Partial Differential Equations
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Lecture – 22
Second Order Partial Differential Equations Canonical Form for an Equation of Parabolic Type

In this lecture we are going to see given an equation of parabolic type, how to get it into its canonical form. For a parabolic type equation, remember we want to model it after heat equation. How does heat equation look like? It is u t = u xx. So, in the second order part for example, let us write down.

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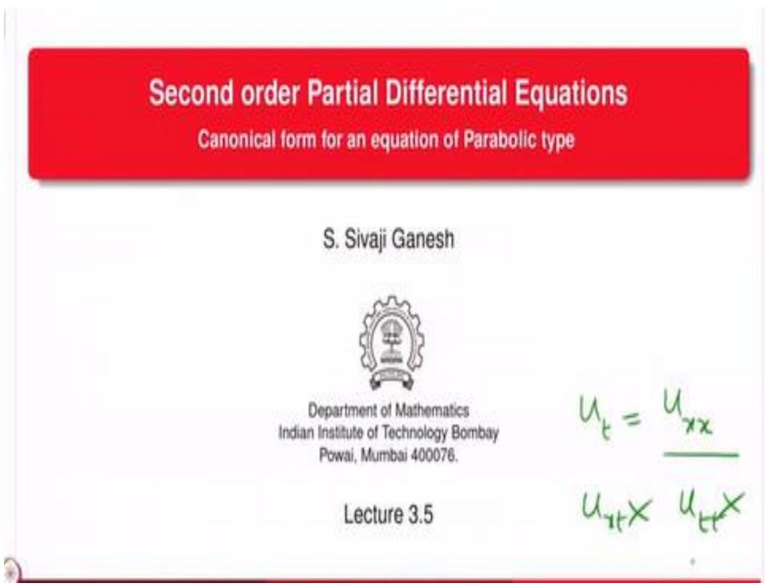

u t equal to u xx is the heat equation. In this if you look at the part where the second order partial letter will disappear that is simply u xx. In particular, u xt is not present and u tt is also not present. So, this is what we are planning to do given an equation a parabolic type equation, which is parabolic at every point in some domain, we want to change coordinates. So, that the transformed equation we will feature exactly one such second order partial derivative where it is with respect to same variable x and x. The other variable is completely absent.

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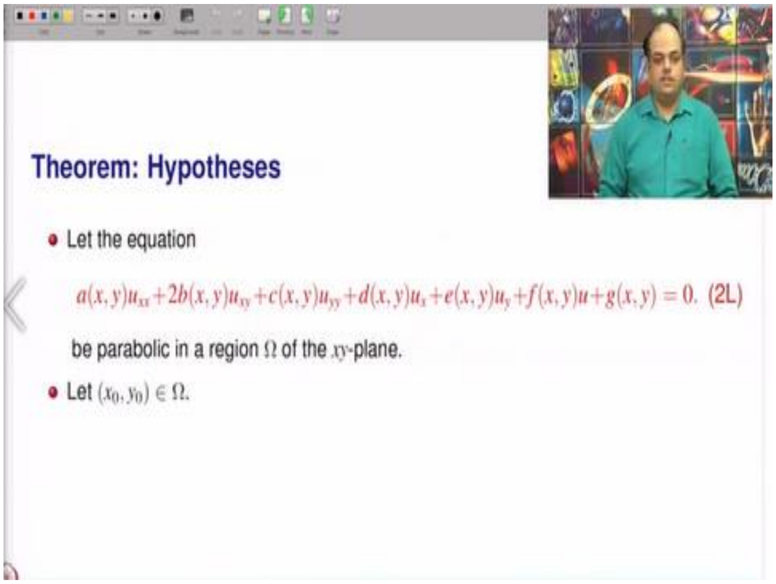

So, let us take the hypothesis which is required to do this there is not really any hypothesis here. It is very standard very simple and what is to be expected. So, let this equation be parabolic in a region omega of the xy plane, this is a second order linear equation. We assume these coefficients abc are at least continuous, we need that and take a point in omega.

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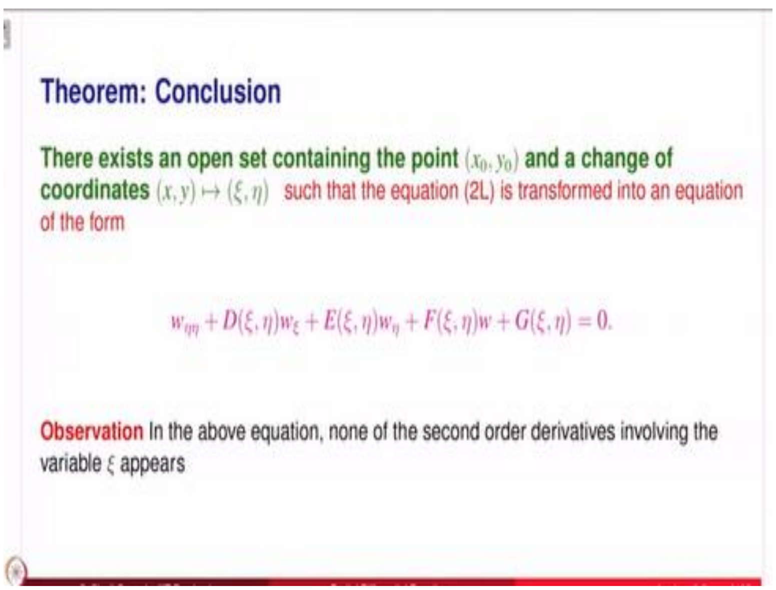

So, the theorem as before for the case of hyperbolic equations, it is going to be of local nature. That is why we take a point x 0, y 0 in the domain omega where the equation is parabolic. Conclusion is that there is an open set containing this point x 0, y 0 and the change of coordinates x, y going to psi eta and there is a new coordinate system. When the equation 2L is expressed in the coordinates, it will look like this.

If you notice here w eta eta, these are only second order partial derivative which appears. So, none of the second order derivatives involving the variables psi appears.

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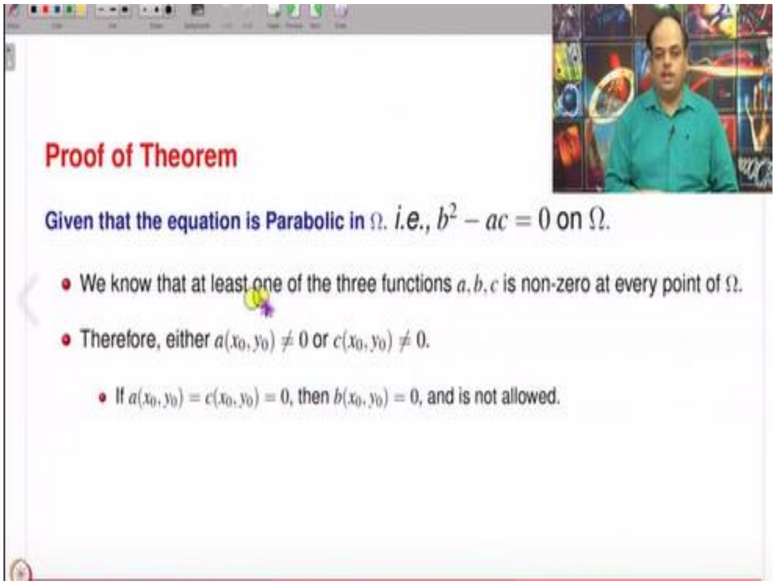

So, given that the equation is parabolic in omega. What does that mean? b square - ac is 0 in omega. So, to have a second order partial differential equation, we need that at least one of the functions a, b, c is not 0 at every point of omega. So, if you notice all of them cannot be simultaneously 0, we are not allowing such coefficient. Therefore, either a is not 0 or c is not 0. If it does not happen what will happen a and c both of them are 0 at x 0, y 0.

The moment ac 0 this equation tells b square is also 0 at that point that means b of x 0 y 0 is 0. It means all the three namely a of x 0 y 0, b of x 0 y 0 and c of x 0 y 0 are 0 and that is not allowed. Therefore, we have this, either this is not 0 or this is non 0.

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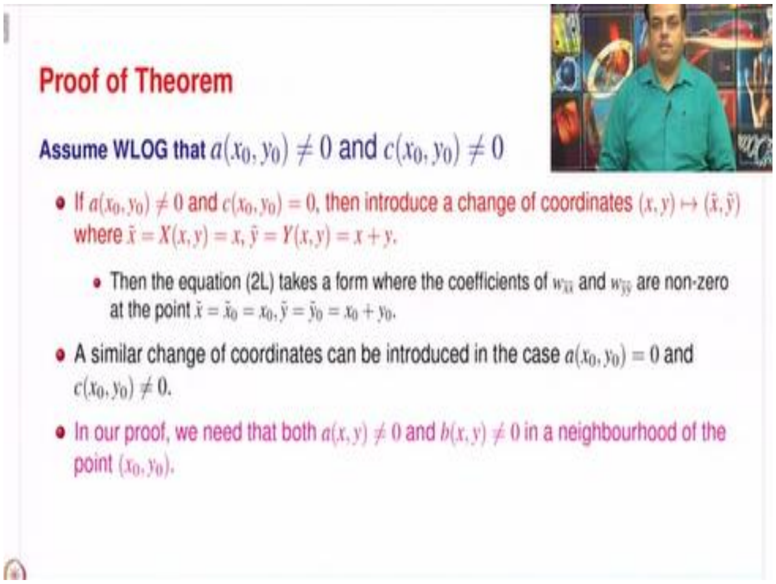

At least one of them happens. So, I assume without loss of generality that a is non 0 and here is a twist and c is not also not 0, we can arrange this. What we are concluding in the previous slide is that given any equation of parabolic type at any point in the domain x 0 y 0, at least one of the a or c is non 0. But now, we are saying that we can make it happen that both of them are non 0, but after a change of variables.

Imagine a is non 0 on c 0 at the point x 0 y 0. Then let us introduce a change of coordinates x, y going to x tilde or y tilde where x tilde is defined as a function of x, y is just X and y tilde as a function of x, y is x + y. So, then the equation 2L takes the form where the coefficients of w x tilde x tilde and w y tilde y tilde are non 0. At this point x tilde = x 0 + y tilde = x 0 + y 0. We have already introduced change of coordinates hand how a PDE changes under change of coordinates into a new equation.

Use those formulae and conclude this, very simple exercise. I will not be giving the details on this. So, therefore, we can assume that a is not 0 and c is not 0. So, there is one more case it might happen that a 0 and c is not 0 then also we can introduce this change of coordinate such that we have w x tilde x tilde and w y tilde y tilde their coefficients are non 0. But can be arranged.

Why are we doing this? Because in our proof we need that both a is non 0 and b is non 0. Because if a is non 0 it may happen that b is 0. What are the conditions for parabola city? We have is b square - ac 0. So, a is non 0, c may be 0 in which case b is 0, we do not want that in our proof. In our proof will not work we require that both a and b are non 0 in a neighbourhood of x 0, y 0.

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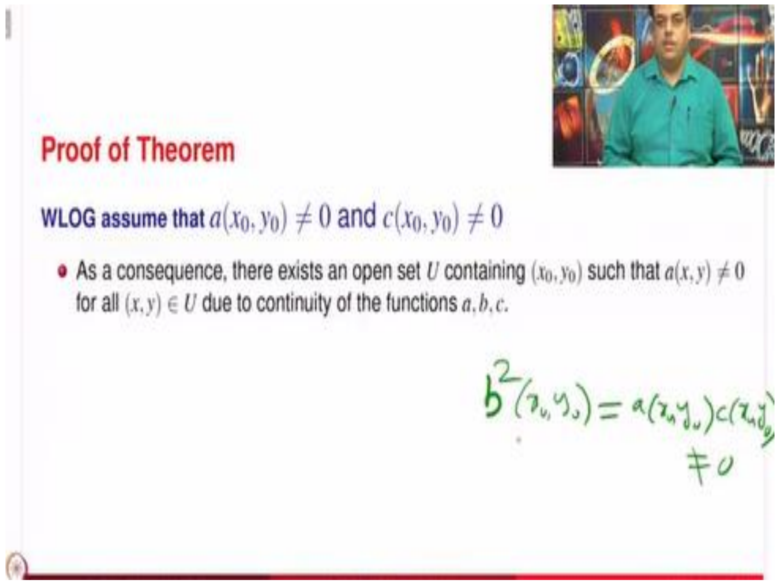

So, the moment we are now we decided we can there is no loss of generality in assuming this. After assuming this is a continuous function and it is not 0 at a point. Therefore, in an open set it will continue to be non 0 because of the continuity. Same thing will hold for c and for b also. If c is also not 0, what do we have? b square - ac is 0 a is not 0 and c is not 0. What we know is? This b square at x 0, y 0 b square - ac is 0.

Therefore, I can write this is equal to ac, this is not 0 that means b is also not 0. Once b is also not 0 at the point x 0, y 0 it will be non 0. Thereafter all three functions a, b, c so, we can take the same open set on which all of them are non 0.

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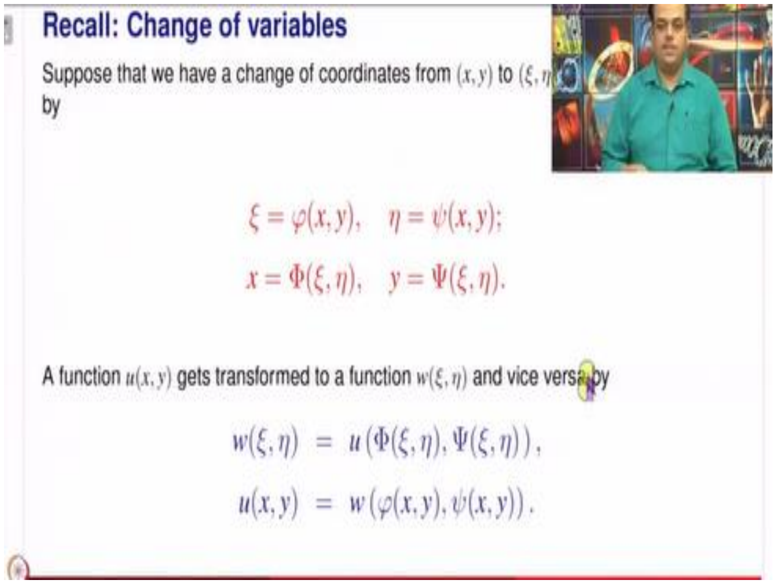

Recall the change of variables psi and eta by the functions phi x, y and psi x, y then here the inverse functions for that. And the function u x, y in terms of w is w of phi x, y psi x, y and w in terms of u is given by this equation.

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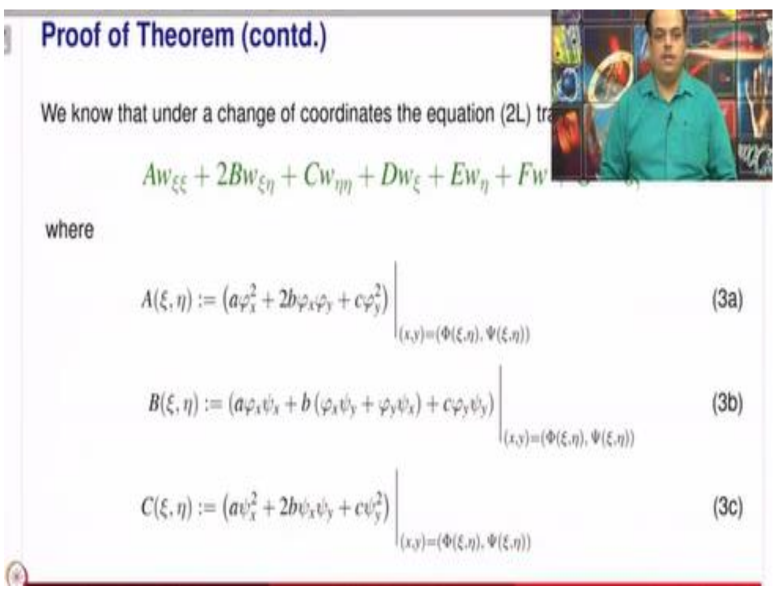

And we have seen that the 2L the second order linear equation and this change of coordinates becomes this where the coefficients A, B, C which matter for the type of equation are mentioned here. So, for proving a theorem what do we need? We need that only this should appear. So, C should be non 0, A and B must be 0. Suppose we choose the coordinate system such that A and B are 0.

Once A and B are 0 c cannot be 0, because if all of them are 0 there is no differential equation, we are not allowing that. So, A B 0 automatically means C is not 0, once C is not 0, you can divide this equation with C and get w eta eta with coefficient 1.

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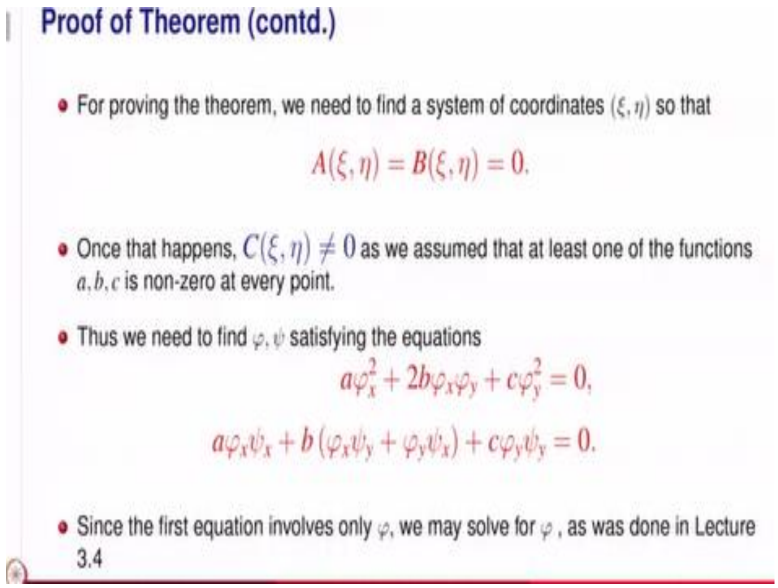

So, this is what we want A and B to be 0. And once that happens C is automatically non 0 because at least one of the three functions a, b, c are non 0 at every point. So, thus we need to find the functions phi and psi such that we satisfies this equation, this is basically a equal to 0 and phi satisfies this equation that is equation b = 0. If you observe this equation, the first equation involves only phi and second equation involves both of them.

So, we may say that there is a coupling between phi and psi, the system couples both of them but in some kind of weak coupling. Because one equation does not have psi at all, the other one has psi. But then once you know phi is an equation only for psi. This is something similar not exactly same, analogous to what we have in linear algebra. We have a linear system of equations which are triangular.

They are coupled but nicely coupled just like that. So, since the first equation involves only phi, we must solve for phi as we have done in the previous lecture.

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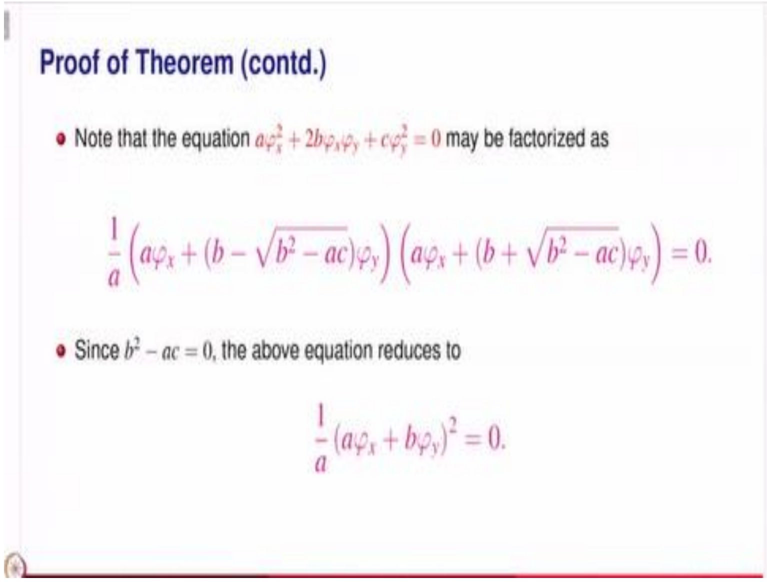

Exactly the same way we factorize. But now, what happens, b square - ac is 0. If once b square - ac 0, this factor in this bracket parenthesis is same as this one. So, this equation reduces to this. So, therefore we need to solve for phi such that a phi x + b phi y is 0.

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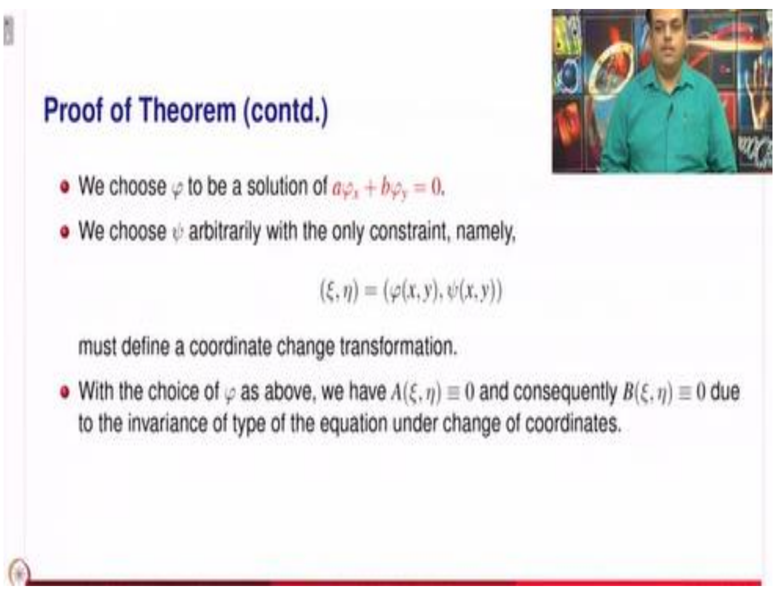

And choose psi arbitrarily. We do not want to go back and solve that equation; we will see in a moment why that is so? What all we want is a certain change of coordinate system, that is all we want. And if you notice, if a is 0, b is automatically 0. So, there is no need to check that that is satisfied, b square - ac is 0. So, if a is 0, then ac is 0, therefore b square is 0, therefore b must be 0. So, there is no need to check the equation for b.

Because b or psi eta equal to 0 is satisfied by any function psi. As long as phi satisfies the equation a of psi eta equal to 0. That is why we have lots of choices for psi, because the only constraint on psi now is that the Jacobian of phi and psi is non 0 which will give that psi equal to phi x, y eta equal to psi x, y we will define a coordinate chain transformation. So, we choose psi arbitrarily with only one constraint namely, this should give rise to a change of coordinates which means certain Jacobian non 0.

And with the choice of phi as above, we have A equals 0 and consequently B 0. Because the equation type is invariant under change of coordinates.

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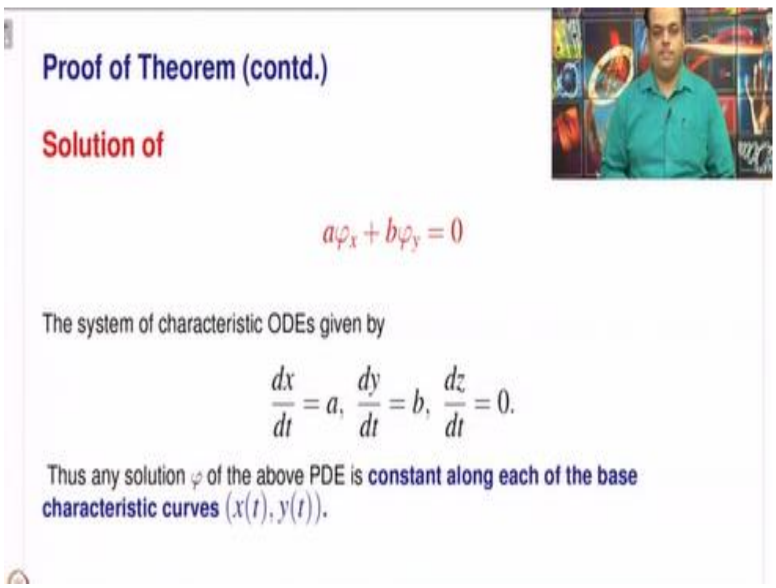

Let us look at the solutions of this equation a phi x + b phi y = 0, system of characteristic ODE is given by dx by dt = a, dv by dt = b, dz by dt = 0. So, this says that any solution of this PDE is constant along solutions of the base characteristic, solutions of this ODE. Along each base characteristic solution is going to be constant.

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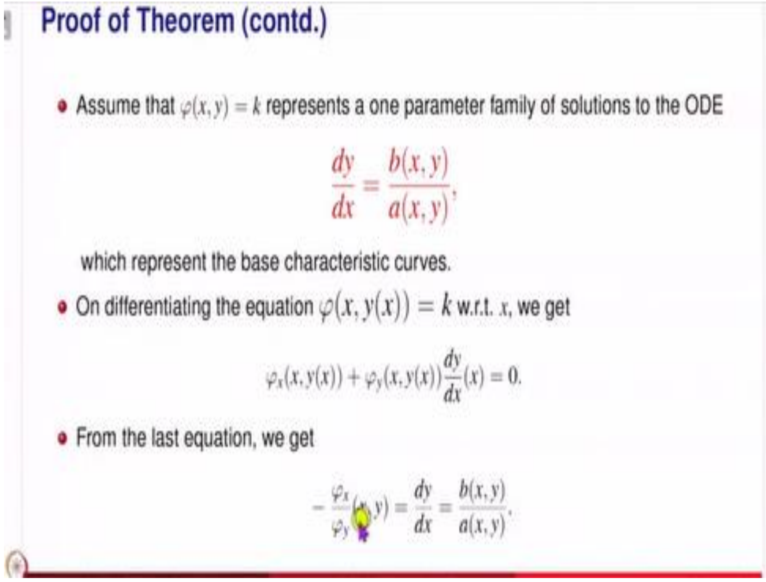

So, assume that phi of x, y = k represents a one parameter family of solutions to this ODE. Of course, this ODE represents base characteristic curves. On differentiating this equation phi of x, y at x = k, we get a relation. This is by chain rule we get this identity and then we get dv by dx = b by a and other hand we get minus phi x by phi y.

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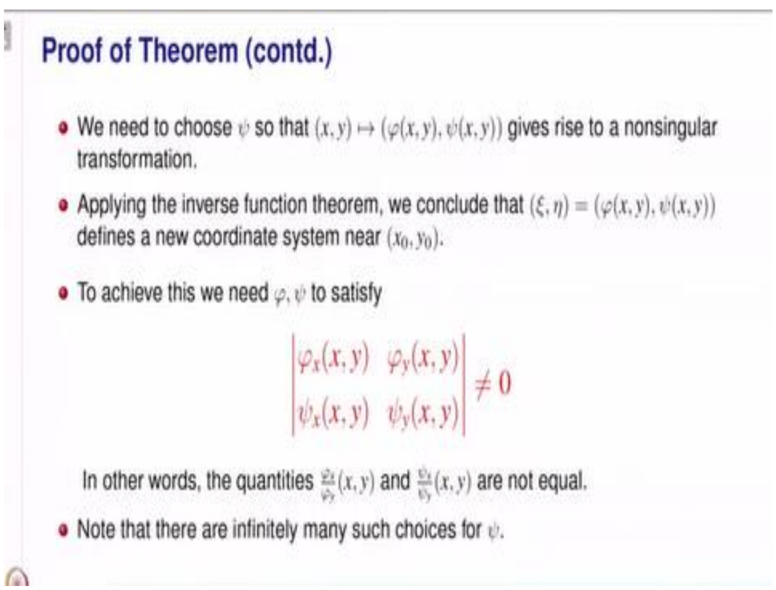

Now we need to choose psi, such that this transformation x, y goes into phi x, y psi x, y is a non singular transformation namely the Jacobian is not equal to 0 at the point x 0, y 0. We are looking at only x 0, y 0. So, applying the inverse function theorem, we conclude that psi eta equal to phi x, y psi x, y defines a new coordinate system near x 0, y 0. We are here to find psi, such that this condition is satisfied non singularity of this transformation that rest will follow.

Now, to achieve this phi and psi need to satisfy this Jacobian to be non 0. Then you can apply inverse function theorem right if the Jacobian is non 0 at x naught y naught then you can invert the transformation which is coming here x, y going to phi x, y psi x, y can be inverted back. So, psi eta is going to capital phi of psi eta and capital eta of psi eta we can do that, sorry capital psi of psi eta. So, what do you mean by this is equals 0 or not equals 0?

It just means that this quantity phi x by phi y and psi x by psi y are not equal. Because this is 0 means they are not equal to 0 means they are not equal. So, we have to find psi. Now, we will use this condition this is not equal to phi x by phi y, there are infinitely many such choices.

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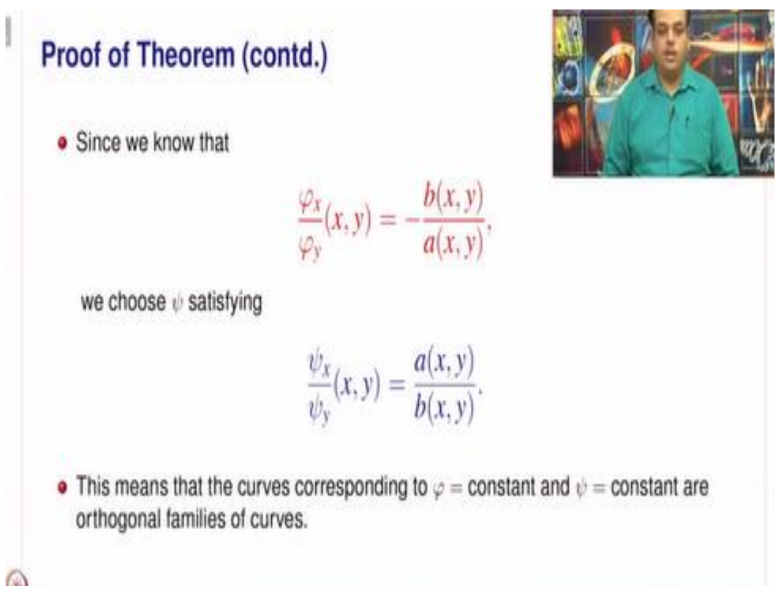

But we make a nice choice nice looking choice for theoretical purpose. When we are dealing with an example, we may not necessarily do like this. So, this is what we know phi x by phi y. So, therefore, we choose psi says that psi x by psi y is a by b. If you look multiply these two things you get minus one, minus b by a into b by b is minus one. So, this means that the curves corresponding to phi equal to constant and psi equal to constant are orthogonal families of course.

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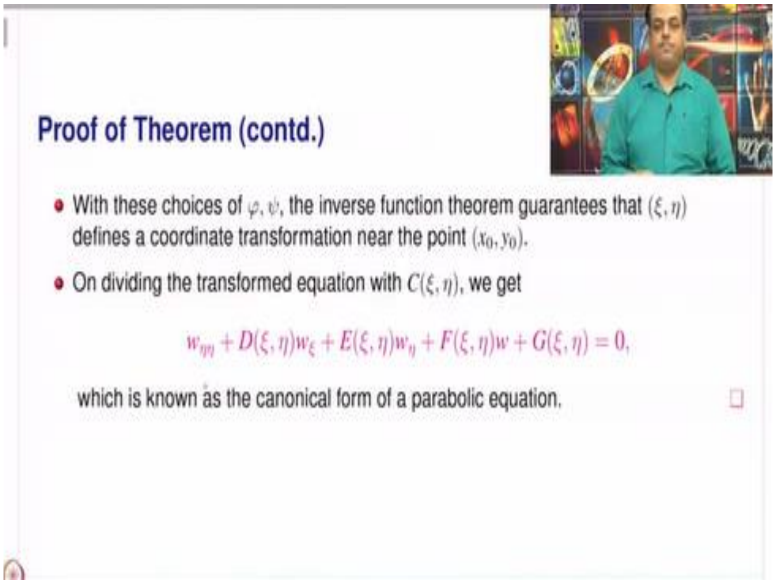

So, with these choices of phi and psi inverse function theorem guarantees that psi eta defines a coordinate transformation near the point x 0, y 0. And therefore, the a and b are 0, C is definitely non 0 C cannot be 0. Now we will divide the equation with C, you get what you want, w eta eta plus these terms do not matter what they are, this alone matters. And this is known as a canonical form for a parabolic equation.

So, canonical form for a parabolic equation means the second order derivatives exactly one of them appears and one variable is missing which means it should be w eta eta and w eta psi, w psi psi do not appear.

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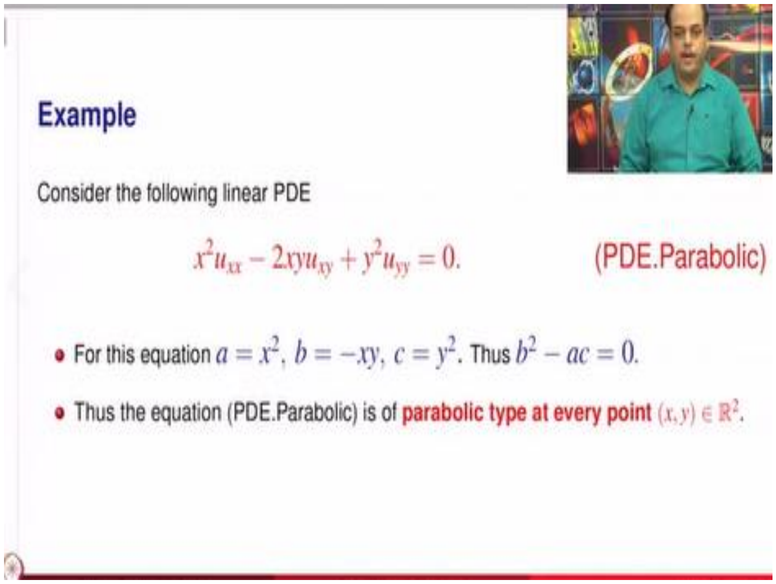

Let us look at an example. Here of course, all the derivatives of u are appearing is not here. So, we have to find what is the type of this equation, a is x square, b is minus x y, c is y square. So, therefore, bc - a square is 0. So, the equation is parabolic at every point in R 2 parabolic everywhere in R 2.

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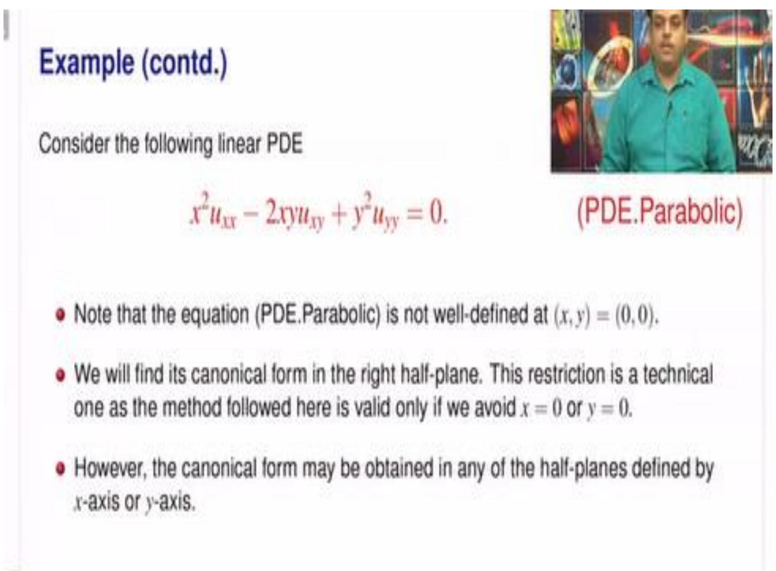

This equation is not well defined at 0, 0 x, y equals 0, 0. Because, all a, b, c vanish. Therefore, we do not want to include such a point in our consideration. So, we will find its canonical form in the right half plane and this restriction is a technical one as I pointed out. Because a method followed here is valid only if we avoid x equals 0 or y equals 0. So, canonical form can be found in any of the half planes not necessarily right half plane, you can do the left half plane or you can do upper half plane or lower half plane.

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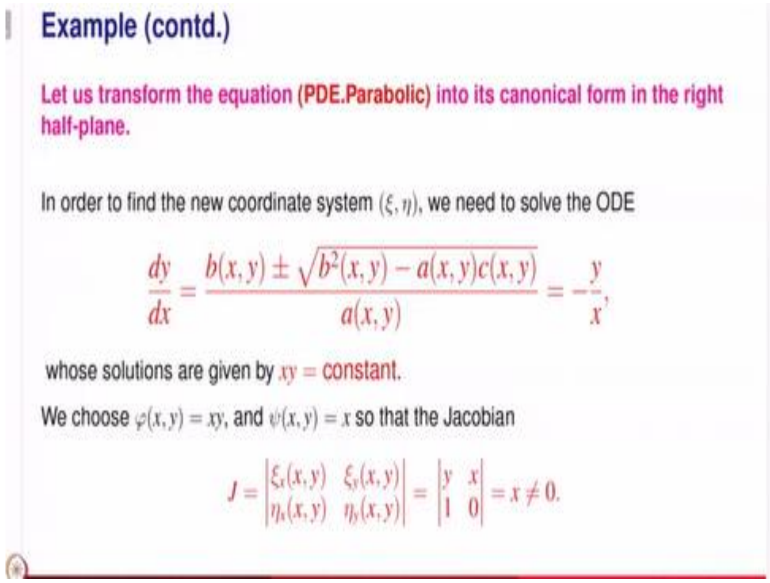

So, let us transform the equation into its canonical form in the right half plane. So, we need to find a new coordinate system. For that we have to solve this will give us phi, dy by dx = - y by x will give us phi half x, y and then we have to make a choice for psi. That is a procedure in the parabolic equations. So, solutions are very easy here. So, they are given by xy equal to constant. So, we need to choose psi such that Jacobian is non 0.

Once you plug in the values of y in this, this should be phi actually not psi and eta but phi, phi x phi y. So, here it will be y here it will be x so, y and x are there you need to fill with somebody. So, that Jacobian is non 0 I have chosen it to be x. Because I will get this and x is not 0 in my domain, in the right half plane x is never 0.

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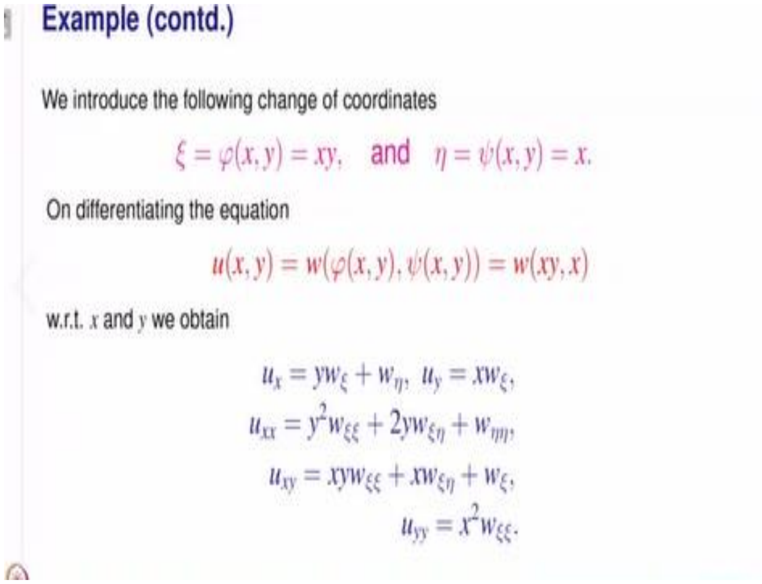

So, we introduce this change of coordinates. On differentiating this equation u x, y equals to w of xy, x and compute the derivatives and then go back and substitute in the given equation. So, u x u y compute like this, u xx is this and u xy is this u yy is this. Please do the computations on your own, pause the video, do the computations.

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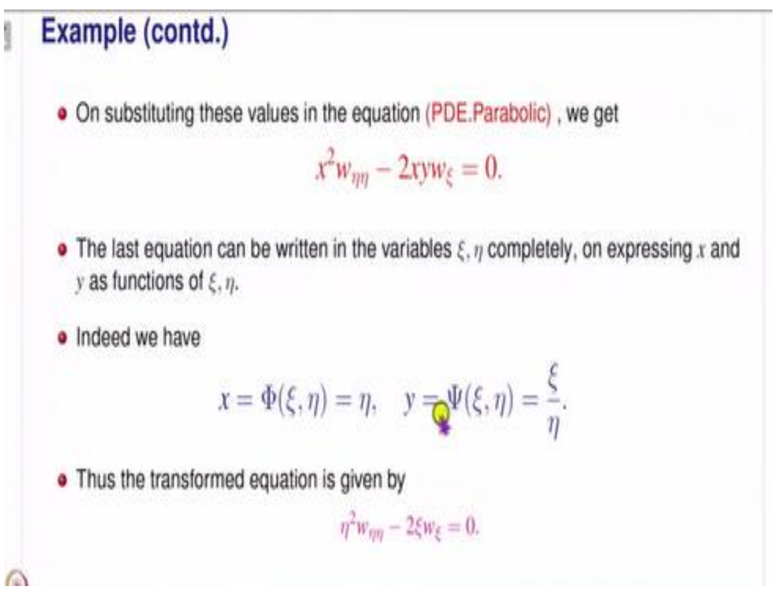

And substituting these values in the given equation we will get this of course, this equation still has x and y. So, it can be written in completely in terms of psi eta expressing x and y as functions of psi eta that is nothing but writing the inverse function. So, x = capital phi psi eta is eta, y =

capital psi psi eta is psi by eta. This transformation makes sense because it is never 0 in our domain it is 0 if and only if x is 0 and we are working in a domain where x is non 0.

So, therefore the transform the equation will be this and we want this to be one so, divide with eta square.

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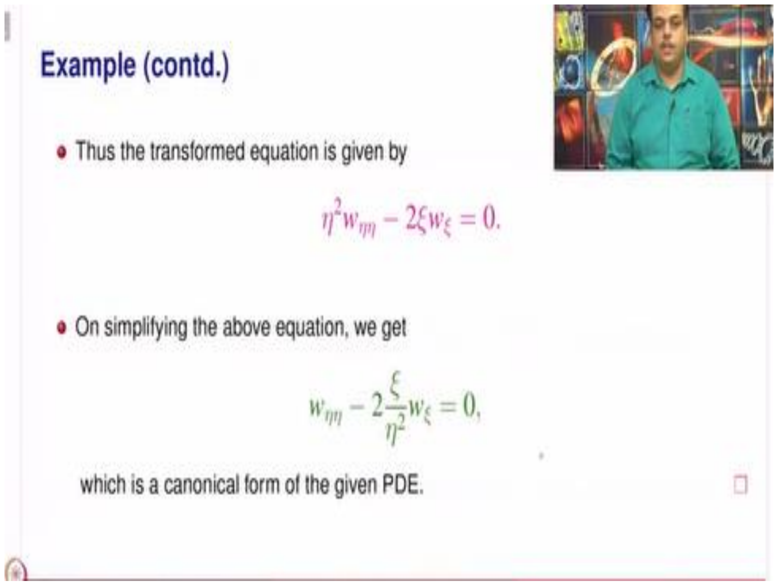

So, we get this expression. So, this is the canonical form of the given PDE. Notice only w eta eta appears, w eta psi and w psi psi do not appear. That is how the canonical form is identified for parabolic equation.

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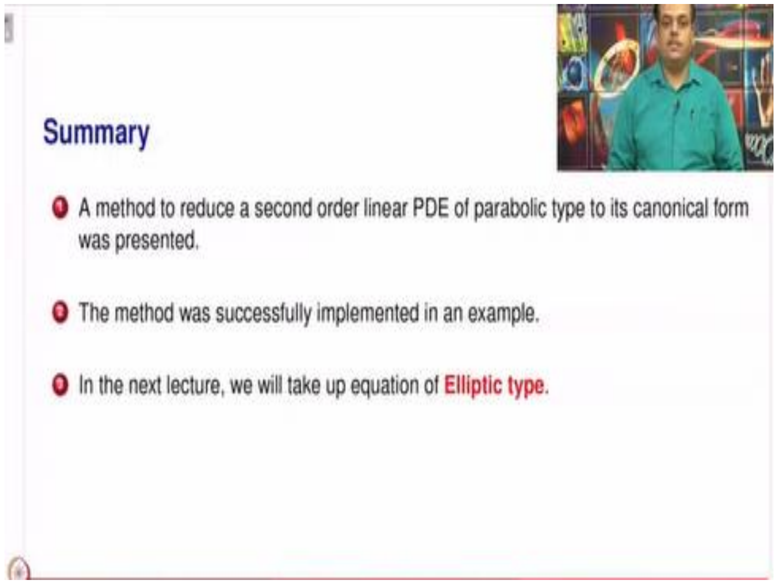

So, what we did today is we have devised a method to reduce second order linear PDE which is a parabolic type to its canonical form and it was implemented successfully in an example. So, in the next lecture we will take an equation of elliptic type and get the canonical form for such equations. Thank you.