Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Science, Bombay

Lecture – 23
Second Order Partial Differential Equations Canonical Form for an Equation of Elliptic Type

In this lecture we are going to consider second order partial differential equations linear, which are identified as elliptic equations, and try to obtain a canonical form for the same. Recall we have seen one example of an elliptic equation.

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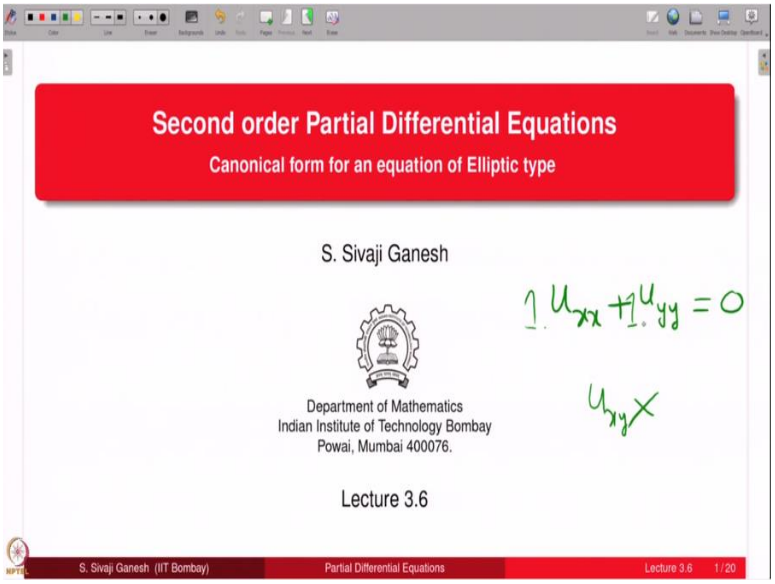

Namely the Laplace equation. So, the canonical form for an elliptic equation we would like to resemble this. If you observe here in this equation u xy does not appear and u xx and u yy appear with coefficients 1. So, this is going to be the model for us for an equation of elliptic type.

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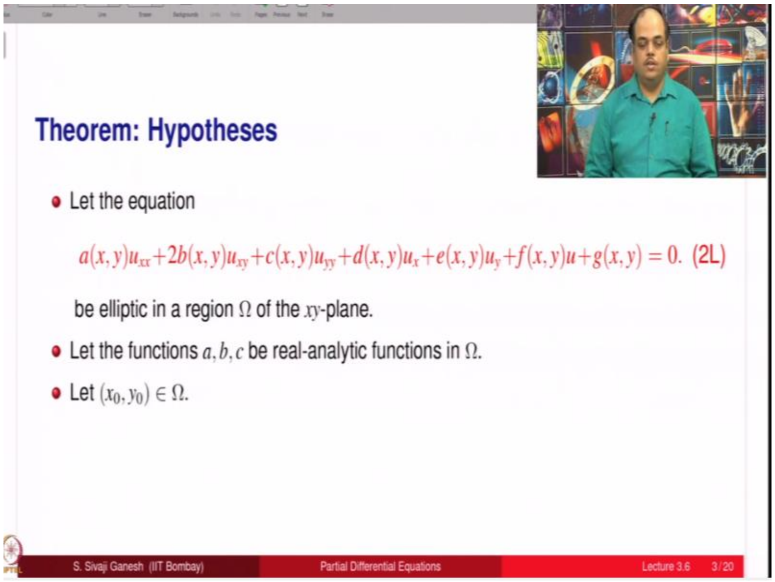

So, let us consider the second order linear equation 2L which is elliptic, assumed to be elliptic in a region omega. Assume further that the coefficients a, b, c are real analytic functions, this is a requirement for the proof of our theorem. In the examples that you want to do, you may simply follow the procedure you may even still be successful. So, let x 0 y 0 belong to omega.

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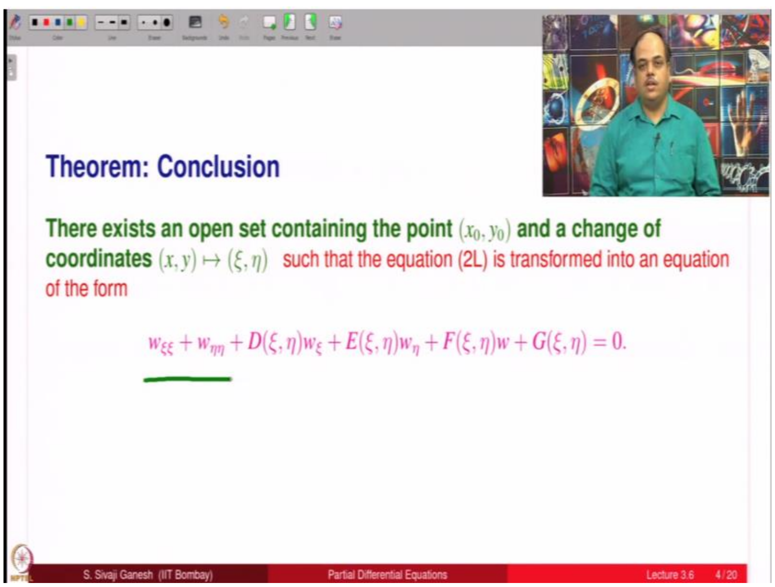

Now, there is an open set containing the point x 0, y 0 and a change of coordinates. Such that the; 2L gets transformed into this equation observed this part, where the second order partial levers up here. w psi psi + w eta eta looks like u xx + u yy an unknown mix partial derivative w psi eta it does not appear in this equation. So, this is what is called a canonical form for elliptic equations.

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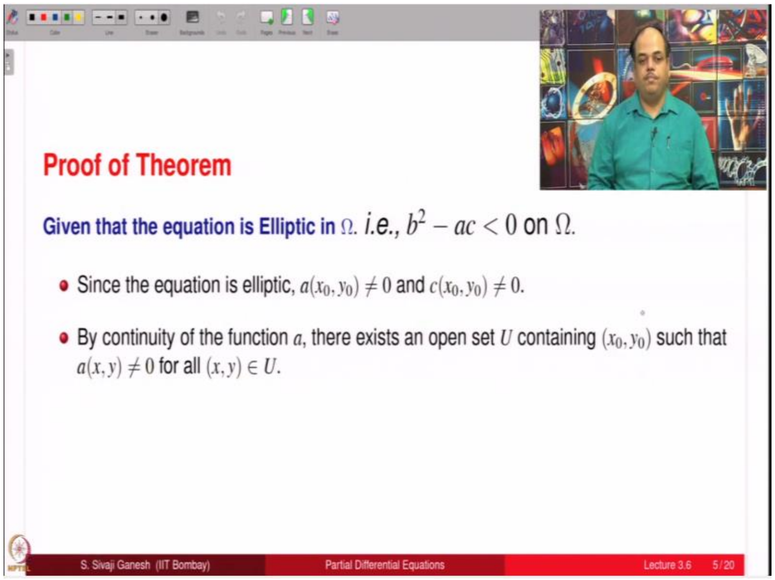

So, equation is given to be elliptic, it means b square - ac is negative less than 0 on omega. So, the question is elliptic, we cannot have a to b 0 a f x 0, y 0 if it is 0. What we have is b square? b square strictly less than 0 is not correct, because b square is always greater than or equal to 0 square of a real number is always got some equal to 0. Therefore, necessarily a should be non 0 and of course, c also cannot be 0 for the same reason.

So, neither a nor c can be 0 at the point x not at any point in omega in particular at x naught y naught. And we are assuming they are really elliptic, definitely continuous. So, by continue to the function a, the function a will be not 0 in some open set U which contains x 0, y 0 by continuity.

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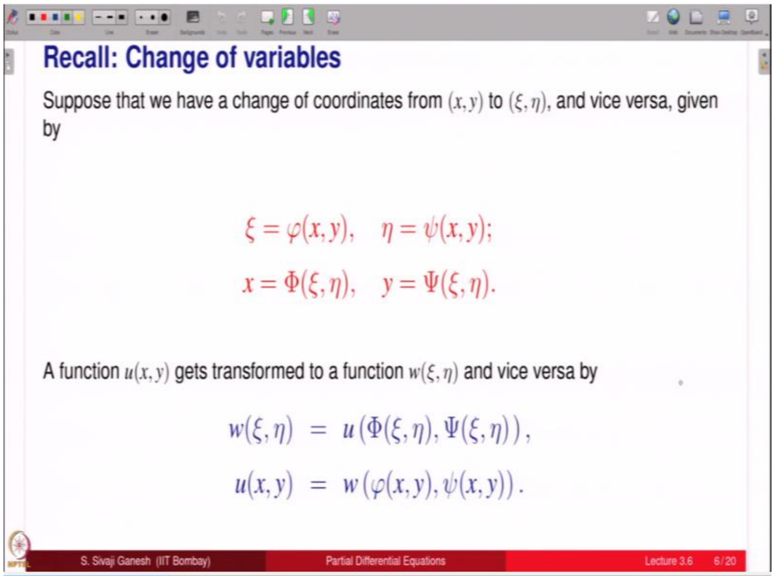

Recall a change of variables psi eta are given by two functions phi and phi if this gives rise to change variables, you can invert back x and y you can write in terms of psi eta we use capital phi and capital psi for that are a function of x, y can be identified with the function of psi eta and they satisfy these two relations.

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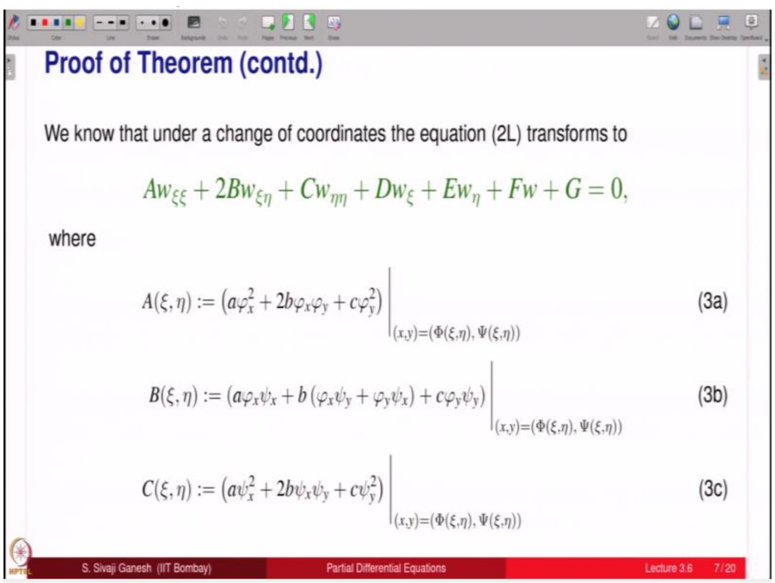

And the 2L equation second order linear equation gets transformed to this equation, where the equations A B C alone are listed here because they are the only things which are important as far as a type of an equation is concerned. So, what do we need for proving the theorem? We want the w psi psi and w eta eta should appear with coefficient one in particular a must be equal to C and equal 1 and B is 0.

Once B is 0 and A equal to C we can divide with the A then I will get w psi psi + w eta eta anyway. So, the condition is A = C and B = 0.

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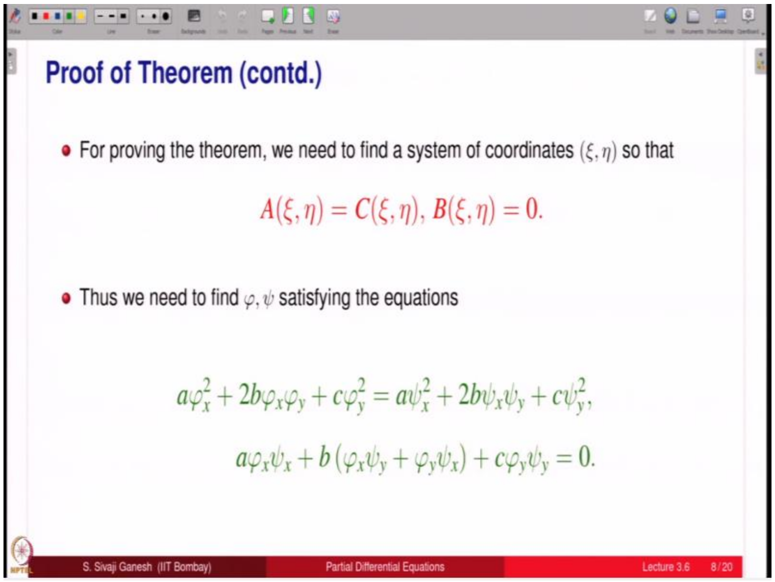

That is enough, so, A = C this is a and this is c. So, A equal to C means this equation must be satisfied and B = 0 means, this equation is satisfied.

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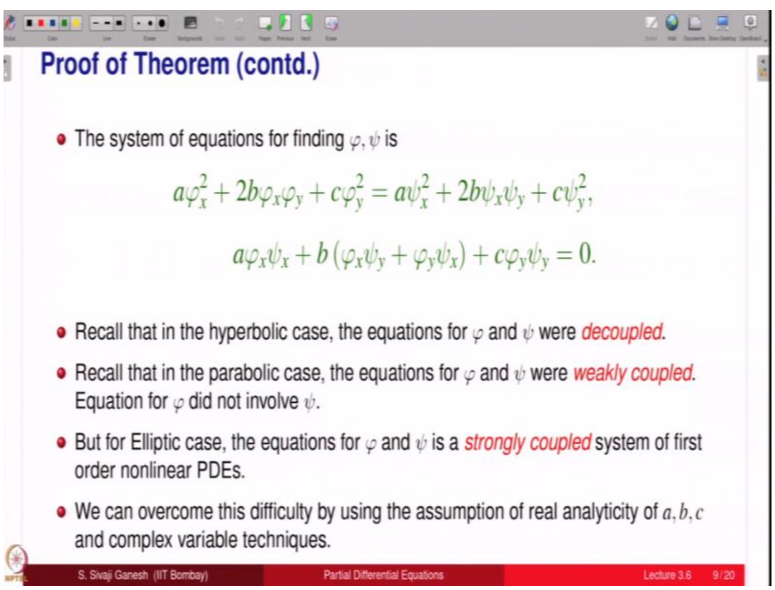

The system of equations for finding phi and psi remark on that, recall it in hyperbolic case, when we want to find phi and psi the equations were decoupled. They were decoupled phi and psi could be solved separately. In the parabola case, they are weakly coupled. The equations for finance are weakly coupled in the sense that the equation for phi in did not involve psi at all. Unfortunately, once we solve for phi and when we go and substitute in the second equation, the equation reduces to identity 0 = 0.

Maybe it is a good thing, because now we can choose psi arbitrary really, as long as the Jacobian of phi and psi is non 0. This can also be explained in the following way. Once you solve for phi, which is a solution of the equation here psi, eta is equal to 0 we did not even solve for b psi = 0, because it is automatically satisfied by the invariance of the classification type under change of coordinates.

Not only that, it is satisfied by any function psi b psi = 0 is satisfied by any function psi. Therefore, we had lots of freedom. In choosing sides, the only requirement was that the Jacobian of phi and psi is non 0. But for elliptic equation what is happening? In the here also, you have phi and psi mixed here also phi and psi mixed. Therefore, it is a strongly coupled system for phi and psi of first order nonlinear PDEs.

We can overcome this difficulty by using the assumption of the real analogue city of a, b, c on some complex variable techniques. You will see that a crucial part of the proof we will skip we will not do the proof.

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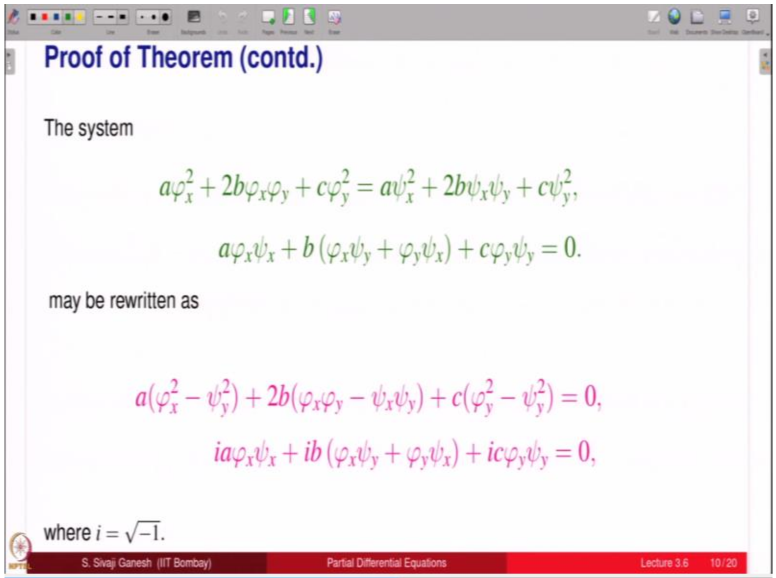

This system may be rewritten as this write the first equation, I simply transfer all these terms to the left-hand side and I take a common so get I get psi x square psi x square 2b common so that I get psi x psi y - psi x psi y and c common so, the phi y square – psi y square. It is exactly the same equation rewritten. Second equation, I am just multiplying with i because I have some idea what I want to do in the next slide. That is why I put it otherwise the same equation, i multiplied with i. It tells us that some complex things are going to enter.

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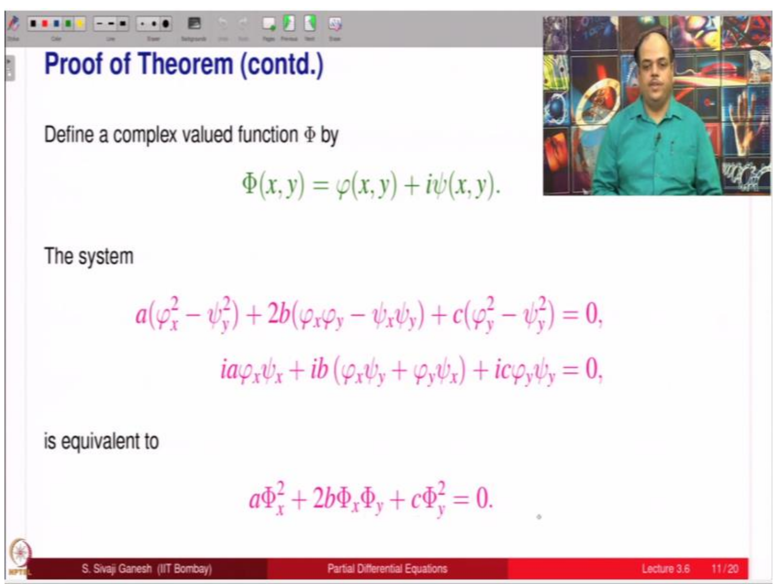

So, define a complex valued function capital phi by a small phi of x, y + i psi of x, y. So, this system the system that we want to solve for phi and psi is equivalent to this one single equation now, a phi x square + 2b phi x phi y + c phi y square equal to 0.

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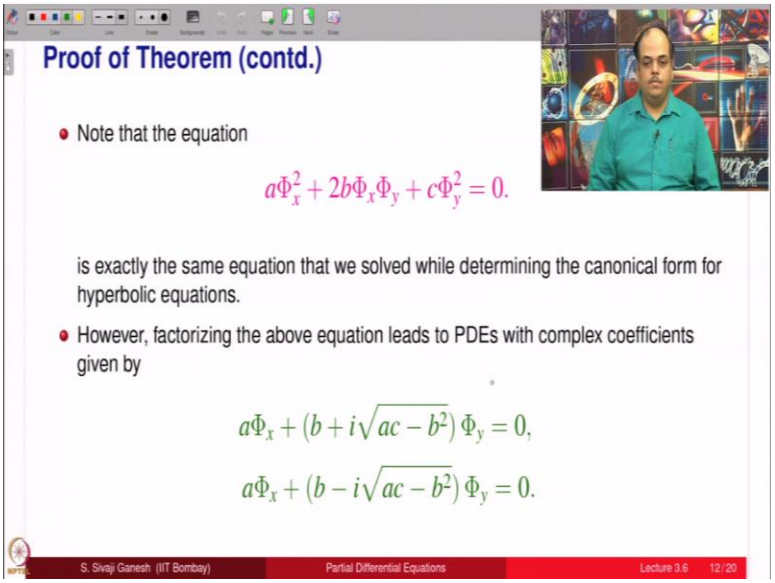

This equation we have seen earlier, this is the same equation that we solved while determining canonical form for hyperbolic equations. Of course, the difference was there that we could factorize as real equations factors were real, but here it will not happen. It leads to PDEs with complex coefficients given by this equation and this equation. In the case of hyperbolic equation, there was no i first of all, an inside thing was b square - ac. So, these are the two equations we have.

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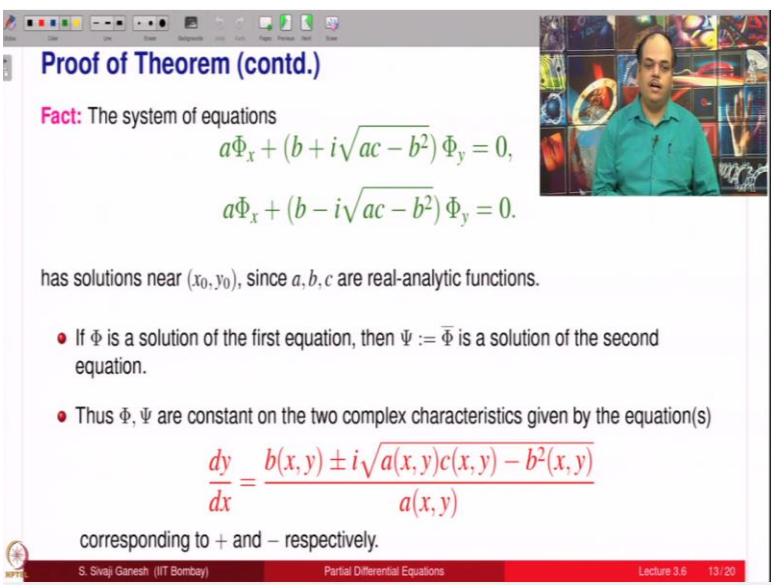

Now, in fact, the system of equations has solutions near x, 0 y, 0. This is where a, b, c real analytic functions is used. This we can observe if phi the solution of the first equation.

Remember a, b, c they are all real valued functions. Therefore, if phi the solution or the first equation phi bar the conjugate of phi that let us call it as capital psi that is a solution to the second equation. Therefore, it is enough to solve one equation, essentially there is only one equation.

But that will give us what we want because phi is proposed as a small phi plus psi times psi. So, you can identify a real part and imaginary part and hopefully that forms a coordinate system that defines a coordinate system. Therefore, phi and psi are constant on the two complex characteristics; these are right hand side is a complex valued function; we have not studied in our Picard's theorem how to solve such equations. This is where it is important that we use our hypothesis and show that solutions exist.

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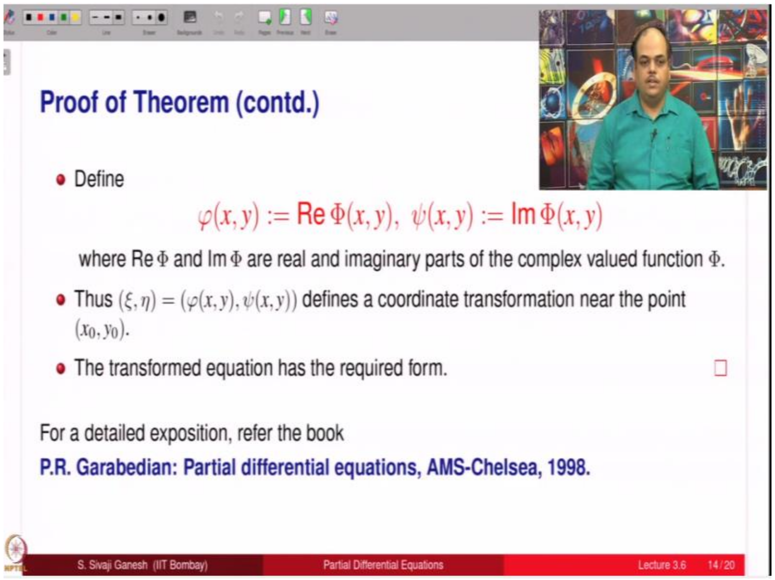

Define a small phi of x, y equal to real part of capital phi small because we have obtained phi after this right this phi exists is what somebody told us this fact. So, I have phi and then real part I will call small phi and imaginary part I call psi. And this defines a coordinate transformation near the point that is left as an exercise very easy and the transformer the equation has the required form. Because we made what we want A= C and B = 0.

So, therefore, we will get this. If you want to see a detailed proof, please look at this book by P.R Garabedian on partial differential equations, you will find full details.

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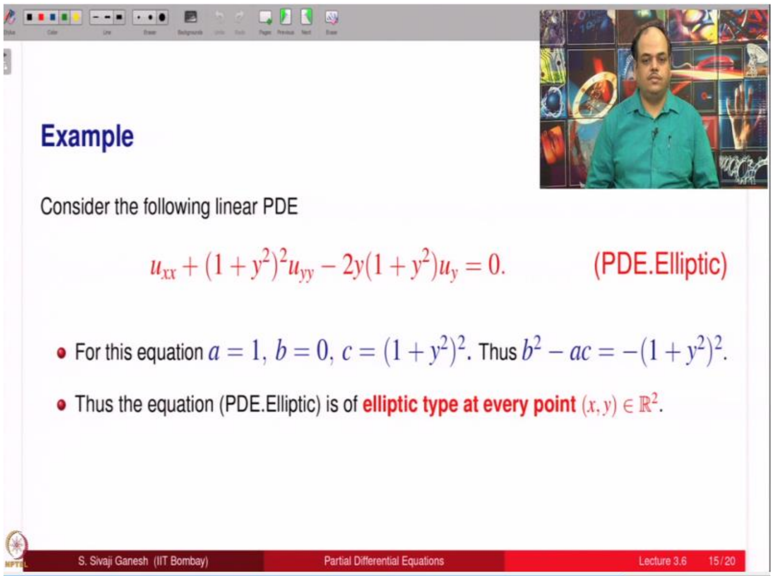

Let us solve an example. Here you see we do not care whether it is elliptical or not of course, here these functions here it is a constant function one here is 1+ y square whole square is a polynomial here also a polynomial. Of course, it does not matter, we are worried only about coefficient of u xx, u yy and u xy. So, b squared - ac is negative strictly at every point. Therefore, the equation is of elliptic type everywhere in the plane R 2.

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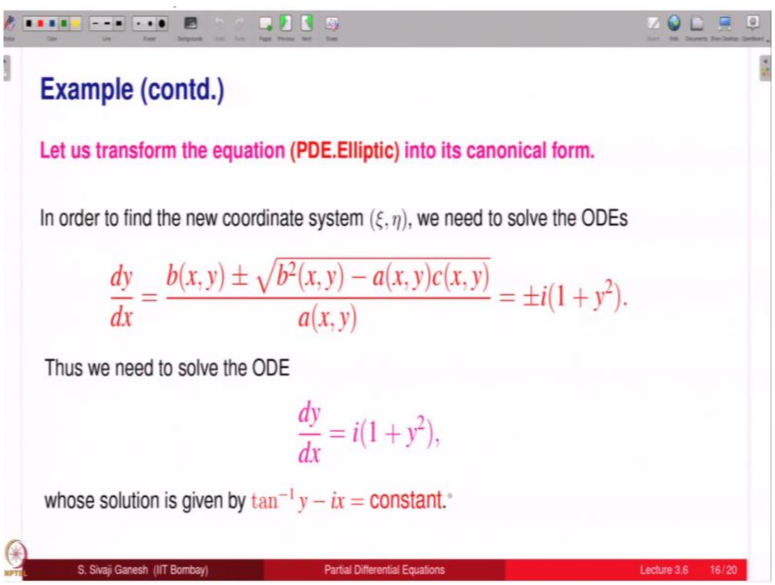

So, let us transform now, the given equation into its canonical form. So, we need to solve these ODEs dv by dx = + or - i into 1 + y square. So, we need to solve this ODE i times this because minus i times does not matter it will be the conjugate. So, this solution is given by tan inverse y- i x = constant. Please accept this that this is solution.

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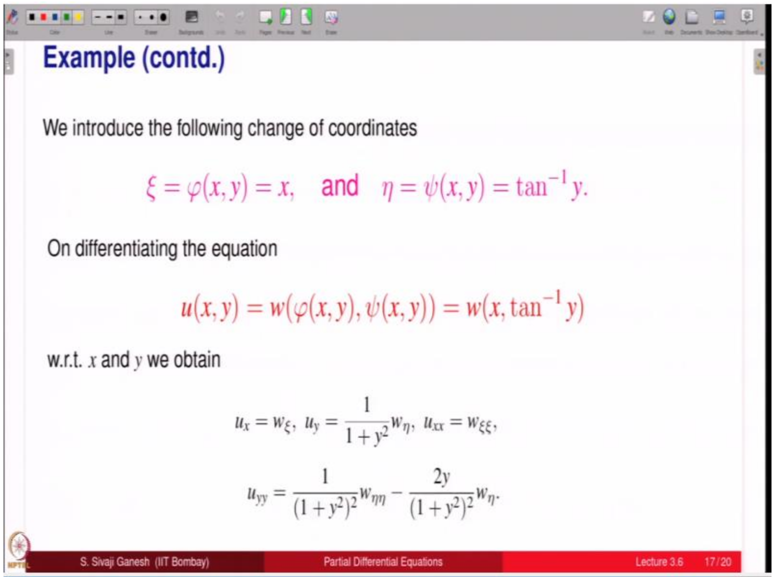

And then real part is x imaginary party tan inverse y are vice versa. Here of course, if this is constant, I can multiply this with i also again right. So, therefore, it does not matter what I call the variables phi and psi in the theorem that we presented this was called phi and this was called psi. But we are interchanging here. It does not matter because there is no preference for the variable x or y or psi are eta.

So, therefore, we propose you have x, y equal to this w of x, tan inverse y differentiate u x is x appears only here. So, it is w psi and derivative of x is 1, u y will be w eta and derivative of this is 1 by 1 + y square. So, you can continue like that.

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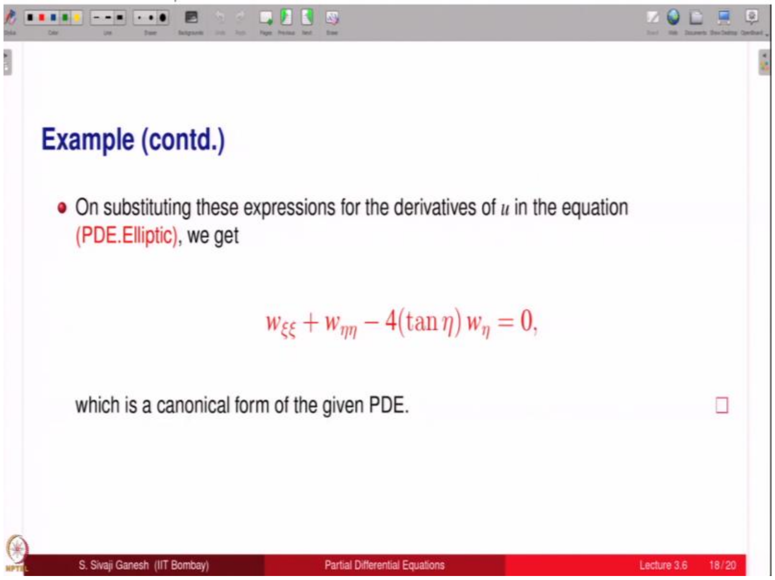

Compute the derivatives go back and substitute in the given equation we get this. So, that is the canonical form of the given PDE.

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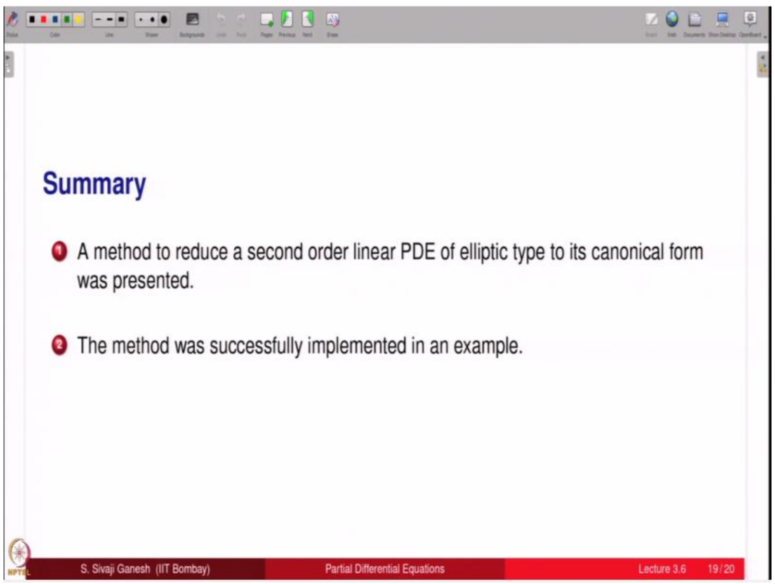

So, summarizing what we did is that we have presented a method to reduce a second order union PDE which is of elliptic type to its canonical form. We have of course assumed very high assumptions on the coefficient a, b, c. But you may generally, if you are given a partial differential equation of elliptic type, you want to find its canonical form, you may simply follow the procedure do not bother about the hypothesis checking for the theorem.

And you will still be successful, if you are able to solve the ODEs which are coming on the way. And we have seen the method you successfully implemented in the example. Thank you.