Partial Differential Equations
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Module No # 07
Lecture No # 31
Cauchy problem for Wave Equation is 2 Space dimensions – Hadamard's method of descent

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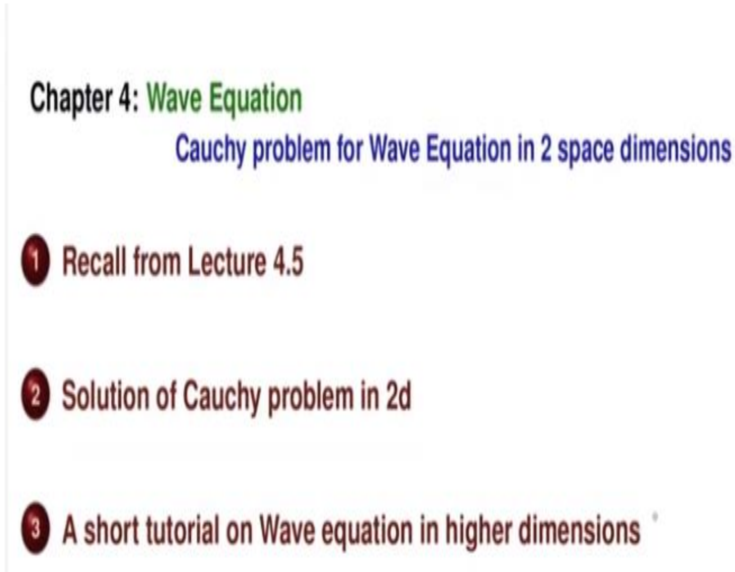

Welcome to this lecture on Cauchy problem for wave equation 2 space dimensions. We are going to use Hadamard's method of descent to find solution to the Cauchy problem.

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Cauchy problem for three dimensional wave equation

$$\Box_3 u = 0, \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, t > 0,$$
$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3,$$
$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3.$$

Assume that Cauchy data satisfies $\varphi \in C^3(\mathbb{R}^3)$ and $\psi \in C^2(\mathbb{R}^3)$.

The outline of the lecture is as follows first we recall from lecture 4.5 where we have solved Cauchy problem for 3 space dimensions using that we will find a solution of Cauchy problem in 2 dimensions and toward the end of this lecture we will give a short tutorial on wave equation in higher dimensions where we are going to solve few Cauchy problems. Let us recall from lecture 4.5 where we consider the Cauchy problem for 3 dimensional wave equations.

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Poisson-Kirchhoff formulae: F3, F4

$$4\pi c^2 u(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S(\boldsymbol{x}, ct)} \varphi(\boldsymbol{y})\, d\sigma\right) + \frac{1}{t}\int_{S(\boldsymbol{x}, ct)} \psi(\boldsymbol{y})\, d\sigma. \quad \textbf{(F3)}$$

$$4\pi c^2 u(\boldsymbol{x}, t) = \frac{1}{t^2}\int_{S(\boldsymbol{x}, ct)} \{t\psi(\boldsymbol{y}) + \varphi(\boldsymbol{y}) + \nabla\varphi(\boldsymbol{y}).(\boldsymbol{y} - \boldsymbol{x})\}\, d\sigma. \quad \textbf{(F4)}$$

This box 3 stands for the wave equation in 3 dimensions equal to 0 that means homogenous wave equation and this is the Cauchy data u of x, 0 is 5 and u d of 0 is psi. Assuming that the Cauchy data satisfies phi is a C 3 function and psi is the C 2 function we have obtained an expression which is called Poisson Kirchhoff formulae which reprints the solution we have obtained in fact 4 versions of the same formula.

So we derived F 1 from their we have got F 2 by expanding this derivative which give rise to first 2 terms the third term is as it is in F 1, F 2 is same.

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And then these are integrals on the unit sphere with the center of origin when we change back the variables we get an expression in terms of the sphere with center x radius ct. So that gave rise to another 2 formulae F 3, F 4. So they are all called Poisson Kirchhoff formulae. Now we are going to use them to obtain a solution in 2 dimensions the idea is called Haradamard’s method of descent. We are solving 3 d problem first and then we are going to 2 different problems so it is a descent. It is coming down from higher dimensions to lower dimensions.

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Cauchy problem for two dimensional wave equation

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2}\right) = 0, \;\; \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2, \; t > 0,$$

$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2,$$

$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2.$$

Assume that Cauchy data satisfies $\varphi \in C^3(\mathbb{R}^2)$ and $\psi \in C^2(\mathbb{R}^2)$.

So this is the 2 dimensional wave equations Cauchy problem where the equation is same Cauchy this is the homogenous wave equation in 2 space variables x 1 and x 2. Initial conditions are phi and psi respectively which are u and u t and we also assume that this phi is smoothness is C 3 and psi is smoothness is C 2 on R 2. So pi and psi are given we want to solve a function we want to solve this wave equation.

And obtain function which is the solution to this homogenous equation and satisfying the Cauchy data which is given here u x, 0 is phi x u t s x, 0 is psi x.

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Hadamard's method of descent

- Cauchy problem for wave equation in 3 dimensions was solved and a solution is given by **Poisson-Kirchhoff formulae** .
- The idea behind **Hadamard's method of descent** for solving Cauchy problem in 2 dimensions is
  - Solutions to 2d Wave equation may be thought of as solutions to the 3d wave equation which do not depend on the third variable, say x_3.
  - Solution $u(x_1, x_2, x_3, t)$ is expressed using **Poisson-Kirchhoff formulae** .
  - $u(x_1, x_2, x_3, t)$ would not depend on x_3 as the Cauchy data does not depend on x_3.
  - Thus u obtained as above solves the Cauchy problem in 2d.
- One can reduce the integrals on spheres in three dimensions appearing in the Poisson-Kirchhoff formulae

So Cauchy problem for wave equation 3 dimensions was solved and solution is given by Poisson Kirchhoff formulae. The idea behind Hadamard's method of descent for solving Cauchy problem in 2 d is solutions to 2 d wave equation may also be thought of solutions to wave equation 3d. Only that the solution do not depend on the third variable let us say x 3 we use x 1 x 2 for the variables in R 2 and x 1, x2, x3 for the variables on R 3.

So it is does not depend on the third variable solution u of x 1, x 2, x 3, t is expressed now using Poisson Kirchhoff formulae and it would not depend on x 3 as a Cauchy data does not depend on x 3. Because we; are starting with a Cauchy problem for the 2 d problem 2 d wave equation. So you obtained as above solves the Cauchy problem this is idea. So; one can reduce integrals of spheres in 3 d because the Poisson Kirchhoff formulae in 3 d feature integrals on spheres in R 3. They can be reduced to integrals on disks in R 2.

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Cauchy problem for two dimensional wave equation

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2}\right) = 0, \quad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2,\ t > 0,$$
$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2,$$
$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2.$$

Let us pose it for 3d wave equation as: Set $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, $(\boldsymbol{x}, x_3) \in \mathbb{R}^3$. Solve

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}\right) = 0, \quad (x_1, x_2, x_3) \in \mathbb{R}^3,\ t > 0,$$
$$u(\boldsymbol{x}, , x_3, 0) = \varphi(\boldsymbol{x}), \qquad (\boldsymbol{x}, x_3) \in \mathbb{R}^3,$$
$$u_t(\boldsymbol{x}, , x_3, 0) = \psi(\boldsymbol{x}), \qquad (\boldsymbol{x}, x_3) \in \mathbb{R}^3.$$

So this is the Cauchy problem that we start with a 2 dimensional wave equation as discussed before. Now let us proceed for 3d wave equation so for that we need to add u x 3, x 3 that is when we get the 3d wave equation still homogenous equation. And u of x, 0 = phi x is given for x in R2 we simply put u of x, x 3, 0 = phi x. x is x 1, x 2 so this is the way we are writing. We reserve this bold phase x notation for x 1, x 2 which is an R 2 as mentioned here.

So we join another variable x 3 another coordinate x, x 3 is in R 3 so phi and psi if you notice this is a Cauchy problem for the 3d wave equation and the Cauchy data depends only on the first 2 variables it does not depend on x 3. Because bold phase x means x 1, x 2.

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Let S denote the sphere in $\mathbb{R}^3$ with center at $(x_1, x_2, 0)$ and radius ct, *i.e.*,

$$S := \left\{ \boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 \,:\, (x_1 - y_1)^2 + (x_2 - y_2)^2 + y_3^2 = c^2 t^2 \right\}.$$

From the Poisson-Kirchhoff formula **(F3)**, we get

$$\begin{aligned} u(x_1, x_2, t) &= u(x_1, x_2, 0, t) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \int_S \varphi(y_1, y_2)\, d\sigma \right) + \frac{1}{4\pi c^2 t} \int_S \psi(y_1, y_2)\, d\sigma. \end{aligned}$$

Since $\varphi \in C^3(\mathbb{R}^2)$ and $\psi \in C^2(\mathbb{R}^2)$, the above formula gives a Classical solution.

Let us introduce a notation let us denote this sphere in R3 which center at x 1, x 2 at 0 and radius ct. That is set of all elements y in R 3 which are at distance ct from the point x 1, x 2 0. Now from the Poisson-Kirchhoff formula F 3 we get expression for u of x 1, x 2 0, t to be this. Where taking x 3 = 0 because the solution does not depend on x 3 variable so therefore I can take x 3 equal to any point that is why I have taken 0 for convenience to compute this quantities.

And that let us call a function of x 1, x 2 t now since phi is c 3 of R 2 and psi is c 2 of R 2 this formula give a classical solution to the Cauchy problem in 3d. Because the phi and psi does not depend on x 3 therefore trivially it is a function of c 3, R 3 and psi is also c 2 of R 3.

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Let us compute the integral $\int_S \varphi(y_1, y_2)\, d\sigma$, and computation of the other integral is exactly similar.

- Let S^+ denote the upper hemisphere.
- Since the integrand is independent of the y_3 variable, we have

$$\int_S \varphi(y_1, y_2)\, d\sigma = 2\int_{S^+} \varphi(y_1, y_2)\, d\sigma.$$

- Note that the upper hemisphere is the graph of the function $h : D((x_1, x_2), ct) \to \mathbb{R}$ given by

$$h(y_1, y_2) = \sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2},$$

where $D((x_1, x_2), ct)$ denotes the open disk with center at (x_1, x_2) and having radius ct.

So let us compute this integral which is appearing here this integral and this integral is exactly same because anyway we have no idea what phi and psi are. So it is a general phi let us simplify this integral which is posed on this sphere to something which is on a disc that is what we are going to do now. So let S+ denotes the upper hemisphere since the integrant is independent of the y3 variable integral on the whole sphere is on the 2 times integral on the upper hemisphere.

Note that the upper hemisphere is a graph of this function which is defined on x 1, x 2 center c 2 radius disc to R and h of y 1, y 2 is square root of c square t square –x 1 – y 1 square –x 2 –y 2 square where d of x 1, x 2, ct denotes. So open disc with center at x 1, x 2 and having radius ct.

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Thus the formula

$$u(x_1, x_2, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_S \varphi(y_1, y_2)\, d\sigma\right) + \frac{1}{4\pi c^2 t}\int_S \psi(y_1, y_2)\, d\sigma$$

reduces to

$$u(x_1, x_2, t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\varphi(y_1, y_2)}{\sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}}\, dy_1 dy_2\right) + \frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\psi(y_1, y_2)}{\sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}}\, dy_1 dy_2$$

Therefore the surface measure R d sigma on S+ which is given as S+ is given by y 3 = h of y 1, y 2 this is what we mean by saying is S+ is the graph of this function h is given by this. This is a fact from multi variable calculus d sigma = square of 1 + norm grad h square d y 1, d y 2. So that means d sigma is equal to ct by square root of c square to square –x 1 – y 1 square – x 2 – y 2 squared y 1, d y 2.

After plugging in this value of norm grad x square we get this now substituting in this formula this formula reduces to this formula or it change changes to the this formula. This formula is posed on a particular sphere in R 3 now this is posed on a disc in R 2 which is what we like because we are saying the solution at the point x 1, x 2 at time t is given by this formula.

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Carrying out the differentiation in the last equation yields

$$u(\boldsymbol{x},t) = \frac{1}{2\pi c t^2}\int_{D(\boldsymbol{x},ct)} \frac{t\varphi(\boldsymbol{y}) + t\nabla\varphi(\boldsymbol{y}).(\boldsymbol{y}-\boldsymbol{x}) + t^2\psi(\boldsymbol{y})}{\sqrt{c^2t^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2}}\, d\boldsymbol{y}.$$

Re-writing the integral in the above equation using a change of variable, we get

$$u(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{D(0,1)} \frac{\varphi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-\|\boldsymbol{z}\|^2}}\, d\boldsymbol{z} + \frac{ct}{2\pi}\int_{D(0,1)} \frac{\nabla\varphi(\boldsymbol{x}+ct\boldsymbol{z}).\boldsymbol{z}}{\sqrt{1-\|\boldsymbol{z}\|^2}}\, d\boldsymbol{z} + \frac{t}{2\pi}\int_{D(0,1)} \frac{\psi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-\|\boldsymbol{z}\|^2}}\, d\boldsymbol{z}.$$

So this formula is also called Poisson Kirchhoff formulae we have to remember that this is in 2 dimensions. So if you compute the differentiation here dou by dou t, here we get this. Rewriting the integral in the above equation using change of variable we get it on the disk of radius 1 center at the origin. Please do these computations by yourselves particularly this change of variables very simple.

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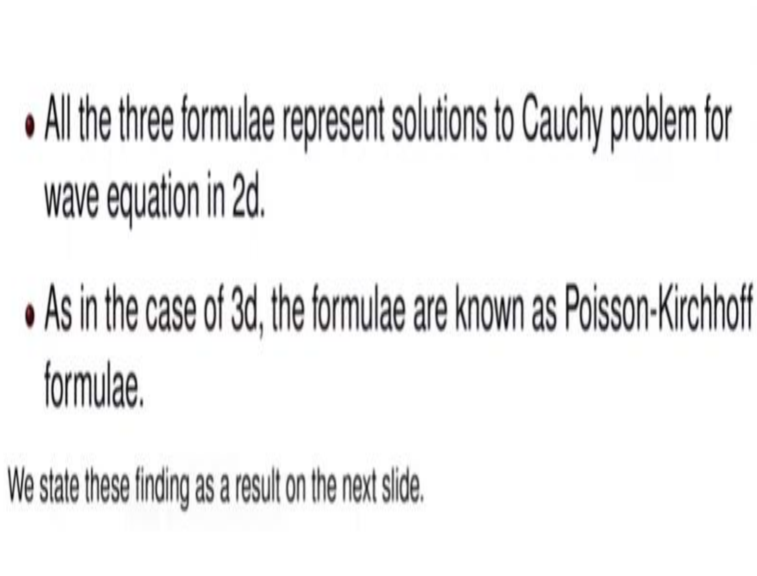

So all the 3 formulae represent solution to Cauchy problem for wave equation in 2d as in the case of 3d the formula are still known as Poisson-Kirchhoff formulae and this finding we state as a result.

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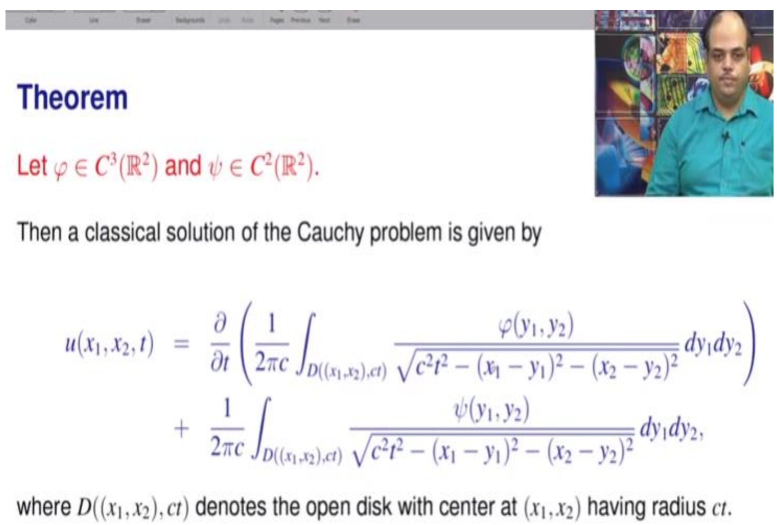

So theorem let phi be C3 on R2 and psi be C2 function on R2 then a classical solution to Cauchy problem is given by this formula Poisson Kirchhoff formula. So now how do I check this it is a matter of verification so that is left as an exercise to you.

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A short tutorial on Wave equation in higher dimensions

So let us begin a short tutorial and wave equation in higher dimensions.

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Problem 1

Solve the Cauchy problem for 3d Wave equation with $c = 1$:

$$\square_3 u = 0, \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, \ t > 0,$$

$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3,$$

$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \|\boldsymbol{x}\| \le 1, \\ 0 & \text{if } \|\boldsymbol{x}\| > 1. \end{cases}, \quad \boldsymbol{x} \in \mathbb{R}^3.$$

Without worrying about the smoothness of Cauchy data, go ahead and compute.

The problem 1 so this is a wave equation we are taking c = 1 so wave equation in 3 space variable homogenous equation we take ux 0 = 0 ut to be 1 inside the unit ball up to the boundary and 0 outside the norm x bigger than 1 it will be 0. Of course immediate vary is the data phi is C3 function yes it is a c infinity function in fact fine. But psi is not C2 it is actually discontinuous so as before we did the further wave equation in 1d do not bother about this that this is not a smooth function as required for a classical solution being given by Poisson-Kirchhoff formulae.

Go ahead and compute the solution with quotes because now we are not sure it is a classical solution but still go ahead and compute that is what we are doing.

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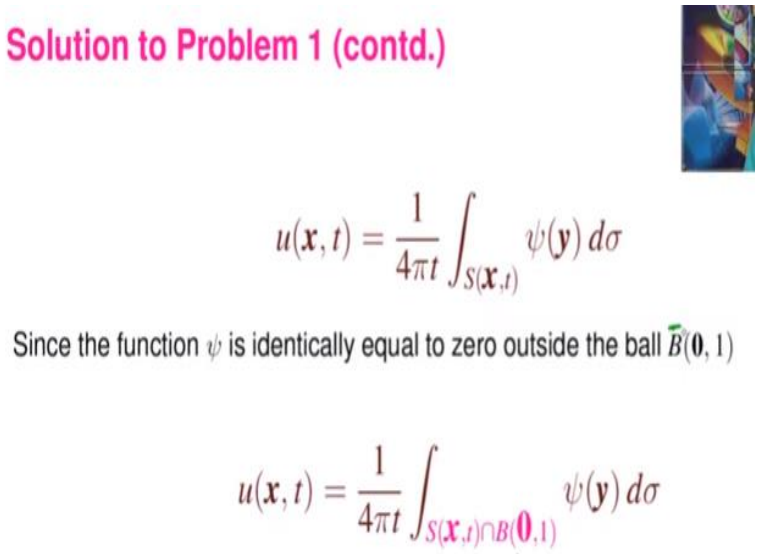

So since phi is 0 the first term drops out what you have is only this so therefore it is this so we need to compute what this integral is for the given sign. Since the function psi is identically equal to 0 outside this ball or the closure of the ball u x t the integral reduces to integral on S x t intersection B 0, 1.

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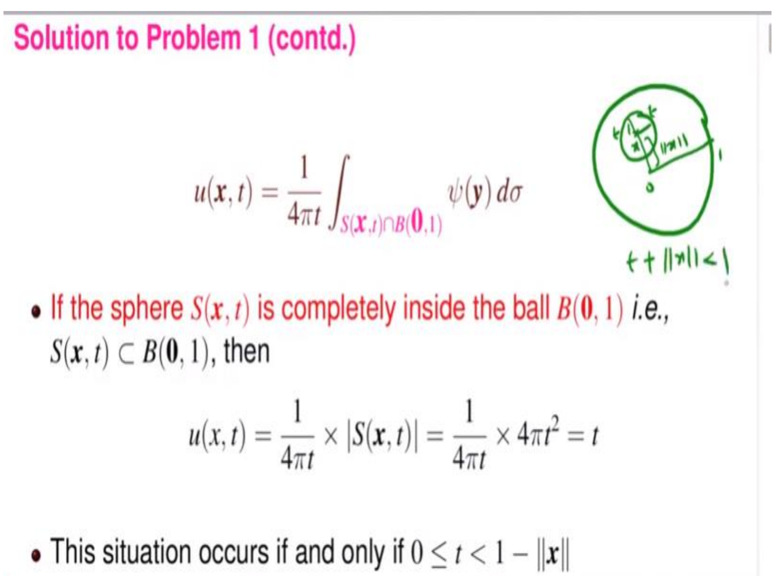

Now if this sphere is completely inside the ball so let us write this ball this is the ball in 3d of radius 1. Suppose my x is here and this is the sphere S of x, t so if the sphere is completely inside the ball that is the situation here then what happens? What about the intersection S x t intersection B 0, 1 it is simply S x, t? Therefore u x t is 1 by 4 pi t integral over S x, t of the function size 1 on the ball.

Therefore what we get is surface are of S x, t which is 4 pi t square we are in R3 S x, t has 4 pi t square is the surface area. So therefore u x, t equal to t and this situation occurs if and only if t is less than 1 – norm x. Just a rough picture 0, 1 this is x and the radius is t this distance is t. So you can just draw this line so this distance is norm x this distance is t that is the farthest point. So t+ norm x should be less than 1 therefore t is less than 1 – norm x.

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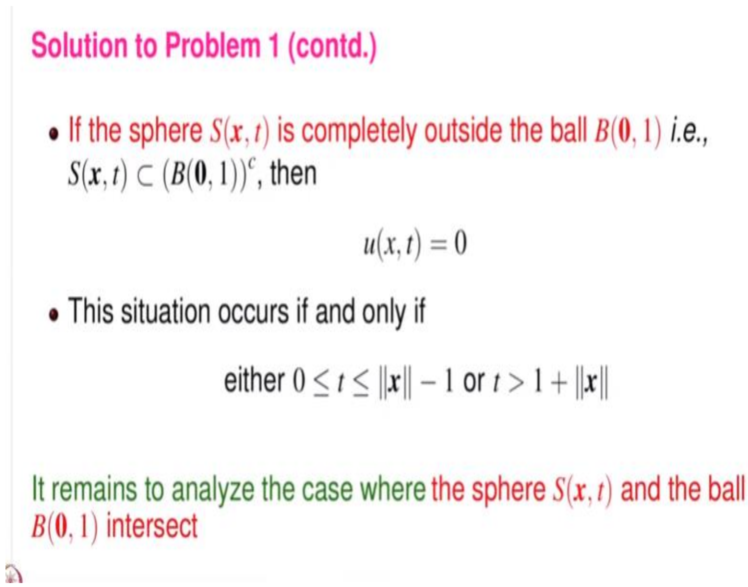

Now let us consider the second situation sphere is completely outside the ball how will that happen? You have this ball let us use a different color so I am x here so this is one way that this sphere of S x is t does not intersect is outside completely outside or another way is I have a x here it could be this. So in this case also it does not intersect the ball is here the sphere is this blue color thing. So it does not intersect then in that case u x, t is 0 because the intersection is empty set so u x, t is 0.

The integral is taken on domain which is empty set so integral is 0 and this situation occurs either if t is less than or equal to norm x – 1 or when t is bigger than 1 + norm x. So please deduce these from the picture draw the picture and you can deduce this. Now something else is remaining what remains is to analyze the case where the sphere and the ball intersect partially.

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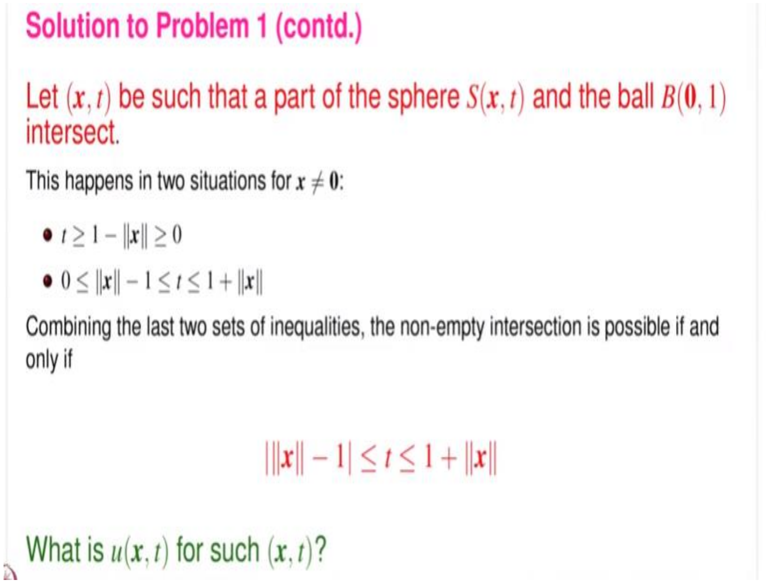

What we have considered is that never insects out total intersects so let x t be such that a part of this sphere and the; intersect and this happens in 2 situation for non-zero we have to be careful with x = 0. So let us consider for x non-zero t is greater than or equal to 1 minus norm x that is one situation coming from the first picture coming from the later pictures it is this zero less than or equal to norm x – 1. Less than or equal to t less than 1 + norm x now; combining the last 2 sets of inequalities.

The non-entities intersection is possible if and only if this happens modulus of norm x -1 should be less than or equal to t t less than or equal to 1 + norm x is the situation. So what about u x, t for such x t there are 2 things every time. We have to find out what is u of x t and what is the range of x, t for which that u x, t is valid.

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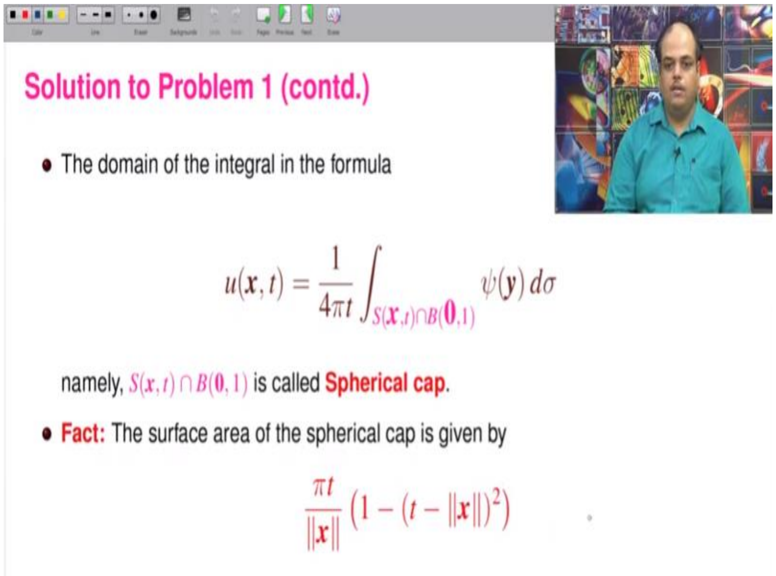

The domain of the integral in this formula S x, t intersection b 0, 1 namely this it is called spherical cap. Of course on the intersection psi is still 1 therefore the computation is becoming very easy so what we get is u x, t equal to 1 by 4 pi t into this surface area of the spherical cap. And surface area of the spherical cap is known to be given by this formula.

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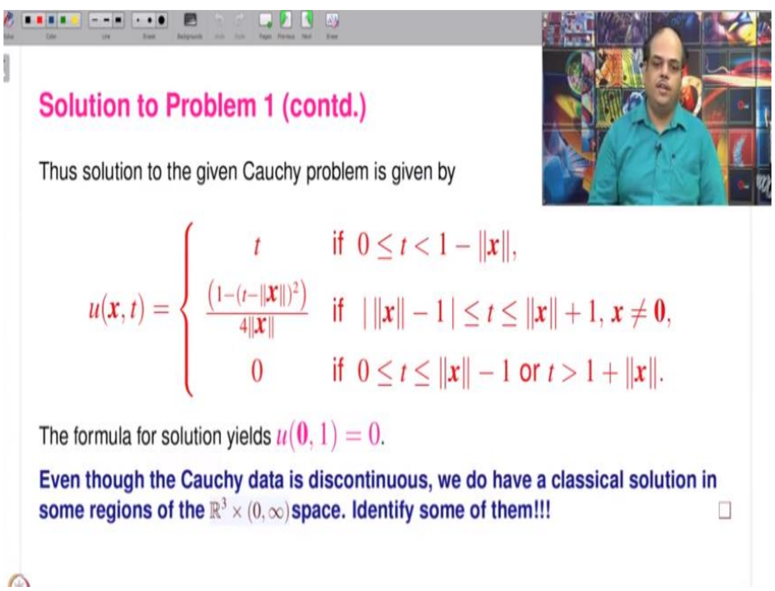

Therefore the solution we write down in this form this is coming from the first picture this is coming from the second picture this is from the other case for x non-zero. You see why we put x non-zero because of norm x is coming in the denominator that is natural it is coming. So at x = 0 u of 0, 1 t = 1 is a problematic point we will discuss this example again later when we are going to discuss how the confined disturbances propagate or be are propagated by wave equations in various dimensions.

This is an example which will be very useful to understand that so there is some problem at equal to 1 if you recall the Cauchy data has a trouble at sphere 1 right. It is discontinuous across norm x = 1 that finally leads to this trouble u of 0, 1 = 0 and something else happens as t goes to 1 at the point x = 0 which we will discuss later. So even though the Cauchy is discontinuous we do have a classical solution in some regions.

Identifying what are those some of them this is the same philosophy we applied in the case of Burgers equation they are also we worked with discontinuous initial conditions. We found of formula then we realized that there are regions where it actually solution to the burgers equation and regions where it is solutions to the Cauchy problem for the burgers equation. So, same thing here so please identify some of them.

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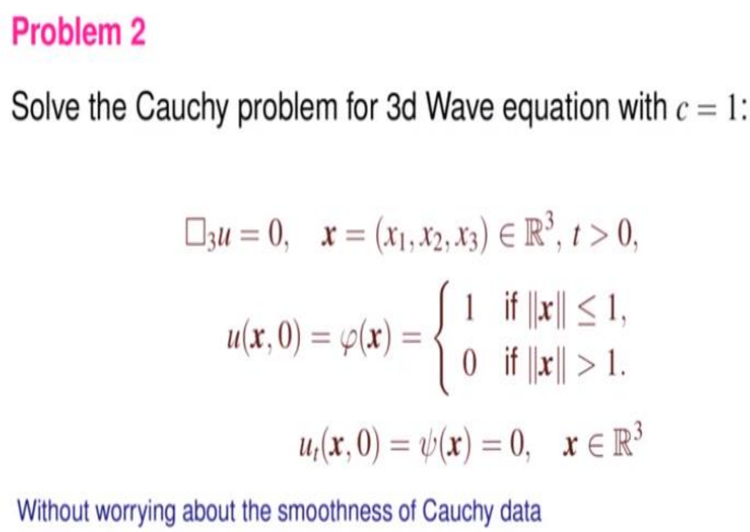

Let us move on to problem 2 where we have now change the sign the previous problem to phi. Now phi is 1 norm x is less than equal to 1 0 if norm is greater than 1 but psi is 0. Again as before do not worry about the smoothness of the Cauchy data go ahead and compute.

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Solution to Problem 2

The Poisson-Kirchhoff formula for a solution to the Cauchy problem in 3d

$$4\pi c^2 u(\boldsymbol{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S(\boldsymbol{x},ct)} \varphi(\boldsymbol{y})\,d\sigma\right) + \frac{1}{t}\int_{S(\boldsymbol{x},ct)} \psi(\boldsymbol{y})\,d\sigma. \quad \textbf{(F3)}$$

reduces to

$$u(\boldsymbol{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S(\boldsymbol{x},t)} \varphi(\boldsymbol{y})\,d\sigma\right)$$

In **Problem 1**, we already computed the integral term. Need to compute its derivative only.

And this Poisson formula now reduces to this earlier it was given by second term now it is even given by the first term. But if you notice who is inside this dou by dou t this integral term is precisely same as before in fact the function phi now is the same the psi in problem 1. So we know what this is therefore it is a matter of just finding the derivative and get solution to this Cauchy problem.

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Solution to Problem 2 (contd.)

Solution to Problem 1 is given by

$$u(\boldsymbol{x},t) = \begin{cases} t & \text{if } 0 \le t < 1 - \|\boldsymbol{x}\|, \\ \frac{(1-(t-\|\boldsymbol{x}\|)^2)}{4\|\boldsymbol{x}\|} & \text{if } |\,\|\boldsymbol{x}\| - 1\,| \le t \le \|\boldsymbol{x}\| + 1, \boldsymbol{x} \ne \boldsymbol{0}, \\ 0 & \text{if } 0 \le t \le \|\boldsymbol{x}\| - 1 \text{ or } t > 1 + \|\boldsymbol{x}\|. \end{cases}$$

Therefore solution to **Problem 2** is given by

$$u(\boldsymbol{x},t) = \begin{cases} 1 & \text{if } 0 \le t < 1 - \|\boldsymbol{x}\|, \\ \frac{\|\boldsymbol{x}\|-t}{4\|\boldsymbol{x}\|} & \text{if } |\,\|\boldsymbol{x}\| - 1\,| \le t \le \|\boldsymbol{x}\| + 1, \boldsymbol{x} \ne \boldsymbol{0}, \\ 0 & \text{if } 0 \le t \le \|\boldsymbol{x}\| - 1 \text{ or } t > 1 + \|\boldsymbol{x}\|. \end{cases}$$

So and we get this this is the solution in problem 1 now in problem 2 this is the solution there will be troubles at this boundary points of this domains of validity and that is where there are possible singularities for the solutions. Otherwise in the interior of these ranges it must be, find and these are the solution.

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Problem 3

Solve the Cauchy problem for 3d Wave equation with $c = 1$:

$$\Box_3 u = 0, \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, \, t > 0,$$
$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}) = x_1^2 + x_2^2, \quad \boldsymbol{x} \in \mathbb{R}^3,$$
$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3$$

Now let us look at problem 3 which is again a homogenous 3 dimensional wave equation with c = 1 where we have specific nice initial conditions phi x is x1 square + x2 square it means this does not depends once again on x3 and psi 0 of course. So it does not depend on any variable it is a constant. So because phi does not depend on x3 we have to 2 waves of solving this problem one using Poisson Kirchhoff formula in 3 space dimension. Second one is using in 2 space dimension we do both.

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Solution to Problem 3

The Poisson-Kirchhoff formula for a solution to the Cauchy problem in 3d

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{\|\nu\|=1} \varphi(\boldsymbol{x} + ct\nu)\, d\omega\right) + \frac{t}{4\pi}\int_{\|\nu\|=1} \psi(\boldsymbol{x} + ct\nu)\, d\omega \quad \textbf{(F1)}$$

reduces to

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{\|\nu\|=1} \varphi(\boldsymbol{x} + t\nu)\, d\omega\right)$$

Because psi is 0 the Poisson Kirchhoff formula f1 reduces to this one so what we have to do is really to compute this integral.

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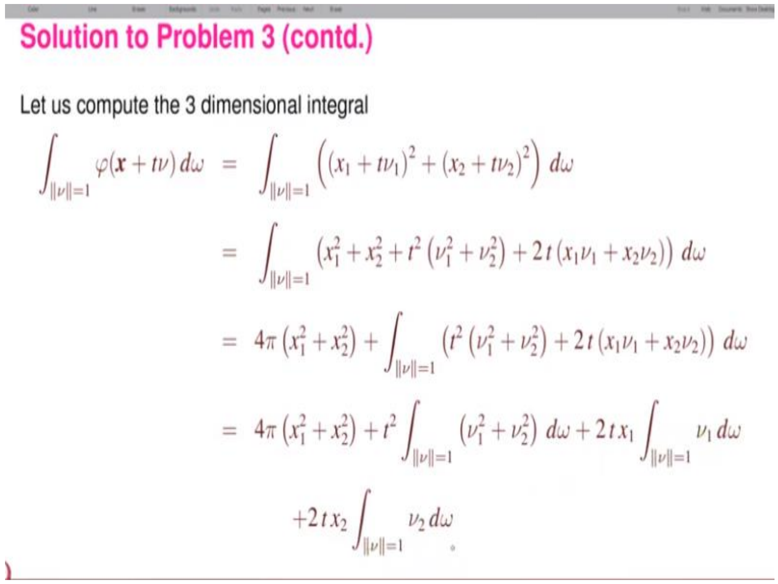

So let us compute this 3 dimension integral phi of x + t nu is the first coordinate of x + t nu square plus second coordinate of x + t nu square which is here. Let us expand this it is will give us x1 square + x2 square + t square nu 1 square t square nu 2 square which is here plus the 2 a, b terms that is 2t x1 nu1 2 t x2 nu same. Now here x1 square + x2 square does not depend on nu so it comes out as constant and with the multiplied with the surface area of the unit sphere.

Here t square also comes out but there is a nu1 square + n2 square remember that is not 1 because here u1 square + nu 2 square + nu 3 square is one. Therefore we have to do some computation let us see how this term split? As I have mentioned earlier this comes out as the constant times the surface area of this which is 4 pi this is done. The second term and third term are still kept together one at a time.

Now we have separated them into 2 terms in fact 2 terms 2t x1 nu 1 is here extremities here. Now we need to compute this integral and this integral it looks like this and this integrals will behave similarly and this integral we have to compute again.

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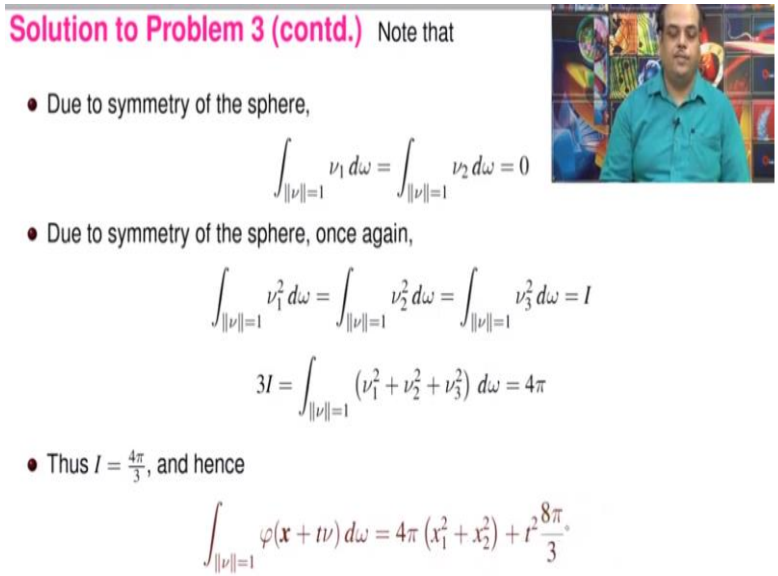

Due to symmetry of this sphere this is 0 because the monitor of time this is nu is positive it is also negative the unit sphere that is the rotational symmetry because of that this is 0 and this 0. Please convince yourself about this now once again due to symmetry what we know is this nu1 square integral is same of nu 2 square is same as integral nu2 square. Let us call all of them to be I now we are going to play trick 3I is not but summation.

Earlier we noticed that nu 1 square + nu 2 square may not be 1 is not 1 in general right except at few points. But now nu1 square + n2 square + n3 square is here which we know is 1 therefore what this integral evaluates to just surface area of the unit sphere which is 4pi therefore I is 4pi by 3. Therefore we know this integral it is 4 point x1 square + x square + t square 8pi by 3 because there are 2 terms. So 8 pi by 3 c.

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Solution to Problem 3 (contd.)

Thus solution to the Cauchy problem is given by

$$\begin{aligned} u(\boldsymbol{x},t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{\|\nu\|=1}\varphi(\boldsymbol{x}+t\nu)\,d\omega\right) \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(4\pi t\left(x_1^2+x_2^2\right)+t^3\frac{8\pi}{3}\right) \\ &= \frac{\partial}{\partial t}\left(t\left(x_1^2+x_2^2\right)+\frac{2t^3}{3}\right) \\ &= x_1^2+x_2^2+2t^2 \end{aligned}$$

Therefore we have computed integral that we wrote multiplied which this t so this is the expression we have and finding the derivative we get answer to be x1 square + x2 square + 2d square. So this is solution one can easily check now that this is a solution to be given Cauchy problem.

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Solution to Problem 3: 2nd Method

The Cauchy problem for 3d Wave equation with $c = 1$:

$$\Box_3 u = 0, \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3,\ t > 0,$$
$$u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}) = x_1^2 + x_2^2, \quad \boldsymbol{x} \in \mathbb{R}^3,$$
$$u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3$$

may also be solved using the formula for 2d wave equation as $\varphi(x)$ does not depend on the variable x_3 and $\psi \equiv 0$.

And now we are going to discuss about a second method which I have outlined at the beginning and realizing that the phi does not depend on x3 variable we think that the solution to this can be obtained using the expression for 2d wave equation solutions. Of course psi does not depend on x3 because 0.

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Solution to Problem 3: 2nd Method Solution to Cauchy problem reduces to

$$u(x_1,x_2,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{D((x_1,x_2),ct)} \frac{\varphi(y_1,y_2)}{\sqrt{t^2-(x_1-y_1)^2-(x_2-y_2)^2}}\,dy_1dy_2\right),$$

where $D((x_1,x_2),ct)$ denotes the open disk with center at (x_1,x_2) having radius t.
After a change of variable $z = \frac{y-x}{t}$, the above formula becomes

$$u(x_1,x_2,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{D((0,0),1)} \frac{\varphi(x+tz)}{\sqrt{t^2-t^2z_1^2-t^2z_2^2}}\,t^2\,dz_1dz_2\right),$$

where $D := D((0,0),1)$ denotes the open disk with center at $(0,0)$ having radius 1.

So this is the formula u of x1, x2 t later on we are going to write it as u of x1, x2, x3 t solution does not depend on x3. Now let us do a change of variable z = y – x by t so then y become x + t z from here and dy1 and dy2 that is a measure that becomes t square dz1 dz2. And here I have substituted and therefore that denominator changes and the domain which is the disc radius ct center x becomes radius once center 0.

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Solution to Problem 3: 2nd Method

$$u(x_1,x_2,t) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{D((0,0),1)} \frac{\varphi(x+tz)}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2\right),$$

$$= \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{D} \frac{(x_1+tz_1)^2+(x_2+tz_2)^2}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2\right),$$

Let us compute the integral

$$\int_D \frac{(x_1+tz_1)^2+(x_2+tz_2)^2}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2 = \int_D \frac{x_1^2+x_2^2+t^2(z_1^2+z_2^2)+2tx_1z_1+2tx_2z_2}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2$$

Now we have to compute this integral and then substitute so after substituting formula for phi we get this. So let us compute the integral which is here I have written d for, d disc of radius one which center origin just for notational convenience otherwise this integral you see how much space is coming. So it will fill up the spaces to avoid that I have done d. So d stands for the disc of radius 1 center at the origin. So this on expansion numerator we will get this.

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Solution to Problem 3: 2nd Method

Exercises: Show that

1. $$\int_D \frac{z_1}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2 = \int_D \frac{z_2}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2 = 0$$

2. using polar coordinates,
$$\int_D \frac{1}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2 = 2\pi$$

3. using polar coordinates,
$$\int_D \frac{z_1^2+z_2^2}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2 = \frac{4\pi}{3}$$

Now exactly as before we are going to compute these integrals so which I leave it as a exercise this in both integrals are 0 using polar coordinates show that this integral is 2 pi and this integral is 4 pi by 3.

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Solution to Problem 3: 2nd Method

Solution to Cauchy problem is given by

$$\begin{aligned} u(x_1,x_2,t) &= \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\left(2\pi(x_1^2+x_2^2)+\frac{4\pi t^2}{3}+0\right)\right) \\ &= \frac{\partial}{\partial t}\left(t(x_1^2+x_2^2)+\frac{2t^3}{3}+0\right) \\ &= x_1^2+x_2^2+2t^2 \end{aligned}$$

Now we are going to go and substitute in the expression for the solution please do these computations on your own. Once you compute the integrals nothing is left everything is easy so we get of course the same solution and it does not depend on x3. Because the problem is posed as a solution to 3d wave equation we have to write u of x1, x2, x3, t equal to x1 square + x2 square + 2 t square.

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Let us summarize using Hadamard's method of descent solution to Cauchy problem for wave equation in 2d is found and we solved 3 casual problems we used very nice functions as Cauchy data to make computation very easy, We can only imagine the scale of difficulties in dealing with more general Cauchy data thank you.