Partial Differential Equations
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Lecture - 3.1
Second Order Partial Differential Equations
Special Curves Associated to a PDE

We are going to begin the study of second order partial differential equations. In this lecture, we are going to study about special curves associated to second order partial differential equation.

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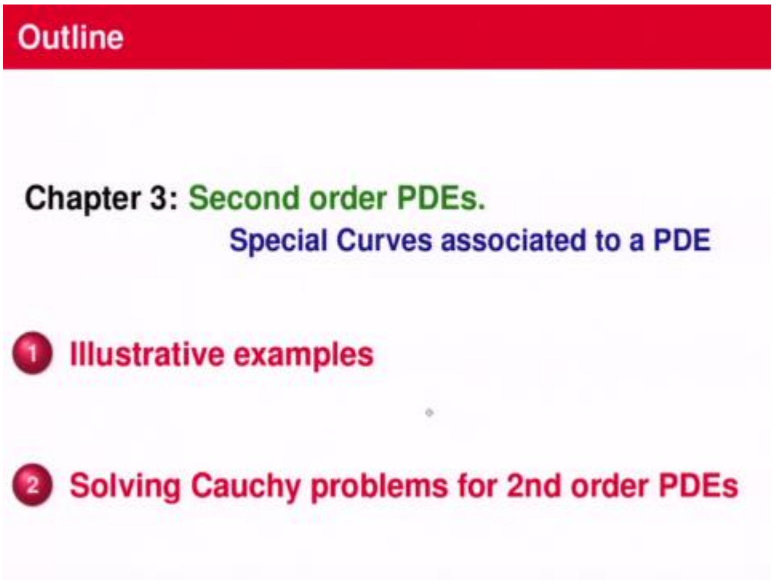

So, the outline for today is we start with some illustrative examples exactly like how we started off our study of first order partial differential equations. Wherein Lecture 2.1, we looked at 3 Cauchy problems which exhibited all the 3 possibilities for the number of solutions. Unique solution, when the data is datum curve is of certain type. And when it is of another type, it was either 0 solutions are infinitely many solutions.

Then we will make an attempt to solve a Cauchy problem for a second order Quasilinear PDE.

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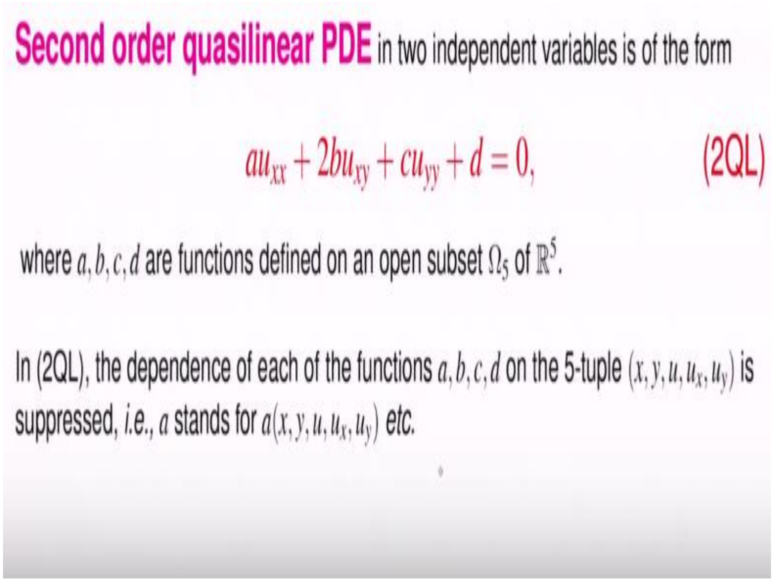

So, second order Quasilinear PDE in 2 independent variables, the most general such equation is of this form au xx plus 2bu x y plus cu y y plus d equal to 0, where a, b, c, d are functions of 5 variables. So, they are defined on an open subset omega 5 of R 5. So, in the above equation, we refer to that as to 2QL second order Quasilinear equation. We suppress the dependence of a, b, c, d on x, y, u, u x, u y. Otherwise, the equation will be very long.

When it is understood, there is no need to repeat it. So, that is why the dependencies suppressed here. So, A stands for a of x y u u x u y etc.

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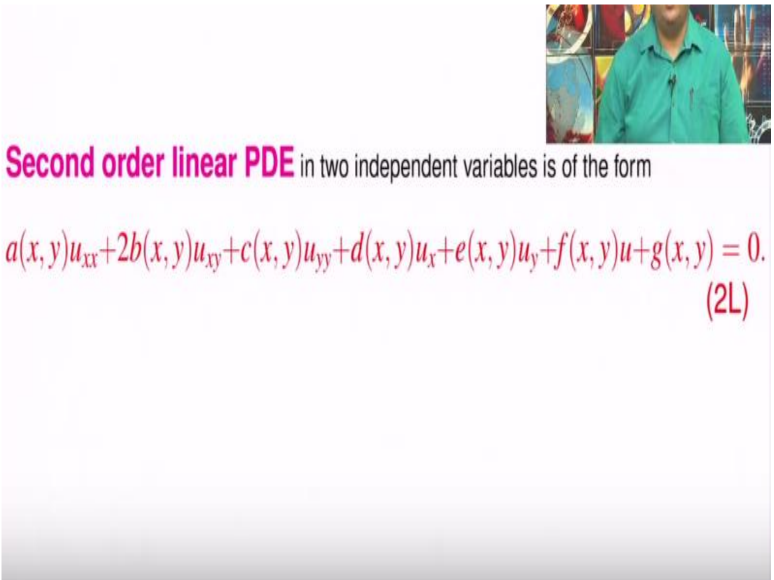

A second order linear PDE we will consider this also in this chapter. In fact, the second part of this chapter will be exclusively dealing with second order linear PDE and a general such general form of such equation is here. So, we refer to that as 2L, second order linear partial differential equation.

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So, let us look at some illustrative examples.

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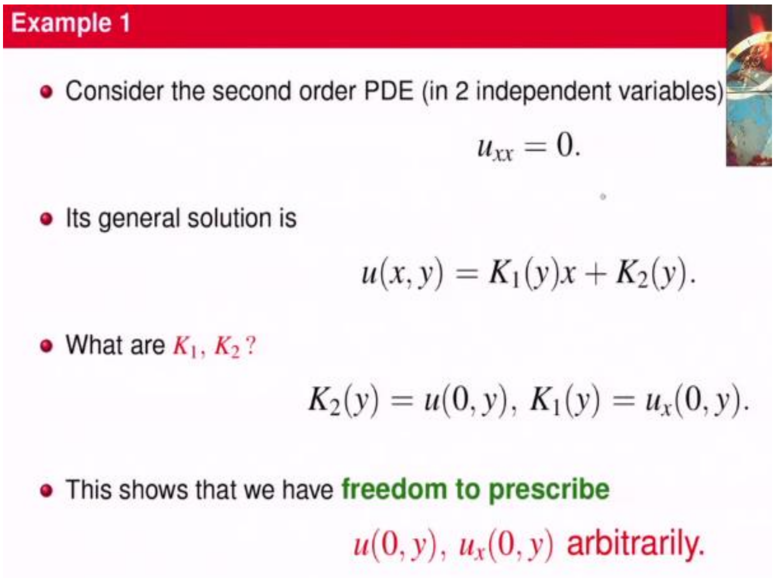

Let us start with this second order partial differential equation u xx equal to 0. I have to mention it is in 2 independent variables because it is not clear from the equation. So, its general solution we can write down this is actually a ODE with respect to x. Therefore, when you integrate first time u x of x y will be constant with respect to x. So, it will be some arbitrary function of y that is K 1 y.

Integrate once more, you get u x y equal to K 1 y x plus K 2 y. See solutions of u xx equal to 0 or straight lines if it is a ODE. Straight line will have a they look like a x plus b. Now, because this there is a y variable involved a and b will be functions of y also. Now, what are K 1 and K 2? K 2 of y if I want to find out, I need to finish this term kill this term. So, I put x equal to 0. That means u of 0 y. This term is gone. What remains is K 2 y.

So, K 2 y is nothing but u of 0 y. And, what is K 1 y? That looks obvious. You are to differentiate with respect to x. And u x of x y, in fact, is K 1 y. But I am taking x equal to 0 to be uniform with this because later on we are going to look at prescribing conditions to solve u xx equal to 0, conditions like our Cauchy data. That we have seen in first order PDE. Therefore, let it be u 0 y and u x 0 y. Then what we get is K 1 y and K 2 y.

So, therefore, this analysis shows that you have complete freedom to prescribe u of 0 y and u x of 0 y if you want to solve u xx equal to 0.

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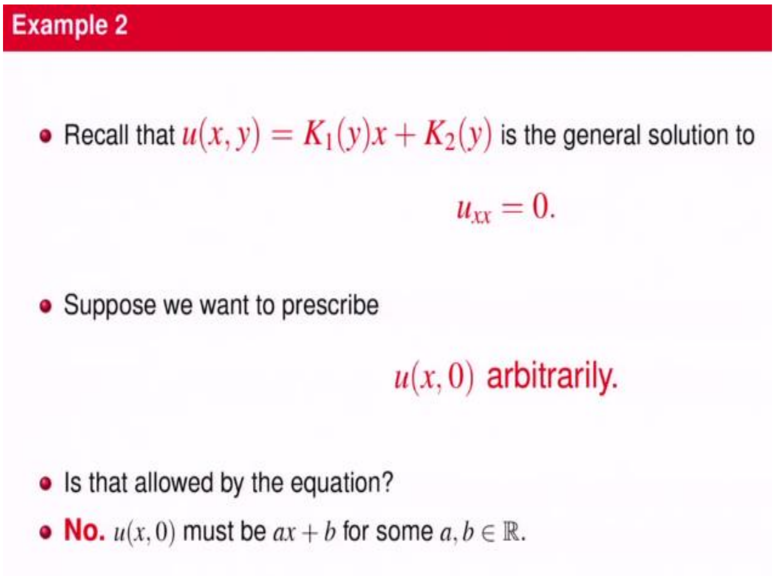

Now, let us go to the second example. Equation is the same. We are not changing the equation u xx equal to 0. Therefore, the general solution continues to be the same. Now suppose we want to prescribe u of x 0 arbitrarily, what should be that? Can we do it? Is that allowed by the equation? Because this equation immediately the solution is coming like this. So, therefore, asking this question is (()) (05:04) asking whether equation (()) (05:05).

So, what is u x 0? When I put u x 0 y equal to 0 I get K 1 0 times x plus K 2 0. So, it is not allowed by the equation you cannot have arbitrary functions. u x 0 must look like ax plus b for some real numbers a and b.

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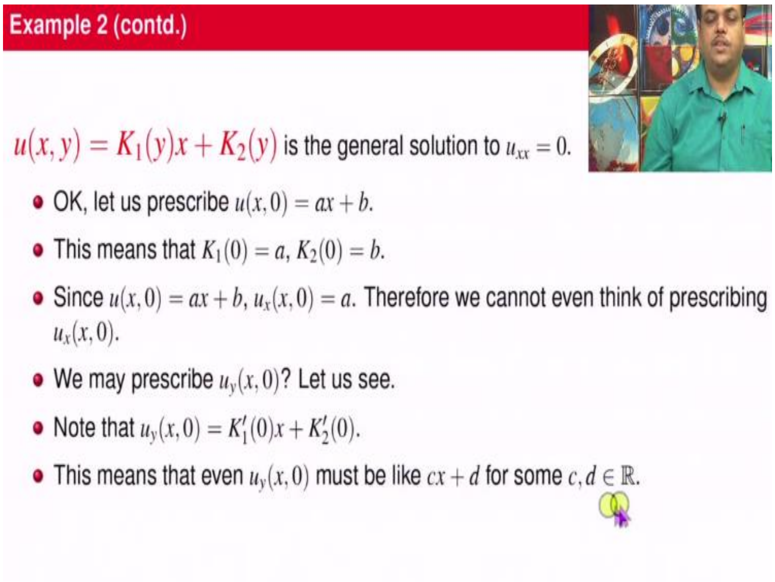

Let us prescribe u x 0 equal to ax plus b. Prescribing u x 0 equal to ax plus b means that K 1 0 is a, K 2 0 is b. We just saw that on the previous slide. So, since u x 0 equal to ax plus b, the derivative of u with respect to x is determined on the x axis. u x x 0 will turn out to be a. So, there is no way that we can prescribe u x at on the line x 0. It is not possible. Then we asked the question, can we prescribe u y in that case, u y of x, 0? Let us find out.

So, u y of x 0 from this general solution will look like this now, derivative with respect to y. That means K 1 dash y x plus K 2 dash y. When y equal to 0, it is K 1 dash 0 times x plus K 2 dash 0. This means even u y of x, 0 must be like cx plus d. It must be linear function for some constant c, d in R.

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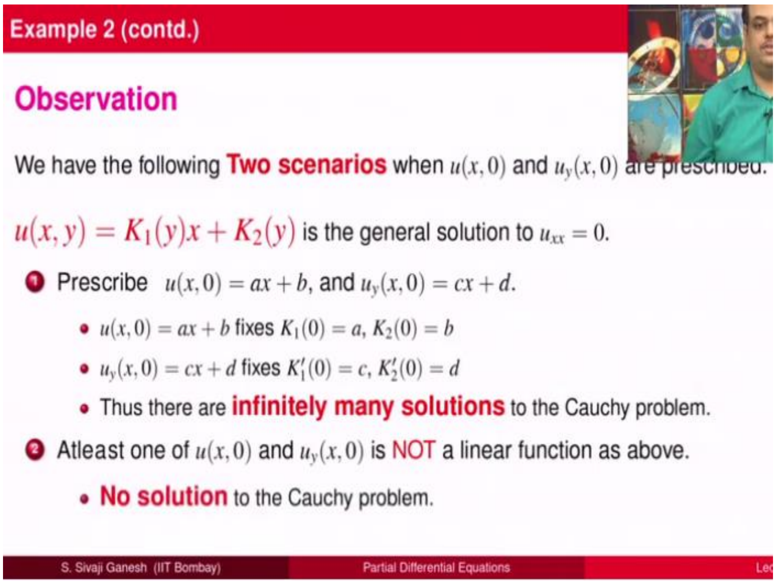

So, therefore, to conclude we have the following 2 scenarios when u x 0 and u y x 0 are prescribed. What are they? This is just to recall that the general solution is this for u xx equal to 0. What are the 2 scenarios? Prescribe u x 0 equal to ax plus b and u y x 0 equal to cx plus d. Now, prescribing u x 0 equal to ax plus b fixes the values of K 1 and K 2 at the point 0, x equal to 0. K 1 0 is a and K 2 0 is b.

Now, the other condition, u y of x 0 equal to cx plus d. That fixes K 1 dash 0 equal to c, K 2 dash 0 equal to d. Therefore, there are infinitely many solutions to the Cauchy problem, because these 2 conditions namely u x 0 equal to ax plus b and u y x 0 equal to cx plus d does not determine both K 1 and K 2, the functions. What are all the things which are determined by these conditions are simply the values of the function and the derivative at the point 0 for both K 1 and K 2.

So, you have so many functions which satisfy these criteria. That is why you have infinitely many solutions. Now, the second scenario is at least one of them you have not prescribed as a linear function, what will happen? No solutions. No solution to the Cauchy problem.

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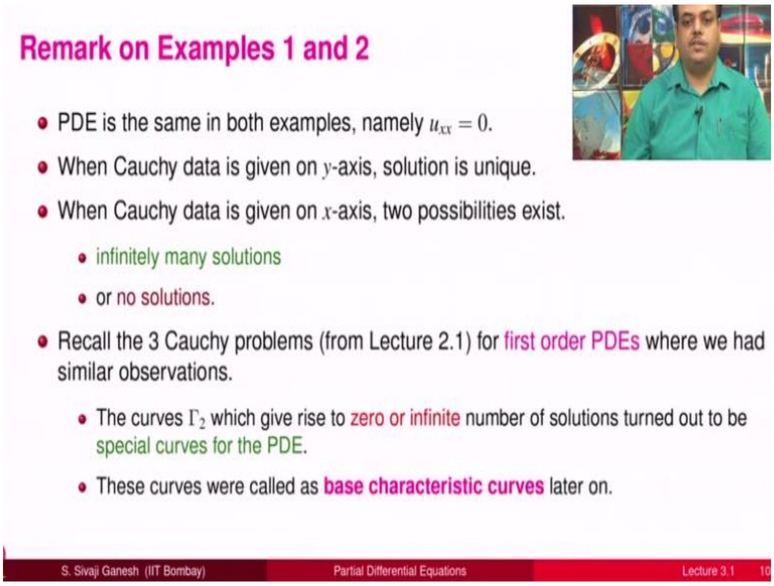

So, if you compare both the examples, PDE is the same in both examples namely u xx equal to 0. When Cauchy data is given on y axis, solution is unique. When Cauchy data is given an x axis, 2 possibilities exist, infinitely many solutions or no solutions. So, recall the 3 Cauchy problems that we considered in Lecture 2.1 for first order PDEs where we had the similar observations.

The curves gamma 2 which give rise to 0 or infinite number of solutions turned out to be a special curves for the PDE. And these curves were called base characteristic curves later on.

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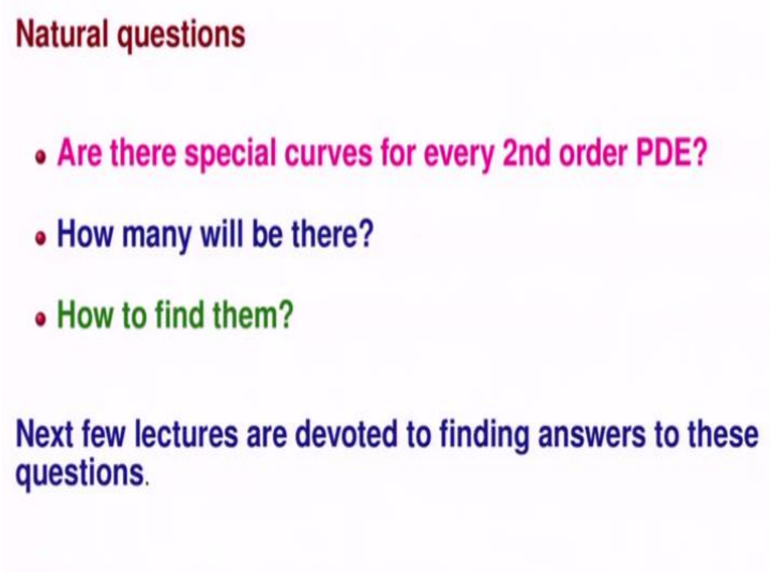

Now, the natural questions are, are there special curves for every second order PDE? How many will be there? How to find them? Next few lectures are devoted to finding the answers to these questions.

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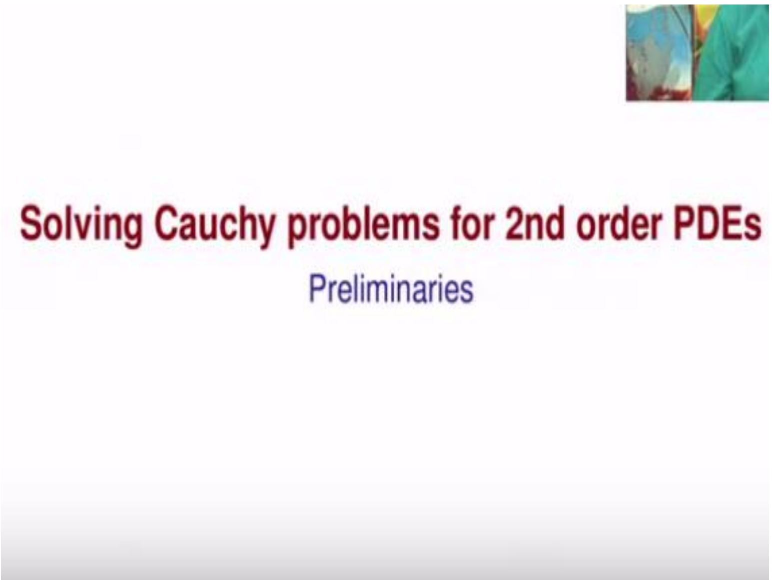

So, solving Cauchy problems for second order PDEs, we will cover some preliminaries. So, we are going to pose a Cauchy problem and then implement a classical strategy to solve that. Before posing Cauchy problem, we need to introduce few terminology.

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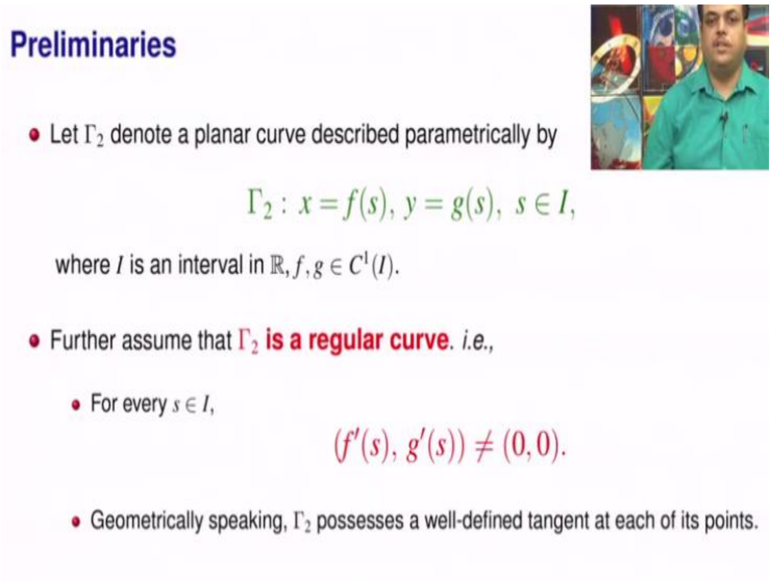

We are going to do that. So, let gamma 2 denote a planar curve described parametrically by gamma 2, x equal to f s, y equal to g s, s in I, where I is an interval in R and f, g are C 1 functions. Further, assume that gamma 2 is a regular curve. What does that mean? For every s in I, f dash, g dash is not equal to 0 0. Geometrically speaking, gamma 2 possess the well-defined tangent at each of its points. We have come across the notion of a regular curve even in the context of first order PDEs.

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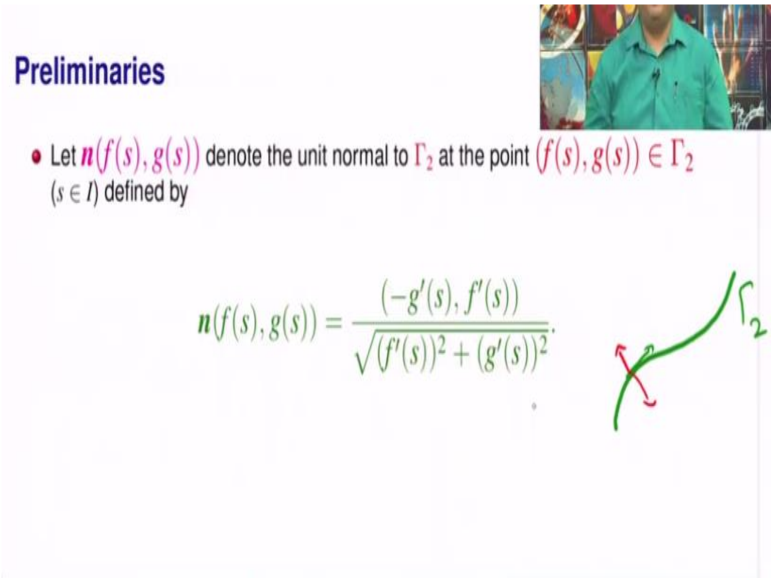

Let n of f s g s denote the unit normal to gamma 2 at the point f s g s in gamma 2. f s g s is a point on gamma 2 and n denotes the unit normal to gamma 2 at that point defined by this because unit normal is not unique. It will be there will be 2 choices. For example, if this is your gamma 2, this is the tangential direction. And what you have here is the normal direction. So, plus or minus of each other, we do not care which one we are taking for this problem.

It should be given by this. We are giving the formula here, n f s g s equal to this. Notice this is well defined if the denominator is not 0. That is precisely the assumption of the regular curve.

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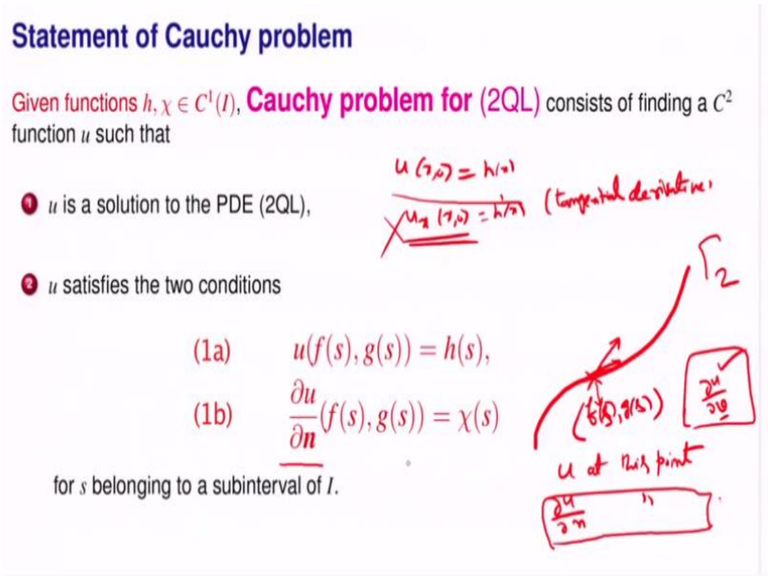

So, we are now ready to state the Cauchy problem. Given 2 functions, we will see where they appear, h and chi. Cauchy problem for the second order Quasilinear equation consists of finding a C 2 function that is twice continuously differentiable function such that u is a solution to the PDE which is a second order Quasilinear equation, 2QL. And u satisfies the 2 conditions, u of f s g s equal to h s.

That means u is prescribed on point at points of gamma 2 as h of s. And dou u by dou n which is called a normal derivative of u is also prescribed as chi s at every point of gamma 2. So, suppose this is your gamma 2 and you take a point here. At this point, this point is like f s g s. These how points look on gamma 2. You are prescribing what should be the value of u at this point. And you are also prescribing dou u by dou n at this point, the normal derivative.

Why not any other derivative? That question, we will discuss at the end of this lecture. Why not any other derivative? So, actually if you see the normal direction is like that or maybe any of the directions. Let us take this side. You can actually define the directional derivative in any direction that can be prescribed. So, what can be prescribed is dou u by dou v. That can be prescribed. That is also fine.

What all you should not prescribe is the direction of the tangent you should not prescribe. If you recall if you consider u x 0 equal to some function h x, u x is already determined. So, therefore, you cannot prescribe this with freedom. And, what is u x x 0? It is a directional derivative of u in the direction 1, 0. That is the direction of the x axis. So, that is what is called a tangential derivative.

And at every point, the direction of the tangent and the normal they will be linearly independent. So, you can prescribe 2 derivatives. But one derivative tangential derivative is already determined if you have prescribed the function u. Therefore, there is this you cannot prescribe. Therefore, any other directional derivative we can prescribe where v is independent of the direction 1 0.

But, to be very clear, we are prescribing on a normal because tangent on a direction which is immediately connected to a tangential direction is a direction perpendicular to that which is a normal direction. So, that is why we prescribe dou u by dou n. That is the secret. You can prescribe any other derivative also. And we require as usual, the condition should be met for s belonging to a subinterval of I which means we are looking at local with respect to data kind of solution.

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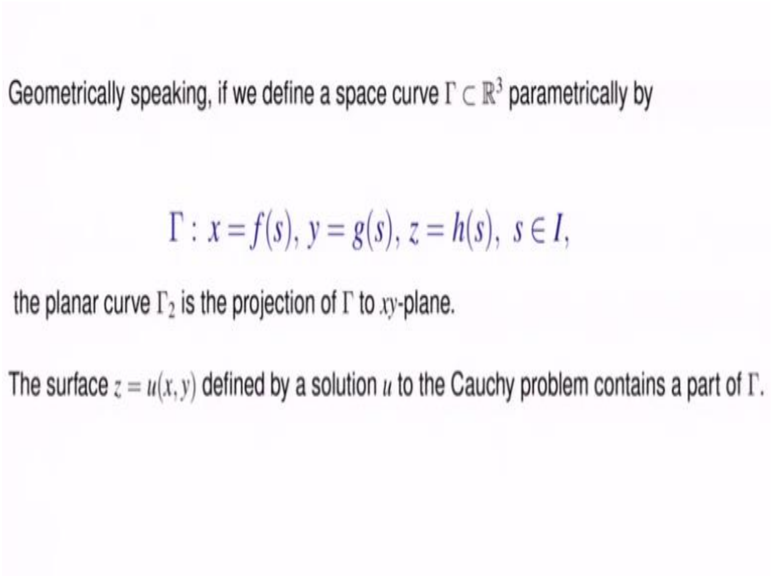

Geometrically speaking, if you define a space curve, gamma in R 3 by putting z equal to h s, we get gamma 2 will be the projection of gamma 2 xy plane. And the surface z equal to u x y defined by solution of the Cauchy problem will contain a part of this gamma.

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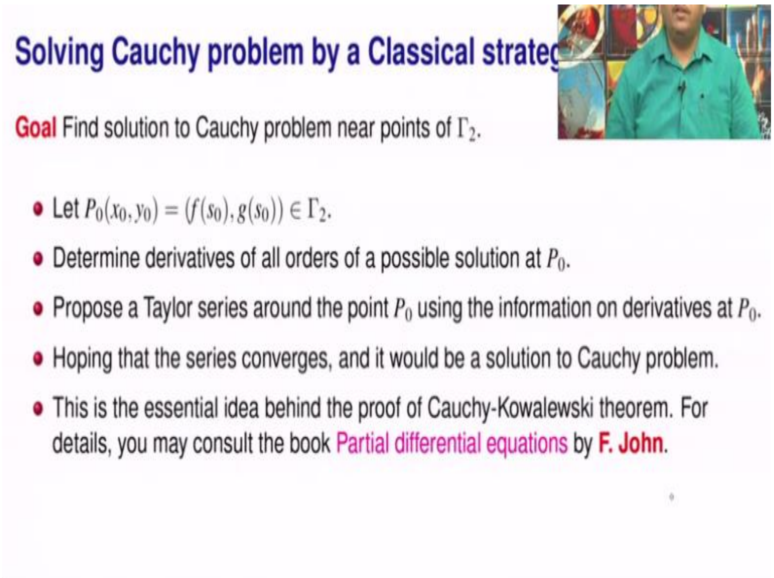

Now, we discuss a classical strategy to solve the Cauchy problem. The goal is to find solution to Cauchy problem near points of gamma 2 because functions will be defined in a neighborhood of gamma 2. The same thing was true for the first order partial differential equations also. So, take a point on gamma 2, P naught. In terms of x and y, you may call x naught y naught, in terms of the parameter running on the gamma 2, f s naught g s naught.

Determine derivatives of all orders of a possible solution at P naught. Determine derivatives of all orders. Propose a Taylor series around the point P 0 using the information on derivatives at P naught. What information do you need to propose a Taylor series of a function? All the derivatives at a particular point in this case P naught that you have determined in this step determine all the derivatives. So, Taylor series can be proposed.

And hoping that the series converges, and it will be a solution to the Cauchy problem. That is the hope. One needs to prove that. This is the essential idea behind the proof of Cauchy Kowalewski theorem. For details, you may consult the book of Partial Differential Equations by Fritz John. You will find that details there. So, in other words, somebody comes to you and tells you that boss, I know that this Cauchy problem has a solution, which can be expressed in Taylor series format.

It has a Taylor series expansion. In other words, he is telling you that solution is real analytic. Now, your job is only to find that. To find that, what all you need to do is find all partial derivatives of the function at the point P 0. If you can determine them uniquely, then you caught hold of all the derivatives and propose that series Taylor series. And since somebody told you that he has a Taylor series expansion, you hope that this will be solution (()) (17:32).

But to implement the strategy, we would require the data in the problem namely the a b c d to be smooth to be as many times differentiable as we want. Similarly, f g and h which are prescribed functions or maybe h and chi, f g are determined defined by gamma 2. Gamma 2 is defined by f g, and then we are given the Cauchy data in terms of h and chi. So, all of this of course, we need to assume are C infinity functions. Then only we can implement this strategy.

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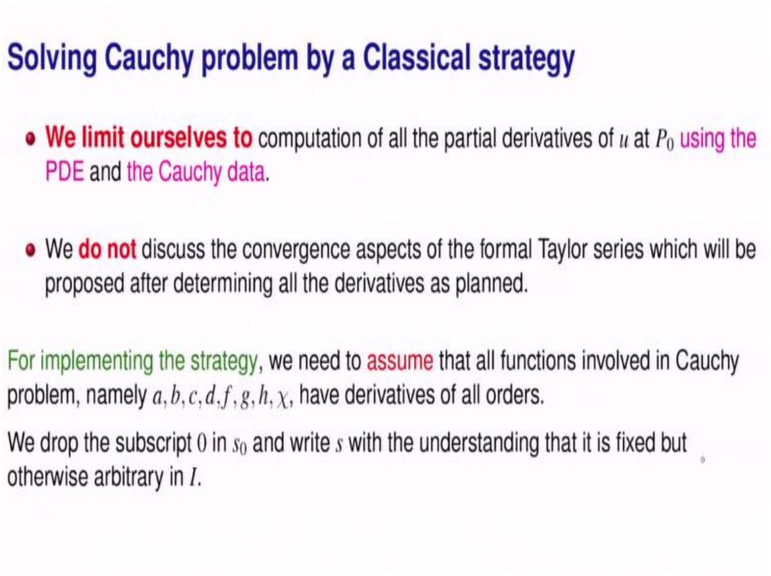

We limit ourselves to just computation of all derivatives. We are going to enquire into the possibilities of computing all derivatives at a point P 0. Can we do it or not? Whether somebody stops us from doing that? If so, who is that? We will identify. Of course, we have no a priori knowledge of the solution. The person who told there may be a solution, he is not given us a formula so that I can compute derivatives using the function.

No, it is not that you are given a function and then find its Taylor series. It is not the case. You are thinking that there is a solution which has a Taylor series expansion and you are trying to find out. If such is the case, what are the derivatives? And, what is available to you is only the Cauchy data and the PDE. These are the only 2 things that you can use no nothing else.

So, we do not discuss the convergence aspects of the formal Taylor series which needs to be proposed after computing the partial derivatives of all orders. So, for implementing this strategy, we need to assume that all the functions involved in the Cauchy problem namely a b c d f g h chi (()) (19:25) C infinity that means, they have derivatives of all orders. Now, we are going to drop the subscript 0 in s 0. And we write s with understanding that it is fixed but otherwise arbitrary in I.

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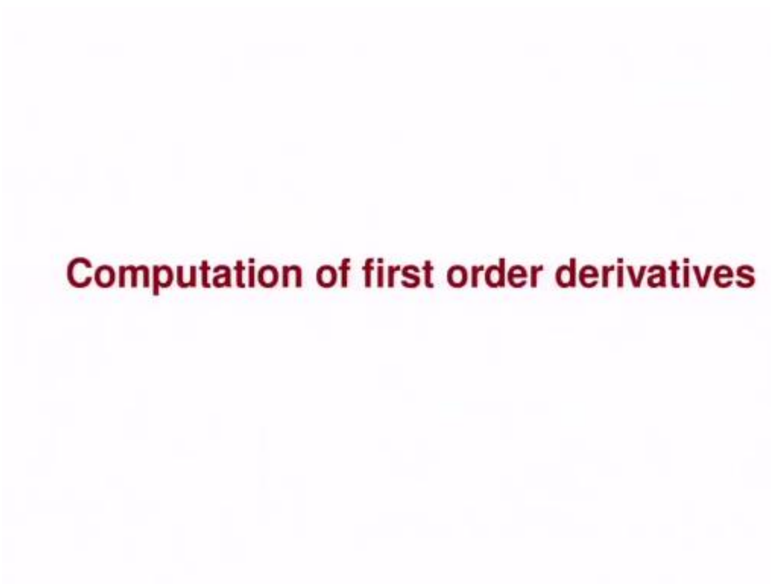

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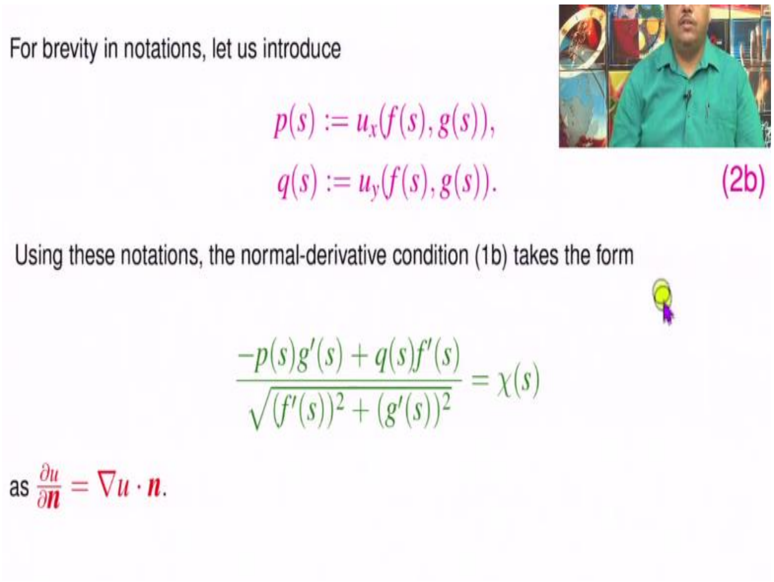

Now, let us see the computation of first order derivatives using only the Cauchy data and the PDE. And for brevity in notations, let us introduce u x at a point f s g s on gamma 2 as p s. Similarly, u y at a point f s g s as q s. So, these functions are defined on gamma 2. We are just introducing. We do not know u x u y yet. We need to determine u x u y. So, using these notations, the normal derivative condition takes this form.

Minus p g prime plus q f prime by root f dash square plus g dash square equal to chi s. And dou u by dou n is gradient u dot n. Gradient u is u x and u y dot n is minus g prime f prime divided by root f dash square plus g dash square. That is why we get this equation.

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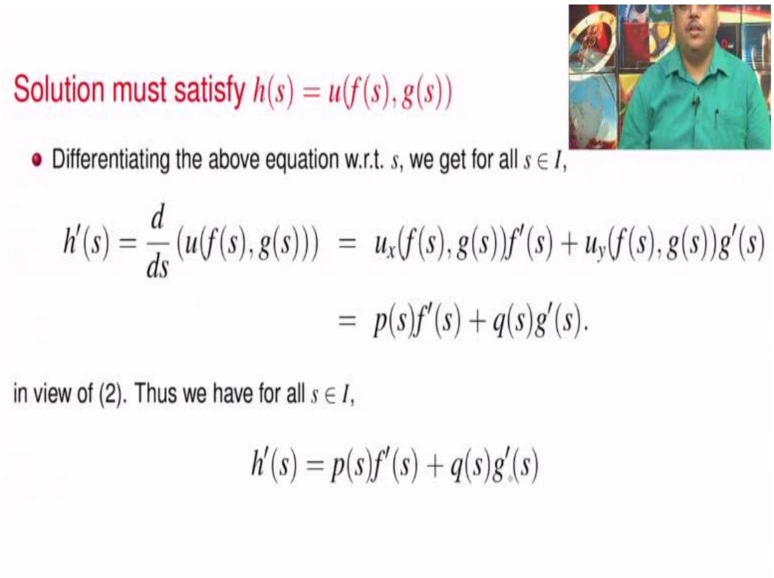

Solution must satisfy h s equal to u of f s g s f s u of f s g s also. So, let us differentiate this equation, because we want to get an equation for p s and q s. So, we differentiate this. Apply chain rule. So, from here, we get this. But u x is p s, u y is q s. Therefore, that is p s f dash plus q s g dash s. So, we have got one more equation. So, we had one equation on the previous slide and one more equation on this slide. Both of them are linear with respect to p s, p s q s.

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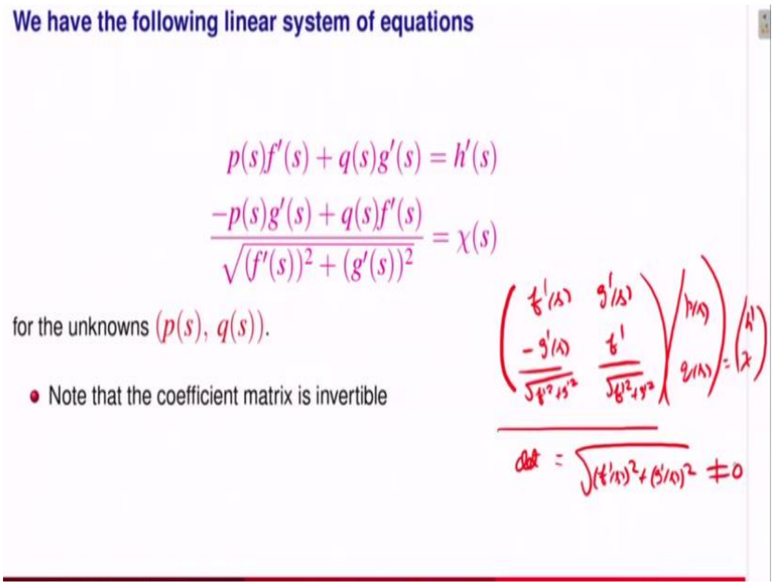

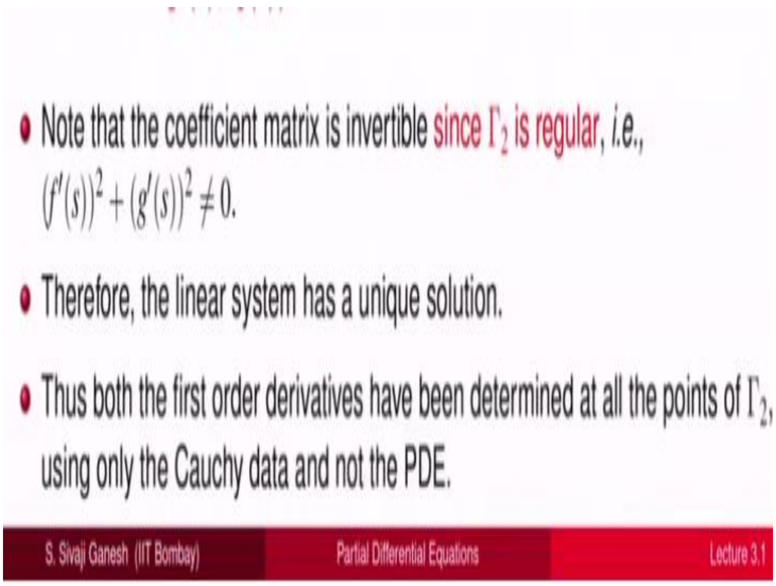

So, let us recall. Both the equations in one place, look we want to determine p s and q s. This is one linear equation featuring p s q s f prime g prime h prime are known. Here also g prime f prime and chi are known. So, this is also a known linear equation in p and q. The coefficient matrix is invertible. What is the coefficient matrix? The first equation is f prime of s g prime of s. Second one is minus g prime of s divided by square root of f prime square plus g prime square.

And here it is f prime by square root of f prime square plus g prime square into p s q s. This is the system equal to h prime into chi not chi into h prime and chi. Now, what is the determinant of this? It is here prime square plus g prime square divided by square root of f prime square g prime s square. Therefore, determinant is equal to square root of f prime square plus g prime square and that is not equal to 0 due to the regularity of the curve.

So, therefore, there is exactly one solution for p s and q s. So, we can find p s and q s uniquely. So, both the first order partial derivatives have been determined at all points of gamma 2 with this. In fact, we are interested in determining at some point of gamma 2 that we have fixed. Since the point is arbitrary, we are saying at any point in gamma 2. Using only the Cauchy data, the PDE did not play any role.

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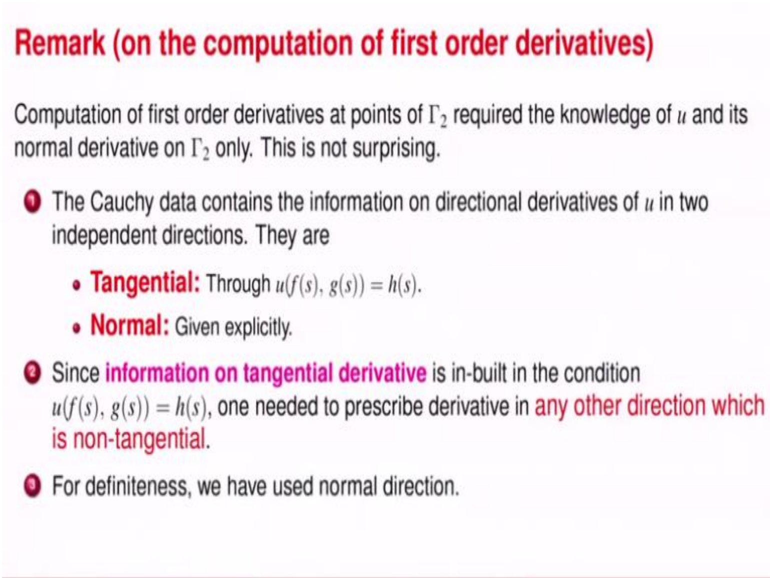

So, the remark, computation of first order derivatives at points of gamma 2 required the knowledge of u and its normal derivative on gamma 2 only. This is not surprising. The Cauchy data contains information on directional derivatives of u in 2 independent directions. What are they? They are tangential through u of f s g s equal to h s. Normal through the normal derivative which is given explicitly.

Since information on tangential derivative is in-built in this condition u f s u of f s g s equal to h s, one needed to prescribe derivative in any other direction which is non-tangential. This what we discussed in the beginning of this lecture. For definiteness, we have used normal direction. That is all.

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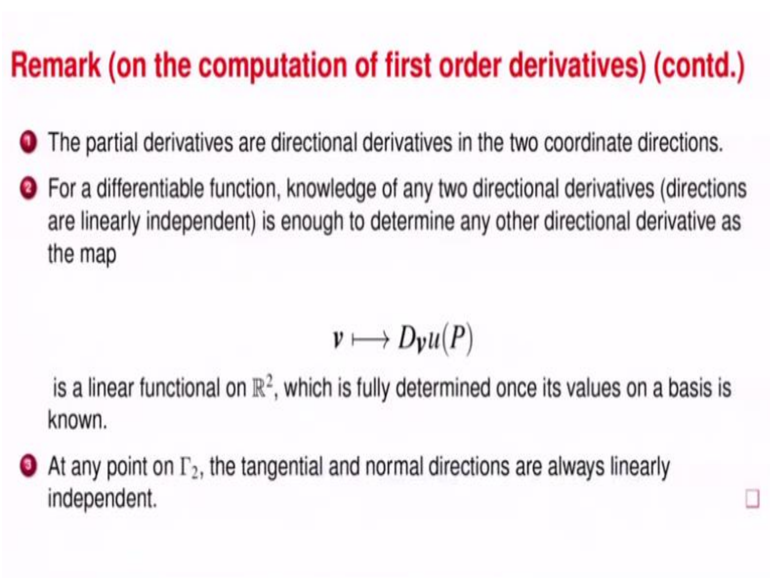

The partial derivatives are directional derivatives in 2 coordinate directions. For a differentiable function, knowledge of any 2 directional derivatives, of course, directions must be linearly independent. That is enough to determine any other directional derivative. As this map is a linear functional on R 2, v going to D v u. That is a directional derivative of u in the direction of v at the point P. So, P is fixed.

Then the mapping v going to D v u at P is a linear functional, which is fully determined once its values on a basis is known. At any point on gamma 2 the tangential and normal directions are always linearly independent.

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Now, let us look at computation of second order derivatives. Here we need to use the PDE.

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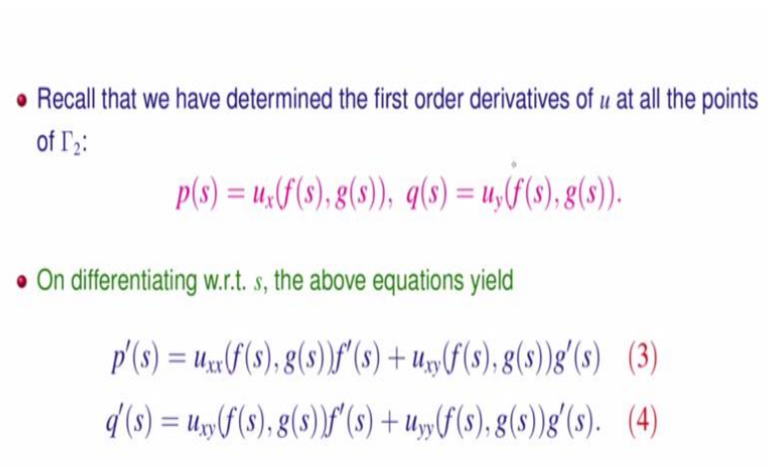

We have determined the first order derivatives at all points of gamma 2. That is u x of f s g s and u y of f s g s. We call them p s and q s. On differentiating with respect to s, we get p prime of s equal to u xx into f prime s plus u xy into g prime s. Similarly, we get this expression for q prime s. Note here that the LHS p prime s and q prime s is known, because p and s are known functions. This is known. f prime and g prime are anyway known.

So, what are unknowns here? u xx, u xy and u yy. Note that we are not making any distinction between u xi, u xy and u yx. Why? Because we are planning to compute all the derivatives. And then propose a formal power series expansions for the solution. Therefore, we are assuming that solution is smooth. And for smooth functions, the mixed or mixed partial derivatives do not depend on the order in which you take the derivatives.

So, thus, in conclusion, 3 and 4 represent 2 equations, 2 linear equations in the 3 unknowns. Therefore, it will be nice to have one more equation so that we can hope to determine the unknown quantities. Namely, the second order partial derivatives of u along the curve gamma 2.

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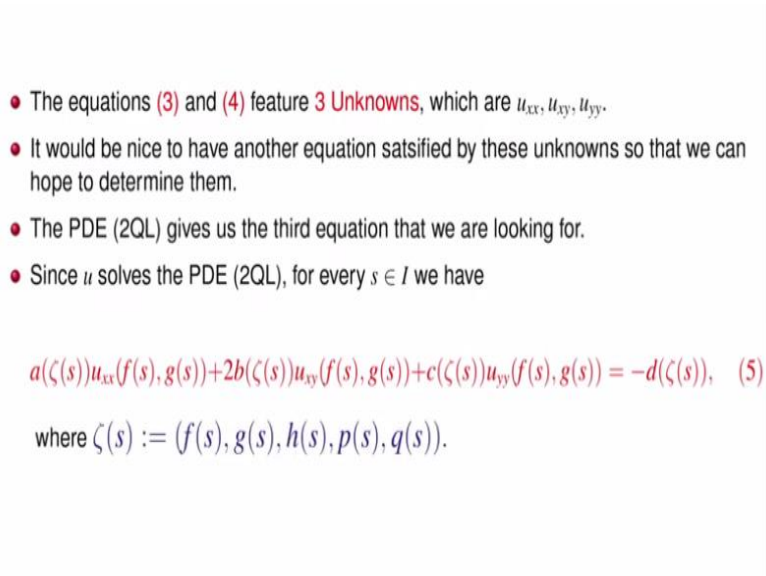

So, the equations 3 and 4 feature 3 unknowns which are u xx, u xy, u yy. It would be nice to have another equation satisfied by these unknowns so that we can hope to determine them. The PDE, the second order Quasilinear PDE gives us a third equation because PDE is an expression for some combination of second order partial derivatives. Thus, we have a u xx plus 2b u xy plus c u yy equal to minus d. I have written in this form.

I have taken d to the other side because I wanted to write finally a system of linear equations for the unknown quantities, where zeta s is f s g s h s p s q s.

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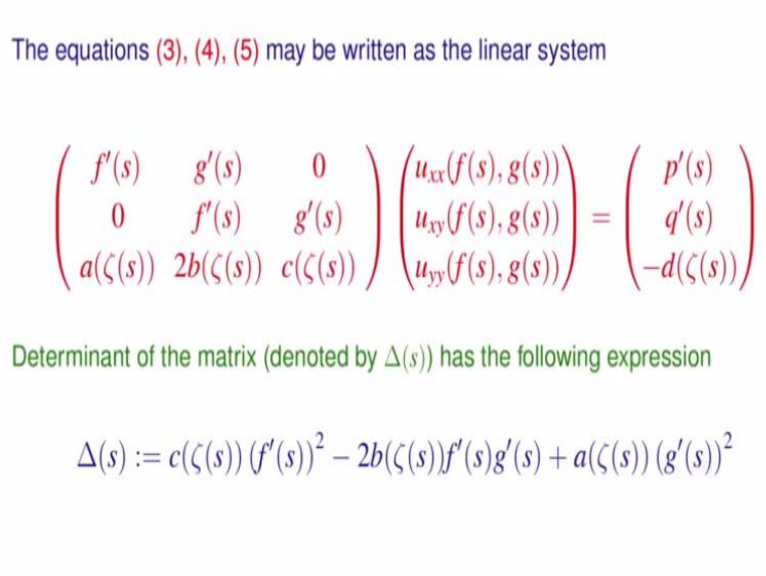

So, the equations 3, 4, 5 may be written as the linear system. In this linear system, notice this is these are known quantities. The matrix on the right hand side is known function of s. Therefore, we can determine uniquely these quantities provided this determinant is nonzero. What is the determinant of this matrix? Let us denote it by delta of s, because it keeps coming throughout this lecture. It has this expression. Once you expand this determinant it turns out it is this is it is this.

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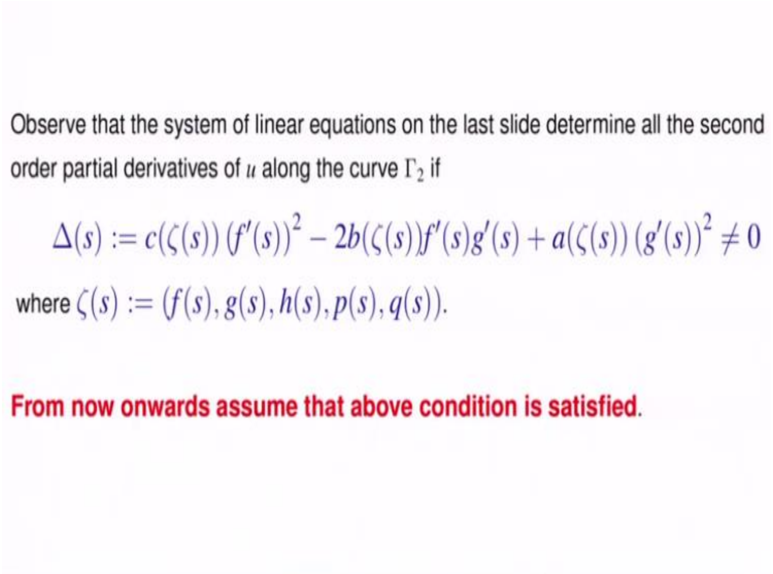

Observe that the system of linear equations on the last slide determine all the second order partial derivatives if delta s is not equal to 0 at points of gamma 2. Wherever it is nonzero you can determine the second order derivatives at that point. So, from now onwards assume that the above condition is satisfied.

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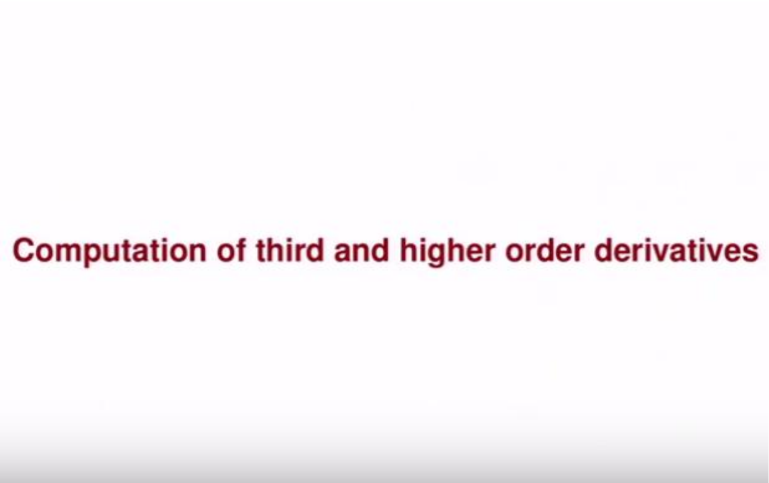

Now, let us go to the competition of third and higher order derivatives.

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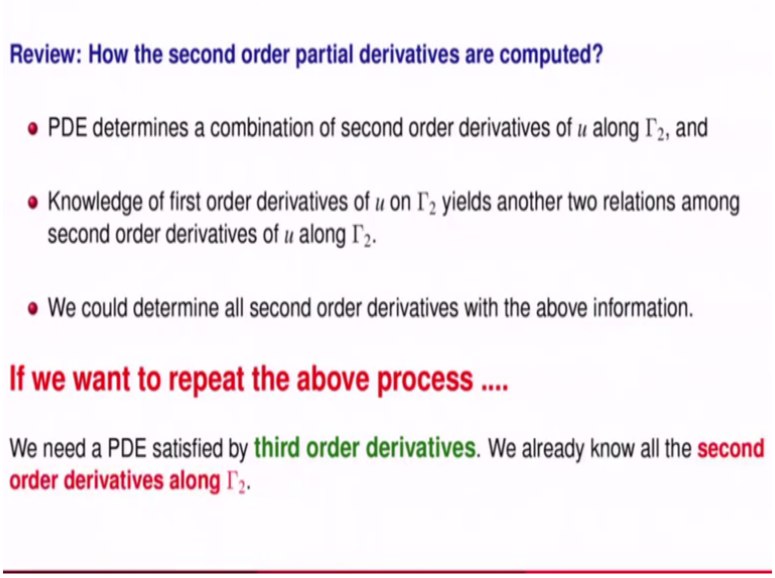

How the second order partial derivatives are computed if you look at. PDE determines a combination of second order derivatives of u along gamma 2. Second thing is knowledge of first order derivatives of u on gamma 2 yields another 2 relations when we differentiate that with respect s. So, that is how we got the 3 equations. And we could get all the 3 second order partial derivatives.

Now, if you want to repeat the above process, what you need is a PDE which gives a combination of third order derivatives. How will you get that? Differentiate the PDE. If you differentiate the PDE with respect to x or y, you will get a new PDE which has the third order derivatives in it. And now, second order derivatives we know on gamma 2. Therefore, if you differentiate that that will give you some more relations which involves third order derivatives of u.

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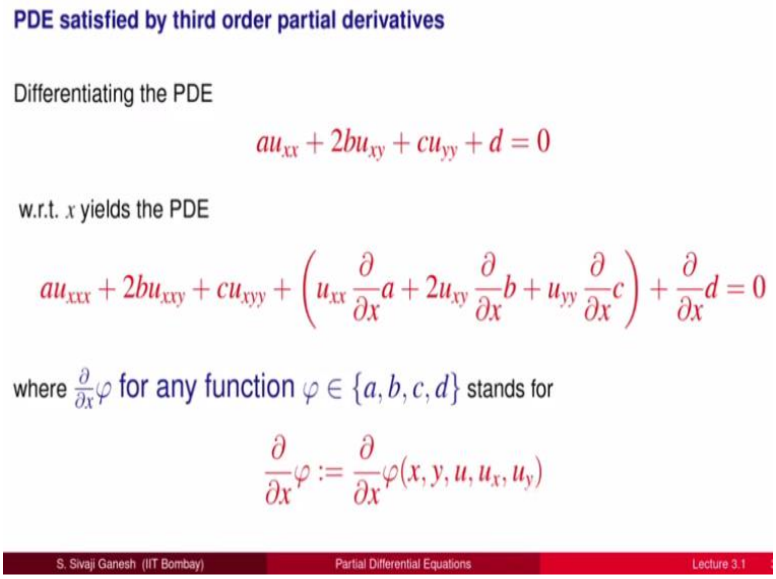

PDE satisfied by third order partial derivatives we want to find. So, let us differentiate the given Quasilinear equation with respect to x. So, by product rule, it turns out to be this. Notice the third order partial derivatives are appearing here and their coefficients are a, 2b and c which are exactly the coefficients in the given equation. And this is nothing special for differentiation with respect to just x.

It will also be the same when you differentiate this with respect to y. There will be a third order derivatives a different third derivatives, but coefficients are a, 2b and c. Even if you differentiate it 10 times even then you will get if suppose you differentiate 10 times, then you get an equation which is 12th order derivatives, but with same coefficients a, 2b and c. This will not change. So, after differentiating we get this equation.

Now, we need to explain slightly what this notation stands for. Notice here a or a, b, c, d, they are all functions of xy, u of xy, u x of xy, u y of xy. So, that we are differentiating with respect to x. That is why we have written this kind of notation. So, let us introduce what this notation is. So, it is dou phi by dou x for any function phi a or b or c or d. What it stands for is this, dou by dou x of phi of x y u of x y u x of x y u y of x y.

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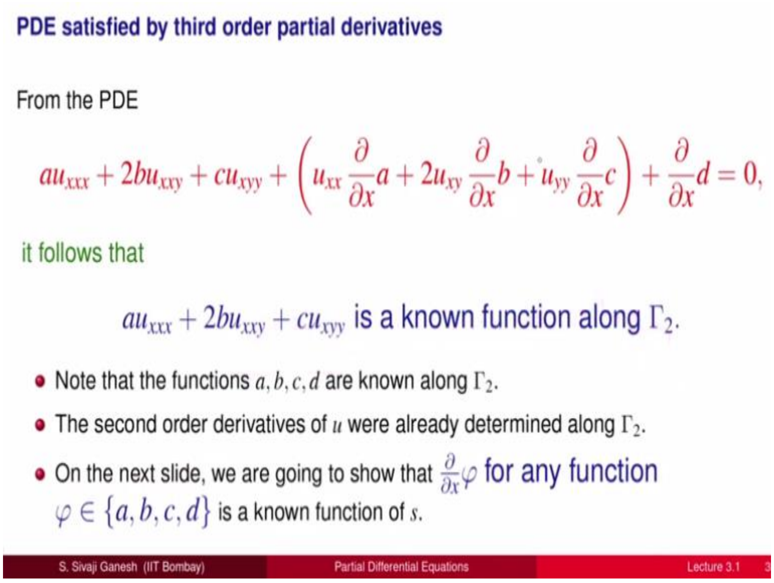

So, from the PDE that we obtained after differentiating the given PDE with respect to x, it follows that au triple x plus 2b u double x y plus c u x double y, namely this. This is a known function along gamma 2 or this is an known function of s provided the rest of the things are known functions of s. Notice here a, b, c, d are known functions. On the second order partial derivatives u xx, u xy, u yy have already been determined along gamma 2.

Therefore, they are known functions. And on the next slide, we are going to show that dou by dou x phi for any of these functions a, b, c, d is a known function of s. And then it follows that these are all known functions of s. And hence, this quantity is a known function of s. That means, this particular combination of u triple x u double x y and u x double y is known function of s.

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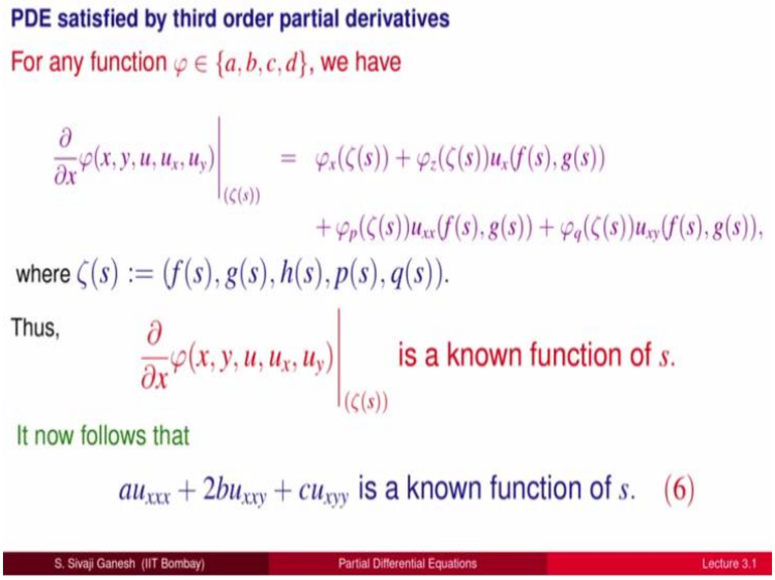

Take any function phi in a, b, c, d. What is this? This by chain rule is exactly this. Differentiate phi with respect to x at this point zeta s. Differentiate phi with respect to the third variable which we are calling z and then that is u. So, u, derivative of u with respect to x. That is why u x. And phi, this is p, this is q. So, phi p and phi q and this is u x. Therefore, it is u double x. This is u y. Therefore, it is u xy. Therefore, this is a known function of s.

The conclusion is that u a u triple x plus 2b u xxy plus c u xyy is a known function of s. What is remaining is u triple y. That is not appearing here. For that, we need to work separately. We will discuss that later.

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The following system of equations holds for third order derivatives at the point $(f(s), g(s)) \in \Gamma_2$:

$$\frac{d}{ds}\left(u_{xx}(f(s), g(s))\right) = u_{xxx} f'(s) + u_{xxy} g'(s) \quad (7a)$$

$$\frac{d}{ds}\left(u_{xy}(f(s), g(s))\right) = u_{xxy} f'(s) + u_{xyy} g'(s) \quad (7b)$$

$$\frac{d}{ds}\left(u_{yy}(f(s), g(s))\right) = u_{xyy} f'(s) + u_{yyy} g'(s). \quad (7c)$$

Here, all 3rd order partial derivatives of u are evaluated at the point $\zeta(s) := (f(s), g(s), h(s), p(s), q(s))$. The functions on LHS are known.

S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations

The following system of equations holds for the third order derivatives at the point f s g s in gamma 2. See, we knew u xx, u xy, u yy as a function of s. These are been already determined. Therefore, we can differentiate them, d by ds, d by ds, d by ds. So, earlier for the first order derivatives we called p q as u x and u y. Now, we can call r, s and t. But, it will introduce new notations. I want to avoid that. That is why I am retaining it as it is.

But, by chain rule, this quantity is given in terms of a combination of third order partial derivatives of u. So, this we can get this expression. So, here, all third order partial derivatives are evaluated at this point zeta of s. The functions on the LHS are known because we know all the second order partial derivatives of u along gamma 2. So, they are all known functions of s and hence their derivatives.

And on the right hand side, we know f prime, g prime. The only thing we do not know is the third order partial derivatives which we are trying to determine.

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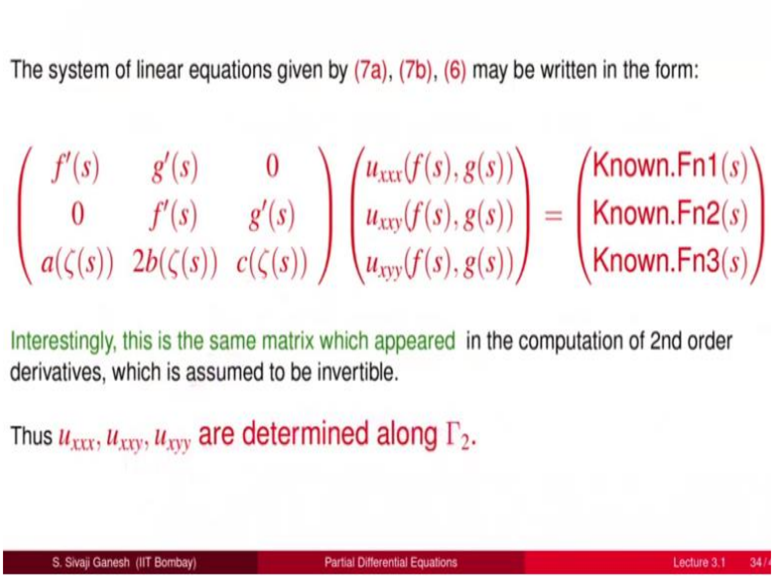

So, the system of linear equation is given by 7a, 7b and 6. 6 is the equation that we obtained after differentiating the given equation with respect to x. And the 7a and 7b are the first 2 equations on this slide. And the right hand sides are all known functions. I am not writing explicitly what they are because that is not important for us. What we need to know is these are known functions. These are the unknown functions which we are trying to determine.

And this matrix is interesting, because exactly the same matrix that appeared in the computation of second order partial derivatives. And we have assumed that is invertible. Therefore, we can determine all these 3 derivatives. Now, we had an option of writing equations u triple y also featuring. So, we can write 4 equations. But, for this reason, I avoided that. This is convenient for us. u triple y, we will do similarly. How to find u triple y?

So, thus, u triple x, u double x y, u xyy are determined along gamma 2. What remains is to find u triple y.

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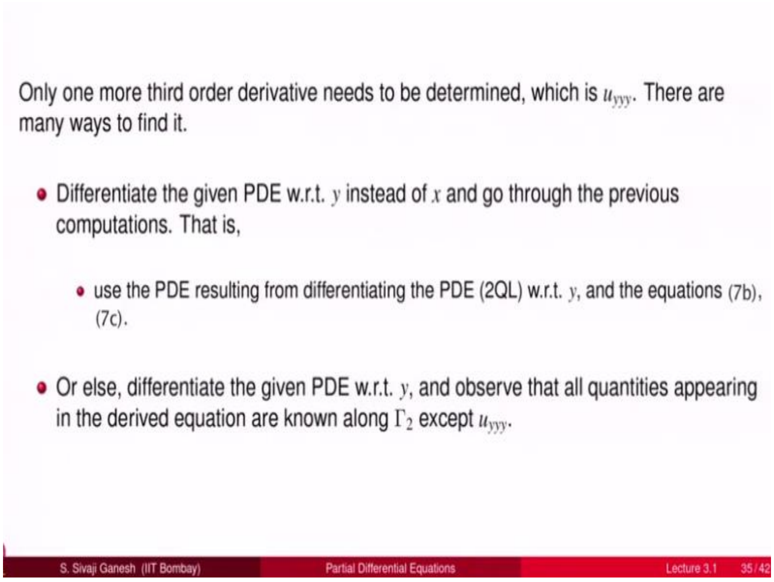

How do you get that? Many ways. One of them is do the same procedure instead of differentiating the PDE with respect to x differentiate with respect to y. And consider the last 2 equations on that slide where we had the 3 equations. The equations which came out of differentiating second order derivatives along gamma. So, exactly same computations you repeat. Otherwise, differentiate the given PDE with respect to y.

And then in that only u triple y will be unknown. Rest of the third order derivatives have already been determined. So, therefore, you can determine u triple y. Of course, you would need that the coefficient multiplying u triple y is nonzero.

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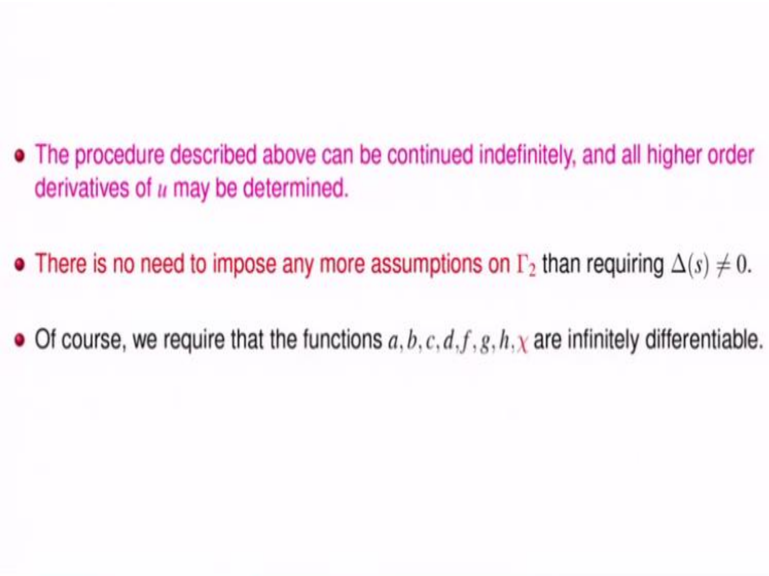

What that can be done. So, the procedure described above can be continued indefinitely. And all higher order derivatives of u may be determined. There is no need to impose any more assumptions on gamma 2 other than requiring delta s not equal to 0. That is important. Of course, you need all these functions to be infinitely differentiable.

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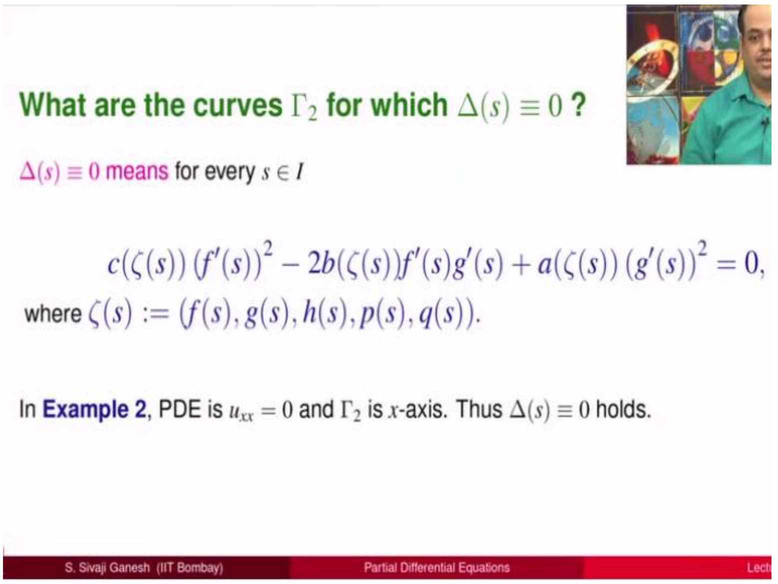

Now, what are the curves gamma 2 for which delta s is identically equal to 0? This is a natural important question because such curves will prevent you from doing these computations. We may not be able to determine the second order derivatives if delta s is identically equal to 0. Or, even if you are able to determine, it is not unique. So, we do not say it is determined uniquely. So, there is a trouble if delta s is identically equal to 0.

What does that mean? It is just means this. This is the equation equal to 0 for every s in I. In Example 2, PDE is u xx equal to 0, gamma 2 is x axis. Delta s is identically equal to 0 holds. And we saw the trouble there. Either there is no solution or infinitely many solutions.

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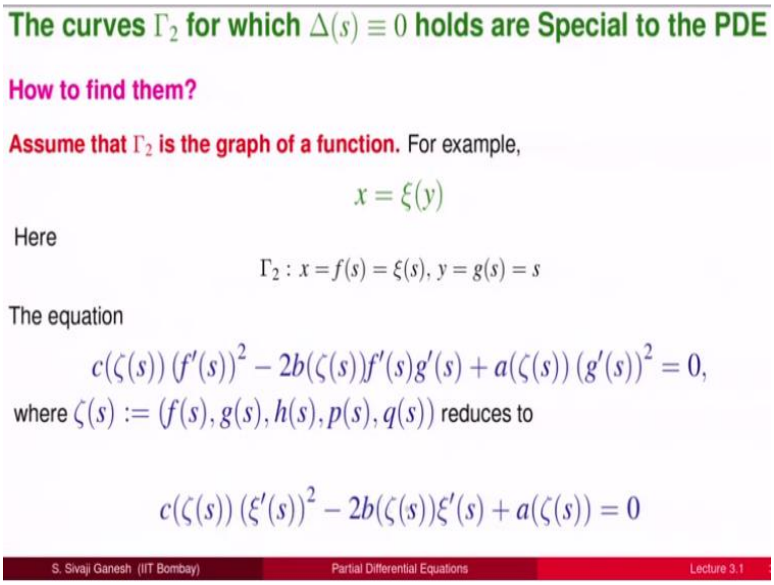

Now, how to find these curves? How to determine these curves? Of course, here the equation that we have is in terms of the parameter s. So, now, how do I find such curves in xy plane? Unfortunately, it involves h s, p s and q s, because we have considered the Quasilinear equations. So, therefore, if you consider, it is a linear equation, much easy. Assume that gamma 2 is a graph of a function.

For example, x equal to xi of y, then gamma 2 will be x equal to f s which is now xi s. And, y equal to g s which is s. Then this equation becomes this equation. Of course, still zeta s is there.

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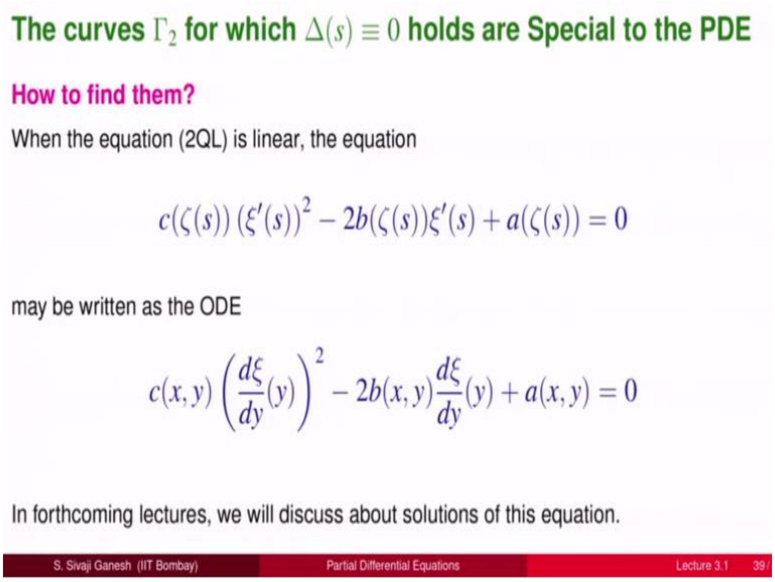

So, when the equation is linear, then c zeta s does not depend on 5 quantities. It only depends on the first 2 quantities which is x and y f s and g s. So, writing in terms of x and y we get this equation. This is an ordinary differential equation, first order, but degree 2. There is a power 2 here. So, dxi by dy, maybe one can compute using the formula of the solutions of quadratic equations. And you are likely to get 2 equations, likely to. You may not get.

We will see that later. In forthcoming lectures, we will discuss about solutions of this equation.

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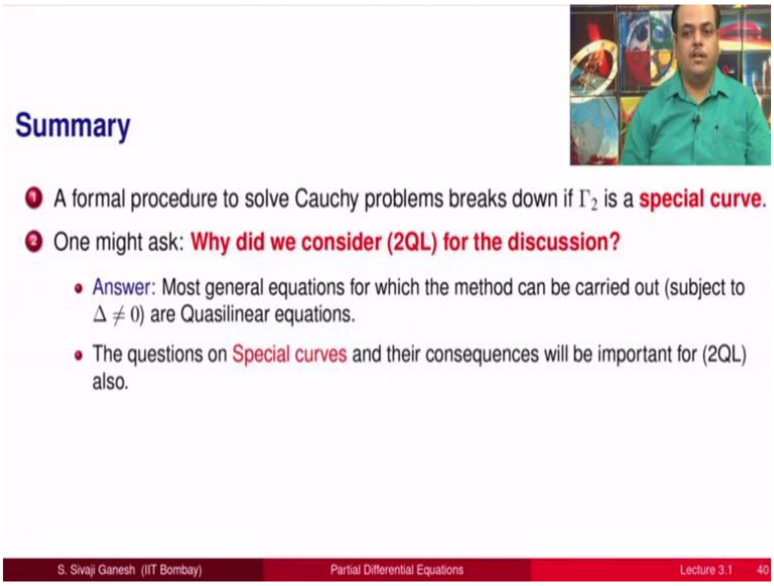

So, let us summarize what we did. A formal procedure to solve Cauchy problems breaks down if gamma 2 is a special curve. One might ask, why did we consider second order Quasilinear equation for this discussion? Simply because most general equations for which the method can be carried out are Quasilinear equations. That is why we have done for Quasilinear equations. For general nonlinear equations, we cannot do.

That a, if you remember a depended only zeta s. Zeta s is already known the moment you compute the first order derivatives. So, whenever you differentiate as many number of times as you want the partial differential equation, the highest order partial derivatives are always multiplied with a, 2b and c which are known functions. And as a result, the linear system that we may write from time to time will be the same whose determinant will always be delta s.

So, that is the advantage. Since, we could do, we have done it for Quasilinear equations. And the questions on special curves and their consequences will be important for the second order Quasilinear equations also.

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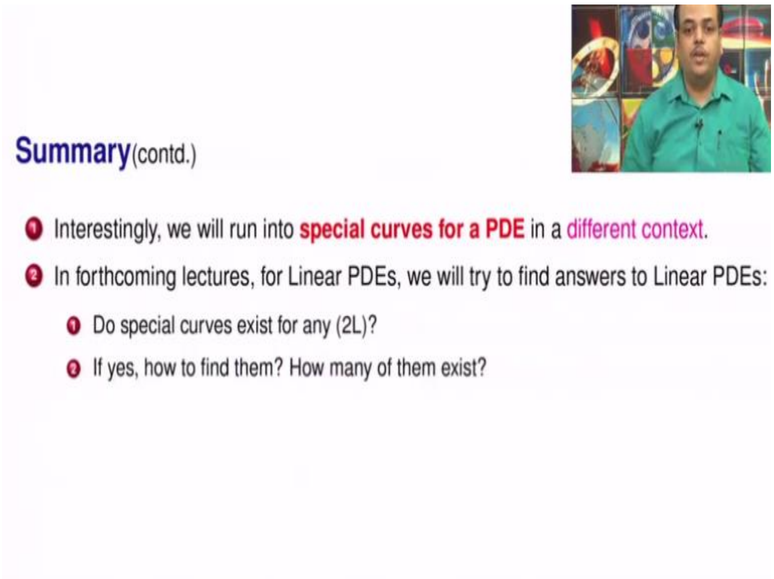

We will come across them later. Interestingly, we will run into special curves for PDE in a different context also which we will see in the next lecture. In forthcoming lectures for linear PDEs, we will try to find answers. We will try to find answers to the following questions. What are the do special curves exists for any second order linear equation? If yes, how to find them? How many of them exist, etcetera?

So, in the next lecture, we will take up another context where delta s makes an (()) (41:41). Delta s identically equal to 0 will become important. Thank you.