Partial Differential Equations

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Module No # 06
Lecture No # 28
Tutorial on One dimensional wave equation

Welcome to the tutorial on one dimensional wave equation in this tutorial we are solve a few problems based on the Cauchy problem for the homogenous one dimensional wave equation.

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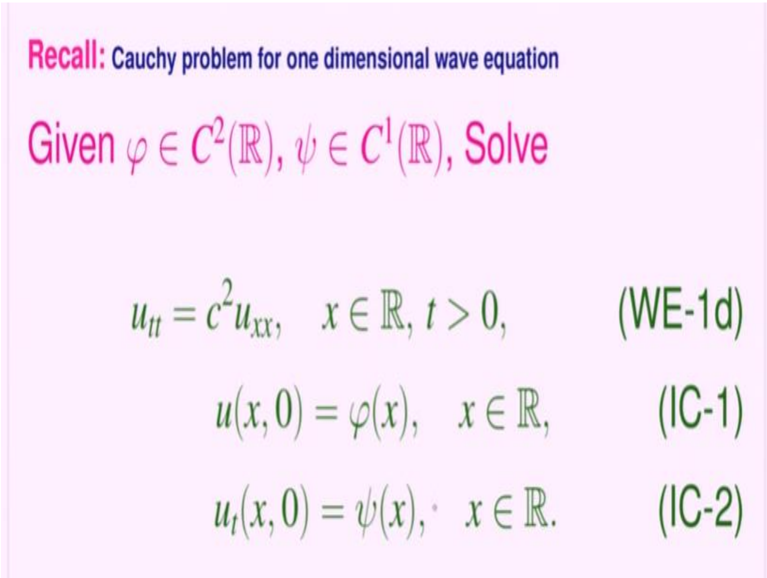

Recall what is a Cauchy problem given functions phi and psi phi is C 2 psi is C 1 defined on whole of r real line. We need to solve this partial differential which is a wave equation u t t = c square u xx for x in R and t positive such that it satisfies this 2 initial conditions u x, 0 = phi x and u t x, 0 = psi x whole for every x in R.

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Problem 1

Show that solutions to Cauchy problem for the homogeneous wave equation have the following properties.

1. If the Cauchy data consist of even functions, then solutions $u(\cdot, t)$ are even functions for each fixed t.
2. If the Cauchy data consist of odd functions, then solutions $u(\cdot, t)$ are odd functions for each fixed t.
3. If the Cauchy data consist of periodic functions of the same period, then solutions $u(\cdot, t)$ are periodic functions for each fixed t.

So the first problem show that solutions to Cauchy problem for the homogenous wave equation have the following properties, what are they? If the Cauchy data consist of even functions that is function phi and psi are even functions then solution also is even function at every time instant fixed time instant t. Similarly if the data is consisting of odd functions solution is also odd for each fixed t. And moreover if the Cauchy data is also periodic functions phi and psi are periodic function are same period then solutions are also periodic.

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Solution to Problem 1

Let us use the **d'Alembert formula** for solution to the homogeneous wave equation, which is

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

1. φ, ψ are even functions. It means

$$\varphi(-x) = \varphi(x), \ \ \psi(-x) = \psi(x), \ \ \forall x \in \mathbb{R}.$$

We want to check: For each fixed $t > 0$,

$$u(-x,t) = u(x,t) \ \forall x \in \mathbb{R}$$

holds.

So let us use the d'Alembert formula for solution to the homogenous equation which is this u x, t = phi at this point x- ct and x + ct we take the arithmetic mean. And then we take this mean of psi on the interval on the interval x – ct x + ct 1 by 2c. Not exactly mean it is a t times mean anyway. So this is the d'Alemnbert formula so phi and psi are even functions we want to show that u is also an even function what is the meaning of even functions?

Phi of –x = phi x for every x in r similarly psi of –x = psi x for every x in r even with respect to x = 0 that is a correct statement. So we want to check what u is also even function that means u of –x, t = u of x, t this holds.

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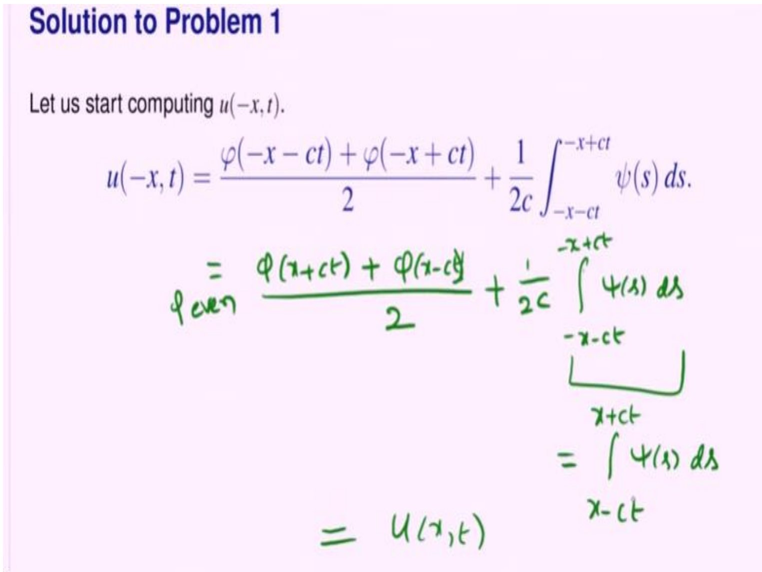

Only starting point is a d'Alembert formula so let us write the formula for u of –x, t it is this. Now we have to use that the functions phi and psi are even functions and show that this expression indeed equal u of x, t. So now phi is a even function therefore this equal to phi of x + ct right minus of this is this and here it is phi of x – ct by 2. Because now we are using phi is even this is a first term then we have a second term 1 by 2c integral minus x I just write as it is from the above.

We will handle this separately I think it is better to work with the 2 terms separately so now let us showed that this integral is actually equal to x – ct 2x + ct of psi s ds. Then what will happen? This precisely the expression for u x, t that means we would have shown that the function u is even function. So we need to deduce this from this is equality to these 2 things. So that naturally suggests a change of variable to be used.

Because even, odd functions means some change of variable x going to –x naturally comes into place.

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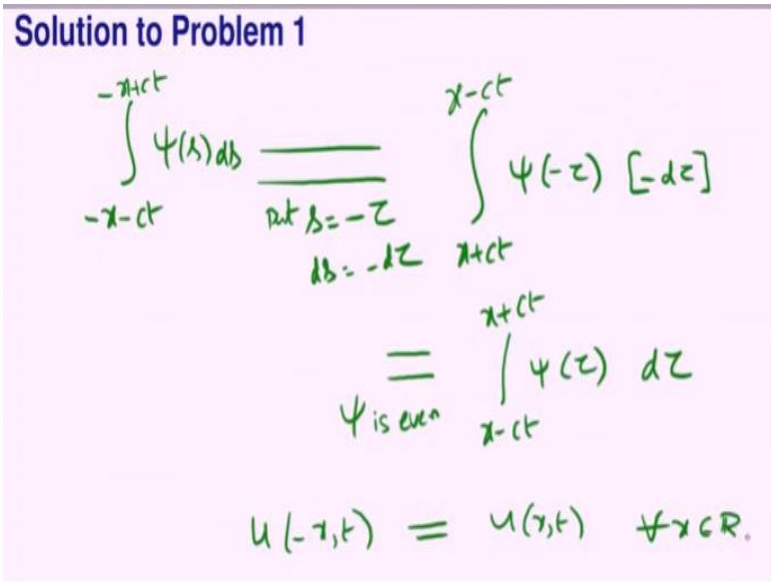

Let us do that so what we have is –x –ct –x + ct this is integral that we have now we apply change of variable put psi =minus tau psi s ds. So then what will happen inside it is psi of minus tau and d s is –d tau fine. And now the limits when s = -x – ct what will be tau it will be x + ct. Because tau is –s and when s is equal to –x + ct tau will be –f therefore this will be x – ct So now we have a minus sign here that minus sign corresponds to this change of limits upper and lower limits.

So I get x – ct to x + ct d tau –d tau as given rise to this change of limits and psi is even therefore psi of minus tau equal to psi of tau. So therefore we have shown that u of –x, t we have started with the formula and we have shown it is equal to u of x, t. This holds for every x that means u is an even function.

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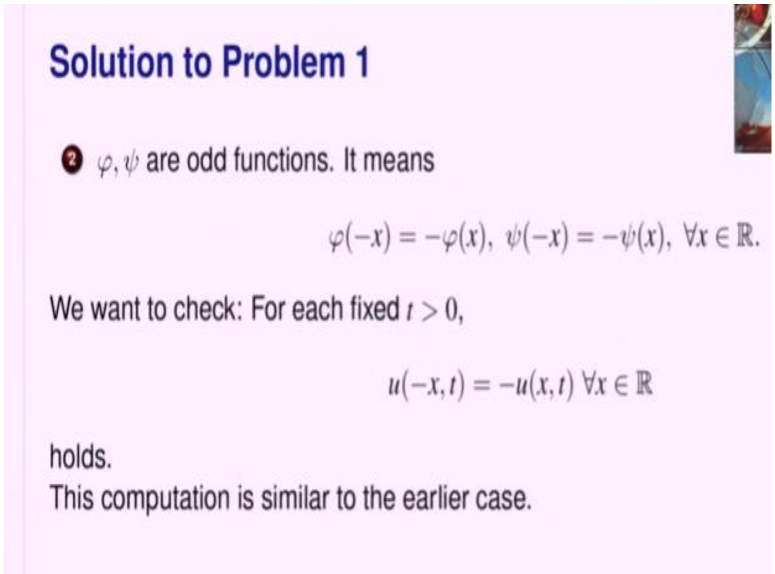

So let us move on to the next part if the data is hard functions phi and psi is hard functions we want to show that u is hard function. Hard function means odd about 0 phi of –x = minus phi x psi of - x = minus psi x holds for every x naught. We want to check that u of –x, t equal to u of –x, t holds for every x in x naught. Computations are exactly similar to the earlier case.

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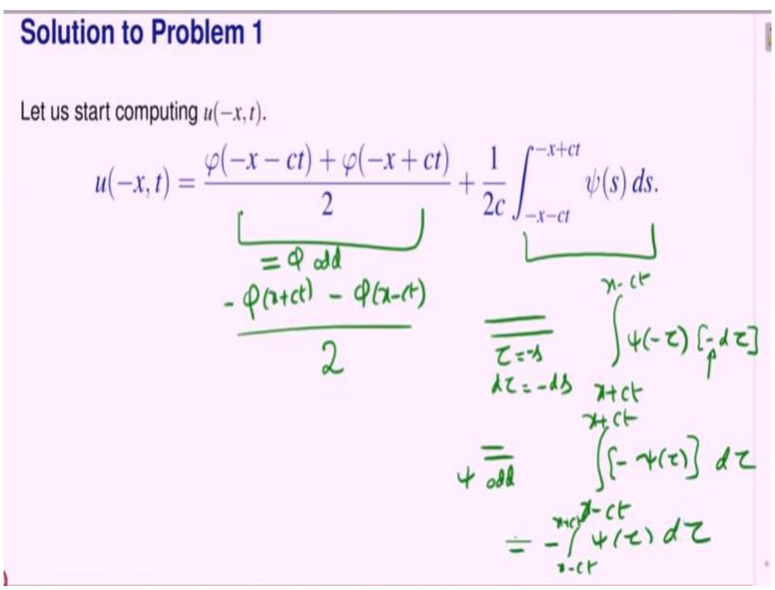

So let us start computing u of –x, t the d'Alembert formula gives us this and now we need to use that phi and psi are odd functions. So therefore the first thing here is equal to phi of x + c, t with a minus sign and phi of x – c, t with a minus sign because phi is a odd function by 2. And here also we are going to use that psi is an odd function so let us compute this integral. Once again use s = tau = -s so d tau = -ds therefore the integral would become it is x = -tau.

So when s is equal to this tau will be minus of that will be x + c, t 2 x – c, t ds is –d tau as before, and psi of minus tau. But now psi is odd therefore the integral psi of minus tau is minus psi of tau. Now this –d tau the minus here corresponds to change of this limits upper and lower d tau and that is nothing but –x – c, t to x + c, t psi tau d tau.

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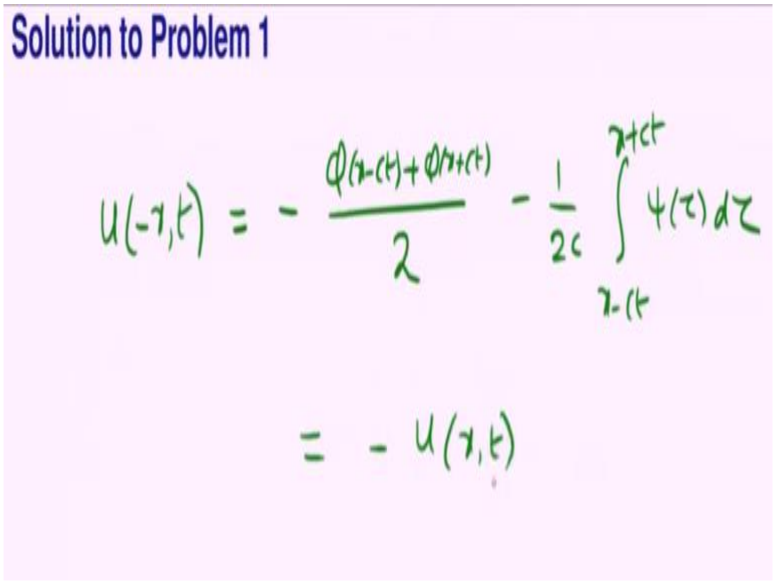

Therefore what we have obtained is u –x, t = minus phi of x- c, t+ phi of x + c, t by 2 that is the first term from here this one and the second term is here 1 / 2 c is here with a minus sign and x – 2 this is x + c, t sorry this is for the top one. This is x – c t this is x + c t psi so this is the first term and then we have 1 = 2 c x – c t to x + c t psi tau d tau which is precisely –u of x, t. Therefore if the data is odd solution is odd at all times.

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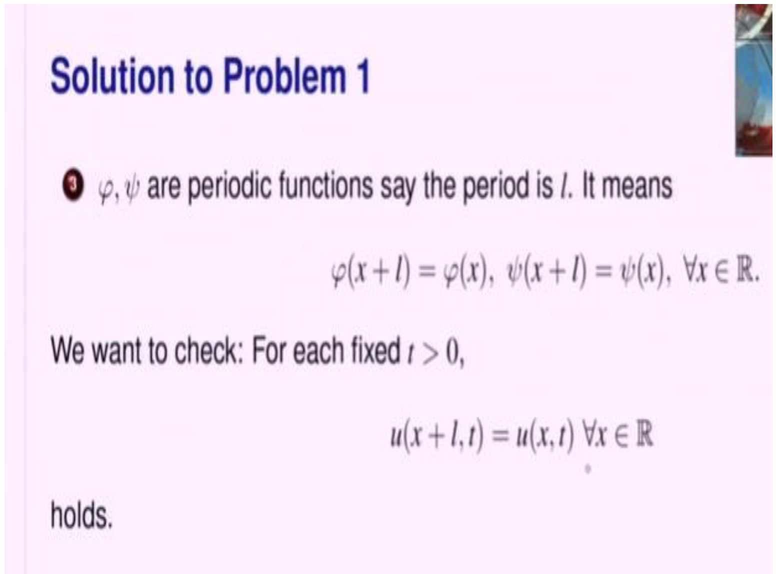

Now let us turn our attention to the last part where the function is period say the period is l which means that phi of x + l equal to phi x and psi of x + l equal to psi x this holds for every x in R. We want to check that u of x + l, t = u of x, t.

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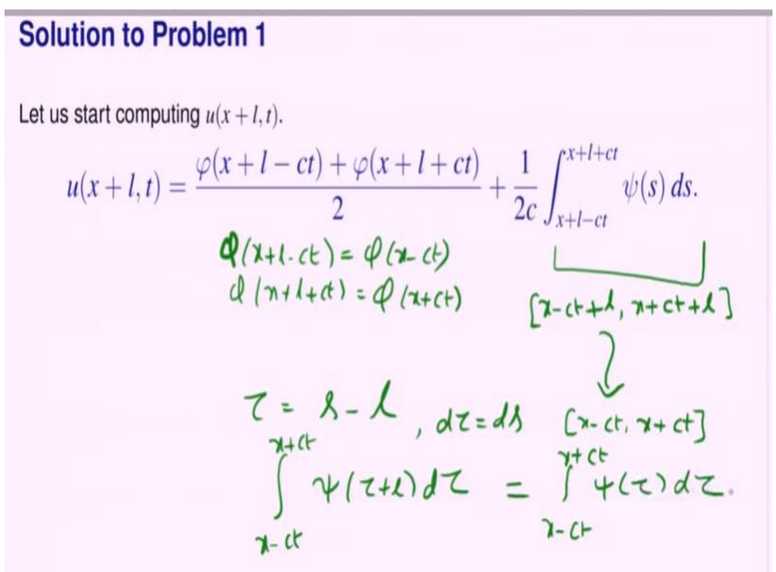

So let us start with an expression for u of x + l, t by the d'Alembert formula it is this now phi is periodic or periodic l. Therefore phi of x + l – ct is actually phi of x + x – ct similarly phi of x + l + ct is nothing but phi of x + ct. And if you look at this integral once again what is this integral it is actually on what which interval? x – ct + l to x + ct + l this so this under translation by l will go to x – ct, x + ct this is what we want right.

So this suggests what change of variable we have to put let us put tau = s – l then what will happen d tau is ds and this integral will become when s = x + l – ct tau will become x – ct. When s = x + l + ct tau will become x + ct and integrated psi of s psi s is tau + l d tau. But psi is periodic of p radial therefore this is equal to x – ct to x + ct psi of tau d tau.

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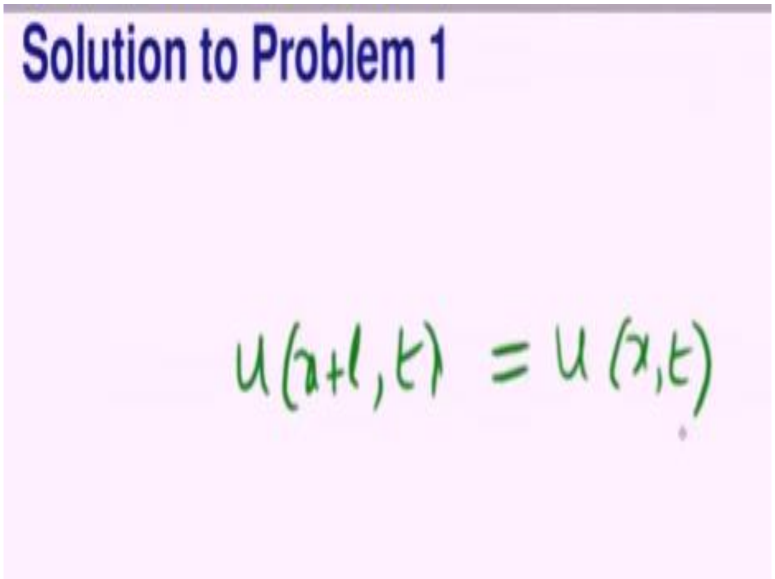

So therefore what we have shown is that u of x + l , t = u of x, t that means u is periodic.

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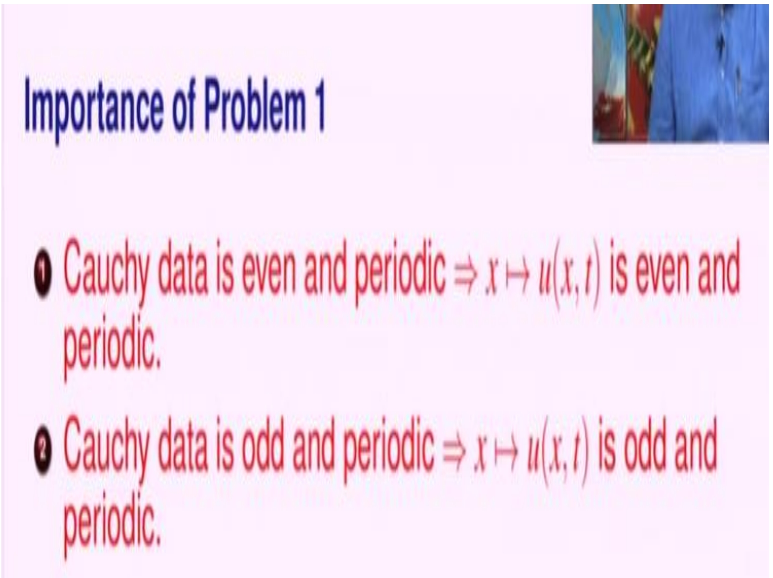

So what is importance of this problem 1 is the Cauchy data is even un-periodic then the solution is also even because Cauchy data is even solution also periodic because Cauchy data is periodic. Similarly when the Cauchy data is odd and periodic then the solution is odd and periodic.

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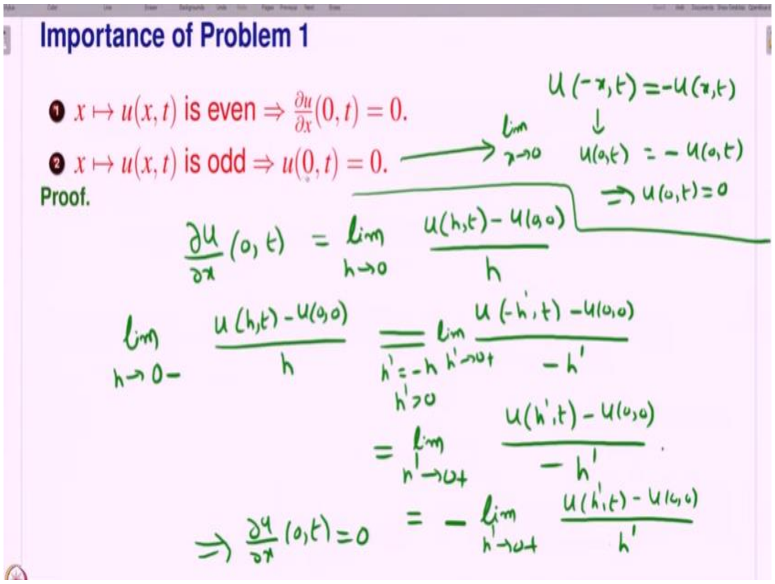

If x going to u x, t is even the function u is even then dou u by dau x at 0, t 0 we will show that and with the function is odd u of 0, t is 0. So let us prove this how we prove that? Let us show dou u by dou x of 0, t let us compute and show that it is 0 what is it by definition limit which goes to 0 u of h, t – u of 0, 0 by h. And since we have we know that the function is even so naturally it is, suggest that we have to consider h positive and h negative.

So we can compute the limits as h coming to h goes to 0 from positive side and negative side that is what we are going to do now. So limit h goes to 0 from the negative side let us see what it is? u of h t this is exactly same difference quotient as before. Now we are going to into this information so what I do is that I put h prime is equal to –h. Then what will happen h prime will become positive so then what I have here u of –h dash, t –u of 0, 0 by –h dash.

Of course limit h dash goes to 0 from positive side this, what is this I had to use that u is even function. So this is nothing but limit h dash goes to 0 from positive side is u of h dash, t because u is even with respect to x – u, 0, 0 by –h dash this limit exists. So what is this? This is actually minus this minus can come out outside minus limit h dash goes to 0 + of u of h dash, t –u of 0, 0 by h dash this is precisely the right hand derivative at 0, 0 with respect to x.

This is the left hand derivative and both exist and one is negative of the other that implies that dou u by dou x is 0 both of them are 0 and dou u by dou x is 0. So this shows that if u is an even function of x then the derivative with respect to x is 0 now showing this is much simpler because u is odd function right. So u of –x, t the t does not play any role here you can think t is not there equal to x, t with a minus sign by odd thing.

Now let x goes to 0 then the LHS will go to u of 0, t right hand side will go to minus of u, t. This implies that u of 0, t is 0 so therefore if the function is even derivative is 0 at 0 and the function is odd then the function itself is 0.

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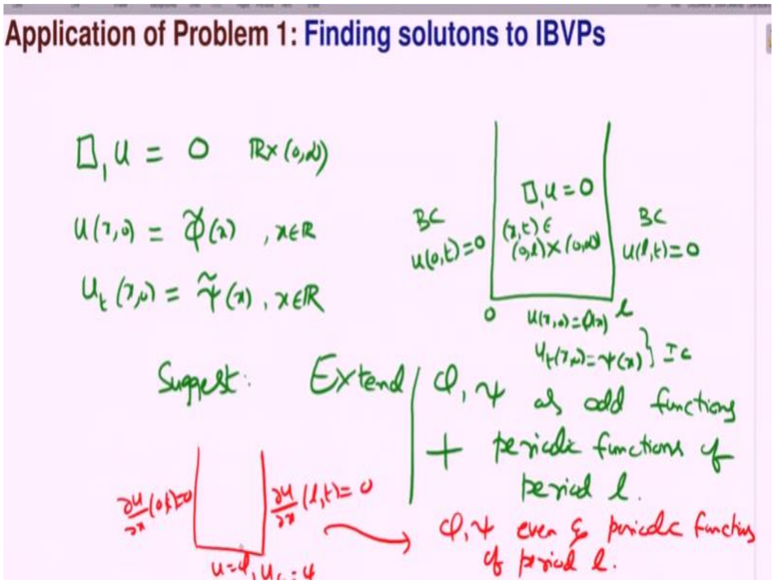

What is the application of this? It will be used in finding solutions to initial boundary value problems. So let me tell you what is initial boundary value problem? We have discussed this in the last lectures 4.1 we have introduced the initial point of value problem. So we want to solve d = 1 obviously and u equal to let us say homogenous wave equation and here we prescribed u of x 0 to be phi x and u t of x, 0 = psi x this is the Cauchy data.

So this let us say 0 and l so we are defining let us say u of 0, t is given to be 0 this is the boundary condition and u of l, t = 0 this is the boundary condition these 2 are initial conditions. That means we have a finite string 0, l and we are interesting in studying solutions of homogeneous wave equation. In this domain which is 0 l cross 0 infinity this is where x, t would belong to and boundary condition is given as u = 0 at x = 0 and x = l for all times.

Now in finding this solution the problem will be useful so if you see here u of 0, t is 0 u of l, t is 0. So one would like to transform this problem to a problem which is posed on full space so what one would be interested in solving is this equal to 0 in r cross 0; infinity. But if we want a Cauchy problem the full space so you have to give what is u x, 0 that I will tell you what is going to be some function v tilde x.

Similarly u, t of x 0 is equal to some function psi tilde of x has to be defined for R so that means we have to extend the function phi and psi which are given in the interval 0, l to functions which are defined on whole of R. So this means we have to extend these functions how do we extend so that these boundary conditions are naturally satisfied. So what is the boundary condition? u of 0, t is 0 that means this function here u of 0, t is 0 therefore when is u of 0, t 0 when the function u of x, t is odd function when u of x, t is odd function you have 0 of t, 0 we know that.

Therefore what we think is to get u of x, t as odd function we need the data to be odd function so this suggest extend phi and psi as odd functions. So this problem 1 actually gives us an idea of solving initial boundary value problems. So extend them odd function so that solution will be odd so that u, 0 t is 0 is satisfied. But how u l t, be equal to 0 will be satisfied that we would like some kind of periodicity u of 0 t is 0 u of l t is 0.

If u is periodic also therefore extend this as odd functions and periodic functions of period l then what will happen? Suppose you are successfully have done this and so that the extended functions have the records smoothness namely phi tilde c 2 cycled by c 1. We can solve this problem solution in the whole space is going to be both odd function and periodic function period l.

Because the Cauchy data is periodic as well as odd functions and then this naturally satisfies this condition boundary condition are automatically satisfied. If the solutions in entire r of course it will be solution on 0, l as well. So that is the way to solve so this is the application of the problem 1. The other part where we have shown that if the function is even dou u by dou x at 0, t is 0 will be applicable for a different initial boundary value problem where instead of prescribing u = 0.

We prescribe dou u by dou x at 0 t is 0 dou u by dou x at l, t is 0 then that suggests that we have to extend phi and psi as even function. So that solution is even and boundary conditions are satisfied 0 boundary condition is satisfied for dou u by dou x. But I also want at x = l therefore I will do periodically also. So in other words what I am saying is for this problem where u is given to be phi here I will be brief it is exactly same as this u, t will be psi here instead of u, 0 t or u l, t what we give is dou u by dou x at l t is 0.

And here dou u by dou x at 0, t 0 in that case the suggestion is extend phi and psi as even and periodic functions of period l. So that solution will be even and periodic and these conditions are satisfied.

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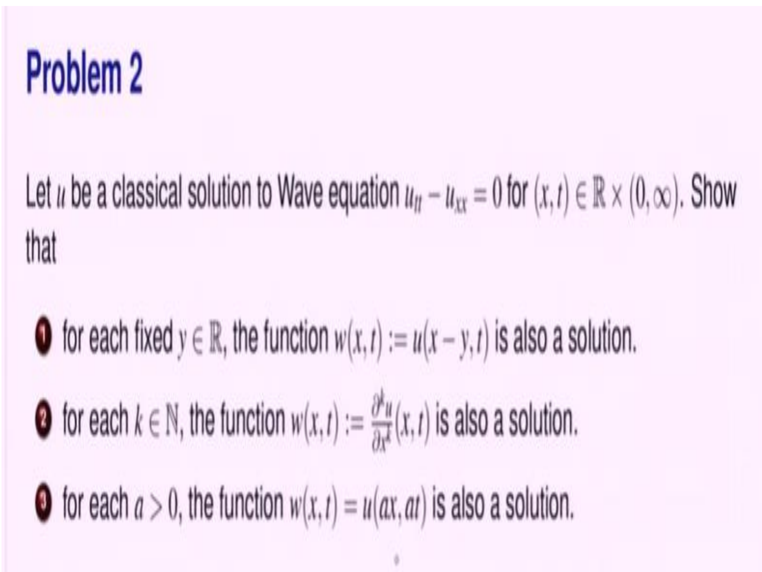

So let u move to the next problem let u be a classical solution to the wave equation show that for each fixed y in R the new function which is defined by translating u by y is also a solution. Similarly if u as kth derivative or kth derivative is c2 then you can also take the kth derivative with respect to x, x and then show that it is also a solution. And this is u of a x, a t is also a solution so this is a translation of the solution of solution this is the dilation of a solution-solution.

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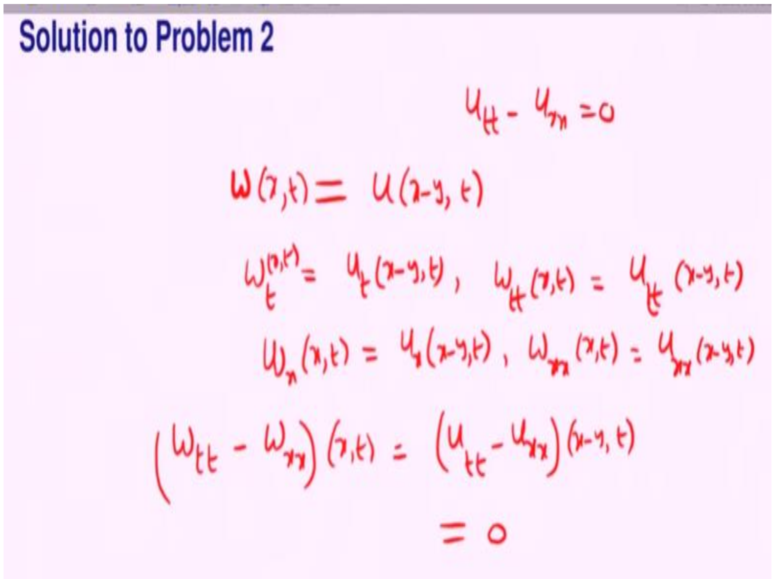

Let us do the first problem so w of x, t = u of x – y, t u satisfies u tt – u xx = 0. So now I have to compute w t and w tt w x and w xx w t is simply u t at the point x – y, t is better to write the arguments all the time. Particularly when there is a chain role involved so w tt of at the point x, t is u tt at the point x – y, t. Similarly w x of x t is u x this y does not play any role is just translation x – y, t and w xx at x, t is u xx at x – y, t.

Therefore w tt – w xx at any point x, t is nothing but u tt – u xx at this point x – y, t and that is equal to 0 because solves the wave equation at any point x, t and hence at point x – y comma t also. So that means w is the solution to the wave equation similarly you can check the other 2 parts of this problem. So equation is translation invariant because the coefficients are constant coefficient therefore solution is also translation invariant.

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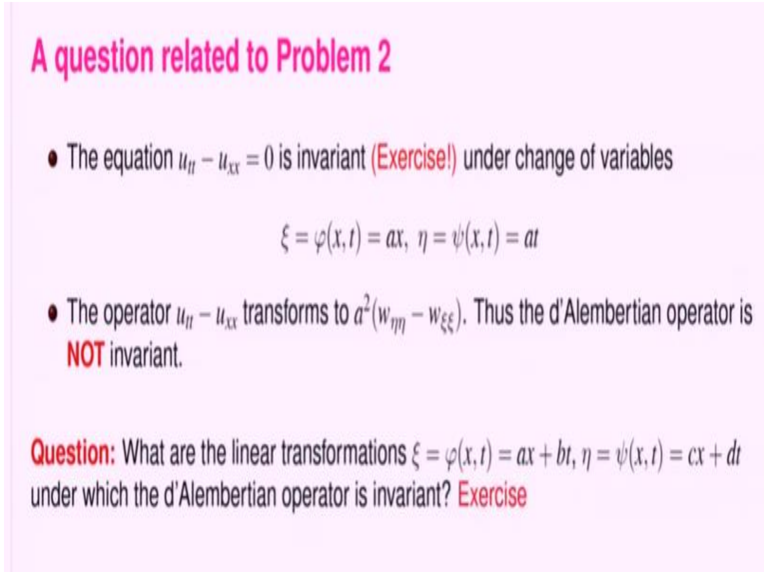

A question related to problem 2 the equation u tt – u xx = 0 is invariant this you please check under this change in variables psi equal to x and eta equal to at. The operator transforms to a square into w eta – w psi-psi. That means the d'Alembertion operator is not invariant under this change of variables. So question is what are the linear transformations psi equal to ax + + b t eta = cx + dt under which the d'Alembert operator is invariant. So that is an exercise.

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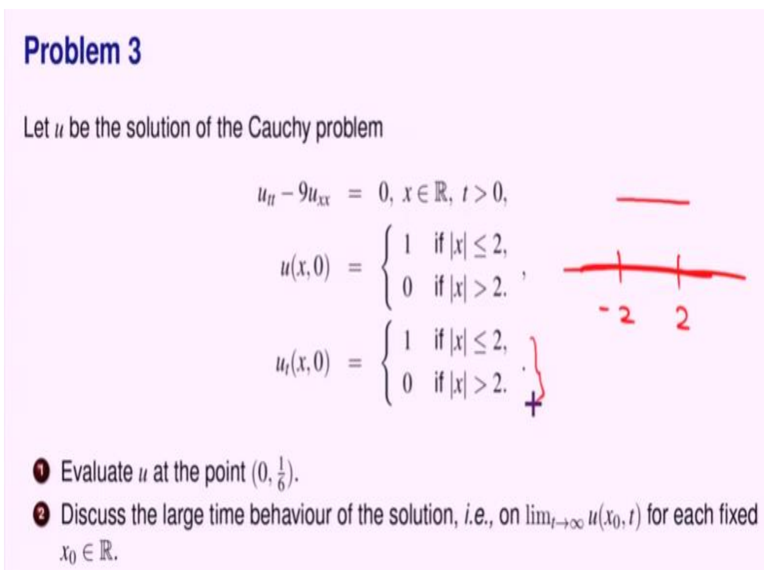

So let us look at the problem 3 let u be the solution of the Cauchy problem which is given here the wave equation which is here that means c = 3. So u tt = 9 u xx is 0 and initial condition u x, 0 is this and u t is this and you are asked to find out the value at the point 0, 1 by 6 and discuss the large term behavior of the solution. That is what happens to u of x naught t as t goes to infinity whether the limit exists if it exists what it is?

At every fixed point x, 0 now if you observe the phi x and psi x phi is discarding us. This is in the interval -2, 2 the function is 1 outside that it is 0 it is a discontinuous function forget about c2. Similarly this function exactly function both phi and psi are same and they are not continuous.

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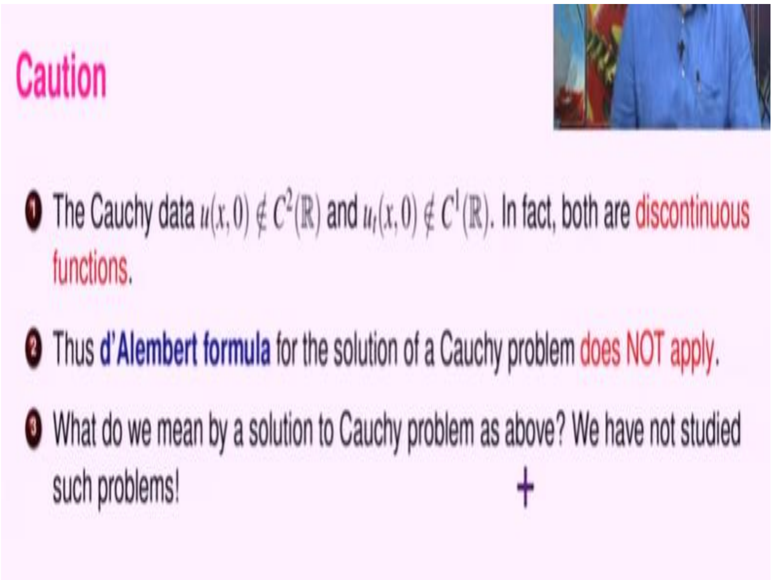

So therefore you have to exercise caution the Cauchy data does not satisfy what is required for d'Alembert formula to be applicable. Therefore the formalism does not apply as such because phi and psi do not satisfy the required smoothness assumptions. So what do we mean by a solution to Cauchy problem as above when the Cauchy data is discontinuous for example as here.

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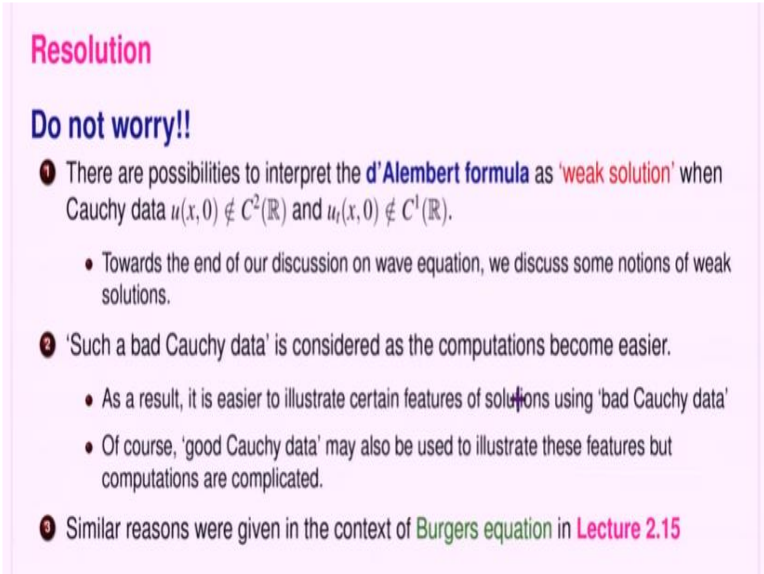

So we are not studied such problems do not give up do not worry there are, possibilities to interpret the d'Alembert formula as a weak solution. So when Cauchy data is not smooth as replied towards the end of our discussion on wave equation we discuss some notions of weak solutions. So do not worry there is some way to interpret this as a solution. So go ahead and compute so such a bad Cauchy data why do we consider?

We consider because the computation becomes easier as a result it is easy to illustrate certain features of solutions that we want to show using bad Cauchy data. Of course good Cauchy data may also be used to illustrate these features but computations become very difficult complicated cumbersome and our attention will be focused on actually computing solution than exhibiting the features.

That is the reason why we use bad Cauchy data but we should be aware that this is not the Cauchy data for which d'Alembert formula is a classical solution that we should keep in mind. We have given similar reasons in the context of Burgers equation also in lecture 2.15.

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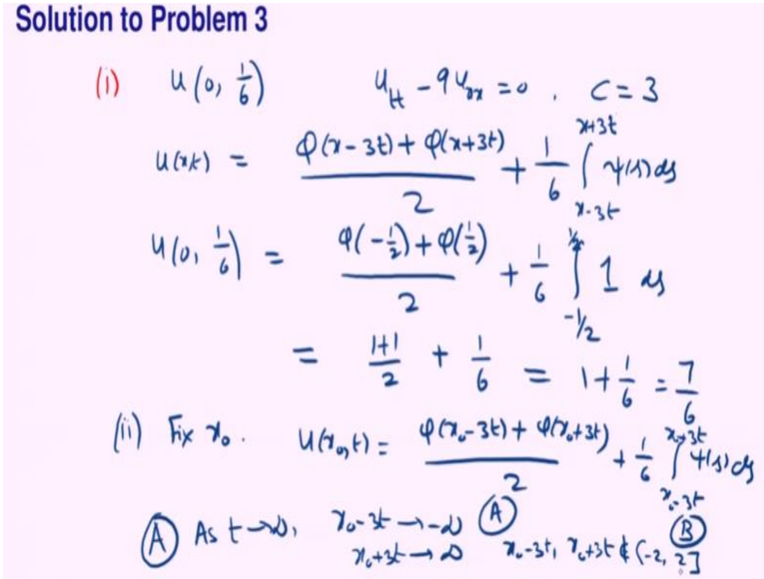

Let us solve this problem very easy we have to simply apply the d'Alembert formula. So first part was to compute u of 0, 1 by 6 so we have to identify what the c is our equation is u, tt -9 u xx = 0 that means c = 3. Because the d'Alembert formula is u of x, t equal to phi of x – ct which is 3 t + x + ct so it is x + 3 t by 2 + 1 by 2 into c that is 6 x – 3 t 2 x + 3 t psi s ds. Now it is very easy u of 0, 1 by psi x is 0 so phi of 3 times 1 by 6 is minus half see how simple the computational has become minus half to half.

But on this interval what is psi? It is 1 so therefore phi is also 1 so 1 + 1 is 2 + 1 by 6 into this is half plus half that is one. So this 1 + 1 by 6 which is 7 by 6 so this is the answer now let us discuss the large term behavior. So fix x naught so u of x naught, t we have the formula above x naught -3 t + phi of x naught + 3 t by 2 + 1 by 6 x naught -3 t x naught + 3 t psi s ds. So here we have 2 terms let us call them A and B analyze them separately.

So what happens as t goes to infinity observe the term A as t goes to infinity what happens? x naught -3 t will go to minus infinity and x naught + 3 t will become infinity go to infinity. In particular after sometime x naught -3 t and x naught + 3 t both of them come of this internal -2, 2. Because they are going to minus infinity and infinity respectively they will be out of this interval after sometime.

We can find out when they will be out so that phi will be 0 after some time therefore for t bigger than some number phi will be 0 so this term will be 0. For the same reason this goes to minus infinity this goes to plus infinity. But psi is defined only I mean it is non-zero only on -2, 2 therefore it does not matter what this interval is after sometime it will be -2, 2 psi s ds. So that is going to be our large time behavior.

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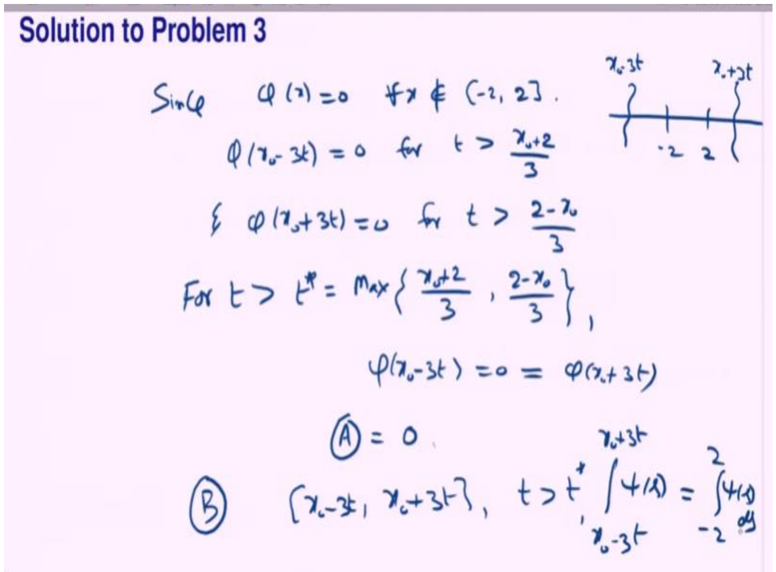

So since phi of x = 0 for all x which do not belong to -2, 2 phi of x naught -3 t will become 0 for t bigger than x naught + 2 by 3. In other words I am just looking at this I am asking when x naught -3 t will become less than -2. Similarly x naught + 3 t when will that become rather than 2 and phi of x naught + 3 t will become 0 for t bigger than 2 – x naught by 3. So whenever t is bigger than both of them phi of x naught -3 as well as phi of x naught + 3 t will be 0.

Therefore for t bigger than let us call t start which is maximum of these 2 numbers phi of x naught -3 t = 0 and that will also be x naught + 3 t. That will also be 0 in other words the term A will become 0 now let us look at the term B. For the same reason the term B which is the integral on x naught -3 t, x naught + 3 t whenever t is bigger than the t star as above this interval is like this. This interval properly contains -2, 2 therefore this integration x naught -3 2 x naught + 3 t psi s ds is actually -2 to 2 psi of s ds.

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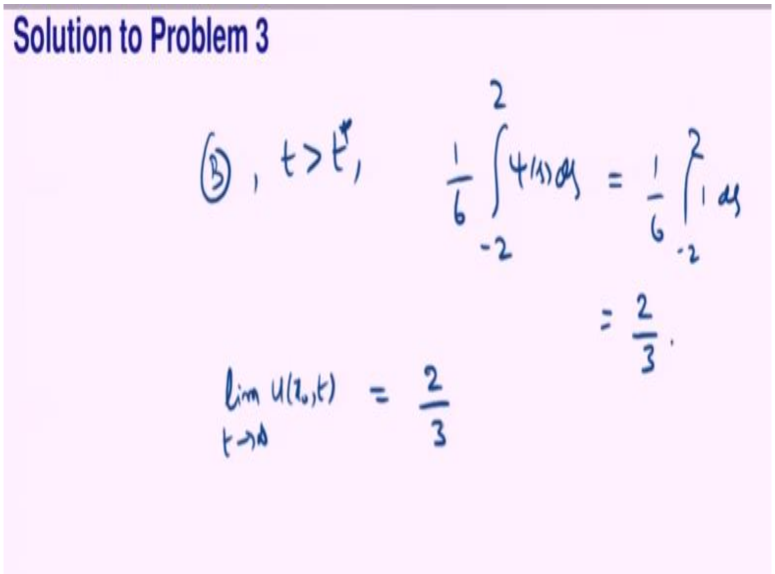

So therefore the term of B for t bigger than the t star as on the previous slide is actually 1 by 6 -2 to 2 psi s ds which is psi s ds is 1 that is 4 by 6 which is 2 by 3. So therefore limit u of x naught, t no matter what x naught is this limit exists and limit is actually 2 by 3.

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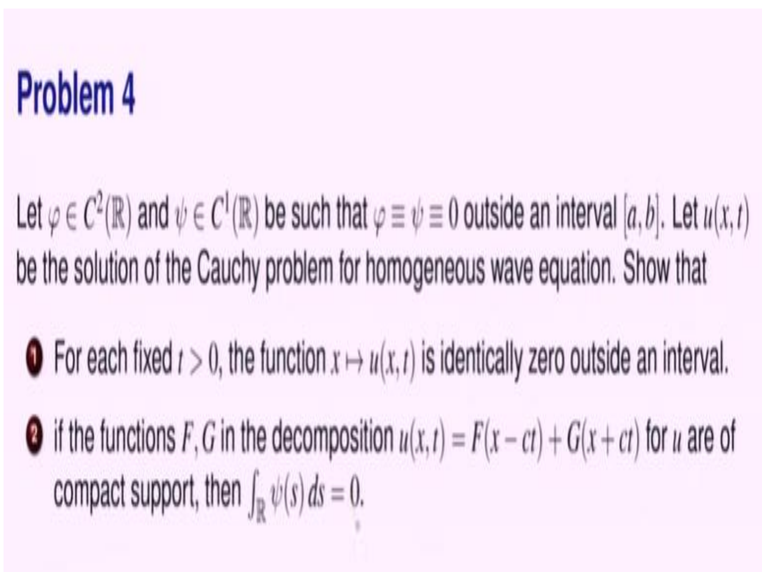

Let us move to the problem 4 now so let phi and psi be C2 and C1 respectively define on the real line such that phi and psi both are identically equal to 0 outside an interval a, b. We have an interval a, b inside interval a, b there may be 0 at some points may not be 0 at some point but definitely that phi is 0 and psi is 0. Similarly here phi is 0 psi is 0.And let u be a solution of the Cauchy problem for the homogenous wave equation show that the function x going to u, x t that means you fix a t then u x, t is the function of x.

There is also identically equal to 0 outside an interval in other words if the data is 0 outside interval a, b solution will also be 0 outside some interval. That interval may not be a, b but some interval so the functions F, G in this decomposition u x t = f of x – ct + G of x + ct remember this is general solution of the homogenous wave equation we derived this in lecture 4.2. So if the functions F and G are such that they are of compact support we will define what is the compact support?

F and G are compact support then the integral of psi over r must be 0 integral of psi over r make sense and need 0.

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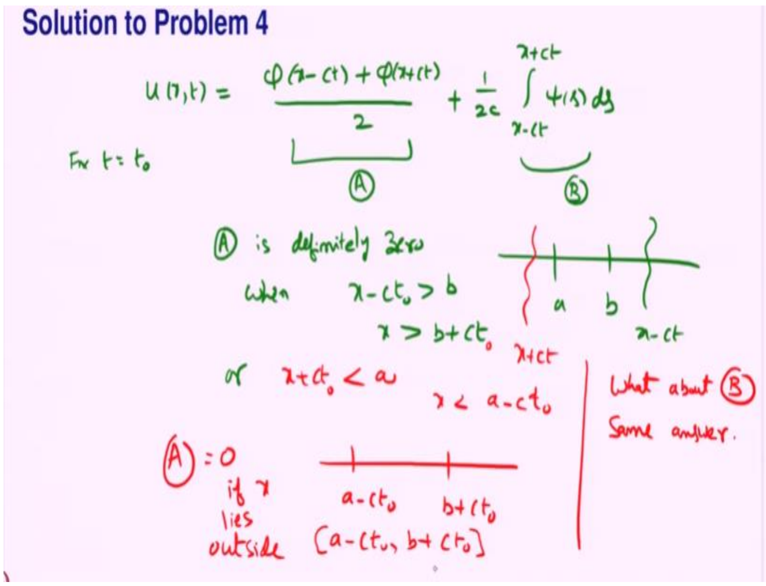

Let us start the solution of problem 4 the starting point as before is the d’Alembert formula. Let us fix a time t = t naught we will show that the function u of x, t naught is 0 outside some interval. So as before let us analyze these terms separately A, and B A is the simpler one to understand. See A is definitely 0 when under 2 conditions any of the 2 conditions our A, and B are here imagine x – ct is here.

Because x –ct is here x –ct will be the right side therefore the term A is 0 because phi is 0 outside interval a, b so what is the condition? One condition x – ct is bigger than ct naught is bigger than d that is the case when x is bigger than b + ct naught. Or the other case is here x + ct is less than

A then also x – ct is to the left side and phi is 0 on the interval x- ct x + ct. Or + ct is less than a ct naught.

So that means that x is less than a – ct naught therefore if you are on this interval a – ct naught b+ ct naught this is t naught u the A is 0 if x is not in this interval. So if x lies outside A – ct naught B + ct naught phi will be 0. Same reason holds for B also psi will also be 0 if you are outside the interval. So what about the term B? Same answer so if x lies outside the interval a – c t naught b + ct naught then both a, and b, are 0.

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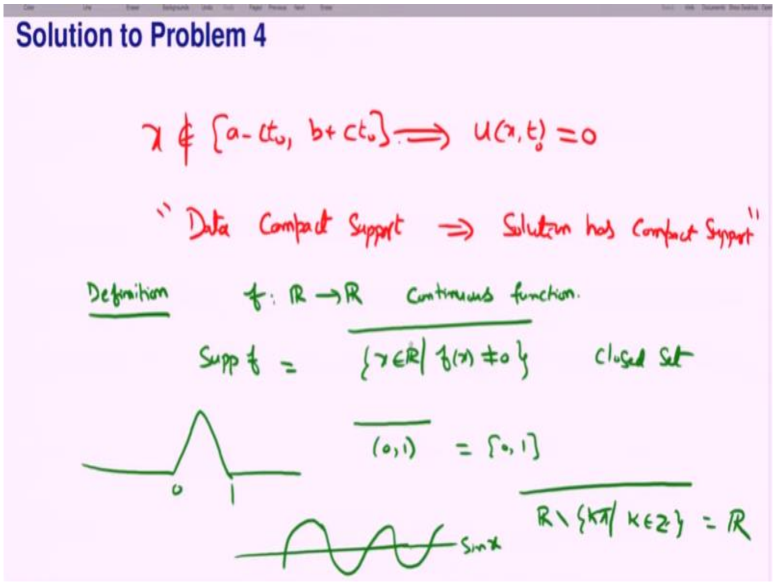

Therefore what we can conclude is if x does not belong to 2 a – ct naught, b + ct naught that implies that u of x, t is 0. u of x, t may be 0 even if x is there inside the interval we are not making any comment on that. What we are saying is? u is definitely 0 if x is not in that interval so this is what is called data compact support implies solution compact support. Of course 1 by 1 want to know what is this support and compact support.

Let us define you can find in any book on topology or real analysis let us consider a continuous function from r to r because as a situation we are now currently in. Let us consider a continuous function then what is the definition of a support of f this is a notation support of, f. What we are look at is? Look at those points where f is not 0 and take the closure of that. So by definition support is a close set by definition and if it also happens to be a compact set it is called compact support.

Support is always closed by definition it is also happen to be a compact set is called support is called compact. We know in r compact sets are closed and bounded sets therefore if the support is boundary then definitely it is compact. Therefore if the function is 0 outside a bounded set then the support is always a compact support. For example let us look at one example 0 up to 0 1 after 1 it is also 0 in between it is something like that.

In this case the set wave is not 0 is 0, 1 there is a set and we have take, the closure for the definition. So that is a closed interval 0, 1 of you notice at the end point 0 the function is 0 but it is still comes under the support due to the closure that is why the end points have come in 0 and 1. Now if you look at another function which is sine x where it is not 0 is precisely accept these points where the sine is 0 which are multiples of pi.

So from r you subtract multiples of pi integer multiples of pi and take the closure and that is r. So the support of the sin x function are Cos x function for that matter is r support of this function here like this hill is actually the interval 0, 1. Of course there are other characterization of support this is the definition of support is look at the points where the function is not zero take the closure.

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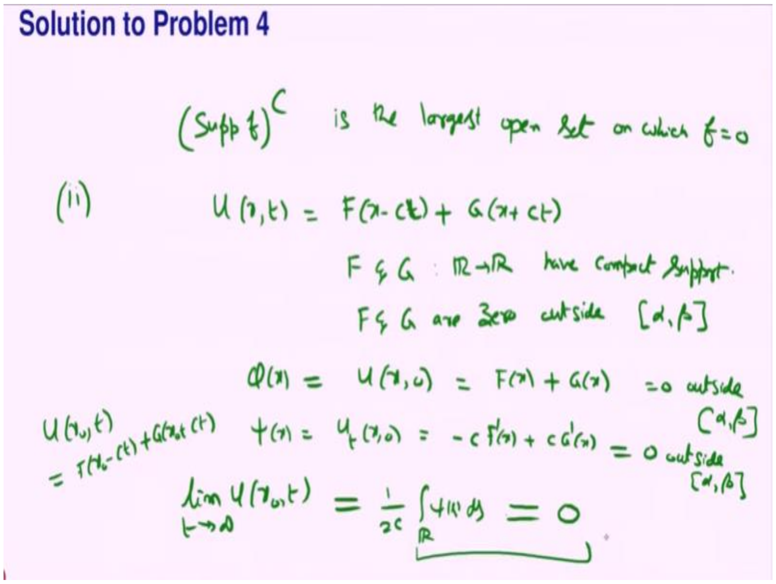

The other way to look is that support of, f complement is largest open set on which f is 0. That means take any open set on which f is 0 keep on collecting such open sets take the reunion therefore you will the largest open set on which f is 0 take the complement that will be the support of the function verify that this coincides in the previous 2 examples that I have given. Now let us turn out attention to the second part said u of x, t is equal to this and it is given that F and G are functions from R to R they are compact support.

That means they are 0 outside some bounded interval in other words F and G are 0 outside some interval which we can take alpha beta. For both of them we can take the same interval because if F is 0 outside alpha beta G is 0 outside alpha dash beta dash by taking the minimum for the left hand point maximum for the right hand points we will get common alpha beta. What is important is that F and G are 0 outside the interval alpha, beta.

Inside what happens we are concerned right now that is given then we are ask to show that integral of psi is 0. Let us see what is phi x to start? Phi x is what u of x, 0 that is F of x + G of x. Of course alpha beta this 0 right so this is 0 outside alpha beta. Similarly what is psi of x that is u t of x, 0 that is nothing but –c F dash of x + c G dash of x. If f and g are 0 outside the intervals for beta F dash and G dash are also 0 therefore this combination is also 0 outside alpha beta.

In other words phi and psi are 0 outside alpha beta so we are in the situation of the part 1 of this problem phi and psi, are 0 outside this interval. Therefore of course solution is of compact support for every fixed time that is fine. Now what we are interested in is the following we want to see what happens to the long term right limit of u of x naught, t as t goes to infinity we did this in a previous problem.

It turned out that this is going to be 1 by 2 c integral over r of psi s ds this is what we saw in the problem. Of course we computed with the specific numbers and we want to say this is 0 why? Beaches what is u of x 0, t you can fix x 0, t what is of x 0, t by this expression it is F of x 0 – ct + G of x, 0 + ct. Now as t goes to infinity x naught – ct x naught + ct will come out the interval alpha beta.

Therefore that limit is 0 therefore we conclude this that integral of psi over the support must be 0 this completes problem 4.

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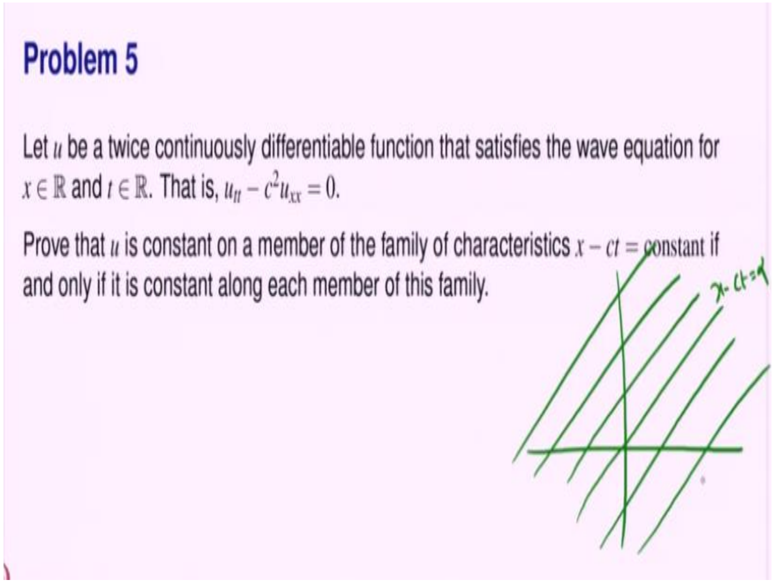

Now let us move on to problem 5 let u be a twice continuous differentiable function which also have wave equation u tt – c square u xx = 0. Prove that u is constant on a member of this family x – ct = constant just write the picture just for reference let me write this. So this is x – ct = 0 these are the lines these are x – ct equal to various constant. So if the solution is constant on any one of them on this it will depend on x and t but that is not the case imagine it is constant then it will be constant on each of those lines.

Take any other line you will constant that will be very simple from the representation formula that we have.

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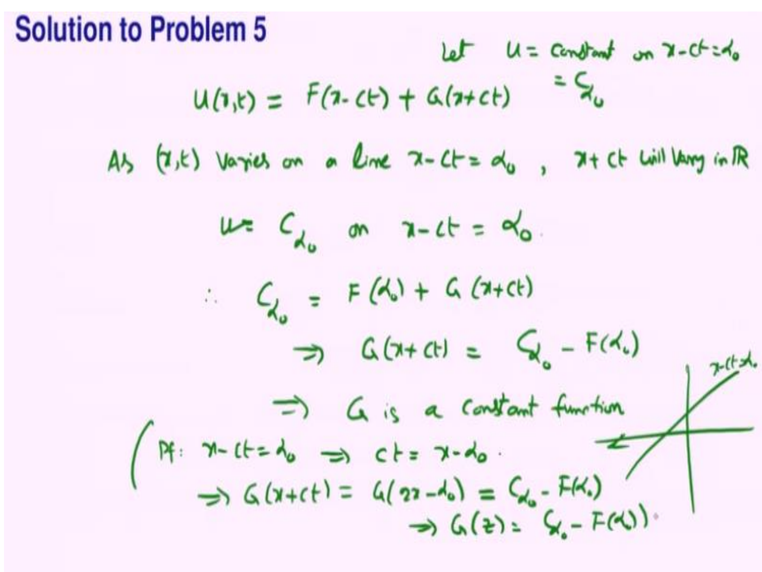

We know that solution of the wave equation is of this form we have this formula now as x, t this point varies on a line x - ct = alpha naught. So let us assume that u is constant on x – ct = alpha naught we are going to show u is constant on any line x – ct = alpha for any alpha also. So as x t varies on this line what happens to x – ct or x + ct? x + ct will vary in entire R but we are assuming us is constant let us assume that constant is c alpha 0.

So u = c alpha 0 on this line x – ct equal to alpha naught therefore what we have is c alpha naught is equal to F of alpha naught + G of x + ct for all the x, t which are on this line that implies that g of x + ct = c alpha naught – F at alpha naught. It means that the function G is a constant function this implies G is a constant function proof x – ct equal to alpha naught that is where the x, t is varying.

That implies what ct = x – alpha naught that implies G of x + t ct equal to G of 2x minus alpha naught and that is equal to this number what is that number c alpha naught – F alpha naught which is a constant number it is a constant so that is what? This implies that G of any value G is identically equal to because as xx varies right if you consider this line x is varying right throughout x – ct what is fixed as alpha naught.

So x varies to minus infinity-infinity so therefore G is a constant function G of z = c alpha naught – F alpha naught. So it is constant function.

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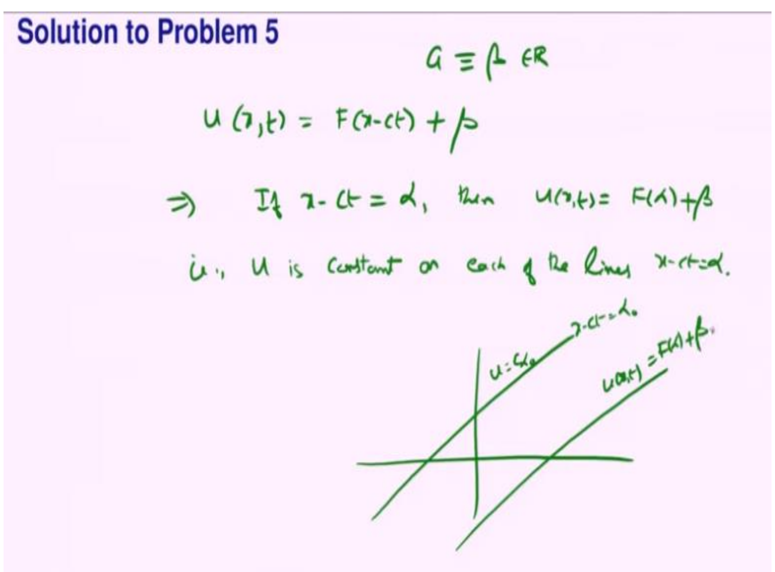

So once it is a constant function what do we have? u of xt = F of x – ct plus let me call beta so let us call G is a constant function so let us call this beta real number this what we have. This implies what if x – ct is equal to some number alpha then u of xt equal to F of alpha + beta this that is u is constant on each of these lines x – ct equals to alpha. Therefore if you found a line on which u is constant for some alpha naught u = c alpha naught.

What we are showing is on this line u of xt for xt on this line will be equal to F of alpha + beta that means constant again thank you.