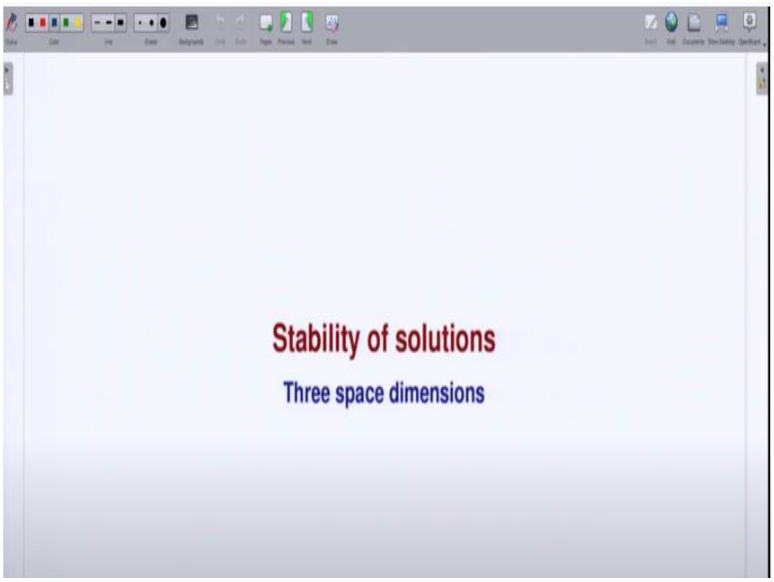

Now move on to 3 space dimensions.

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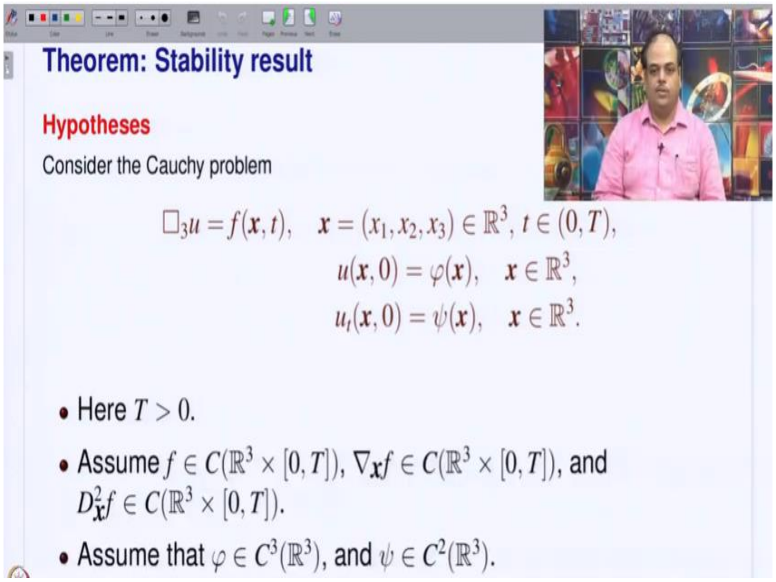

Here we just state the result exactly same as before here T belongs to 0 T. The assumption on f phi psi so that this problem has a classical solution.

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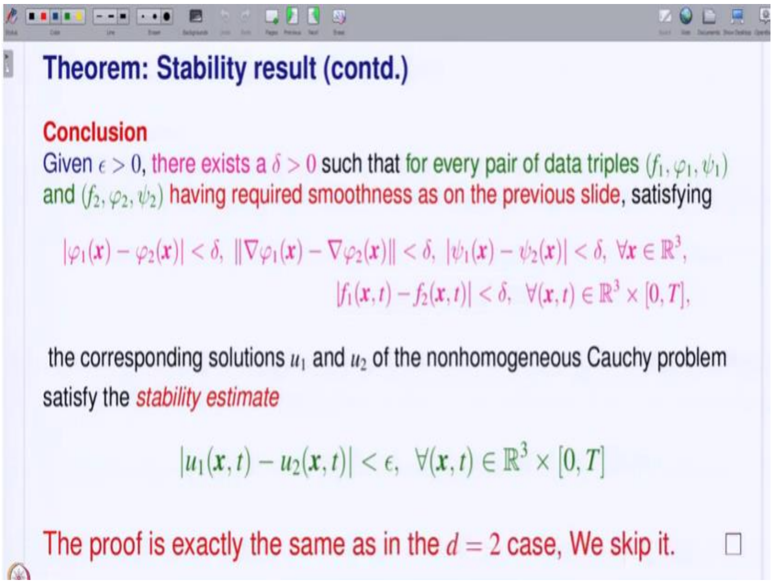

So given epsilon positive, 0 delta positive, such that whenever you have data triple with the required regularity or smoothness on the previous slide and satisfying this that uniformly at a distance of delta at most distance delta and also f 1 and f 2 distance is less than delta uniformly for x, t in R 3 cross 0 T, corresponding solution u 1 and u 2 satisfy the estimate. The proof is exactly the same as in the d = 2 case and we skip it.

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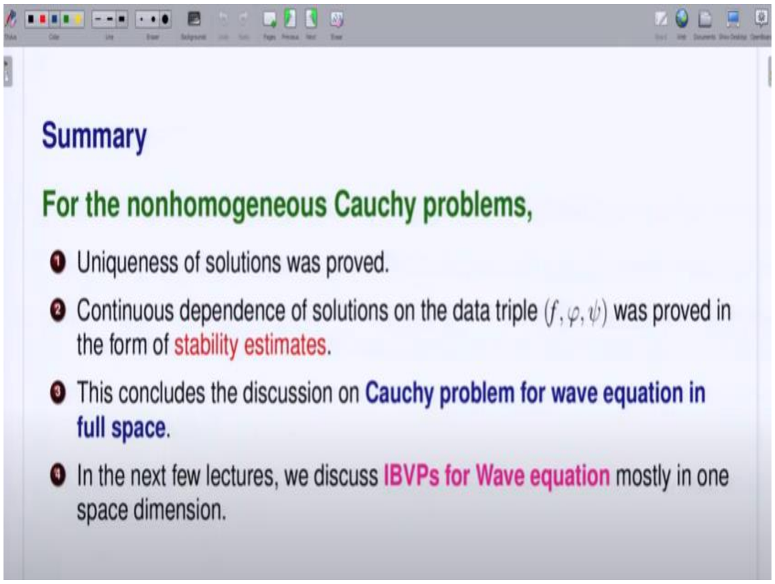

So let us summaries what we did. For the non-homogeneous Cauchy problems uniqueness of solutions has proved. We also going to see another proof of uniqueness later on. Continuous dependence of solutions on the data triple was proved in the form of stability estimate. This concludes the discussion on Cauchy problem for Wave equation in full space that is x belongs to R, R 2 and R 3.

In next few lecture we discuss initial boundary value problems there arise when the wave equation pose is not in the full space R d but on subset of R d. In fact we are going to study subsets of R only mostly in one space dimension. Thank you.