

Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Science, Bombay


Lecture – 23
Second Order Partial Differential Equations Canonical Form for an Equation of Elliptic Type

In this lecture we are going to consider second order partial differential equations linear, which are identified as elliptic equations, and try to obtain a canonical form for the same. Recall we have seen one example of an elliptic equation.

(Refer Slide Time: 00:32)

Second order Partial Differential Equations
Canonical form for an equation of Elliptic type

S. Sivaji Ganesh


Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076.

Lecture 3.6

$u_{xx} + u_{yy} = 0$
 $u_{xy} \times$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 1/20

Namely the Laplace equation. So, the canonical form for an elliptic equation we would like to resemble this. If you observe here in this equation u_{xy} does not appear and u_{xx} and u_{yy} appear with coefficients 1. So, this is going to be the model for us for an equation of elliptic type.

(Refer Slide Time: 01:06)

Theorem: Hypotheses

- Let the equation

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u + g(x, y) = 0. \quad (2L)$$
 be elliptic in a region Ω of the xy -plane.
- Let the functions a, b, c be real-analytic functions in Ω .
- Let $(x_0, y_0) \in \Omega$.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 3/20

So, let us consider the second order linear equation 2L which is elliptic, assumed to be elliptic in a region Ω . Assume further that the coefficients a, b, c are real analytic functions, this is a requirement for the proof of our theorem. In the examples that you want to do, you may simply follow the procedure you may even still be successful. So, let x_0, y_0 belong to Ω .

(Refer Slide Time: 01:39)

Theorem: Conclusion

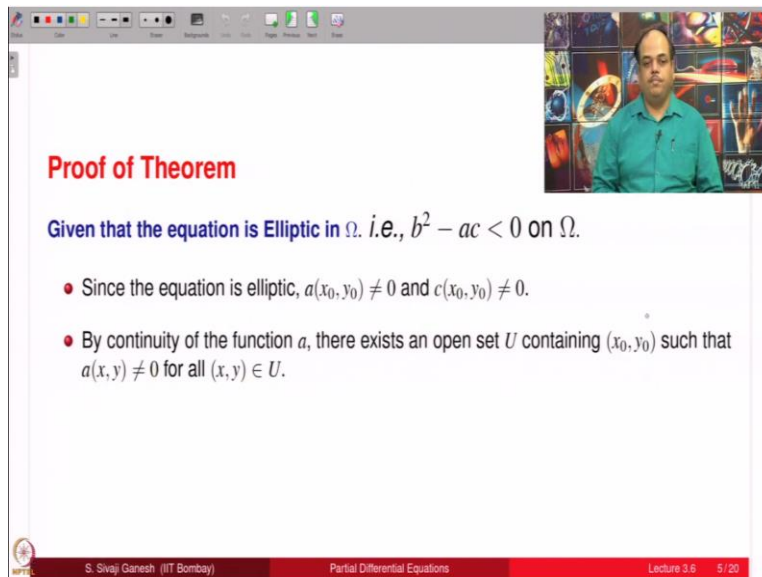
There exists an open set containing the point (x_0, y_0) and a change of coordinates $(x, y) \mapsto (\xi, \eta)$ such that the equation (2L) is transformed into an equation of the form

$$w_{\xi\xi} + w_{\eta\eta} + D(\xi, \eta)w_{\xi} + E(\xi, \eta)w_{\eta} + F(\xi, \eta)w + G(\xi, \eta) = 0.$$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 4/20

Now, there is an open set containing the point x_0, y_0 and a change of coordinates. Such that the; 2L gets transformed into this equation observed this part, where the second order partial derivatives $w_{\xi\xi} + w_{\eta\eta}$ looks like $u_{xx} + u_{yy}$ an unknown mixed partial derivative $w_{\xi\eta}$ it does not appear in this equation. So, this is what is called a canonical form for elliptic equations.

(Refer Slide Time: 02:10)



Proof of Theorem

Given that the equation is Elliptic in Ω . i.e., $b^2 - ac < 0$ on Ω .

- Since the equation is elliptic, $a(x_0, y_0) \neq 0$ and $c(x_0, y_0) \neq 0$.
- By continuity of the function a , there exists an open set U containing (x_0, y_0) such that $a(x, y) \neq 0$ for all $(x, y) \in U$.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 5/20

So, equation is given to be elliptic, it means $b^2 - ac$ is negative less than 0 on Ω . So, the question is elliptic, we cannot have $a = 0$ at (x_0, y_0) if it is 0. What we have is $b^2 - ac$ strictly less than 0 is not correct, because b^2 is always greater than or equal to 0. Therefore, necessarily a should be non 0 and of course, c also cannot be 0 for the same reason.

So, neither a nor c can be 0 at the point (x_0, y_0) at any point in Ω in particular at (x_0, y_0) . And we are assuming they are really elliptic, definitely continuous. So, by continuity of the function a , the function a will be not 0 in some open set U which contains (x_0, y_0) by continuity.

(Refer Slide Time: 03:09)

Recall: Change of variables

Suppose that we have a change of coordinates from (x, y) to (ξ, η) , and vice versa, given by

$$\xi = \varphi(x, y), \quad \eta = \psi(x, y);$$

$$x = \Phi(\xi, \eta), \quad y = \Psi(\xi, \eta).$$

A function $u(x, y)$ gets transformed to a function $w(\xi, \eta)$ and vice versa by

$$w(\xi, \eta) = u(\Phi(\xi, \eta), \Psi(\xi, \eta)),$$

$$u(x, y) = w(\varphi(x, y), \psi(x, y)).$$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 6/20

Recall a change of variables ξ, η are given by two functions φ and ψ if this gives rise to change variables, you can invert back x and y you can write in terms of ξ, η we use capital Φ and capital Ψ for that are a function of x, y can be identified with the function of ξ, η and they satisfy these two relations.

(Refer Slide Time: 03:35)

Proof of Theorem (contd.)

We know that under a change of coordinates the equation (2L) transforms to

$$Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_{\xi} + Ew_{\eta} + Fw + G = 0,$$

where

$$A(\xi, \eta) := (a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2) \Big|_{(x,y)=(\Phi(\xi,\eta), \Psi(\xi,\eta))} \quad (3a)$$

$$B(\xi, \eta) := (a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y) \Big|_{(x,y)=(\Phi(\xi,\eta), \Psi(\xi,\eta))} \quad (3b)$$

$$C(\xi, \eta) := (a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2) \Big|_{(x,y)=(\Phi(\xi,\eta), \Psi(\xi,\eta))} \quad (3c)$$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 7/20

And the 2L equation second order linear equation gets transformed to this equation, where the equations A B C alone are listed here because they are the only things which are important as far as a type of an equation is concerned. So, what do we need for proving the theorem? We want the $w_{\xi\xi}$ and $w_{\eta\eta}$ should appear with coefficient one in particular A must be equal to C and equal 1 and B is 0.

Once B is 0 and A equal to C we can divide with the A then I will get w psi psi + w eta eta anyway. So, the condition is A = C and B = 0.

(Refer Slide Time: 04:19)

Proof of Theorem (contd.)

- For proving the theorem, we need to find a system of coordinates (ξ, η) so that

$$A(\xi, \eta) = C(\xi, \eta), B(\xi, \eta) = 0.$$
- Thus we need to find φ, ψ satisfying the equations

$$a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2,$$

$$a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y = 0.$$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 8/20

That is enough, so, A = C this is a and this is c. So, A equal to C means this equation must be satisfied and B = 0 means, this equation is satisfied.

(Refer Slide Time: 04:40)

Proof of Theorem (contd.)

- The system of equations for finding φ, ψ is

$$a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2,$$

$$a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y = 0.$$
- Recall that in the hyperbolic case, the equations for φ and ψ were *decoupled*.
- Recall that in the parabolic case, the equations for φ and ψ were *weakly coupled*. Equation for φ did not involve ψ .
- But for Elliptic case, the equations for φ and ψ is a *strongly coupled* system of first order nonlinear PDEs.
- We can overcome this difficulty by using the assumption of real analyticity of a, b, c and complex variable techniques.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 9/20

The system of equations for finding phi and psi remark on that, recall it in hyperbolic case, when we want to find phi and psi the equations were decoupled. They were decoupled phi and psi

could be solved separately. In the parabola case, they are weakly coupled. The equations for finance are weakly coupled in the sense that the equation for ϕ did not involve ψ at all. Unfortunately, once we solve for ϕ and when we go and substitute in the second equation, the equation reduces to identity $0 = 0$.

Maybe it is a good thing, because now we can choose ψ arbitrary really, as long as the Jacobian of ϕ and ψ is non 0. This can also be explained in the following way. Once you solve for ϕ , which is a solution of the equation here ψ , η is equal to 0 we did not even solve for $b \psi = 0$, because it is automatically satisfied by the invariance of the classification type under change of coordinates.

Not only that, it is satisfied by any function ψ $b \psi = 0$ is satisfied by any function ψ . Therefore, we had lots of freedom. In choosing sides, the only requirement was that the Jacobian of ϕ and ψ is non 0. But for elliptic equation what is happening? In the here also, you have ϕ and ψ mixed here also ϕ and ψ mixed. Therefore, it is a strongly coupled system for ϕ and ψ of first order nonlinear PDEs.

We can overcome this difficulty by using the assumption of the real analogue city of a, b, c on some complex variable techniques. You will see that a crucial part of the proof we will skip we will not do the proof.

(Refer Slide Time: 06:40)

Proof of Theorem (contd.)

The system

$$a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2,$$

$$a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y = 0.$$

may be rewritten as

$$a(\varphi_x^2 - \psi_x^2) + 2b(\varphi_x\varphi_y - \psi_x\psi_y) + c(\varphi_y^2 - \psi_y^2) = 0,$$

$$ia\varphi_x\psi_x + ib(\varphi_x\psi_y + \varphi_y\psi_x) + ic\varphi_y\psi_y = 0,$$

where $i = \sqrt{-1}$.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 10/20

This system may be rewritten as this write the first equation, I simply transfer all these terms to the left-hand side and I take a common so get I get psi x square psi x square 2b common so that I get psi x psi y - psi x psi y and c common so, the phi y square – psi y square. It is exactly the same equation rewritten. Second equation, I am just multiplying with i because I have some idea what I want to do in the next slide. That is why I put it otherwise the same equation, i multiplied with i. It tells us that some complex things are going to enter.

(Refer Slide Time: 07:24)

Proof of Theorem (contd.)

Define a complex valued function Φ by

$$\Phi(x, y) = \varphi(x, y) + i\psi(x, y).$$

The system

$$a(\varphi_x^2 - \psi_x^2) + 2b(\varphi_x\varphi_y - \psi_x\psi_y) + c(\varphi_y^2 - \psi_y^2) = 0,$$

$$ia\varphi_x\psi_x + ib(\varphi_x\psi_y + \varphi_y\psi_x) + ic\varphi_y\psi_y = 0,$$

is equivalent to

$$a\Phi_x^2 + 2b\Phi_x\Phi_y + c\Phi_y^2 = 0.$$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 11/20

So, define a complex valued function capital phi by a small phi of x, y + i psi of x, y. So, this system the system that we want to solve for phi and psi is equivalent to this one single equation now, a phi x square + 2b phi x phi y + c phi y square equal to 0.

(Refer Slide Time: 07:50)

Proof of Theorem (contd.)

- Note that the equation

$$a\Phi_x^2 + 2b\Phi_x\Phi_y + c\Phi_y^2 = 0.$$

is exactly the same equation that we solved while determining the canonical form for hyperbolic equations.

- However, factorizing the above equation leads to PDEs with complex coefficients given by

$$a\Phi_x + (b + i\sqrt{ac - b^2})\Phi_y = 0,$$

$$a\Phi_x + (b - i\sqrt{ac - b^2})\Phi_y = 0.$$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 9.6 12/20

This equation we have seen earlier, this is the same equation that we solved while determining canonical form for hyperbolic equations. Of course, the difference was there that we could factorize as real equations factors were real, but here it will not happen. It leads to PDEs with complex coefficients given by this equation and this equation. In the case of hyperbolic equation, there was no i first of all, an inside thing was $b^2 - ac$. So, these are the two equations we have.

(Refer Slide Time: 08:32)

Proof of Theorem (contd.)

Fact: The system of equations

$$a\Phi_x + (b + i\sqrt{ac - b^2})\Phi_y = 0,$$

$$a\Phi_x + (b - i\sqrt{ac - b^2})\Phi_y = 0.$$

has solutions near (x_0, y_0) , since a, b, c are real-analytic functions.

- If Φ is a solution of the first equation, then $\Psi := \bar{\Phi}$ is a solution of the second equation.
- Thus Φ, Ψ are constant on the two complex characteristics given by the equation(s)

$$\frac{dy}{dx} = \frac{b(x, y) \pm i\sqrt{a(x, y)c(x, y) - b^2(x, y)}}{a(x, y)}$$

corresponding to $+$ and $-$ respectively.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 9.6 13/20

Now, in fact, the system of equations has solutions near $x, 0, y, 0$. This is where a, b, c real analytic functions is used. This we can observe if ϕ the solution of the first equation.

Remember a, b, c they are all real valued functions. Therefore, if ϕ the solution or the first equation $\bar{\phi}$ the conjugate of ϕ that let us call it as capital ψ that is a solution to the second equation. Therefore, it is enough to solve one equation, essentially there is only one equation.

But that will give us what we want because ϕ is proposed as a small ϕ plus ψ times ψ . So, you can identify a real part and imaginary part and hopefully that forms a coordinate system that defines a coordinate system. Therefore, ϕ and ψ are constant on the two complex characteristics; these are right hand side is a complex valued function; we have not studied in our Picard's theorem how to solve such equations. This is where it is important that we use our hypothesis and show that solutions exist.

(Refer Slide Time: 09:44)

Proof of Theorem (contd.)

- Define

$$\varphi(x, y) := \operatorname{Re} \Phi(x, y), \quad \psi(x, y) := \operatorname{Im} \Phi(x, y)$$
 where $\operatorname{Re} \Phi$ and $\operatorname{Im} \Phi$ are real and imaginary parts of the complex valued function Φ .
- Thus $(\xi, \eta) = (\varphi(x, y), \psi(x, y))$ defines a coordinate transformation near the point (x_0, y_0) .
- The transformed equation has the required form. □

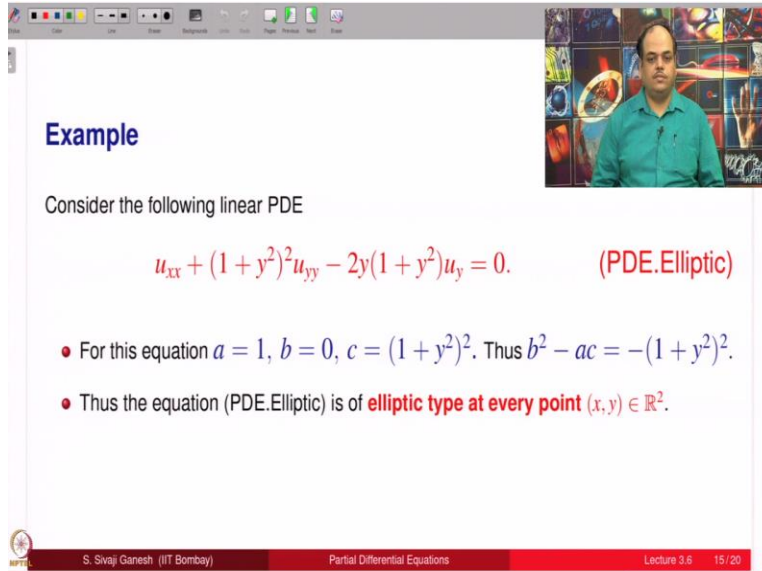
For a detailed exposition, refer the book
P.R. Garabedian: Partial differential equations, AMS-Chelsea, 1998.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 14/20

Define a small ϕ of x, y equal to real part of capital ϕ small because we have obtained ϕ after this right this ϕ exists is what somebody told us this fact. So, I have ϕ and then real part I will call small ϕ and imaginary part I call ψ . And this defines a coordinate transformation near the point that is left as an exercise very easy and the transformer the equation has the required form. Because we made what we want $A = C$ and $B = 0$.

So, therefore, we will get this. If you want to see a detailed proof, please look at this book by P.R. Garabedian on partial differential equations, you will find full details.

(Refer Slide Time: 10:40)



Example

Consider the following linear PDE

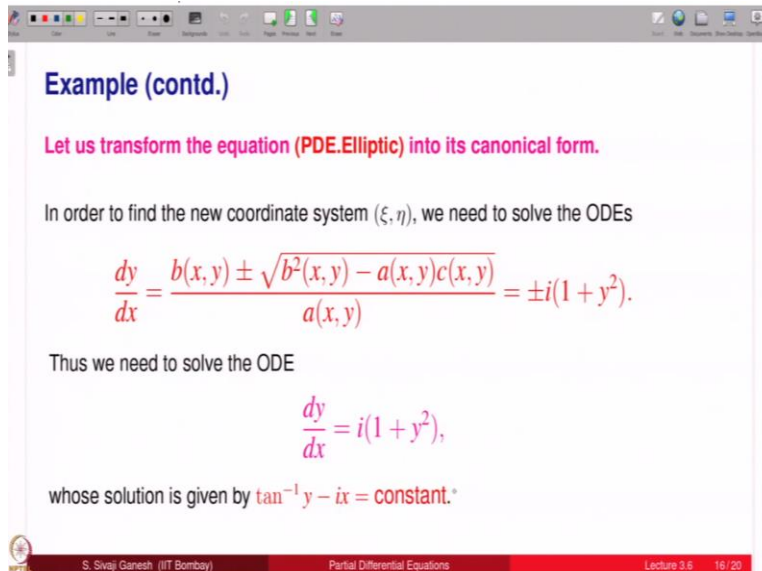
$$u_{xx} + (1 + y^2)^2 u_{yy} - 2y(1 + y^2) u_y = 0. \quad (\text{PDE.Elliptic})$$

- For this equation $a = 1$, $b = 0$, $c = (1 + y^2)^2$. Thus $b^2 - ac = -(1 + y^2)^2$.
- Thus the equation (PDE.Elliptic) is of **elliptic type at every point** $(x, y) \in \mathbb{R}^2$.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 15 / 20

Let us solve an example. Here you see we do not care whether it is elliptical or not of course, here these functions here it is a constant function one here is $1 + y$ square whole square is a polynomial here also a polynomial. Of course, it does not matter, we are worried only about coefficient of u_{xx} , u_{yy} and u_{xy} . So, $b^2 - ac$ is negative strictly at every point. Therefore, the equation is of elliptic type everywhere in the plane \mathbb{R}^2 .

(Refer Slide Time: 11:15)



Example (contd.)

Let us transform the equation (PDE.Elliptic) into its canonical form.

In order to find the new coordinate system (ξ, η) , we need to solve the ODEs

$$\frac{dy}{dx} = \frac{b(x, y) \pm \sqrt{b^2(x, y) - a(x, y)c(x, y)}}{a(x, y)} = \pm i(1 + y^2).$$

Thus we need to solve the ODE

$$\frac{dy}{dx} = i(1 + y^2),$$

whose solution is given by $\tan^{-1} y - ix = \text{constant}$.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 16 / 20

So, let us transform now, the given equation into its canonical form. So, we need to solve these ODEs dy by $dx = +$ or $- i$ into $1 + y$ square. So, we need to solve this ODE i times this because

minus i times does not matter it will be the conjugate. So, this solution is given by $\tan^{-1} y - i x = \text{constant}$. Please accept this that this is solution.

(Refer Slide Time: 11:53)

Example (contd.)

We introduce the following change of coordinates

$$\xi = \varphi(x, y) = x, \quad \text{and} \quad \eta = \psi(x, y) = \tan^{-1} y.$$

On differentiating the equation

$$u(x, y) = w(\varphi(x, y), \psi(x, y)) = w(x, \tan^{-1} y)$$

w.r.t. x and y we obtain

$$u_x = w_\xi, \quad u_y = \frac{1}{1+y^2} w_\eta, \quad u_{xx} = w_{\xi\xi},$$

$$u_{yy} = \frac{1}{(1+y^2)^2} w_{\eta\eta} - \frac{2y}{(1+y^2)^2} w_\eta.$$

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 17/20

And then real part is x imaginary part $\tan^{-1} y$ are vice versa. Here of course, if this is constant, I can multiply this with i also again right. So, therefore, it does not matter what I call the variables ϕ and ψ in the theorem that we presented this was called ϕ and this was called ψ . But we are interchanging here. It does not matter because there is no preference for the variable x or y or ψ or η .

So, therefore, we propose you have x, y equal to this w of $x, \tan^{-1} y$ differentiate u_x is w_x appears only here. So, it is w_η and derivative of x is 1 , u_y will be w_η and derivative of this is $1/(1+y^2)$. So, you can continue like that.

(Refer Slide Time: 12:48)

Example (contd.)

- On substituting these expressions for the derivatives of u in the equation (PDE.Elliptic), we get

$$w_{\xi\xi} + w_{\eta\eta} - 4(\tan \eta) w_{\eta} = 0,$$

which is a canonical form of the given PDE. □

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 19 / 20

Compute the derivatives go back and substitute in the given equation we get this. So, that is the canonical form of the given PDE.

(Refer Slide Time: 12:58)

Summary

- 1 A method to reduce a second order linear PDE of elliptic type to its canonical form was presented.
- 2 The method was successfully implemented in an example.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.6 19 / 20

So, summarizing what we did is that we have presented a method to reduce a second order union PDE which is of elliptic type to its canonical form. We have of course assumed very high assumptions on the coefficient a , b , c . But you may generally, if you are given a partial differential equation of elliptic type, you want to find its canonical form, you may simply follow the procedure do not bother about the hypothesis checking for the theorem.

And you will still be successful, if you are able to solve the ODEs which are coming on the way.
And we have seen the method you successfully implemented in the example. Thank you.