Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology - Bombay

Module No # 06
Lecture No # 30
Cauchy problem for Wave Equation is 3 Space dimensions

Welcome to the lecture on the Cauchy problem for wave equation in 3 space dimensions. We are going to deduce a formula for solution to the Cauchy problem these solution formulas are known as Kirchhoff formulae or Poisson formulae. So instead giving credit to one of them they give to both of them and call them or refer to these formulae as Poisson Kirchhoff formulae.

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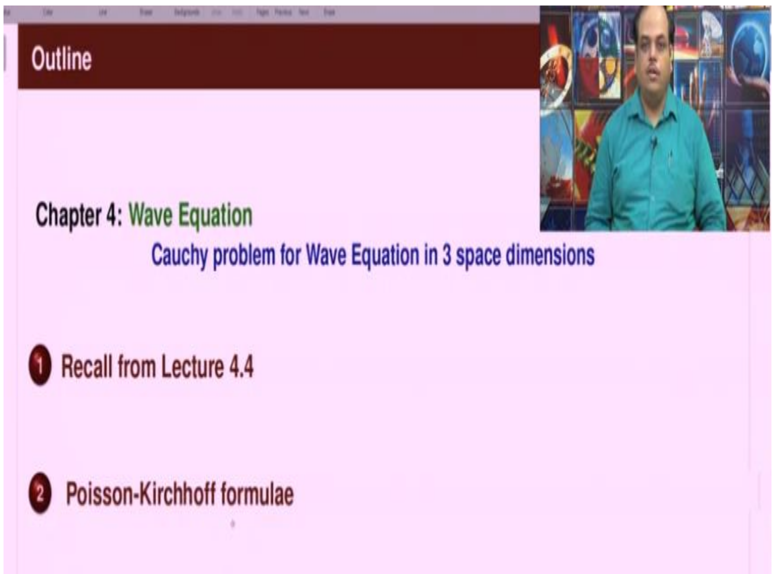

So the outline is very simple we start a recall certain things that we did in lecture 4.4 which his basics about the spherical means and couple of results related to that and then go on to derive Poisson-Kirchhoff formulae.

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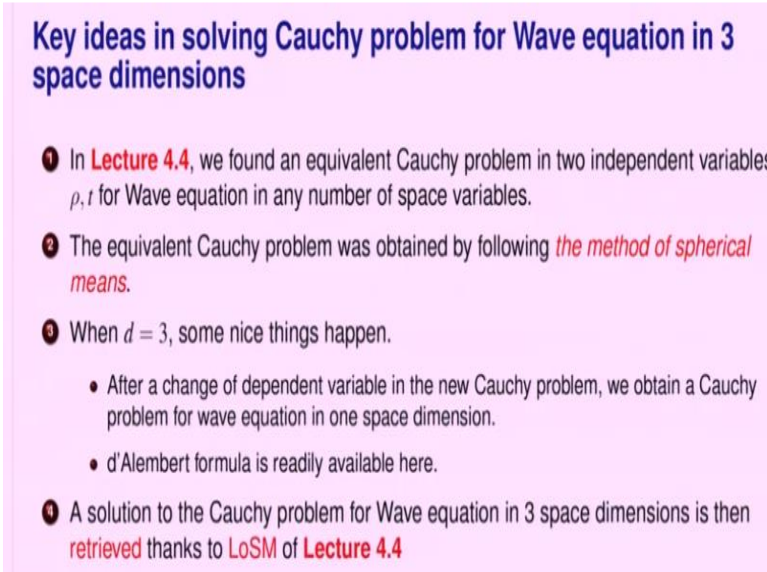

So the key ideas in solving the Cauchy problem in lecture 4.4 we found an equivalent Cauchy problem in 2 independent variable rho and t for any number of space equation in variables. The equivalent Cauchy problem was obtained by following method of spherical means. When d equal to 3 some nice things happens after dependent of change of variable in the new Cauchy problem we obtain a Cauchy problem for wave equation in 1 space dimension.

And d'Alembert formula is a readily available here the solution to the Cauchy problem for wave equation in 2 space dimensions is an retrieved thanks to LoSM of lecture 4.4 LoSM is lemma on spherical means.

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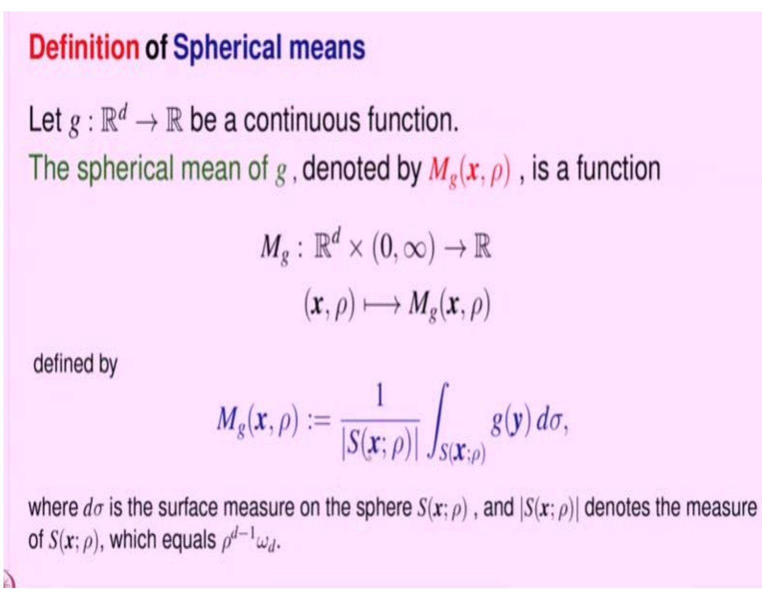

So let us recall from lecture 4.4 definition of spherical means for a continuous function we define spherical mean of g, as a function of x, rho denoted by Mg of x, rho which is integrate g over the sphere S x, rho multiplied by S x, rho is the surface measure of S x, rho.

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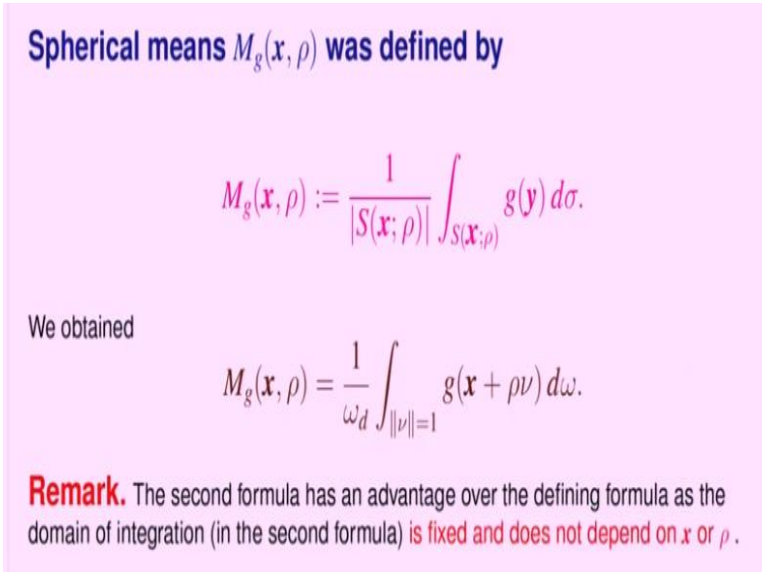

So; we have define the spherical means by this problem and we obtained an alternate formulae and this formula as an advantage over the first formulae because the domain of integration does not depend on x r, n rho.

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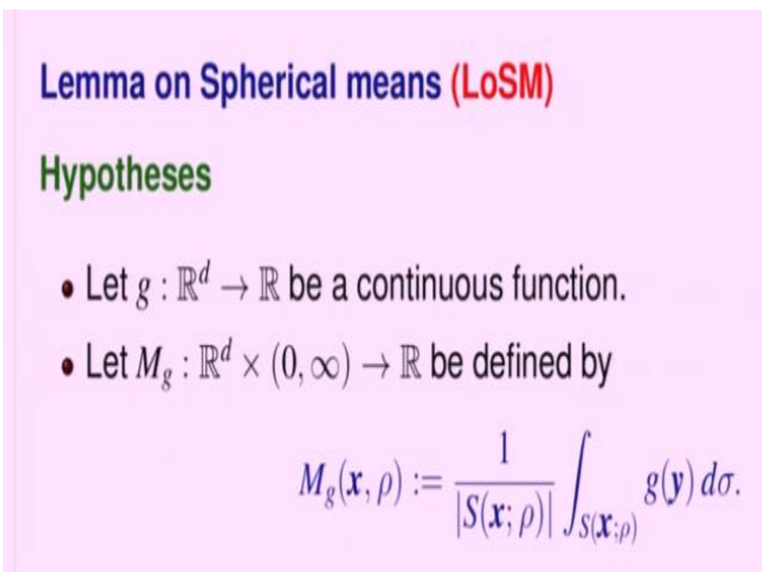

So lemma and spherical means is that hypothesis is that if you have a continuous function and define the spherical means f M g.

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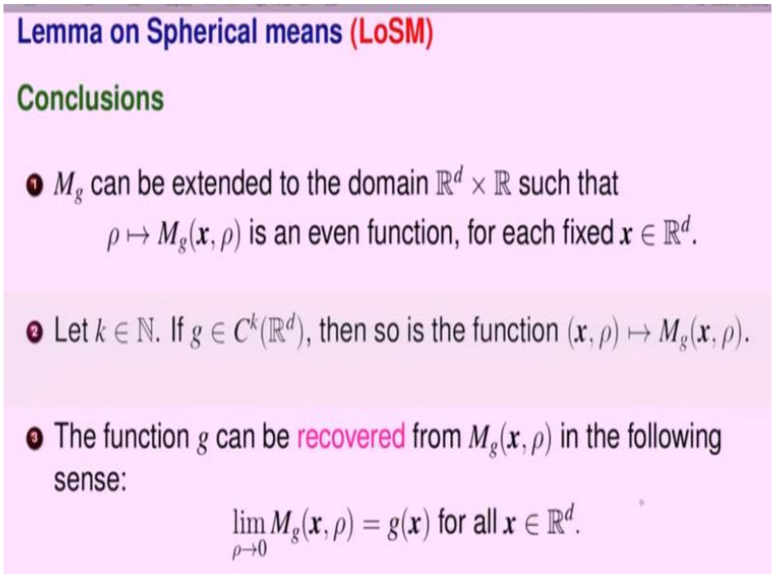

Then M g can be extended to R d cross R such that for each fixed x in R d the M g as the function of rho is an even function. So let k belongs to N if g is a Ck function R d then so is the function x rho mapping to M g of x rho it is a function defined an R d cross R note by conclusion 1 we have already extended the function M g of x rho for rho belonging to R. And the function g itself can be recovered from the spherical means.

So if you know the spherical means over radii rho = 0 then M g of x rho limit as rho goes to 0 will give you g of x. This will happen for every x in R d.

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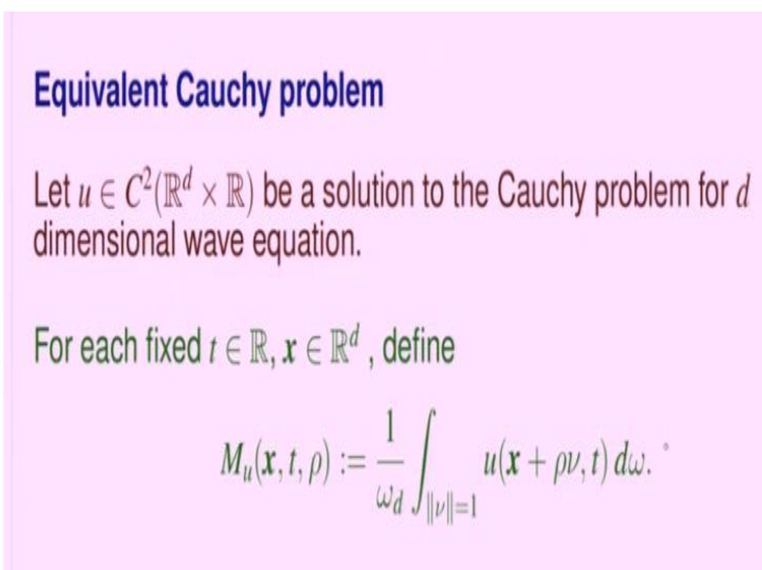

For g in C 2 of R d the Darboux formula is Laplacian of M g with respect to x variable is this dou 2 by dou rho square + d – 2 by rho dou by dou rho acting on M g. So this is the so called radian Laplacian.

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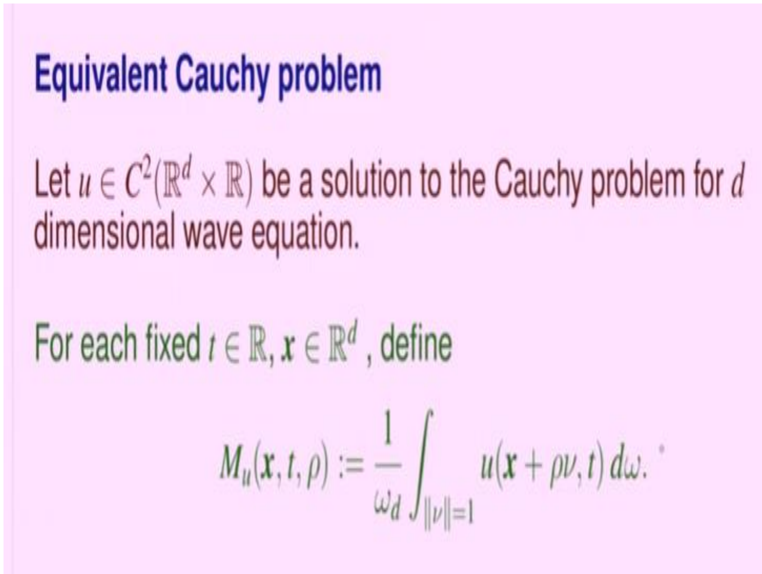

So equivalent Cauchy problem to introduce that, we need to define the spherical means of u. So let u be a C 2 function R d cross R be a solution to the Cauchy problem for d dimensional wave equation the define Mu the spherical mean of u as usually by same formula for the spherical means.

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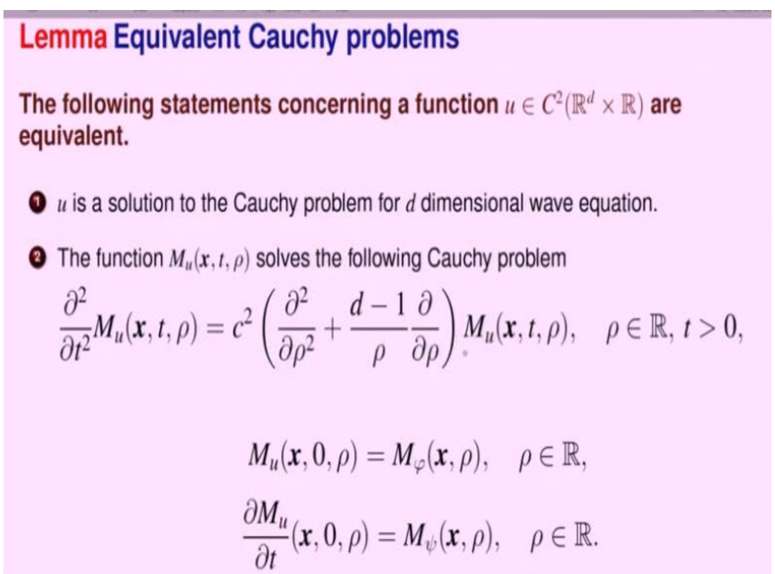

Now this M u satisfies Cauchy problem that is what we are going to say here so equivalent Cauchy problems if u is the solution to the Cauchy problem for the d dimensional wave equation then the M u solves the following Cauchy problem. This is the radial Laplacian here C square and this is second derivative with respect to t of M u and these are the initial conditions for M u and dou by dou t of n.

So this still not the wave equation we need to change the dependent variable Mu to something else then the right hand side will look like the wave equation. That means right hand side looks like dou 2 by dou rho square of the nu quantity.

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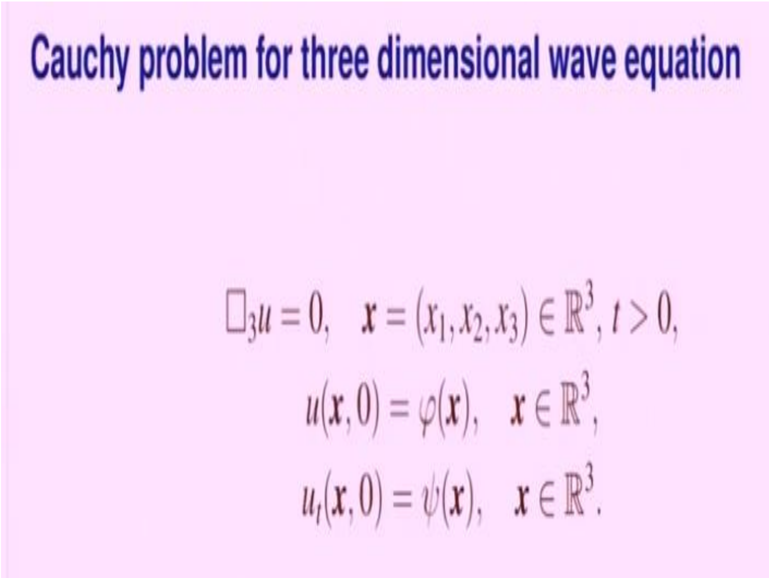

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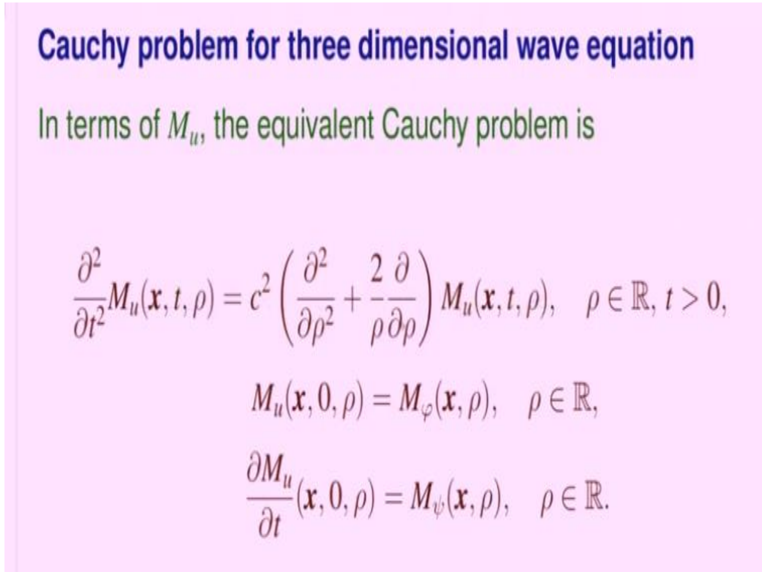

So this is the Cauchy problem for 3 dimensional waves equation now when d = 3 that d -1 is 2 so this is the equivalent Cauchy problem that we have derived in the last lecture. That is lecture 4.4 now we are going to change the dependent variable Mu this is not the way we the usual wave equation one space dimension. So we are going to change it.

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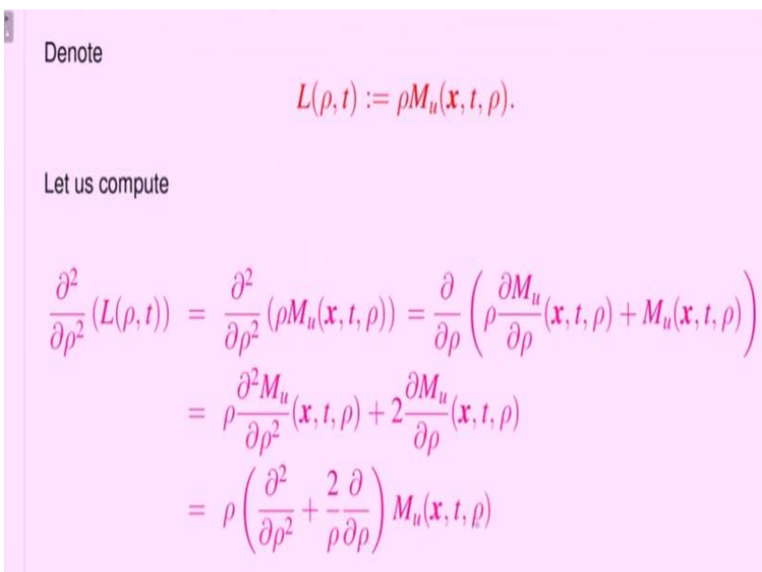

To rho mu that we will call as L of rho t equal to rho M u this L will satisfy 1 dimensional wave equation let us do the computation. So dou 2 by dou square of L rho t is dou 2 by dou square of rho Mu because L = rho M u. So keep 1 dou by dou rho out take 1 / dou rho inside when it differentiate the inside quantity with respect to rho you get this plus this. This is rho times dou M u by rho which is here +M u.

Now differentiate once again with respect to rho you get end up with this so this is actually this rho this operator dou 2 dou rho square + 2 by rho by dou rho acting on M u.

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Using the equation

$$\frac{\partial^2}{\partial t^2} M_u(\mathbf{x}, t, \rho) = c^2 \left(\frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} \right) M_u(\mathbf{x}, t, \rho),$$

we get

$$\begin{aligned}\frac{\partial^2}{\partial \rho^2}(L(\rho, t)) &= \rho \left(\frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} \right) M_u(\mathbf{x}, t, \rho) \\ &= \frac{\rho}{c^2}\frac{\partial^2}{\partial t^2} M_u(\mathbf{x}, t, \rho) \\ &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} L(\rho, t)\end{aligned}$$

So using this equation we get this equation and this in terms of L so what we have is L satisfies dou 2 by dou rho square = 1 by C square dou 2 by dou t square which is nothing but the wave equation in one space dimension.

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Thus L satisfies the following Cauchy problem for a one dimensional wave equation

$$\frac{\partial^2}{\partial t^2} L(\rho, t) = c^2 \frac{\partial^2}{\partial \rho^2}(L(\rho, t)), \quad \rho \in \mathbb{R}, t > 0,$$

$$L(\rho, 0) = \rho M_\varphi(\mathbf{x}, \rho), \quad \rho \in \mathbb{R},$$

$$\frac{\partial L}{\partial t}(\rho, 0) = \rho M_\psi(\mathbf{x}, \rho), \quad \rho \in \mathbb{R}.$$

And let us write down the Cauchy data for L so L of rho 0 will be this and dou L by dou t of rho 0 will be rho M psi. So here we have a one dimensional wave equation and the corresponding

Cauchy data. Therefore using d'Alembert formula we can write the solution. Now the question is the d'Alembert formula going to be a classical solution to get that what we need is this function should be C2 because L of rho 0 should be C2 and dou L by dou t should be C1.

So therefore the question is this C2 this is C2 if and only if M phi C2 with respect to rho. This is just rho multiplication by a rho is very good thing so this quantity if and only if M phi C2 this quantity is C1 if and only if M psi C1.

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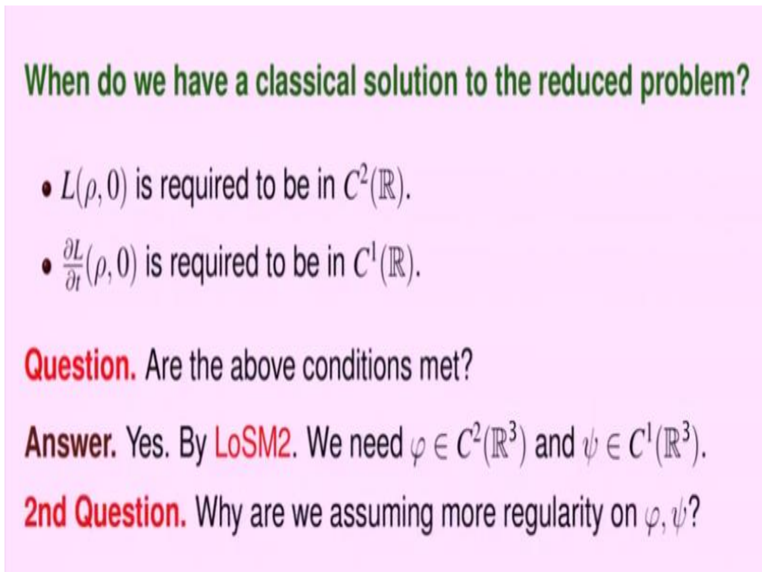

So are these conditions met? Answer is yes by LoSM2 it said if g is C k M g is C k therefore what we need is phi is C 2 therefore M phi will be C 2 and hence rho into M phi will be C 2 which is L rho 0. And we need psi to be C 1 so that dou L by Dou t is C1 at rho, 0 is C 1. Now we ask another question why are, you assuming more regularity and phi, psi we will see this soon. This is not good enough what is not good enough?

This assumption is not good enough for us to deduce a solution to the Cauchy problem for 3 space dimensions from the d'Alembert formula that we get. This hypothesis is good enough to apply d'Alembert formula to the equation for L and get the expression for L of rho, t stops there we will point out later.

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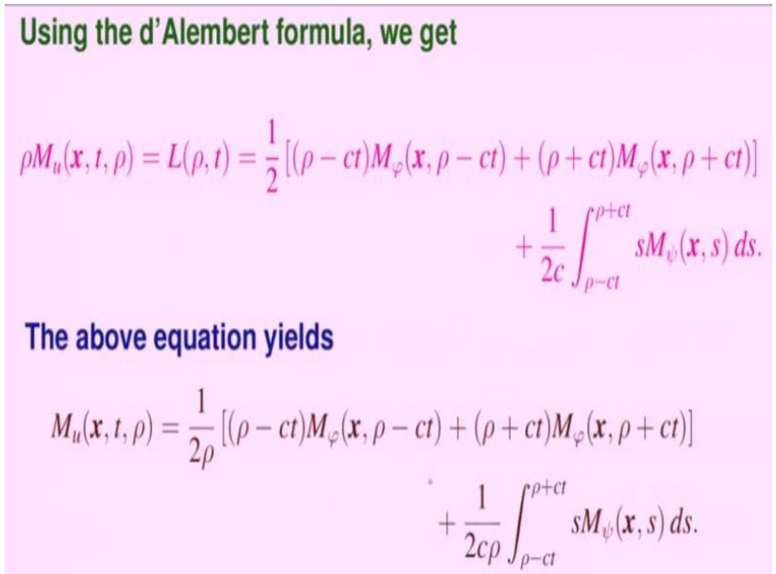

So using the d'Alembert formula this is the L rho M u is L so L = rho – ct M phi + rho + ct M phi + 1 by 2 c into the integral term. This is psi and this is the phi of this problem. So the above equation gives us M u you want therefore you divide everything with a rho so we have this and this. So we have an expression for Mu now I want u therefore L y S m 3 tells me I can get u from M u but I need to pass to the limit as rho goes to 0. Therefore we need to pass to the limit quantities on the RHS as rho goes to 0.

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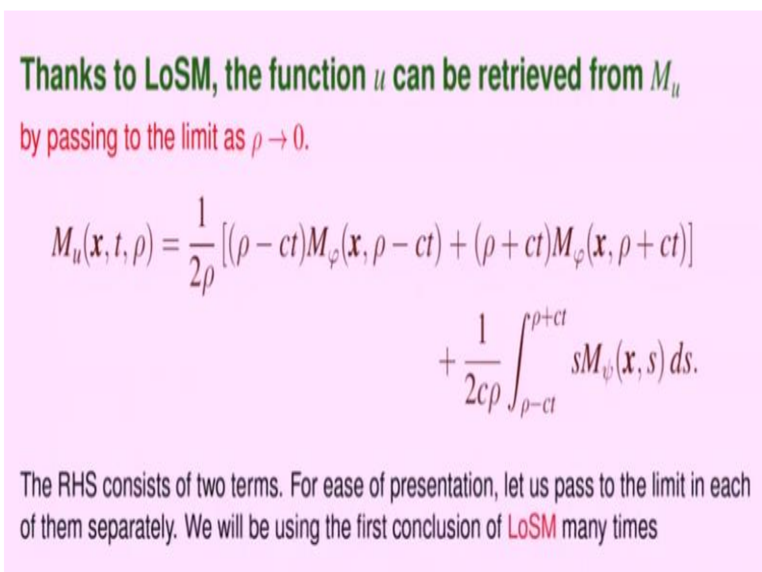

So let us do that this is the expression for Mu so RHS as 2 terms for ease of presentation we handle them separately and we will be using the first conclusion of LoSM many times what is that? Spherical means or even function with respect to rho.

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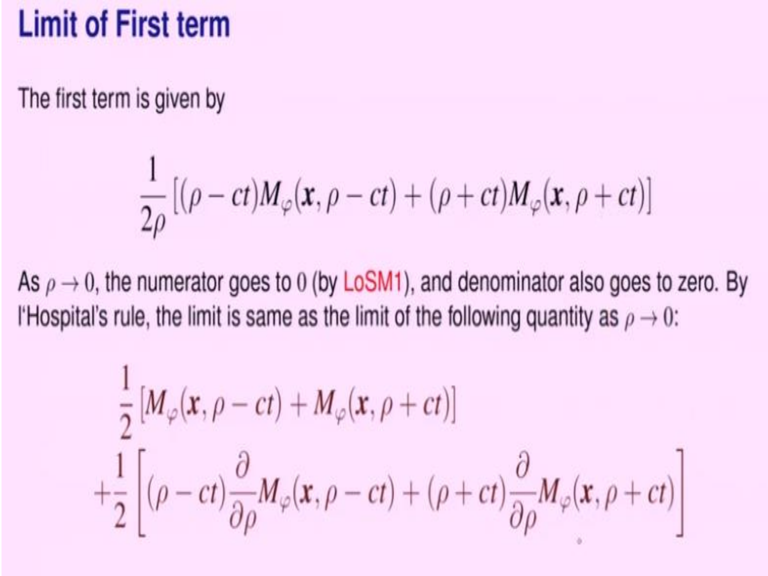

The first term is this we want to pass to the limit in this term if you look at the numerator goes to 0 as rho goes to 0. Because this goes to –ct M phi of x, -ct + ct M phi of x, ct and M phi is even with respect to the radius variable. Therefore M phi of –ct is M of M phi of ct therefore it gets cancelled and you get 0. And of course limit of 2 rho is also 0 so therefore we are in position to apply L hospital rule.

Therefore the limit rho goes to 0 of this is precisely the limit of the derivative of the numerator the denominator is of course just 2. So that is why it is 1 by 2 so the quantity inside the brackets is the derivative of this with respect to rho. So it is very clear how do we get these terms differentiate this with respect to rho then you get M phi of x, rho –ct, Similarly this with respect to rho here this term you get M phi of x, rho + ct.

Product rule so you have to differentiate this with respect to rho that will give you this rho –ct time's derivative of this with respect to rho. Similarly here rho + ct times derivative of this with respect to rho.

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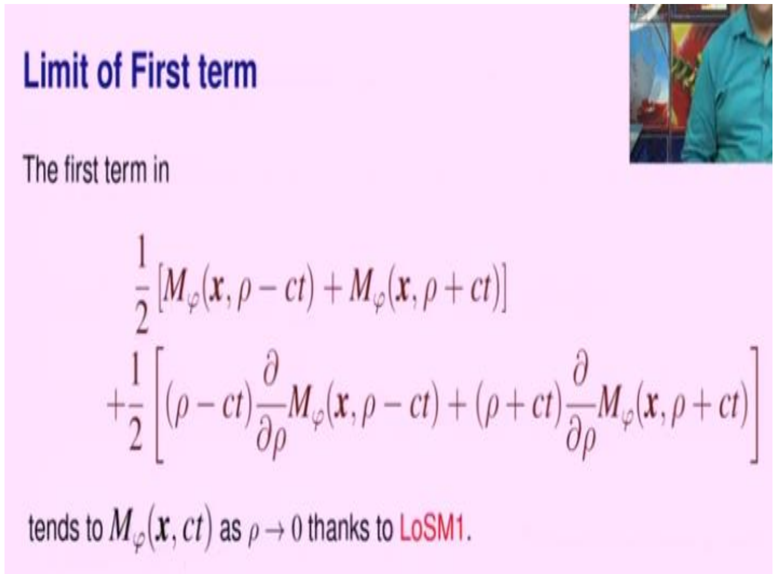

Now the first term tends to M phi of ct as rho goes to 0 because as rho goes to 0 this goes to M phi of x, - ct + M phi of x, ct. But both are same because of LoSM that means I have 2 times and then I have a 1 by 2 here. Therefore I get M phi so it is the easiest term to pass limit.

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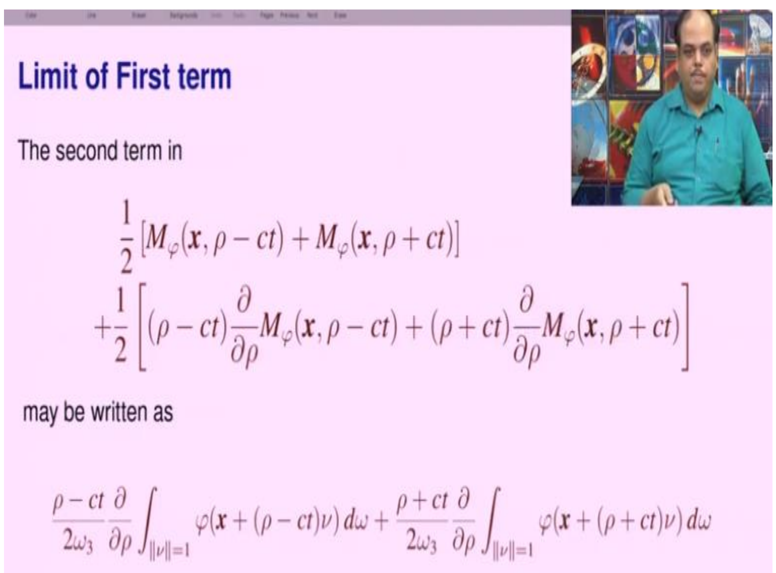

Now let us look at the second term and we just write down the formula for M phi is formula is 1 by omega d whether here it is 1 by omega 3 into integral on norm nu equal to 1 of phi of x + rho –ct nu. Similarly this term also we can write like this.

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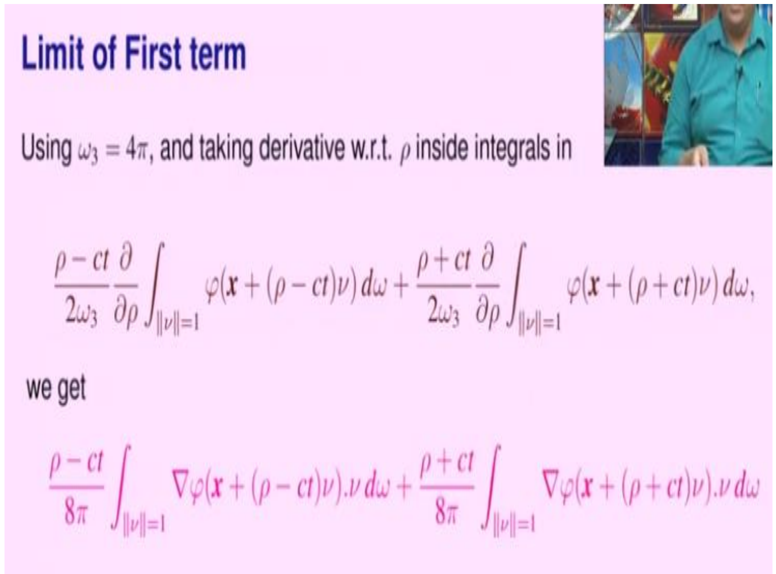

Now omega 3 is 4 phi and we can put that and the derivative term the dou by dou rho is outside the integral. So let us take it inside and differentiate we get this because when dou by dou rho goes here I get grad phi and derivative with respect to rho will give me nu so that is why a nu here. This omega 3 s 4 pi therefore this become 8 pi similarly the second term also we can write after pushing the derivative inside the integral so this is what we have.

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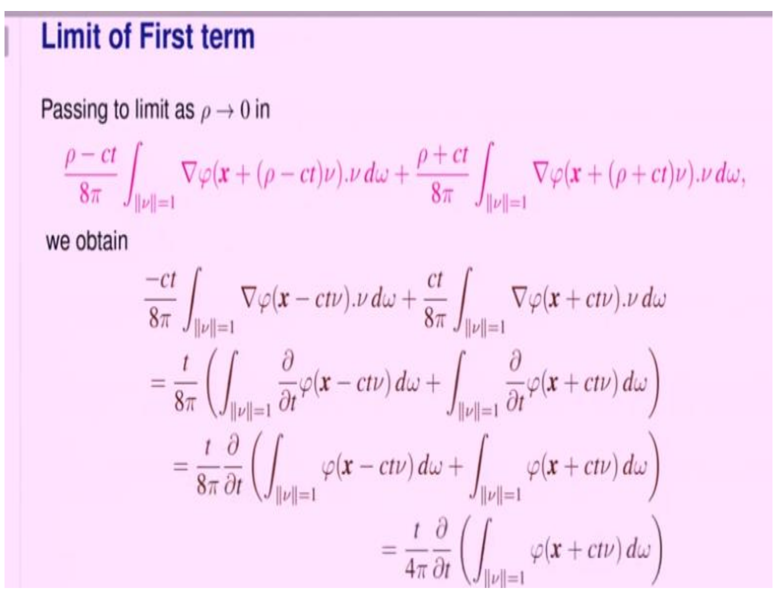

Now we want to pass to the limit in this term as rho goes to 0 we obtain this quantity right just straight forward from here and the second term will give you this. So I take t by 8 pi common I have this expression now I put dou by dou t outside of the 2 integrals and the one which is inside both the terms are equal therefore that is 2 times that. So when it comes out you get t by 4 pi into dou by dou t of this.

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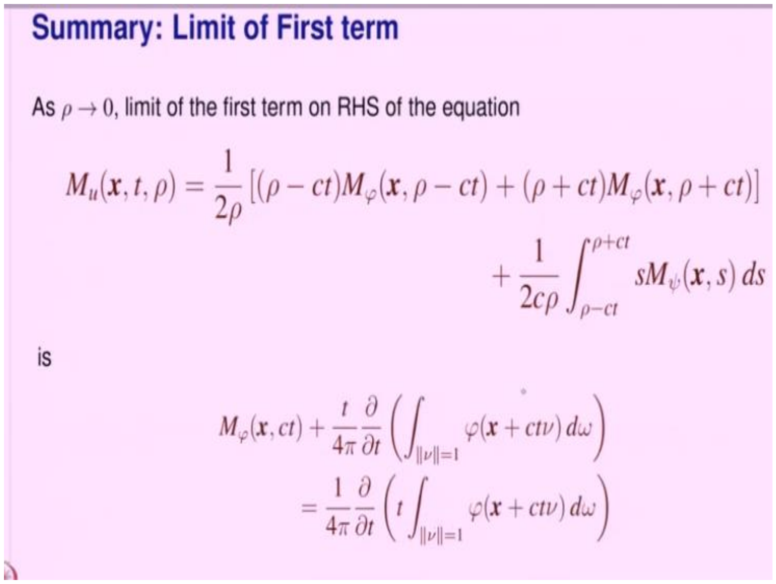

So we have finished obtaining the limit of the first term so as rho goes to 0 limit of the first term and RHS of this equation is this. The first term gave us M phi the second term give us this. Now we need to now pass to the limit in the second term here that is much simpler. This we can club these 2 terms like this if you expand this you get this. So this is more compact notation compact form of this formula.

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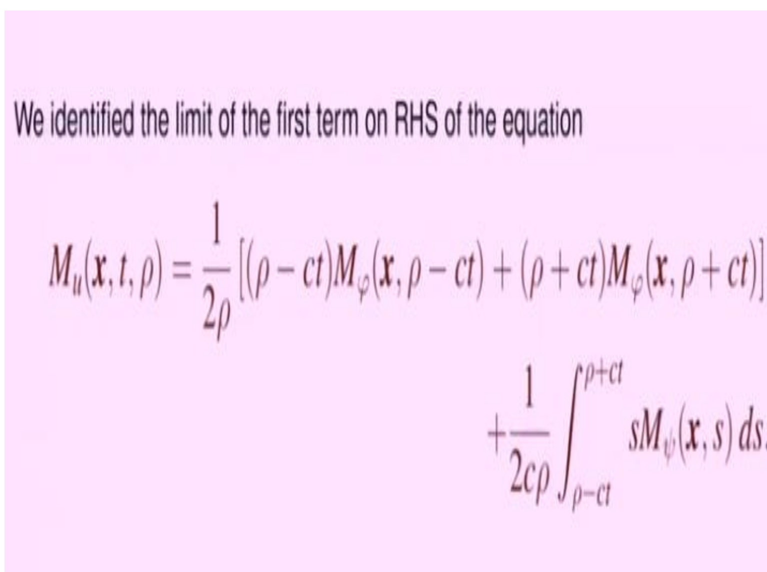

So we identified the limit of the first term let us do identify the limit of the second term now.

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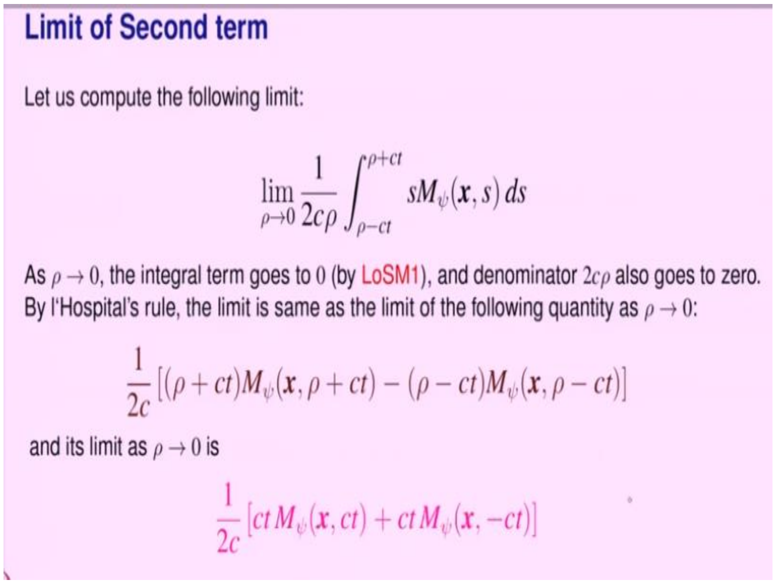

So this is the limit we are interested in the integral term goes to 0 why? As rho goes to 0 this is integral –ct to ct right and M psi is an even function with respect to the variable s. But there is a s which is multiplying therefore the that will be an odd function therefore the integral is 0 when rho is 0. And of course denominator also goes to 0 once again I hospital rule that tell us the limit is 1 by 2c times derivative of this which followed by fundamental theorem of calculus is this.

And now we have to take limit of this as rho goes to 0 and that is very simple now once again M psi is even function.

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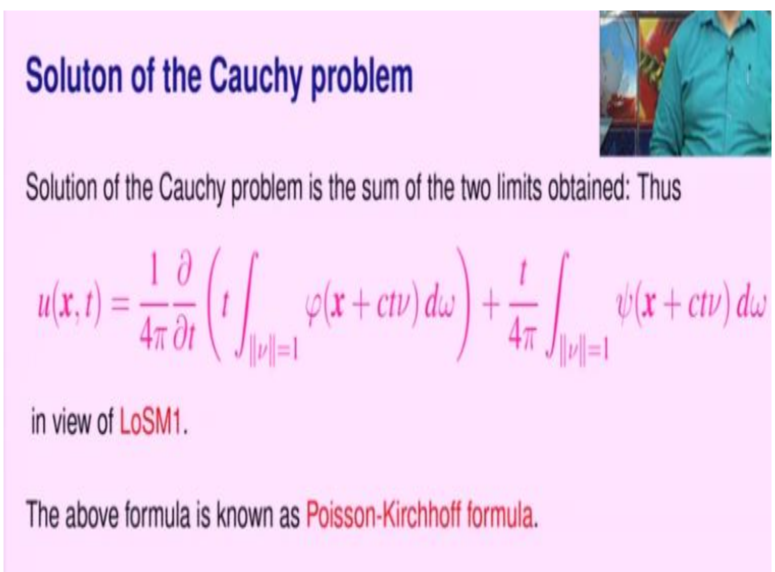

So it further gets simplified to t into M psi now we have obtained all the necessary limits and we are in a position to write the formula for u. So u x, t is equal to this is the limit of the first term this is the limit of second term. So the above formula is known as Poisson Kirchhoff formula some books say these are poison formula some books sat they are Kirchhoff formula we use both the names.

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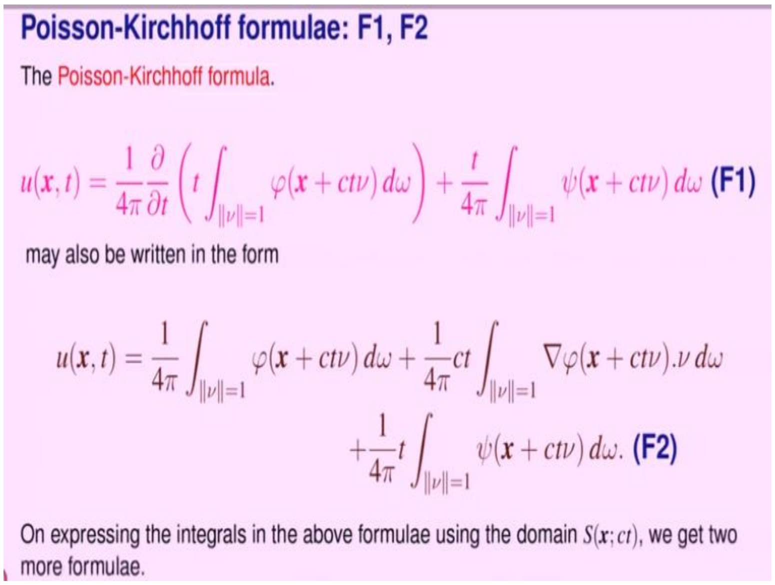

This formula that we divide let us call F1 we are going to derive another formula from this just by expanding this integrals you get some other expression F2. Now these integrals are on norm nu = 1 so we can change back them to integrals on the sphere S of x, ct then we get 2 more formulas. F1 will become some other formula F2 will become some other formula.

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Poisson-Kirchhoff formulae: F3, F4

$$4\pi c^2 u(\mathbf{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S(\mathbf{x},ct)} \varphi(\mathbf{y})\, d\sigma\right) + \frac{1}{t}\int_{S(\mathbf{x},ct)} \psi(\mathbf{y})\, d\sigma. \quad \textbf{(F3)}$$

$$4\pi c^2 u(\mathbf{x},t) = \frac{1}{t^2}\int_{S(\mathbf{x},ct)} \{t\psi(\mathbf{y}) + \varphi(\mathbf{y}) + \nabla\varphi(\mathbf{y}).(\mathbf{y}-\mathbf{x})\}\, d\sigma. \quad \textbf{(F4)}$$

So these are the other 2 formulas F3 and F4 so we have F1, F2, F3 and F4 4 formulae which all gives solution of the wave equation Cauchy problem for the wave equation. We have to slightly careful here when we say that of course there is let us first discuss the advantage of F1, F2 over F3, F4 here the exactly the same reason. The domain of integration does not depend on x, t, here also in the both these formulae whereas in the next 2 formulae they depend.

Now we want to check that you use solution to the Cauchy problem first you have to check whether you use a C2 function. Now you if you look at this formula does not look like it is going to be C2 function. Suppose phi is C2 then grad phi is C1 if psi is C1 this is C1. So the entire quantity may at most look like C1 that is the reason why we put additional hypothesis on phi and psi.

If you recall in one space dimension wave equation the Cauchy data was assumed to be C2 and C1 the initial displacement and initial velocity we have assumed 1 is in C2 and otherwise in C1. Now we have to jack up the smoothness otherwise this function u will not be a classical solution. Perhaps we can say that this will be a weak solution once we define what is the notion of weak solution etcetera which is beyond the scope of this course? Therefore let us put additional hypothesis that is what we are going to do.

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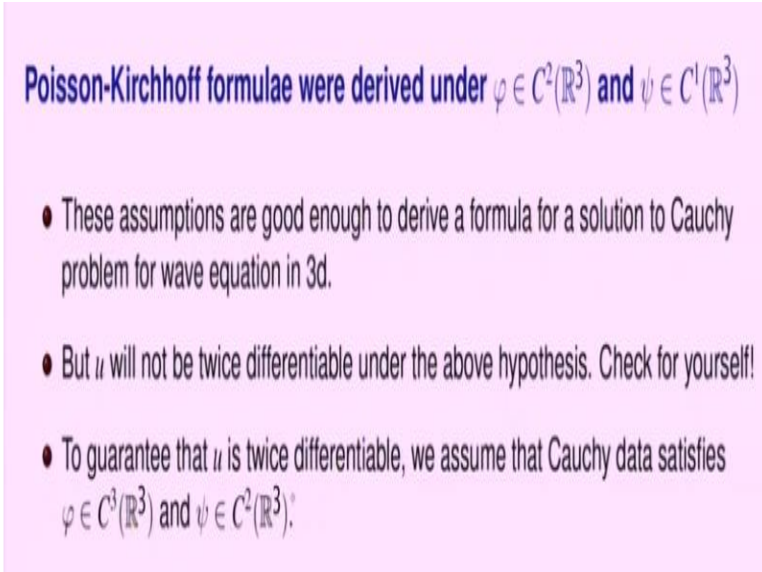

So Poisson Kirchhoff formula where derived under these assumptions phi is C2 psi is C1 that is n f. Because how did we get Poisson formula? Starting from the d'Alembert formula and for the d'Alembert formula we needed just this assumption phi C2 psi C1. So that the d'Alembert formula gives a classical solution to the equivalent Cauchy problem after that we just passed to limits. We do not really require any further regularity.

So these are good enough to derive a formula for a possible candidate we should say solution possible candidate to the Cauchy problem which we have derived. But it will not be twice differentiable under these hypotheses that we have just discussed. So to guarantee that u is twice differentiable we assume that the Cauchy data satisfies phi C3 and psi is C2 that is the reason.

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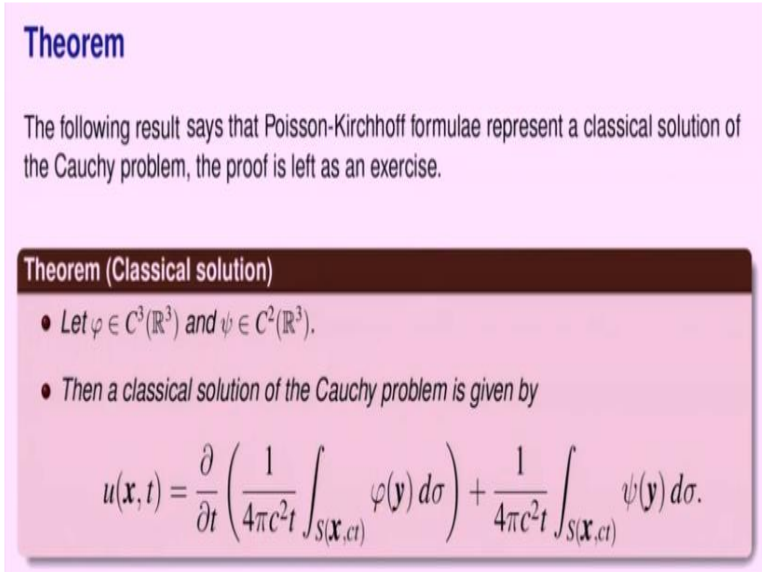

So the following result is Poisson Kirchhoff formula represents a classical solution to the Cauchy problem proof is left as an exercise. Phi C3 psi C2 then this formula is a solution. In fact checking this formula will necessarily go through converting this integrals where domain of integration does not depend on x and t. I guess that is what it is or else you should know a formula straight away what is the derivative of this. That must be derived once.

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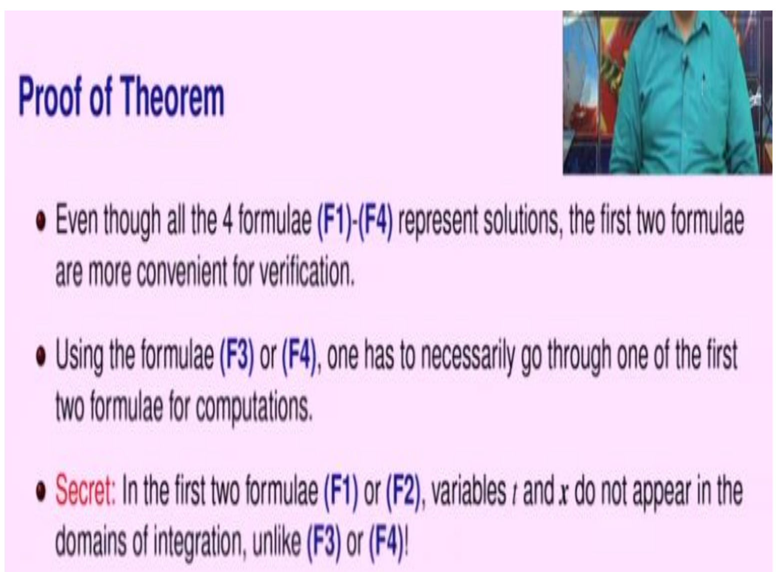

So in any case it is left as an exercise. So even though all 4 formulae represents solutions the first 2 are more convenient for verification using F3 or F4 one has necessarily go through one of the first 2 formulae in first some form or the other. Secret no secret really we have revealed this many times the first 2 formulae F1 or F2 the variables t and x do not appear in the domains of integration unlike F3, F4.

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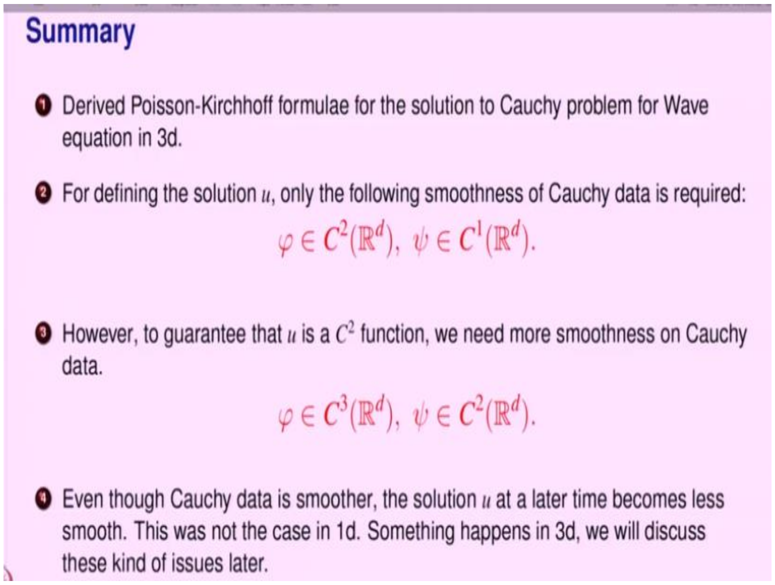

So let us summarize we have derived Poisson – Kirchhoff formulae for the solution to Cauchy problem for wave equation in 3 d. For defining the solution u only the following smoothness is required phi is C2 and psi is C1 only this much is required. However to guarantee that you use a C2 function we need more smoothness on the Cauchy data which is phi is C3 and psi is C2. So even though Cauchy data is smoother the solution u at a later time becomes less smoother.

So this was not the case in one dimension so something happens in 3d and we will discuss these kinds of issues later thank you.