Partial Differential Equations
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Lecture – 2.1
First Order Partial Differential Equations
How They Arise? Cauchy Problems, IVBPs, IBVPs

Today, we are going to start the second chapter. It is called first order partial differential equations. In this chapter we will study first order partial differential equations and Cauchy problem associated to that and what are the possibilities in terms of existence uniqueness and maybe non existence as well, we will discuss. Now in today's lecture, what we are going to discuss is how the first order PDEs appear, how they arise and the Cauchy problem, initial value problem and initial boundary value problems, what are they we will discuss briefly.

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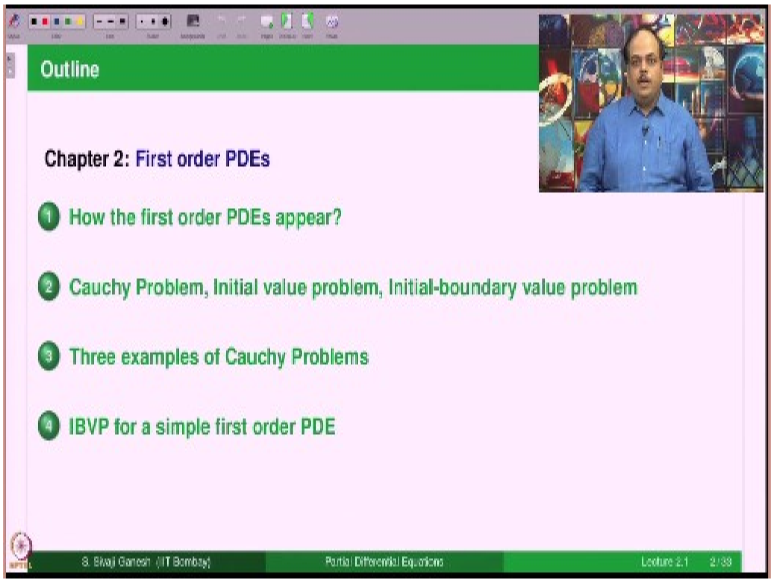

So, outline for today's lecture is, the first point is how the first order PDEs appear. We will see two situations one of them is going to be a physical situation and the other one is within mathematics mathematical situation. And then we define these terms Cauchy problem, initial value problem, initial boundary value problem for first order PDEs. We look at three examples which suggest us what happens or what can happen to a Cauchy problem in terms of existence of solutions, uniqueness.

And then we look at a simple first order PDE for which we consider IBVP that is initial boundary value problem and we study what are the possibilities there.

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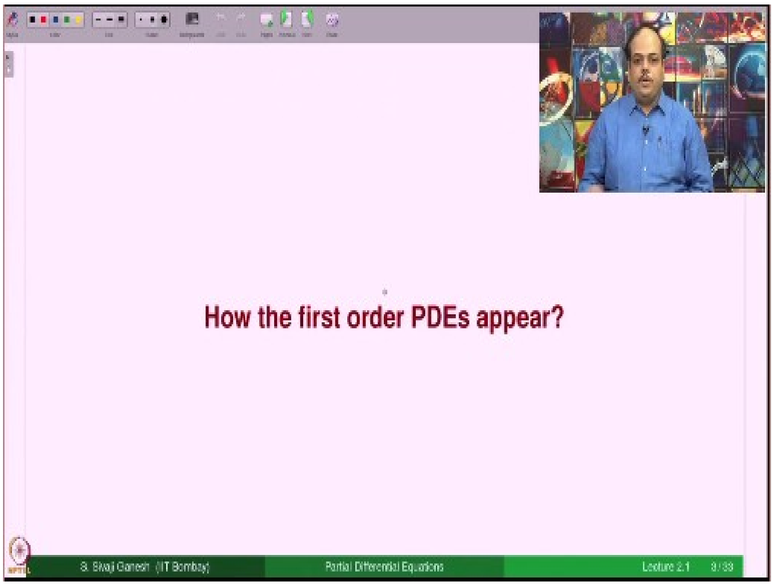

Now, how the first order PDEs is appear.

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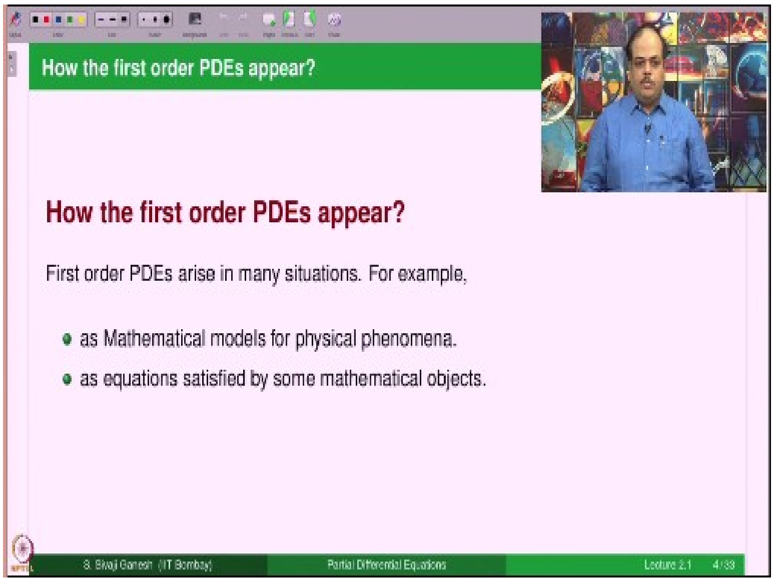

So, first order PDEs arise in many situations. We are going to just see two instances the first one being as a mathematical model for physical phenomena. In fact, what we are going to see is not really a physical phenomena, it just models the traffic on a highway. And then we see as equation satisfied by some mathematical objects.

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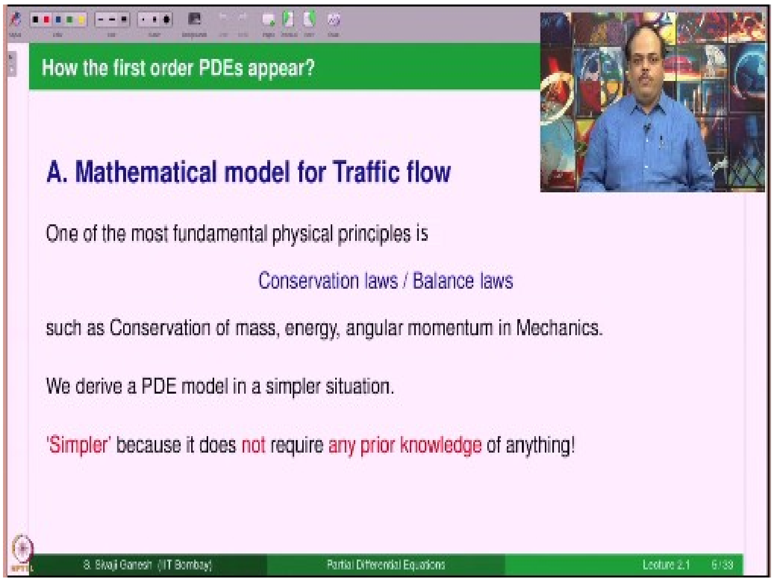

So, this is the physical model, which we are going to look at today is a mathematical model for traffic flow. So, in this model, we are going to have two main ingredients we will briefly introduce them. So, one of the most fundamental physical principles means what are called conservation laws or Balance laws. And one encounters them already in course of mechanics, where we would have seen conservation of mass, energy and angular momentum.

So, we are going to derive a PDE model in a simpler situation. We call it simpler because no apriori knowledge of anything is needed, including physics. As I told you, we are not going to do really a physical model, it is going to actually model of the traffic flow, which can be easily understood.

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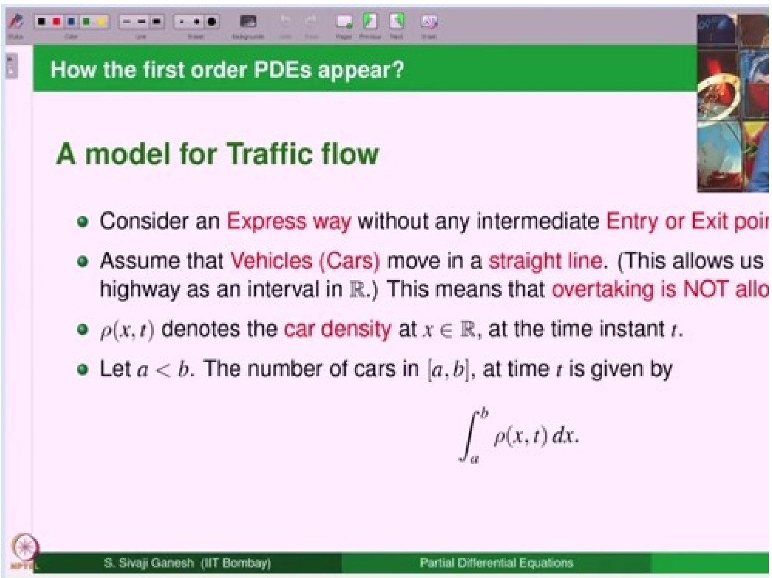

So let us consider an express highway without any intermediate entry or exit points. It has one entry point, one exit point. You cannot join the expressway in between or you can leave, both possibilities are not there. Leaving and joining is not possible. So the traffic what we mean is, let us say vehicles, let us say cars, just to refer very easily. Assume that vehicles are moving in a straight line.

As we know vehicles do not move straight lines, they will be overtaking one another. But what this means is that this gives us a way to identify the highway as an interval in R. And it actually physically means that what we are taking is not allowed, maybe the expressway is very narrow, so you cannot overtake. Now, let rho denote the car density at a point x in R. Yes the road is not infinite.

But then we are going to finally specialise to certain intervals so at the time instant t. What does this mean? Car density means what? We can get the number of cars by integrating with respect to x. So let a be less than b, then the number of cars in the interval a b in terms of the highway, it may be a stretch between point a and point b at time t is given by the integral of rho over the interval a b.

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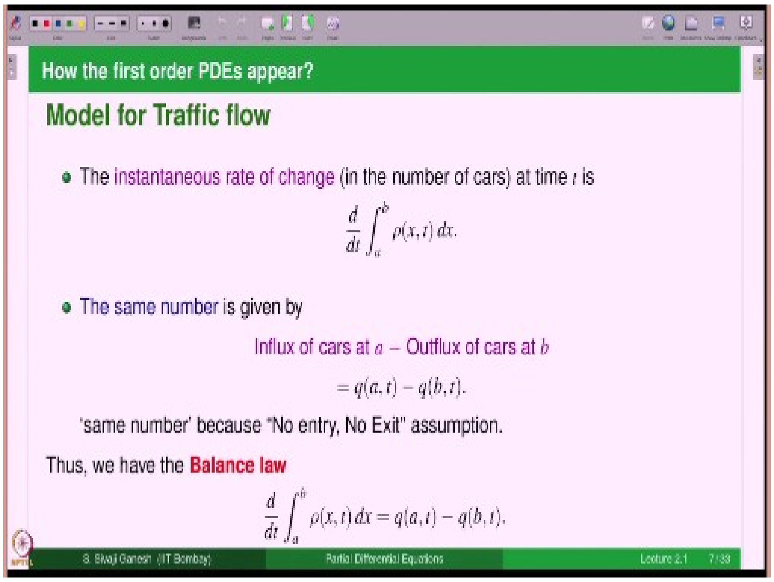

Now we need to know how it changes. So the instantaneous rate of change at time t is given by the derivative of this number d by dt of the integral a to b rho x, t dx. But who is really responsible for this change? It is the cars which are entering at the point a and leaving from the point b. Therefore, the same number is given by the influx of cars at a minus outflux of cars at b, which is q a, t minus q b, t.

Both are same numbers because we made this assumption of no entry and no exit. Thus, we have this Balance law, so both these quantities are balanced. Therefore, d by dt of integral a to b rho x, t is equal to q a, t minus q b, t. That is a balance.

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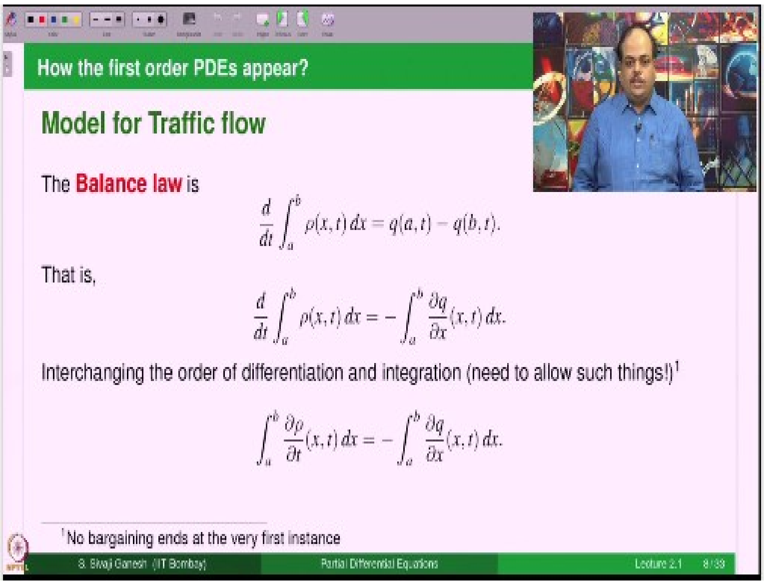

So, the Balance law is now d by dt a to b rho x dy dx equal to q a, t minus q b, t. Now, we are going to modify the right hand side. The right hand side we write it as, as an integral. So, if you use this fundamental theorem of calculus will give you q a, t minus q b, t. So, interchanging the order of differentiation and integration in the first term d by dt is outside the integral we can take it inside.

Yeah, of course, we know that this is not true all the time. It is not allowed. One needs justifications, but this is where I say no bargaining entered the very first instance. So, do not ask the questions right now, they can be asked later. So, now, once the d by dt go inside the integral, the Balance law becomes a to b dou rho by dou t equal to minus integral a to b dou q by dou x.

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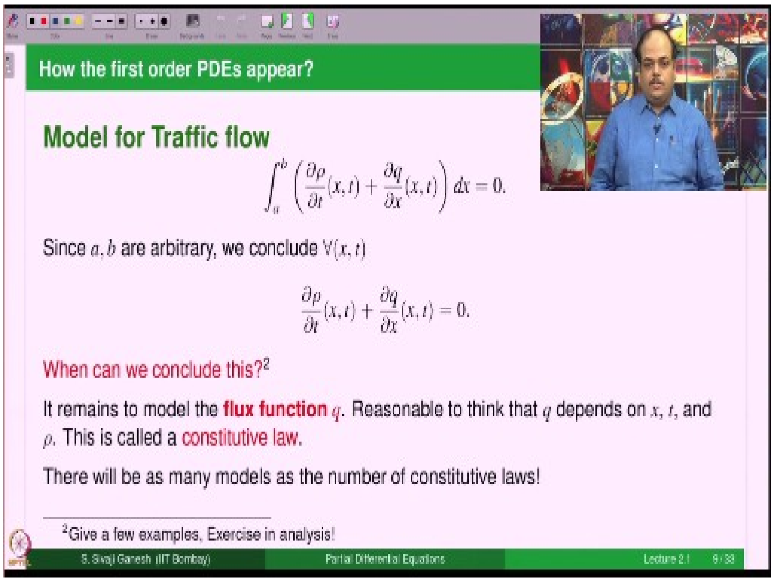

Now, let us take the RHS to the left hand side and then the Balance law becomes integral a to b of this quantity dou rho by dou t plus dou q by dou x equal to zero. Now, this is true for every a and b, such that a less than b. In other words, you have integral of certain quantity zero over every interval a b, it means that integrand has to be zero. This is an exercise in analysis, we would have encountered many such situations.

For example, if you take a continuous function integrate on any interval a b and if the result is zero, then the continuous function has to be zero. And there are generalisations of this, which may be let us say, local integral function. If you integrate on any set and the integral is zero, then the function must be zero. So, these are two typical instances where we can conclude this.

Now, what remains is the equation has a rho that is what is the traffic about. We want to study about this, but there is a unknown quantity q, the flux. It is not known. Therefore, we need to model that. Therefore, it is reasonable to think that q depends on x, time t and the density of cars itself that is rho. So q can be thought can be thought of as a function of x, t and rho.

This kind of proposition of what kind of a function it is of the variables x, t and rho is what is called a constitutive law. Of course, there will be as many models as the number of constitutive laws.

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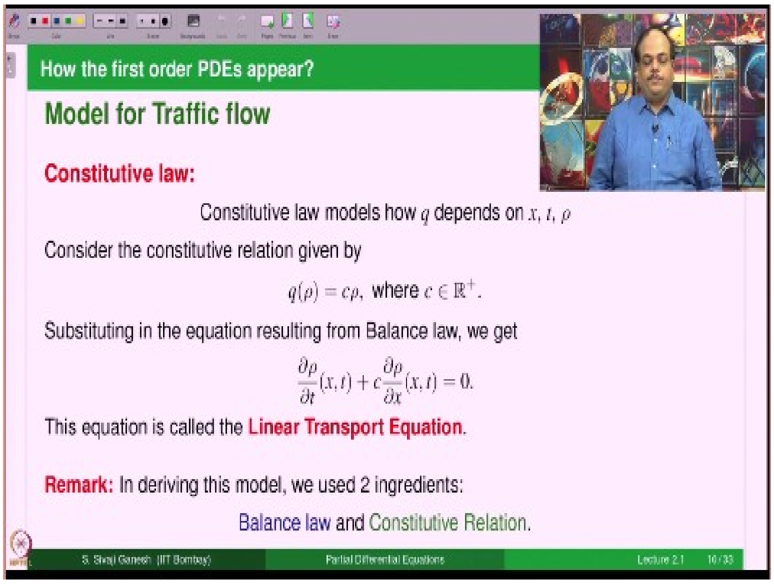

So, constitutive law models how q depends on x, t, rho. Consider a simplest constitutive law which is given by q of x, t rho. Since x, t is not involved in this formula, I have not written x t. So, q of rho equal to c rho, where c is a positive number. This means the flux is proportional to the number of cars. So, substituting this in the Balance law, what we get is dou rho by dou t plus c times dou rho by dou x equal to zero.

This equation is called a linear transport equation. This is a linear equation is clear, because the unknown quantity rho. Rho itself does not appear what appears, it is their first order derivative dou rho by dou t and dou rho by dou x, it appears with the power one. So, this is a first order linear equation. c is a constant. Therefore, It is a linear equation. Why is it called transport equation?

Something must be transported from somewhere to somewhere. It must be from some time t equal something to future times. So, in deriving this model, recall that there were two main ingredients, first one was Balance law and second one is a constitutive law or constitutive relation.

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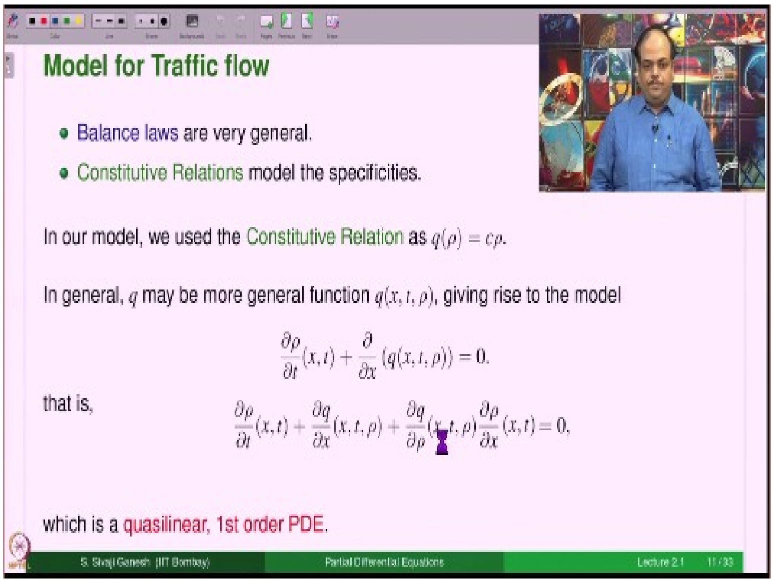

Balance laws are very general. Constitutive relations model the specificness of the situation in this case of the highway. So, in our model we use the constitutive relation as q rho equal to c rho. In general q may be more general function of x t and rho. In that case, the Balance law becomes this dou rho by dou t plus dou by dou x of q of x, t, rho. If we expand this, you will get this one.

So, you need to differentiate x appears here as well as in rho, therefore, need to differentiate q with respect the first variable and differentiate q with respect to the last variable and differentiate rho with respect to x, this is a chain rule. So, a chain rule is a very important rule that one must be very comfortable when you want to do anything with the differentiation.

So, this is a quasilinear equation, It is a first order PDE. Quasilinear because this may depend on rho. That is why these are quasilinear equation.

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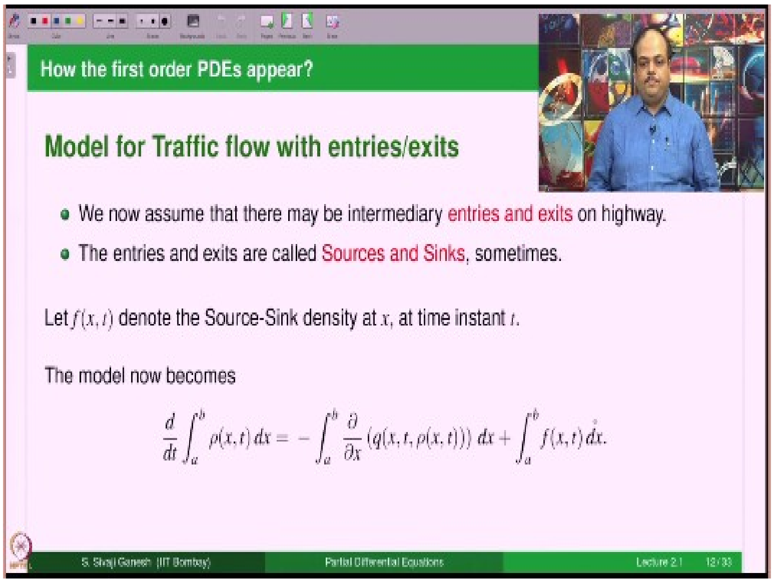

Now, we are going to model a highway where entries and exits are there in between. That means, we will see now how the earlier model derived is going to change. So, the entries and exits are also called source and sinks in the PDE literature. So, let f denote the sourcing density at x at the time instant t. Now, the model becomes this is the rate of change, earlier this was just equal to the flux. But now, this will also have the sourcing term, which is this integral a to b f(x, t)dx.

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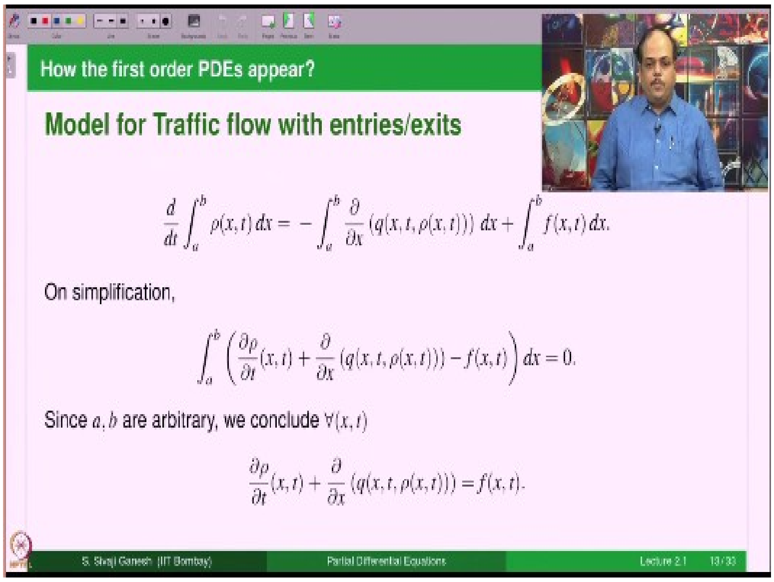

Now bringing everything to one side, what we get is integral a to b of certain thing is zero. And this is true once again as before for every a and b that a is less than b both are orbitrary. Therefore, the integrand must be zero which will be the case whenever the integrand is continuous or local integrable. But It is not unreasonable to assume that integral of over any set is zero implies the integrand is zero.

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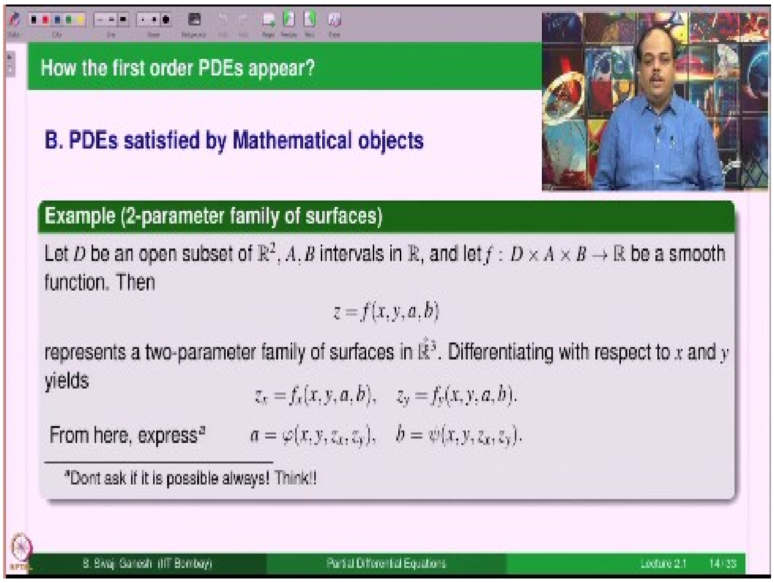

Now, let us look at the how the first order PDEs appear in a mathematical context. So, we are going to see an example. Consider a two parameter family of surfaces. Surfaces in fact, what we are going to consider is surfaces which are graphs. z equal to f of(x, y, a, b). What are a and b? They are some parameters wearing in some intervals. So It is a two parameter family of graphs or functions.

Now, we want to derive a differential equation satisfied by this family and we are saying partial differential equation. Therefore, what we can only do here is to differentiate this with respect to the two independent variables x and y. And this is the expression we get. Now assume that from these two relations, you can solve a as a function phi of x, y, z x, z y and the b as a function of shi x, y, z x, z y. Now, question is, is it always possible these are all questions.

But here if it is possible, we will go ahead and do the next step. What is the next step? I am going to substitute for a and b these formulas inside this. Then what I will have is z equal to f of x y phi of x, y, z x, z y comma shi of x, y, z x, z y which will be a nonlinear partial differential equation.

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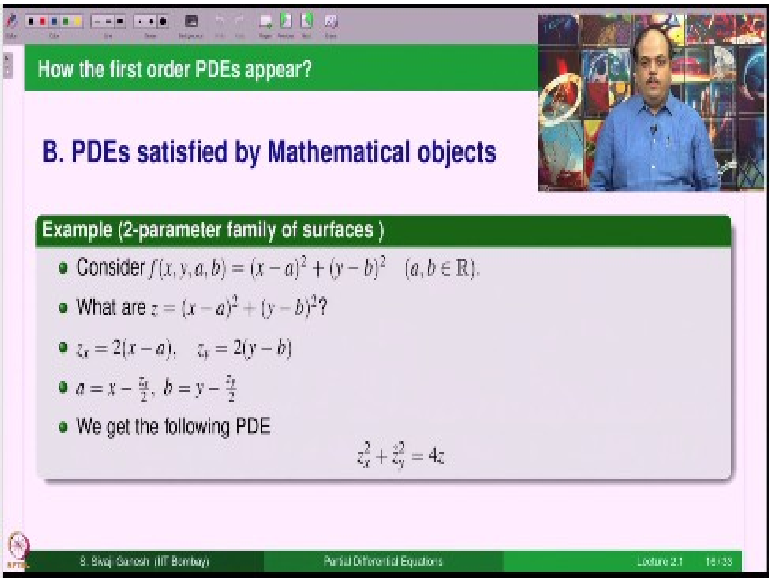

In general Now, let us look at the next specific example where the functions taken to be x – a whole square + y – b whole square, a and b are parameters wearing in real numbers, both of them are real numbers. Now, what are these surfaces, what does this equation represent? Represents cones family of cones. So differentiate this relation with respect to x and y and eliminate a and b. In other words, solve for a and b in terms of rest of the things.

So here it is very simple a = x minus z x by 2, b = y minus z y by 2. Now substitute the values of a and b inside this expression z = x – a whole square + y – b whole square, which is the equation of the family of surfaces. Once you do that and simplify, you get this following PDE. z x actually stands for dou z by dou x. If you want write in terms of x you can write dou u by dou x whole square + dou u by dou y whole square 4z.

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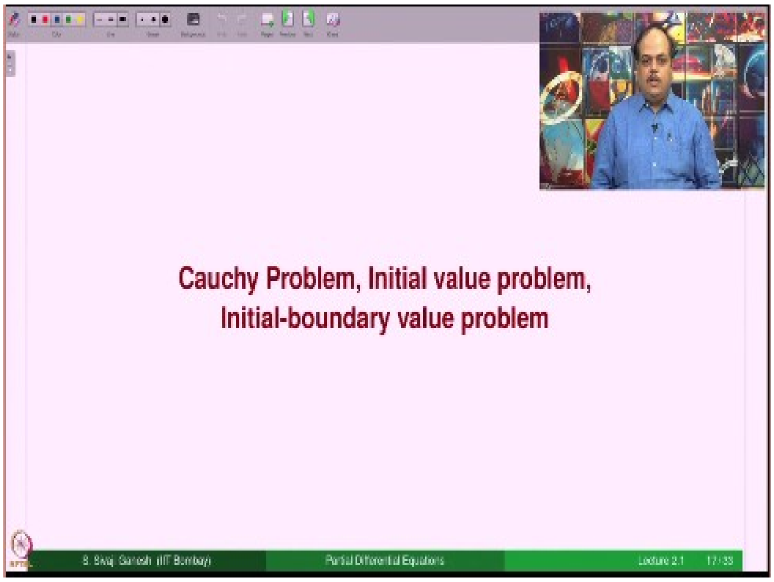

Now, let us see what are the problems by the Cauch code by the name of Cauchy, our initial value, our initial-boundary value problems, what they are.

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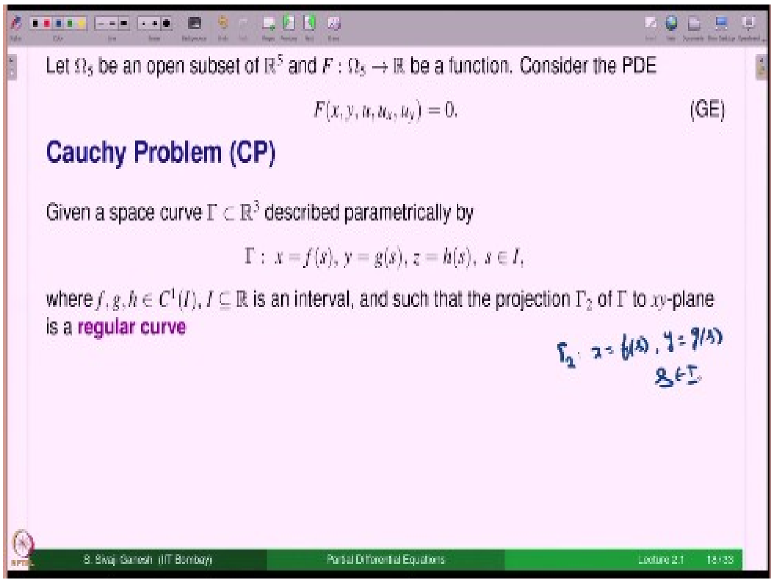

So let us consider a PDE F of x, y, u, u x, u y = 0. It means I am considering a PDE in 2 independent variables. Throughout this chapter, we are going to consider first order PDE in 2 independent variables. And then we will also say what how theory changes if you have one more variable. So, x y and Rare independent variables, u is the dependent variable, these are the derivatives. So, there are five quantities here.

So, F must be defined on a subset of R^5 and that we denote as omega 5, is a subset of R^5s and have function and then this will then define a partial differential equation. So, what is the Cauchy problem? Given a space curve gamma in R^3 that means you are given a curve in R^3 which is described parametrically by this. That is x y z = f(s) g (s) and h(s) , as s varies in some interval I. And f, g, h are C 1 functions on the interval I.

And such that the projection gamma 2 of gamma to xy – plane. Where is gamma? It is in R^3. So, it will have xyz coordinates. Now, you project it to the xy – plane you will have xy coordinates. That is x equal to f(s) and y equal to g (s). That is the curve gamma 2, s in same I. It is a regular curve.

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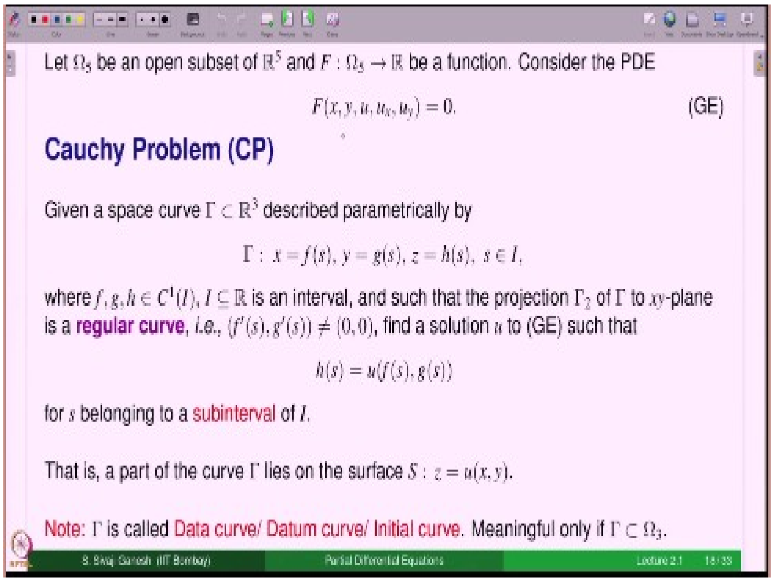

What that means is f dash g dash do not vanish that means, it does not become zero zero at every point of the curve. If one of them is zero the other one is nonzero. If f dash is zero, g dash is nonzero. And find a solution to the PDE such that u of f (s),g (s)= h(s)for s belonging to a sub interval of I. In other words, a part of gamma lies on the surface s denoted by S: equation is set of all x y z so that z equal to u x, y.

On this surface a part of this curve given curve gamma lies. Gamma coordinates are f(s), g (s) and h(s). A third coordinate h s must be equal to u of first 2 coordinates f(s), g(s). That is a condition here. We are not requiring that this should happen for every s in I. We are only asking for s belonging to sub interval of I. So, this can be thought of as a local solution. So, this gamma is called a data curve or datum curve or initial curve.

Note that this problem actually makes sense only if gamma is subset of omega 3. What is omega 3 is a projection of omega 5 which is on R^5 to the first three coordinates x, y, u.

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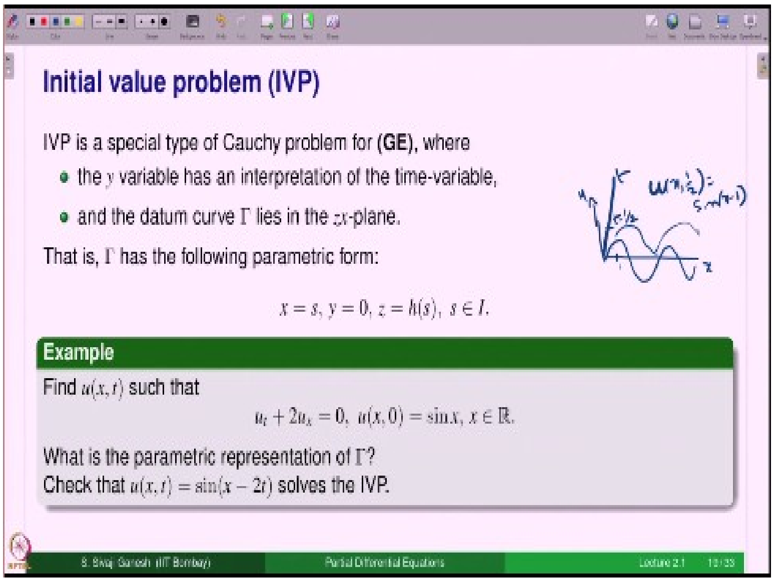

What is initial value problem? It is a special type of Cauchy problem. Initial means that there is something like a time in the problem and initially means at some particular time in the past. Something is happening and then you are interested in studying the equation for future times. So, therefore, one of the variables will have an interpretation of time, let it be y. There is no loss of generality in assuming y. x would equally do the same thing.

So, the y variable has an interpretation of the time variable and the datum curve now, lies in the zx – plane given by this. $x = s$, $y = 0$, $z = h(s)$. In other words, what we will be asking is actually u of x comma zero equal to h of x should be satisfied. This is what we are asking for the question, for the solution of the PDE. So, find u of(x,t) such that $u_t + 2u_x = 0$ and $u(x, 0) = \sin x$. Parametric representation of gamma, what is gamma?

$x = s$, $y = 0$, here $t = 0$. It is not right. It should be t because y has an interpretation of time variable so you can as well write t, so $t = 0$ and z equal to sin s and s belongs to R. That is a parametric representation of gamma. Now, we do not know how to solve this for now, but given a formula we can always check that it is a solution. So you have x t equals sin of $x - 2t$. If you quickly differentiate, you will get that $u_t + 2x = 0$. It solves the initial value problem.

So when $t = 0$, this formula u of(x, 0), what will we get? u of x, zero will be sine x. So, initial condition is satisfied. So we can think of like this. So, this is the time variable, this is the u variable, this is the time, x. For $t = 0$, you are given a sine x like that. So, you have to imagine this in the ux minus plane and then at $t = 1/2$, what you will have is a formula, from the formula, what we have is it is $\sin x - 1$. So u of x comma 1/2 is $\sin x - 1$.

It just means that graph of sin has been shifted to x = 1. Suppose this 1, this is pi there, so, this is pi by 2, so, this is 1.5. So, it is somewhere here. So, the graph will look like this. It is moving. So, it is difficult to write for me in 3 dimensions. But you should imagine that the graph of sine x has moved, graph was like this at t = 0, now it has shifted to 1. But now this I am writing at t = 1/2, so it just moves back. So, now initial boundary value problems. Once again, initial is there, so there should be a time.

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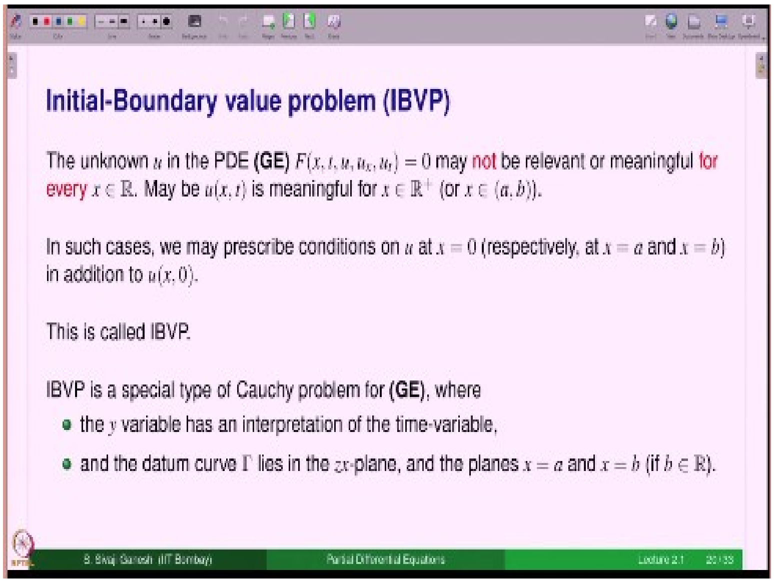

So the unknown u in this PDE x, t, u, u x, u t = 0. I already made this change of y to t because of our initial y boundary. So these equation or what you want to study, it may not be relevant for for all x in R, it may be that it is relevant only for x positive. Or maybe for x in some bounded interval a, b. In such cases, imagine the first case, if t is meaningful only in R +, then x = 0 is a boundary of R + boundary point.

And if you are studying x in the a, b, then there are two boundary points, x = a and x = b, at these points. So you need to prescribe some conditions there. That is what is called initial boundary value problem.

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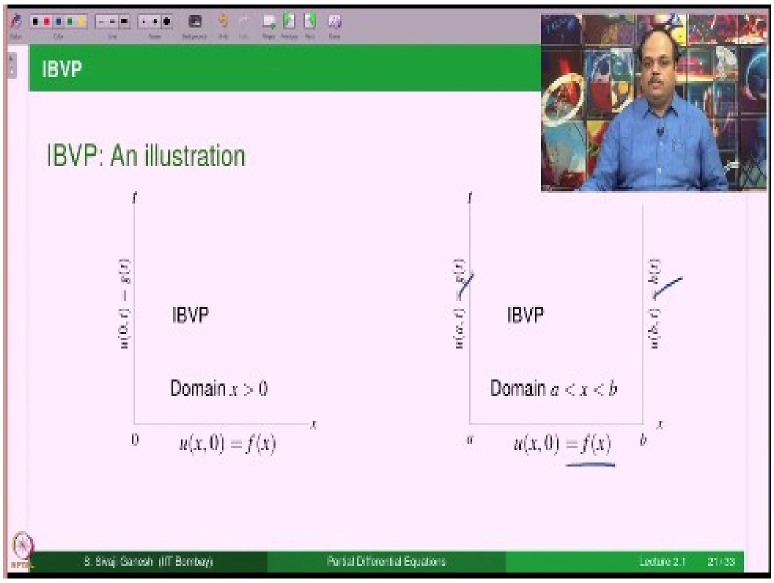

Now let us do a look at a picture illustration. So this is a case where the domain is R +, x belongs to R +, therefore you have a boundary at x = 0. So you prescribe this g. Of course, initial condition you prescribe. Now, when you are doing initial boundary value problem, when x belongs to this interval a b, you have to give initial conditions and also give boundary conditions.

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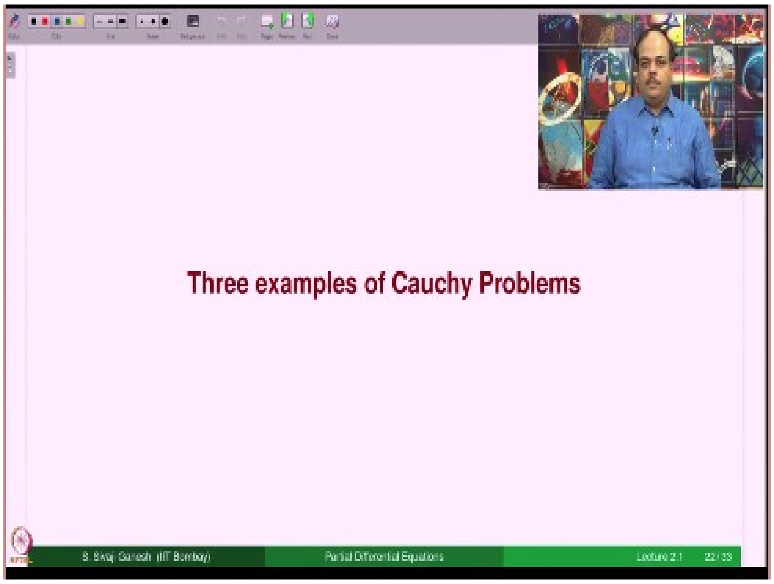

So now let us look at 3 examples, 3 guiding examples for us Cauchy problems, because you can understand the entire theory by using these examples.

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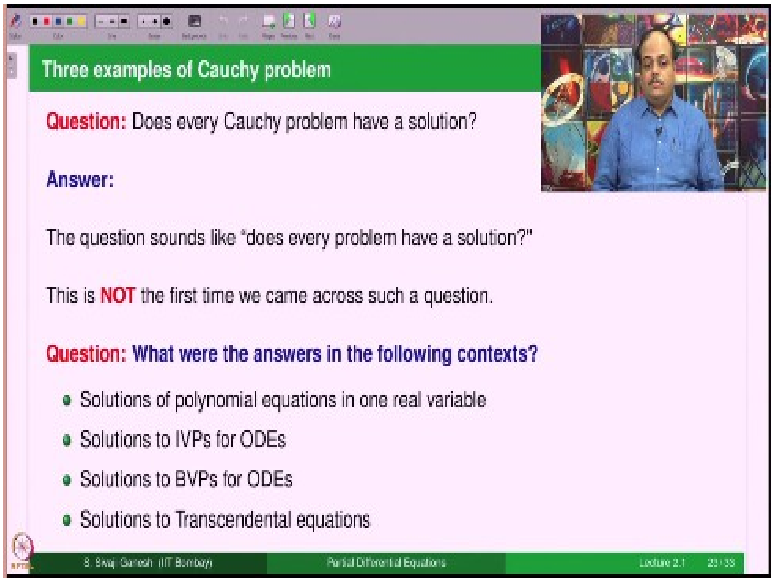

Before that, let us ask the following question. Does every Cauchy problem have a solution? Good question. This question sounds like a very familiar questions that we asked earlier, does every problem have a solution. This is not the first time we came across such a question. It has asked many times. And many times we even had a complete answer. For example, system of linear equations, where a x = b where a is a square matrix there we understand completely.

Under what conditions you have existence, what conditions of uniqueness and non existence. Now let us ask the same question in some few well known situations for us before solutions of polynomial equations in one real variable. What about that a polynomial may not have a solution, polynomial equation, x squared + 1 = 0 has no real solution. It can have exactly one solution. Let us say x – 1 = 0, or x into x square + 1 = 0. They have exactly one solution.

Then you can have finitely many solutions. Let us take very simple x – 2 into x – 3 = 0. Any finite number, let us say a 1, a 2, a m, you will write x – a 1 into x – a 2 into x – a m = 0, precisely these other solutions a 1, a 2 a m. But it cannot have infinitely many solutions for polynomial equations. That is because of fundamental theorem of algebra.

It says given a non constant polynomial, it has exactly the same number of roots as the degree counting multiplicities in the complex plane. Therefore, in real numbers may be less than or equal to that. Then solutions to initial value problems for ODEs. This is where we have seen Picard's theorem and Peano’s theorem. In Peano’s theorem, if the right hand side let us say y dash equal to f x, y, let us write once.

This is the kind of equations for which we have a theory. And this is the initial value problem. If f was continuous, we had Peano's theorem and we said there is a solution. In addition, if f is Lipschitz with respect to the y variable, then Picard's theorem told us that it has a unique solution. And when you do not have unique solution, you can easily show that you have infinitely many solutions. That is the situation about ODEs.

So these under sufficient conditions that we have proved these theorems. So, if these sufficient conditions are not satisfied, we have no idea what will happen anything may happen, you may still have uniqueness, even though the right hand side function f is not Lipschitz that is a possibility. Now solutions to boundary value problems, we have a Bernstein theorem, under some sufficient conditions it guarantees that a second order ODE boundary value problem posed for that will have a unique solution.

At least when the Dirichlet data this if you do not, we are not sharing BVPs, you can look at any good book on ordinary differential equations, you will find. Then solutions to transcendental equation. This is the most difficult one, you have now. It is not like polynomials. So you have to worry with each separate equation is a different story.

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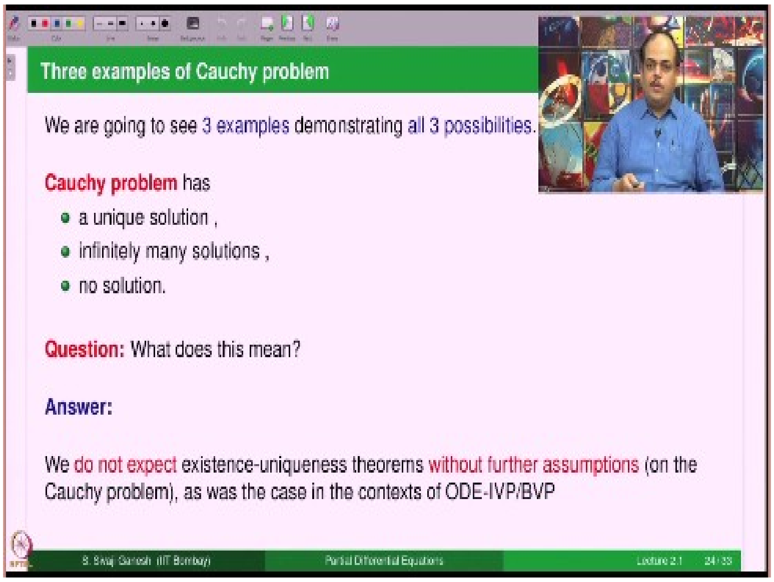

There are once again sufficient conditions. Now, let us see what are the examples? So, examples we are going to see are of Cauchy problems. In these 3 examples, which demonstrate all 3 possibilities, in this case, what are the possibilities? You have a unique solution, infinitely many solutions and no solution. Of course, another option is that finitely many solutions, which are not taught about.

Let us take a take that up in a tutorial session that discussion. So what does this mean now? It means that we do not expect existence uniqueness theorem for free for any first order PDE. We do not expect. So only under some special assumptions, in other words, sufficient conditions, we can expect such theorems, which was true in all our earlier cases. Let us say in the ODE initial value problems or boundary value problems that stood true.

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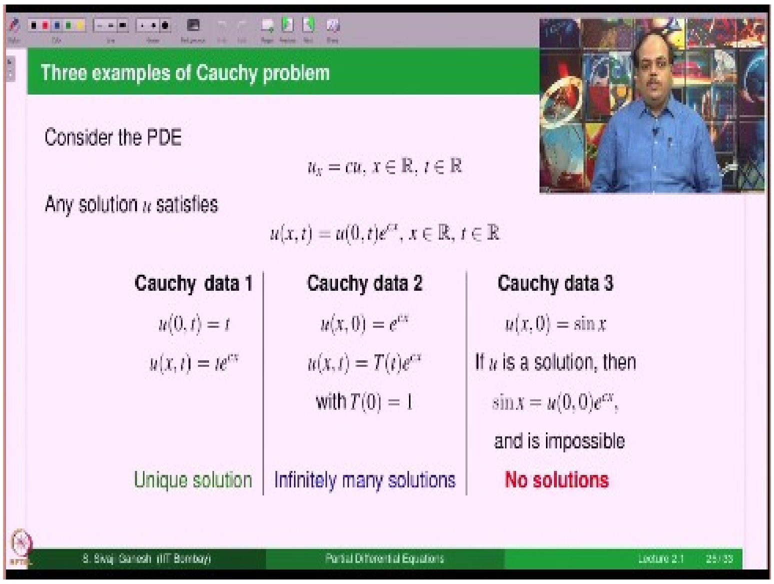

Let us look at a very simple first order PDE it is u x = cu, c is a constant. We will look at a special constants later c = 1 and – 1 later on. In the case of initial boundary value problem, here c can be any number, the real number. So it is posed for x in R and t in R. You wrote c t in the problem. So it is actually ODE. And from our ODE knowledge we can solve this. Any solution must satisfy this equation. u (x, t) = u(o, t)into e power cx.

Now let us look at 3 Cauchy data. What are they? First one is u of (0,t)= t, second is u(x, 0) and here also u (x, 0). That means the initial Cauchy data is given here on t axis, here in other two cases given on x axis. Now using this expression, we will get this u(0, t) you wanted t is supplied, so t into e power cx. That is the solution. In this case, u(x, 0)that is t = 0. That is u zero zero into e power cx, but u zero zero must be 1.

Therefore, any function of t says that T of zero is 1 will do our job. It will be a solution T t into c e power cx is a solution. Whenever T zero = 1. Here, if u is a solution, then what is expected is sin x must be a multiple of e power cx. That is impossible, because sin x and e power cx are always linearly independent as functions on any interval that you consider. So, message is that Cauchy data 1, unique solution.

Cauchy data 2 and 3, one case you have infinitely many solutions, for Cauchy data 3 there are no solutions. So, 3 possibilities have been exhibited here.

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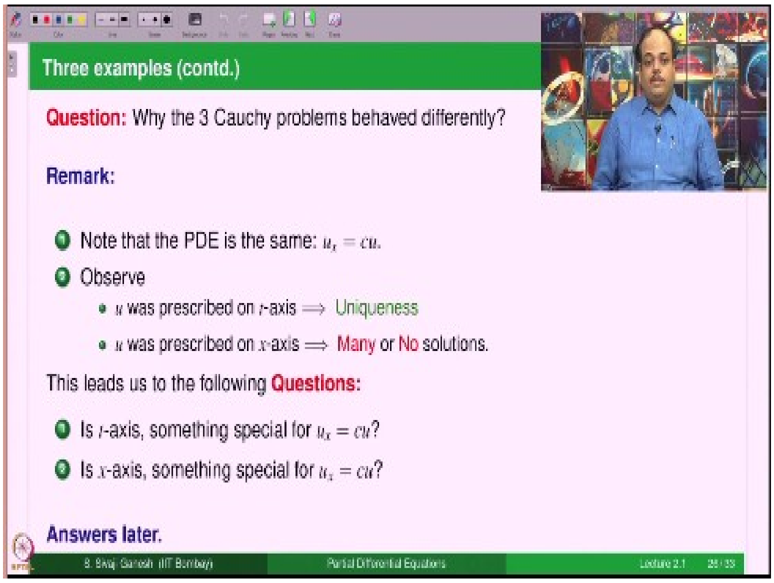

Now the why the Cauchy problems behave differently. PDE was the same, u x = cu, then what is different? u was prescribed on t axis we had uniqueness and when u is prescribed on x axis, either we had infinitely many solutions or no solutions. Therefore, we asked the following question is t axis something special for the equation or is it x axis which is special for this equation?

The reason for the first question is, because we have uniqueness. Second question is we have many possibilities. So, who is really special here? Because special people have to be taken care with a lot of special care. We will see that later on. So, the answers we will see later as we study the first order PDEs.

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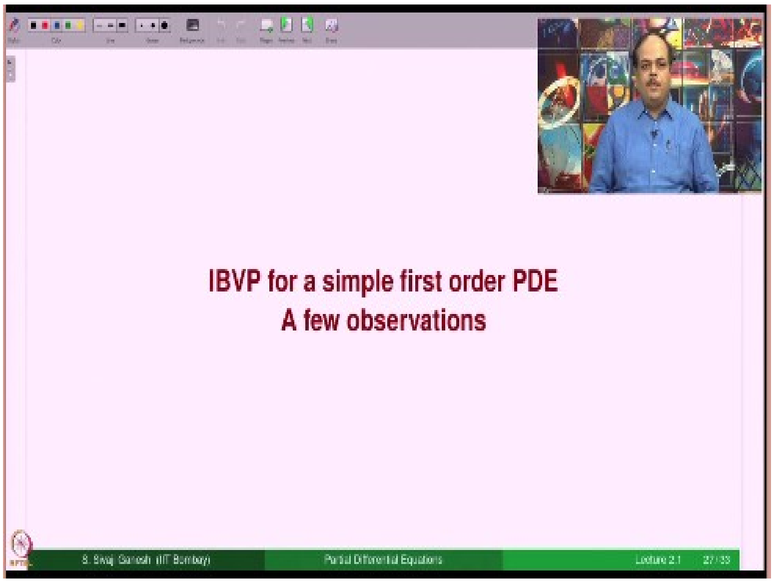

Now, let us look at a simple first order PDE for which we pose IBVP and a few observations. We will not be dealing with this further in our course.

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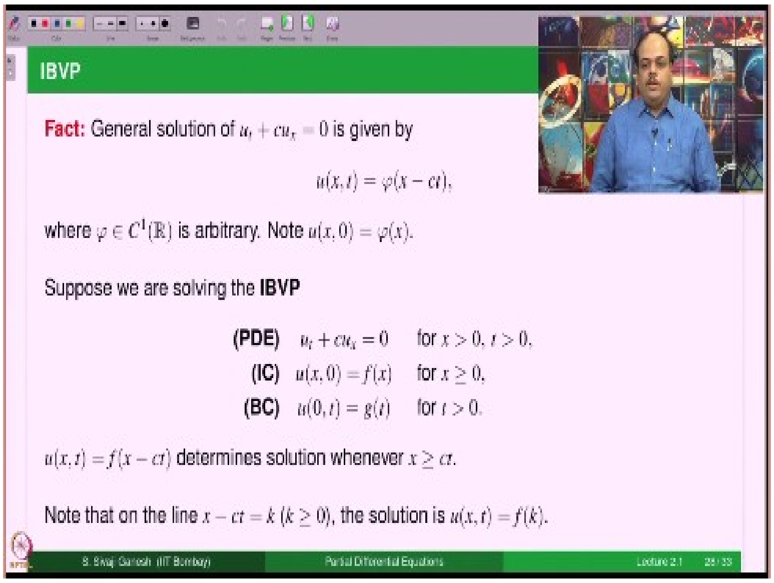

So, the fact is that general solution of u t + cu x = 0 is given by u x, t = phi of x – c t where phi is any C 1 function which is orbitrary. Of course, what is that, t = 0, u(x, 0)becomes phi x. So, this phi is nothing but the initial solution, initial condition. Now, initial boundary value problem as I told you there will be a PDE posed in the first quadrant x, t. x positive, t positive. Initial condition is given and there is only one boundary point namely x = 0.

Therefore, u of (0,t)is also prescribed. Now, u x, t equal f of x – ct. Because, see here we said fees initial data. So, therefore, here also we have initial data, so, how much we can this determines the solution. So, f of x – ct will be a solution surely. No problem. But whenever x

– ct belongs the domain of f, f is defined only for greater than or equal to zero therefore, this determines the solution whenever x is bigger than or equal to CT.

We will see a picture soon Now, on the line x – ct the solution u will be constant. It will be f of k, x – ct = k, k is greater than or equal to zero. Therefore, this formula is applies and u (x ,t) = f k. It is constant.

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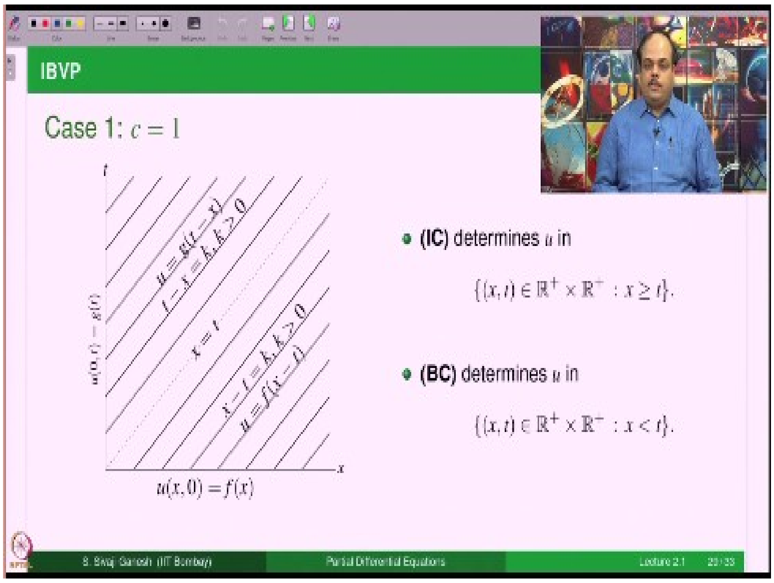

Now, let us look at the case where C = 1, this is the picture. So, this is u (x,0)f (x), this is u 0, t g t. And u in this region in this region the line x = t is in dots to the right side of that. This region corresponds to x bigger than or equal to t. There f of x – ct. Similarly, you can show that here it will be g of t – x that will be solution. So, on this line t – x = k, solution will be g k. And on this line x – ct = k solution will be f of k.

So, initial condition determines u in this region and boundary condition determines in this region. Now, what about x line x = t? What will happen?

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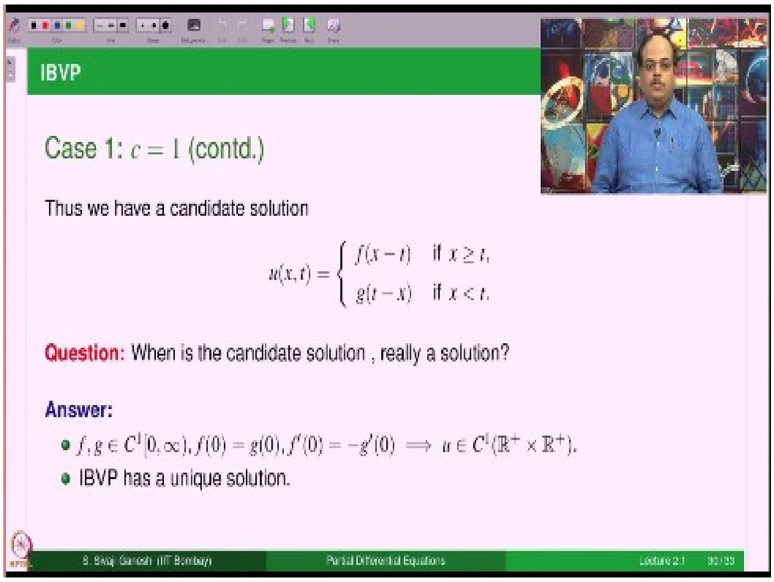

So, let us write the formula. This is a formula that we said. Now, if this has to be solution it has to be differentiable. The first question is , is it differentiable? So, these are candidate solution we say, because we have derived certain possible formula for the solution that is why you propose a candidate we need to verify that it is indeed a solution. So, that requires that f g must be C 1 functions and compatibility of the data is needed.

f zero must be equal to z zero, f dash zero equal to minus g dash zero. How do you get these compatibility conditions? You simply ask at points where x = t write down the differentiability property of u of(x, t)that will tell you that this condition has to be met otherwise, it will not be differentiable. So, once you have these compatibility conditions that implies that you have to use a C 1 function. So, an IBVP has a unique solution.

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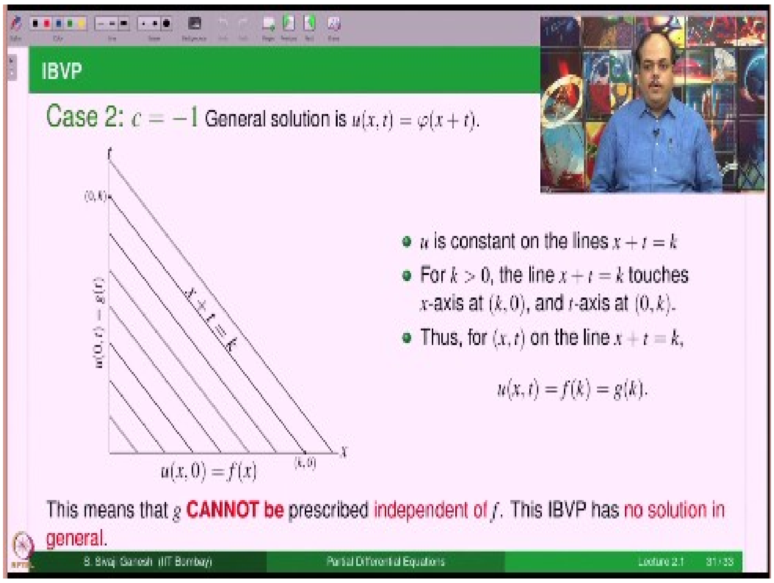

Now, the case c = –1 is totally different. Now, the line is x + t equal to constant. This line touches both the boundary as well as the initial data. So once again f must be constant on this line solution must be constant. Therefore, this line touches at k zero. So, it should the value on this line should be f k and it touches here at g, 0,k. So, the answer must be the solution must be g of k. So, on this line the solution is both f of k and g of k.

What if f of k is not equal to g of k? There is a problem. This means that g cannot be prescribed independent of f. This initial boundary value problem has no solution in general, but there are some generalised notion solutions where under some other condition they admit solutions. So, let us we will not be discussing such generalised notions. This is the normal notion of a solution. There is no solution. That is very clear from the formulae.

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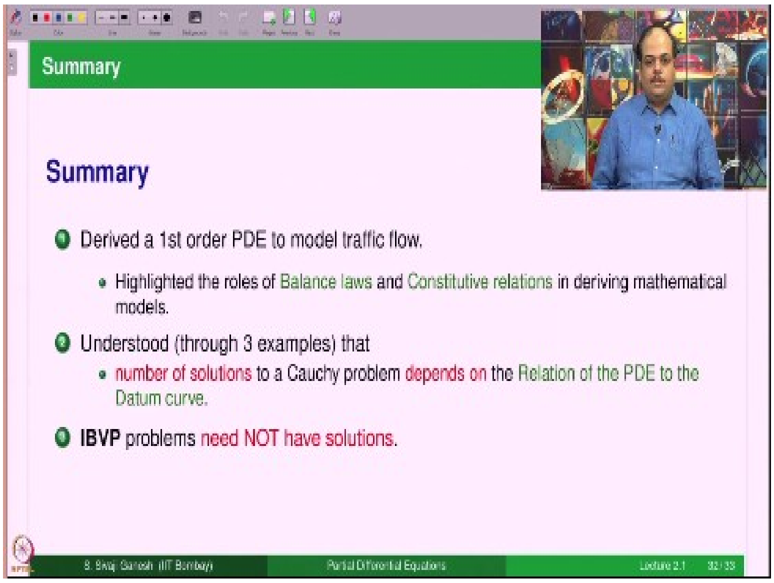

Let us summarise what we did today. We have derived a first order PDE to model traffic flow and also modified to account for sources and things namely entries and exits in the expressway. And we highlighted the rules of Balance law and constitute relations in deriving mathematical models. This is how models in let us say continue mechanics will be derived. These are the two things which are at the heart of any modelling.

And then two three examples, we understood that number of solutions to a Cauchy problem depends on the relation of the PDE to the datum curve and then we saw IBVP problems need not have solutions. So, thank you.

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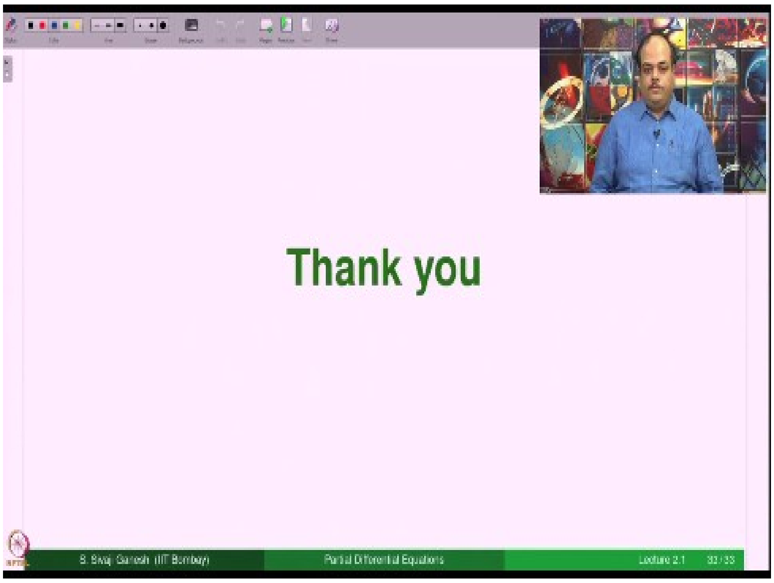
Thank you
S. Sivaji Ganesh (IIT Bombay)
Partial Differential Equations
Lecture 2.1
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