Partial Differential Equations
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Module No # 06
Lecture No # 27
Wave Equation in One Space Dimension
D'Alembert formulae

Now welcome to this lecture from this lecture onwards we start solving wave equations. In today's lecture we solve wave equation in 1 space dimension here we are going to consider only homogenous wave equation for the reasons that we explain in lecture 4.1 the solution is known as d'Alembert formulae.

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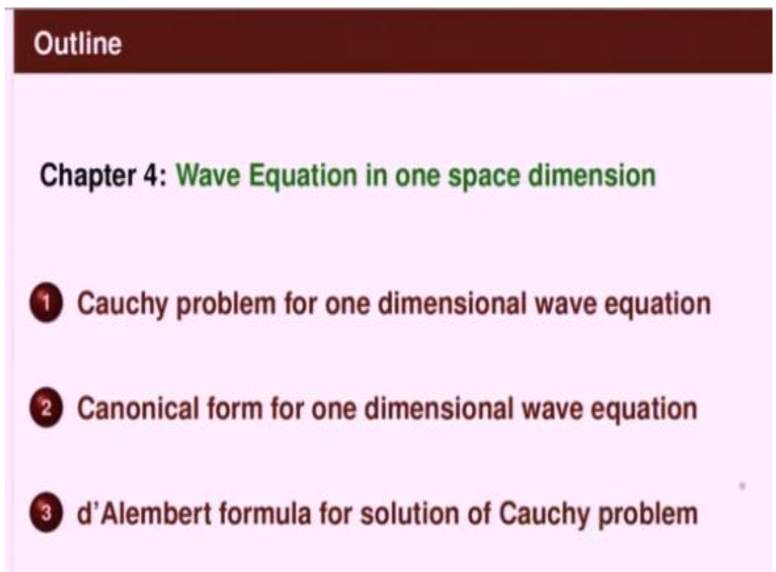

The outline is at we introduce the Cauchy problem for 1 dimensional wave equation and then we look at the Canonical form for the 1 dimensional wave equation using that we obtain a solution which is known as d'Alembert formula for the solution of Cauchy problem.

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So; Cauchy problem for 1 dimensional wave equation and plan of action to solve it.

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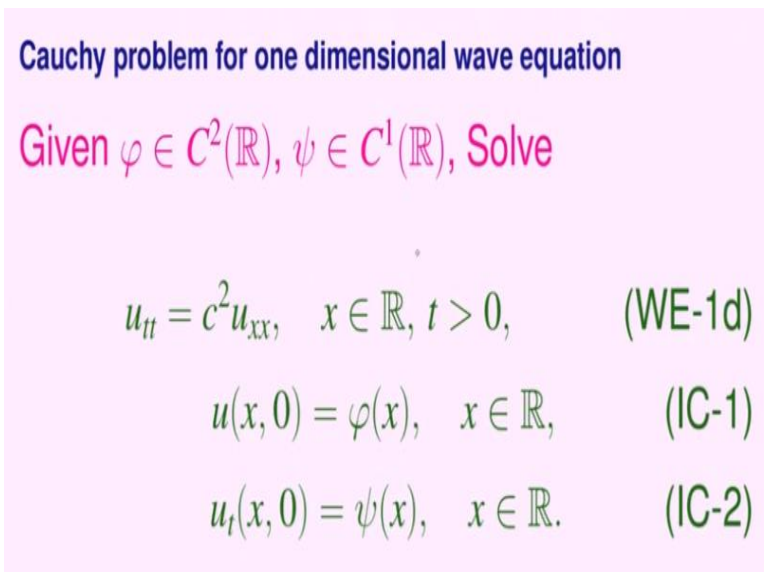

We highlight the key points so the Cauchy points is this given functions phi in C 2 of R that is twice continuously differentiable and psi which is C 1 of R 1 times continuous differentiable on R. We need to solve this problem second equation we write WE wave equation 1d with 1 space dimension. So equation is u t t = C square e xx we are interested in solving x in R and t positive. We could also solve for t in R that we will discuss perhaps in a tutorial.

So u 2 t = C square xx in x belongs to R t belongs to 0 infinity then we have 2 conditions which are called initial conditions are Cauchy conditions that is u x, 0 is phi x u t x, 0 is psi x is given with this much smoothness this is what is called Cauchy problem for 1 dimensional wave equation.

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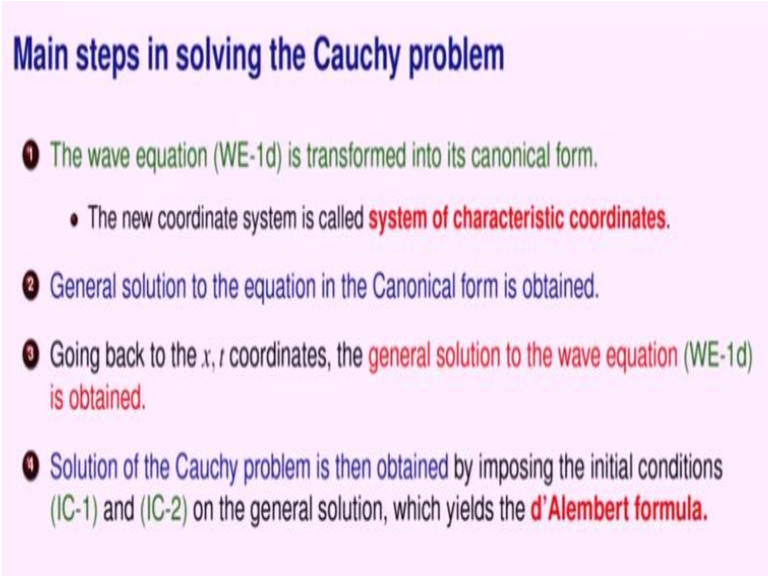

So what are the main steps in solving the Cauchy problems the wave equation that is WE – id is transformed into its canonical form. The new coordinate system is called system of characteristic coordinates. A general solution to the equation in the canonical form is obtained because canonic form is very easy to solve and going back to the x t coordinates. The general solution to the wave equation is obtained as the function of x and t and solution.

Or the Cauchy problem is then obtained by imposing the initial conditions IC-1 and IC-2 on the general solution that we have obtained. The solution that we finally get is what is called d'Alembert's formula.

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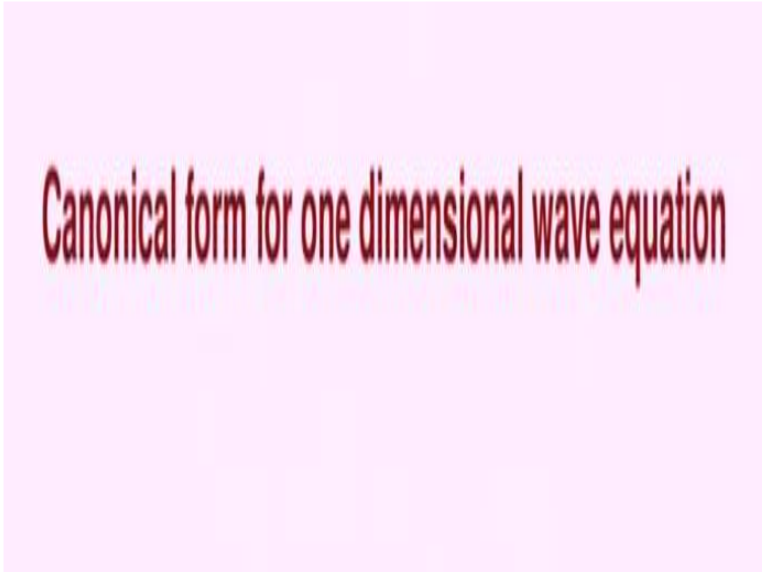

So, first point is to reduce our wave position to canonical form.

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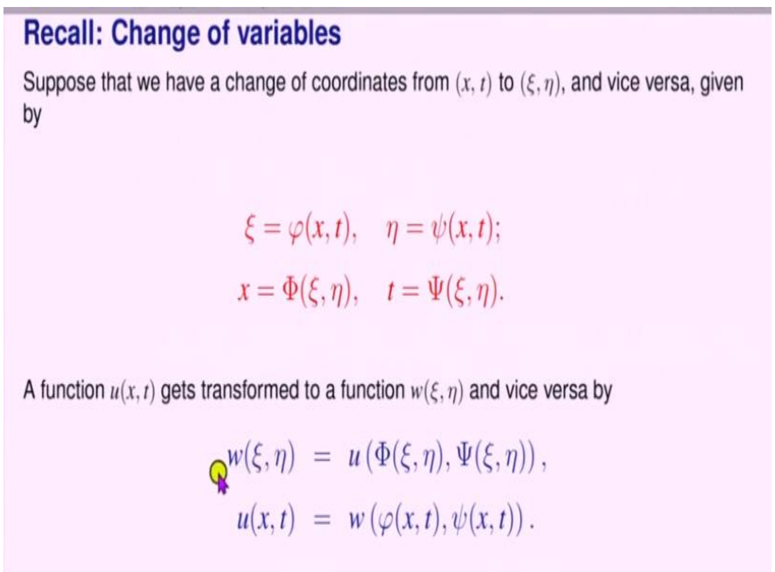

So recall about of change of variables suppose that we have a change of coordinates from x t to psi eta and vice versa given by psi equal to phi x t eta equal to psi x t and x capital phi of psi eta t equal to capital of psi eta. A function u gets transformed into a function in terms of psi eta using these relations and vice versa. So, we are going to find an, equation satisfied by w solve it and get u using the w that is the strategy.

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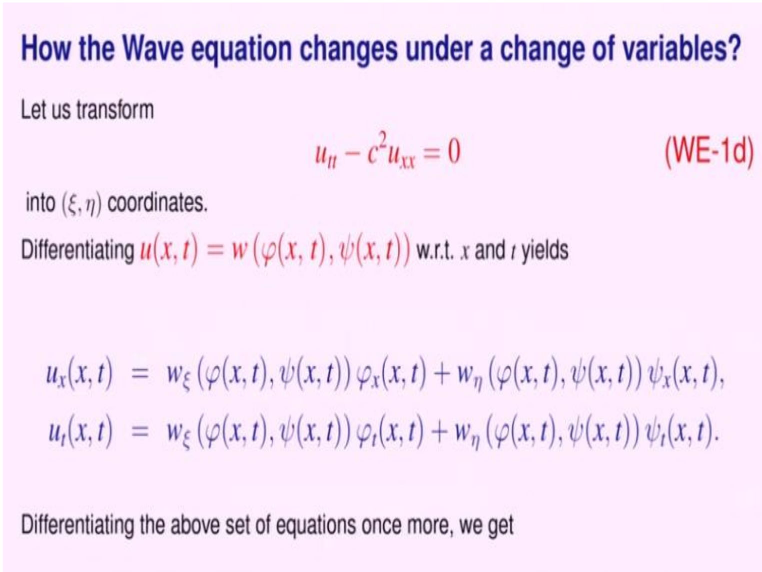

So how does this wave equation change under change of variables? So let us transform WE-1d into psi eta coordinates. We are yet to find this coordinates we will find that so differentiate this relation u x, t equal to w of phi x t psi x, t with respect to x and t you get u x equal to w with respect to the first variable which is psi and whoever is here that is phi with respect to x. Plus w with respect to eta and psi with respect to x this is the chain rule and similarly u, t.

Of course what we need is u t and u xx so that we can go back and substitute the equation. So please do these computations on your own.

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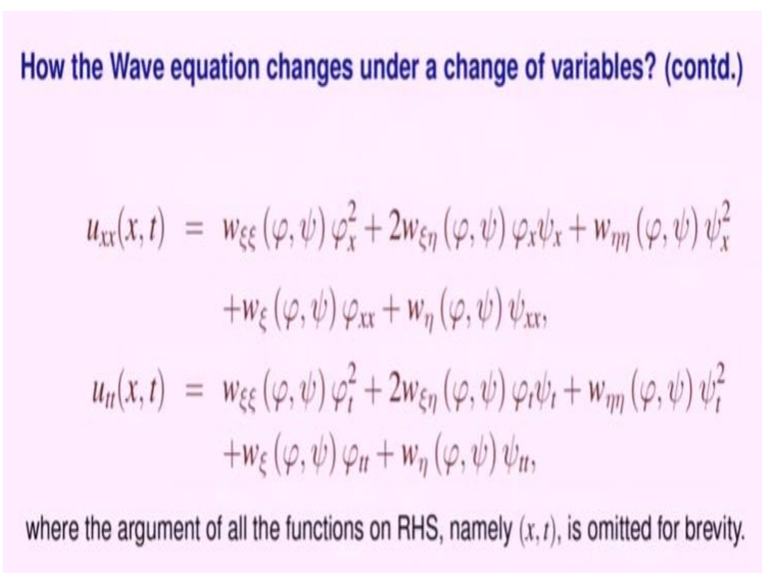

So u xx it will turn out to be having this expression and u tt this expression the argument is s t which, we are suppressing here. Otherwise equations will be in many more lines.

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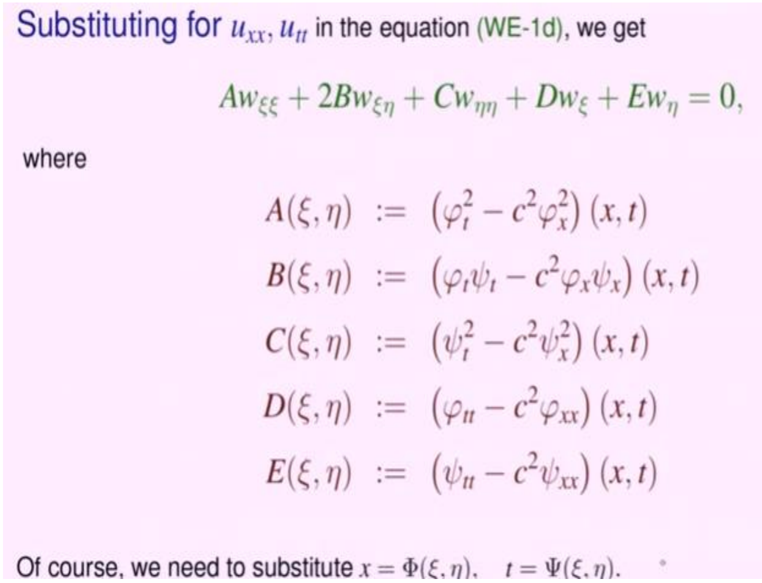

Let us substitute for u x x and u t t in a wave equation and we get an equation of this form x and t we have to substitute capital phi and capital psi.

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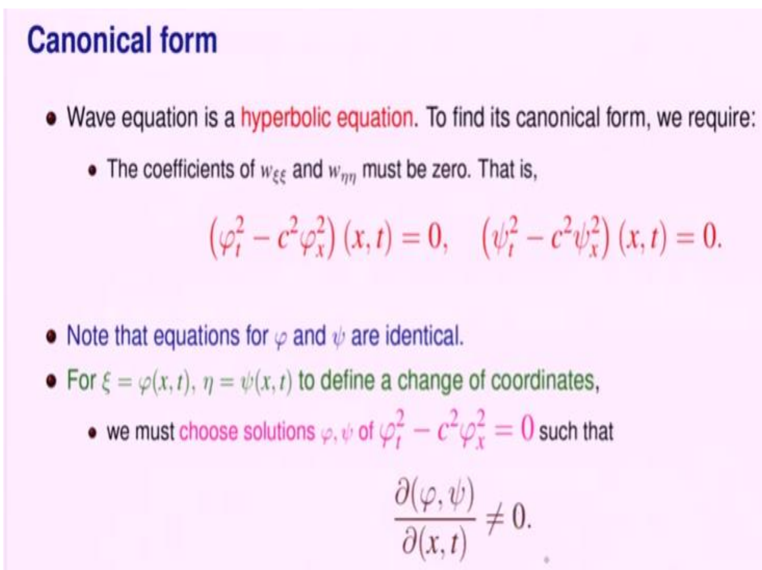

So wave equation is hyperbolic equation to find it is canonical for it require that w psi-psi and w eta-eta their coefficients must be 0. What are they? It is a coefficient of w psi-psi this is for w eta-eta we want that to be 0. But if you observe the equations are same of course this is written for phi this is written for psi otherwise equation itself is a same. So now for psi = phi and eta = psi to define a change of coordinates phi and psi I cannot take it to the same function.

In fact what I need is this Jacobian to be non-zero so I need solutions of this equation phi and psi of this equation with this property. Because that is when I know that there is a change of coordinates, possible thanks to inverse function theorem.

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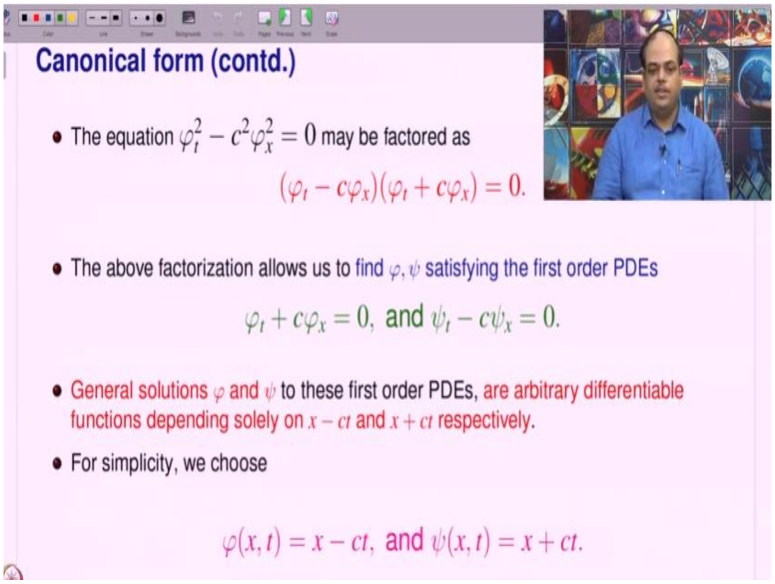

So this equation can be factored into this if you multiply you will see that phi x phi t term will gets cancelled. So this if you multiply precisely this therefore what we do is that we find a fees satisfying one of these equations and psi the other equation. So any one of them is 0 product is 0 that is the logic here so we do this and very easy write down the general solutions of this equations because this is equation with constant coefficients.

So phi is a function of x – ct alone and psi is a function of x + ct alone so please checks for, yourself. We did this in the first order partial differential equations chapter so for simplicity let us take phi = x - ct psi – x = ct.

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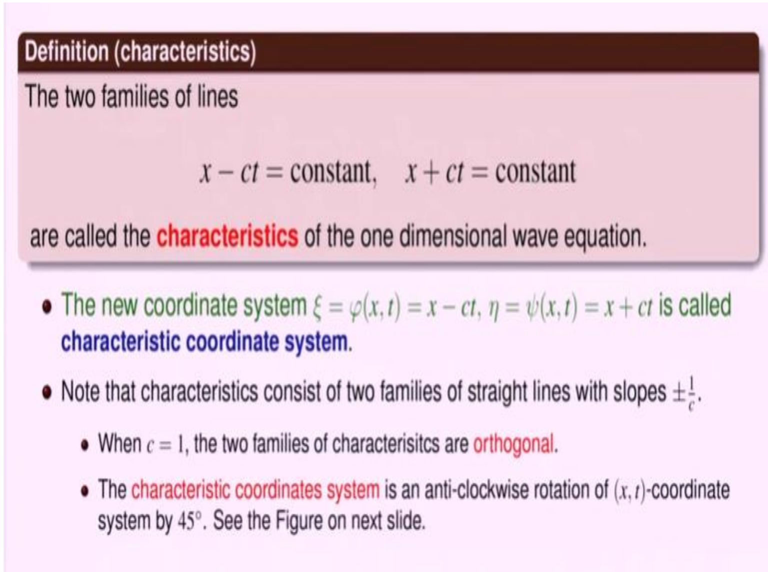

Now this is a definition these are lines x – ct = constant and x + ct = constant they are called the characteristics of the one dimensional wave equation. The new coordinate system psi = x – ct eta = x + ct is called characteristics coordinate system. Note that characteristic consist of 2 families of straight lines having slopes plus or minus 1 by c 1 has 1 by c and the other family has – 1 by c ach family consist of parallel lines.

So when c = 1 the families are actually are orthogonal families the characteristic coordinate system is a anti clock wise rotation of the x t coordinate system. We will see this in the next slide we have a picture.

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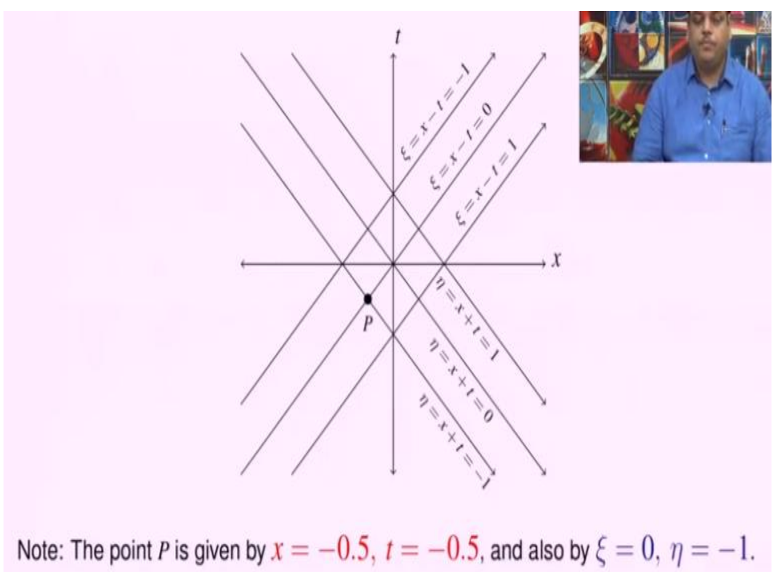

So this is our original x and t coordinate system and this is psi and eta psi = x – t eta = x + t they are orthogonal because c = 1. Look at this point P in terms of x t coordinates is given by x = -0.5 t = -0.5 in terms of psi eta it is lying on this line x = 3 so psi is 0 and eta is – 1.

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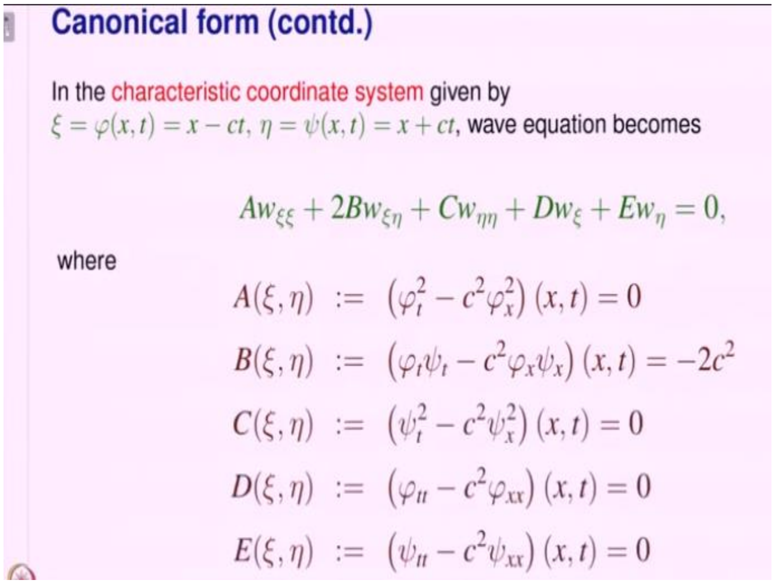

In this coordinate system given by psi = x – ct eta = x + ct the wave equation becomes this we have already noted down. But now we have insisted A must be 0 similarly C is 0. And B turns out to be -2 c square the computation will give C 0 D is also 0.

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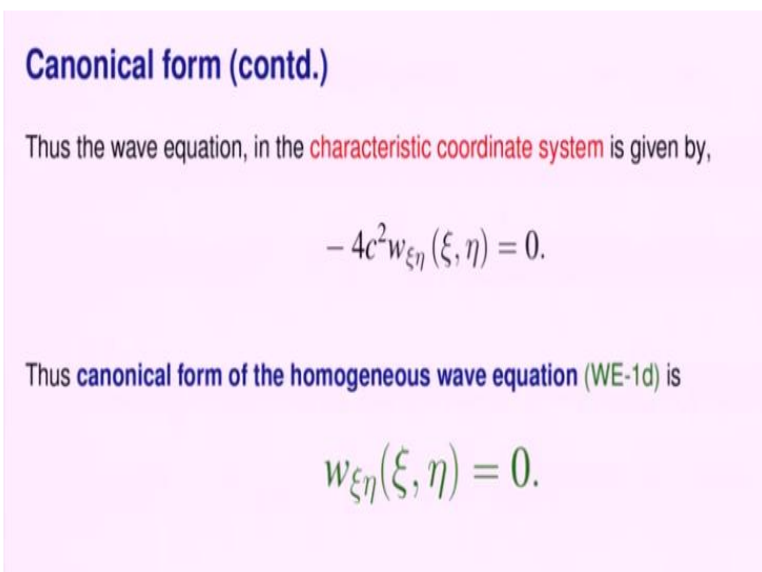

So therefore the equation that we have is -4 c square w psi eta is 0 of course c is non-zero we can cancel this and you get w psi eta is 0. This is a canonical form of the homogenous wave equation.

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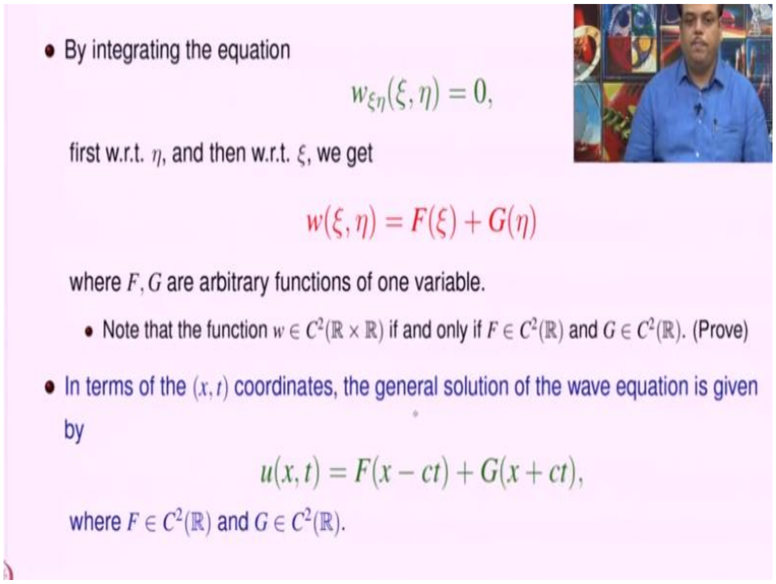

Now we can integrate this with respect to eta first and psi next or even the other order is also fine. And we get this expression if integrate with respect to eta first then what you have is w psi is a function of psi alone and then w eta will turn out to be this w psi eta is F psi + G eta. So note that the function w is C 2 on R cross R if and only if the F is C 2 and G is C2. So w is C 2 function if and only if F and G are C 2 functions.

So in terms of the x t coordinates the general solution of wave equation is given by go back and substitute for psi equal to x- c t and eta = x + ct so we get this where F is C 2 and G is C 2.

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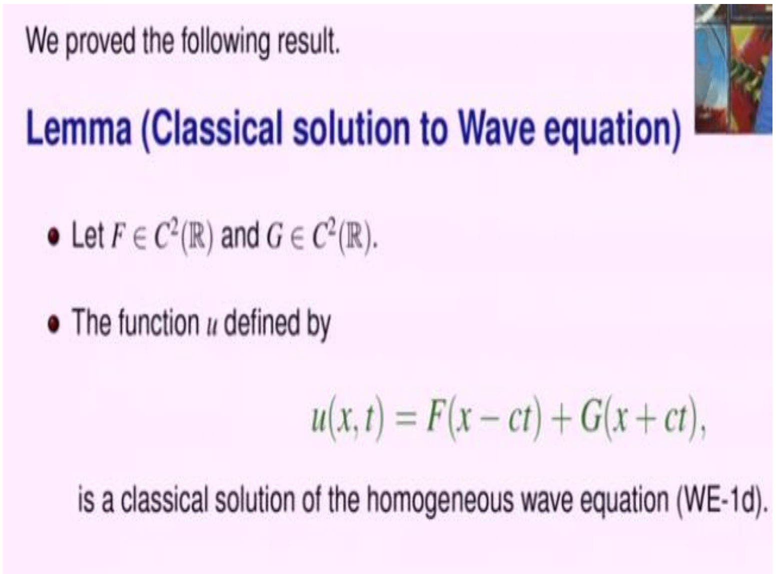

So we prove the following result classical solution to wave equation let F belongs to C 2 and G belongs to C 2 the function u defined by u x, t = F of x – ct + G of x + ct is a classical solution of the homogenous wave equation. No Cauchy problem yet we have just solved the wave equation homogenous wave equation. Now we impose initial conditions on this and we determine what F and G should be in terms of the given function is phi and psi.

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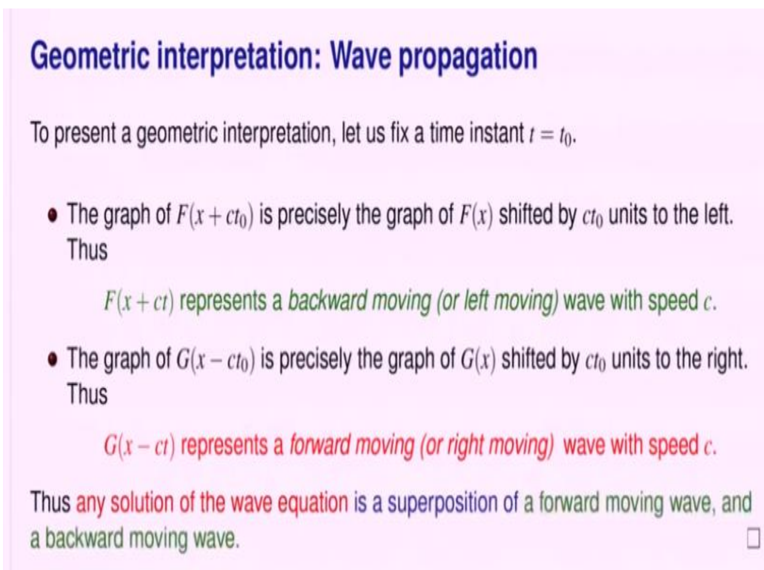

Before that we do a geometric interpretation as a wave propagation let us fix a time instant t = t naught and the graph of F of x + ct naught is precisely graph of F of x shifted by ct naught units to the left. Thus f + x ct represents a backward moving or left moving wave with speed c.

Similarly the graph of g of x – ct naught is precisely the graph of g of x that means shape is not changing but it is getting translated by ct naught units to the right side.

Thus G of x – ct represents a forward moving or right moving wave with speed c thus any solution of the wave equation is a superposition of a wave which is moving forward and another moving backward.

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Now let us derive the d'Alembert formula for solution of Cauchy problem.

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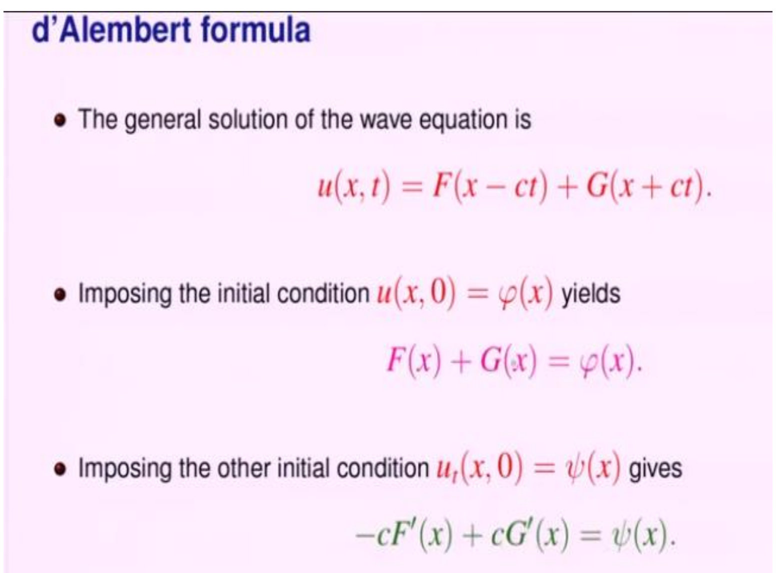

The general solution of the wave equation is u of xt = F of x – ct + G of x + ct now we have to use our given Cauchy conditions u of x, 0 equal to phi x. So I put phi = 0 in this equation I get phi x = F x + G x so this is the equation I have for f and g. Now we have another initial condition that is u t x0 is psi x so differentiate this equation respect to t and then put t = 0. So if you differentiate this with respect to t what you get is F prime into x – ct into -c + G prime at x + ct into c.

And when you put t = 0 what you have is –c F prime x + c G prime x that is equal to psi x therefore you have this. So we have to 2 relations connecting F and G in terms of given phi and psi. But this equation has F dash and G dash so it is better to get rid of the derivative and that is done by integrating. Integrating both sides of this equation so that left hand side becomes a combination of F and G and here already have one combination of F and G linear combination.

So these are system of 2 linear equations you will get from there we can determine F and G that is the strategy.

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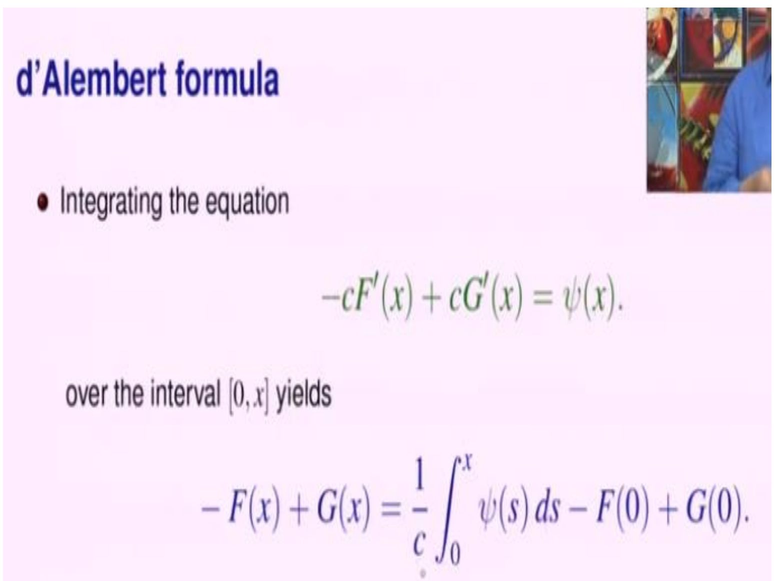

So let us integrate this equation between 0 to x so we get F + G – F + G from here because once you integrate from 0 to x this is actually –c times F prime – G prime. So that is F – G whole dash so if you integrate you get F – G at c – f – g at 0 the 0 terms have been transferred this side. And here you have integral 0 to x psi s ds here and then you divide with a c. So you get 1 by c please do this computation.

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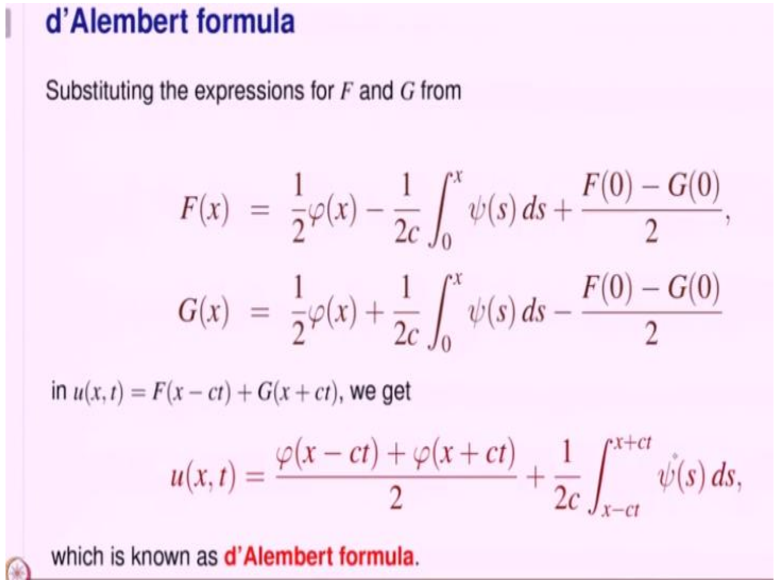

So we have the following 2 equations for F and G this is coming from IC-1 F + G = phi this is come IC-2 after integration –F + G equal to this. Now if you add up you get and expression for G if you subtract you get an expression for F. So we can solve this system of linear equations and obtain F = this expression and G = this expression. Now we take these expressions for F and G and substitute in the solution u which is given in terms of F and G.

So this is the expression that we end up with u x, t = phi of x – ct + phi of x + ct by 2 + 1 by 2c integral x – ct to x + ct. This term this is a constant this is a minus of constant so it actually got cancelled. So this is known as d'Alembert formula for the solution of homogenous wave equation and the Cauchy problem where phi is u x, 0 psi is u t x, 0. So phi is u x, 0 so u, x 0 is called initial displacement psi is u t x, 0 that is the initial velocity.

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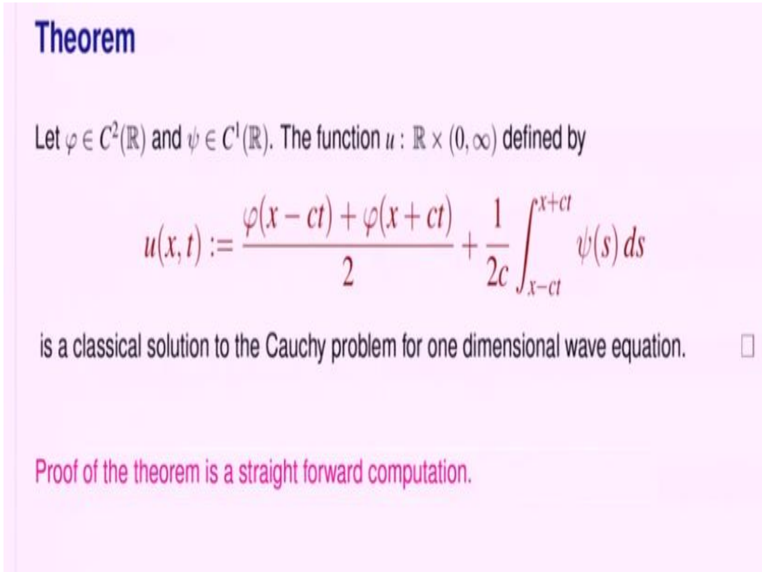

So let phi belongs to c2 of r and psi belongs to c1 of r the function defined by the d'Alembert formula is a classical solution to the Cauchy problem this we have to check right what do you have to check u is a c2 function why is it a c2 function? Because of chain rules we have phi is a c2 function this is a c infinity function x – ct. Similarly x + ct show that phi of x – ct is c2 phi of x + ct is c2 as a function of x and t and psi c1 but we are integrating.

Therefore we gain one extra derivative that is why this term is c2 you can use (()) (16:33) to get third derivatives and then check that u is indeed a solution to the homogenous wave equation and putting t = 0 is very easy put t = 0 this integral is from x to x so this 0 this term is 0. And what we get is phi x + phi x by 2 so you get phi x differentiate with respect to t this equation and substitute t = 0 you should get psi x. So it is a very straight forward computation so I will leave it as an exercise for you to do it.

Please write down in detail why this u is c2 and why it is satisfies the homogenous wave equation and the 2 initial conditions.

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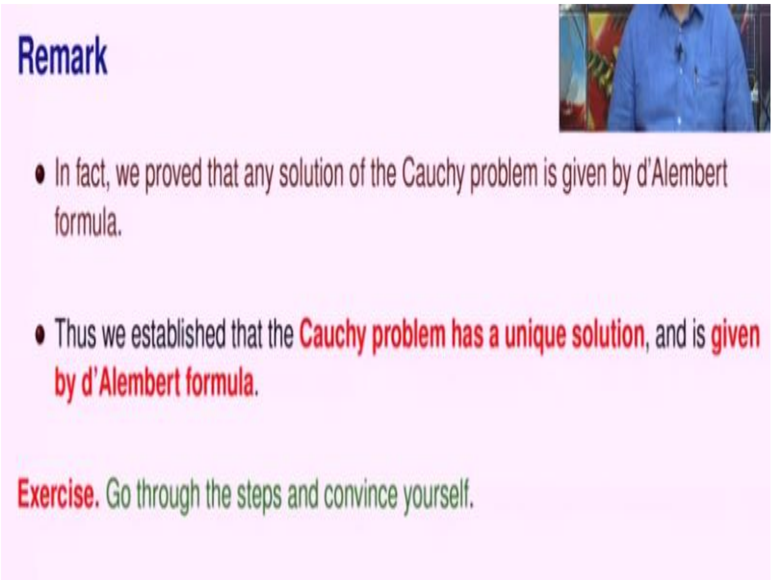

In fact if you observe closely we proved that any solution of the Cauchy problem is given by d.Alembert formula thus we established that the Cauchy problem has a unique solution and is given by d’Alembert formula. So go through the steps and convince yourself what did we do? We are given homogenous wave equation that we reduced to canonical form there is no compromise there.

Canonical form is c2 solution if and only if it is given by F of psi + G of eta and F and G has to be c2 no compromise there. After that we went back and substituted so everywhere there is no loss of information therefore the uniqueness is there in the proof. So please conscience yourself that we have indeed proved uniqueness and in case you are still not convinced we will give another proof later on couple of proofs later on for the uniqueness.

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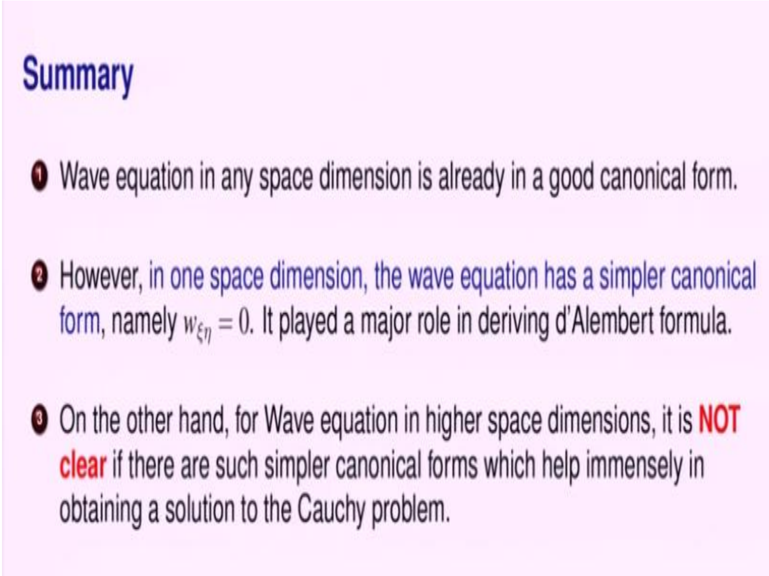

Let us summarize what we did in this lecture wave equation in any space dimension is already in a good canonical form u tt – c square Laplacian u = 0 you cannot ask for anything better than that. However in one space dimension the wave equation is simpler canonical form namely w psi eta = 0 and that played a major role in deriving d'Alembert formula. On the other hand for wave equation highest space dimensions that is dimension 2 onwards.

It is not clear if there are simpler canonical forms these aspects we have discussed in the chapter 3. So it is not clear if there are such simpler canonical which help immensely in obtaining a solution to Cauchy problem.

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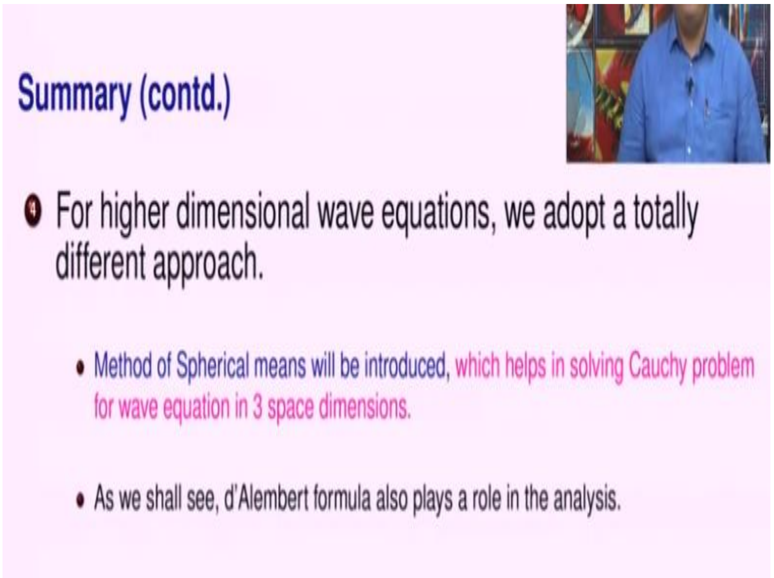

So for higher dimensional wave equation what we do we adopt a totally different approach we use what is called a method of spherical means we will introduce that and that helps in solving Cauchy problem for wave equation in 3d first 3 dimensions not 2d but to 3d. As we shall see d'Alembert formula also place a role in the analysis so using spherical means we kind of reduce the Cauchy problem in 3d to some wave equation in 1d and that is where we can use the d'Alembert formula in the solution thank you.