**Partial Differential Equations**
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**Lecture – 2.15**
**First Order Partial Differentiation Equations**
**Initial Value Problems for Burgers Equation**

In this lecture, we are going to consider a partial differential equation, which is studied very well in the literature. It is called a Burgers equation and we consider certain initial value problems for Burgers equation. The Burgers equation is one of the simplest Quasilinear equations and it exhibits many features of solutions that is common to nonlinear equations. Common in the sense they occur very frequently for many nonlinear equations.

So, the nonlinear effect in the Burgers equation, we already saw the Burgers equation, but we will again see the equation and we will talk about that.

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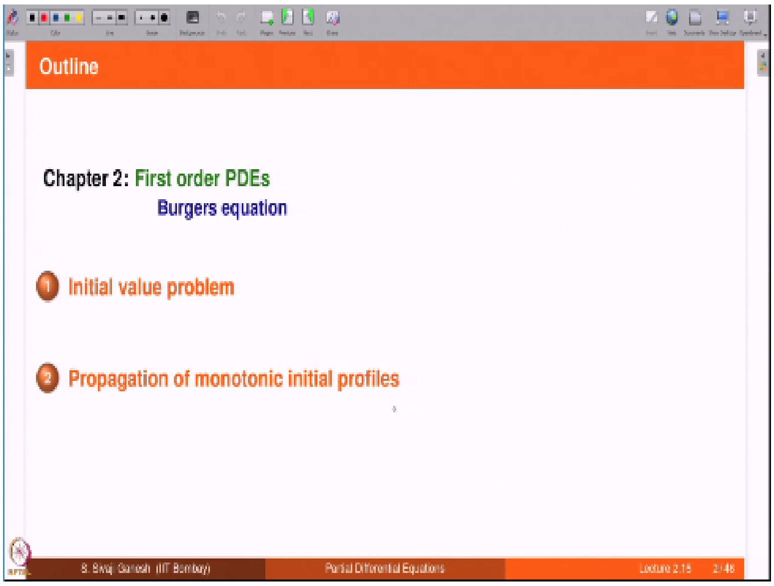


So, we are going to consider the initial value problem It is an equation which has one variable, it is called time variable. So, therefore, its initial value problem. We consider examples basically of the initial value problems, solve them explicitly and then study if I consider an initial condition with a monotonic function what happens to the solution?

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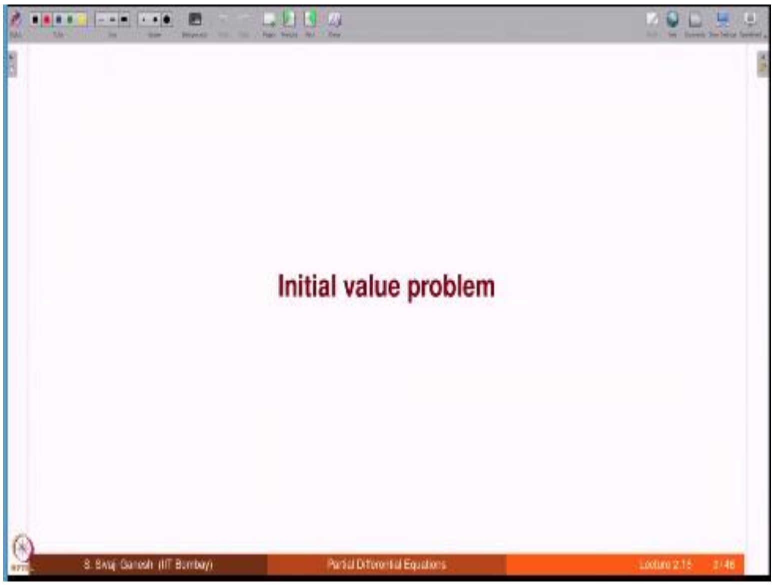


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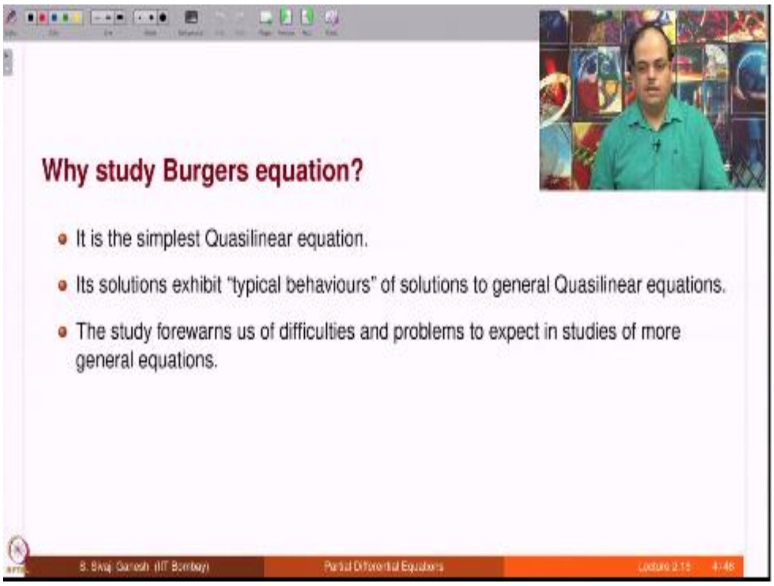


So, question is why study Burgers equation? Why study the special equation? It is a simplest Quasilinear equation. Its solutions exhibit typical behaviours of solutions to general Quasilinear and hence even nonlinear equations. And the study tells us to be, be ready for certain surprises that we may find in studies of more general equations.

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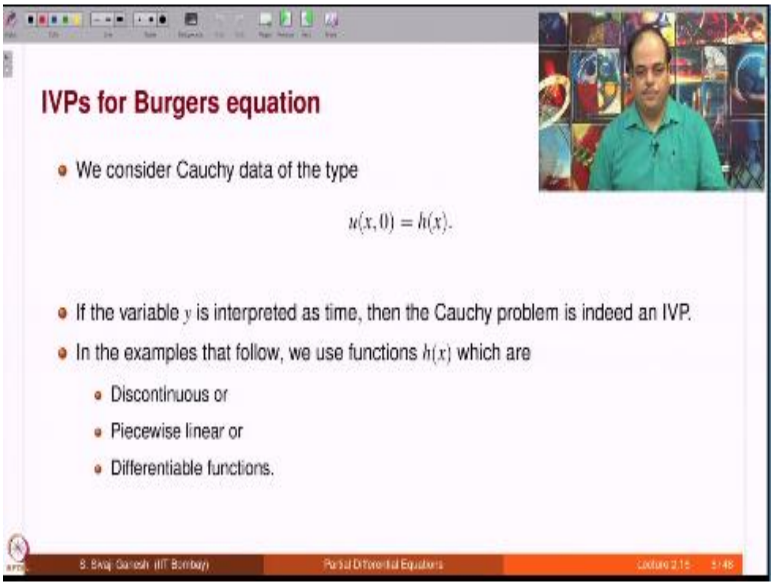


So, we consider Cauchy data of this type u x, 0 = h x. If the variable y, it is a function of 2 variables independent variables x and y. So the variable y is interpreted as time. In fact, it is interpreted as time. So, this is nothing but initial value problem. That is why we write initial value problem for Burgers equation. In the examples that follow we use functions h of x which are monotonic functions, but they are of the following type. They are discontinuous or piecewise linear or smooth functions, differentiable functions.

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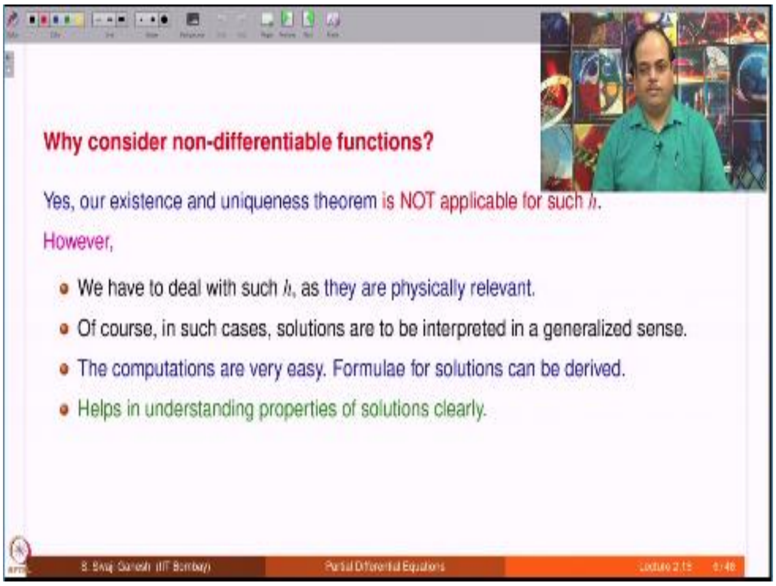


Of course, you may ask this question why consider non differentiable functions? Yes, our existence and uniqueness theorem is not applicable for such h. However, we have to deal with such h, such h, as they are physically relevant. We cannot always say that my theorem allows only smooth functions as the data I will work with only that. No, these are also important. It is also important to study such functions h.

Of course, in such cases, solutions needs to be interpreted differently, not the way that we have been talking about. So far, the ideal solution that we have introduced is what is called a classical solution. So, we have to now generalise the notion of solution, we have to define what do we mean by solution, so that whatever we do is meaningful. And computations are very easy for the kind of functions that we propose to deal with them. And formula for solutions within quotes can be derived.

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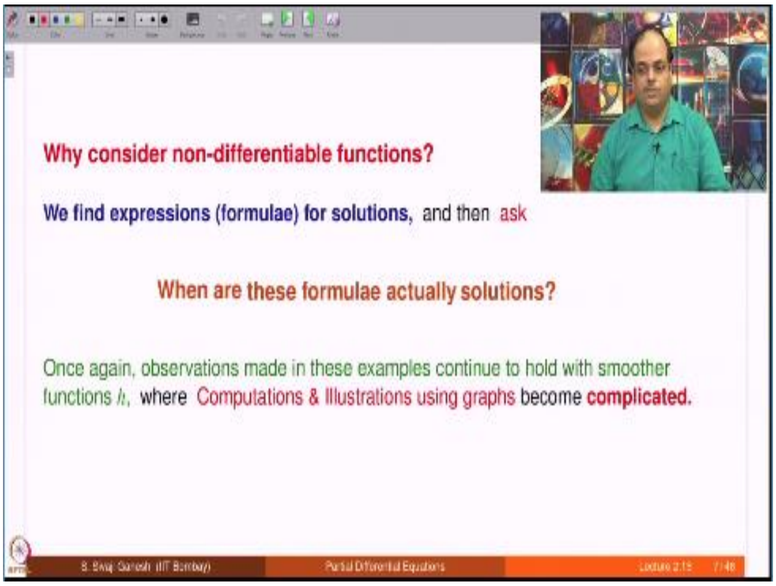


So, it helps in understanding properties of solutions very clearly. So, we find expressions are formula for solutions. We get formulas. Basically some formula like f of x or something like that. Then we understand what is the domain of this f. For what x that f of x makes sense? These questions are to be asked later. So, first we find formulas and then ask, when are these formulae actually give rise to solutions?

Once again observations made in these examples continue to hold with the smoother function, it is not that smooth function things are alright. No. But when you consider smooth functions, you cannot go from one constant to another constant in a smooth way. It takes some time where the function is not constant and computations are difficult and illustrating them by graphs is difficult. But we can be guaranteed or assured that similar difficulties will be there even for such h.

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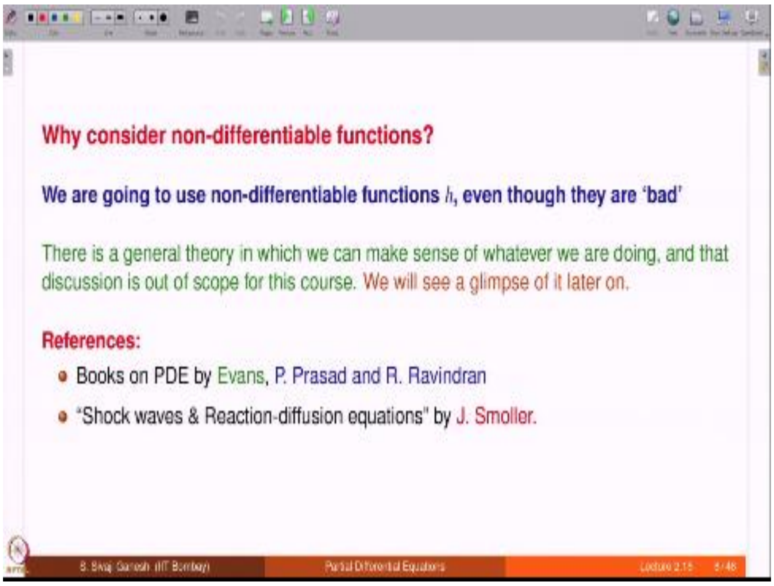


Only thing, it is more complicated. So, why consider non differentiable functions? Again same thing, we are going to use non differentiable functions even though they are bad. There is a general theory, this is some kind of assurance. There is a theory do not worry in which we can make sense of whatever we are doing. So, please go ahead and do not bother about whether it is correct or not. And that discussion is out of scope for this course.

We will see a glimpse of it later on in the next lecture. References or books on partial differential equations by Evans, Phoolan Prasad and Renuka Raveendran. And there is a book by Smoller, Shock waves and reaction-diffusion equations. There are also many more beautiful books written on this kind of material.

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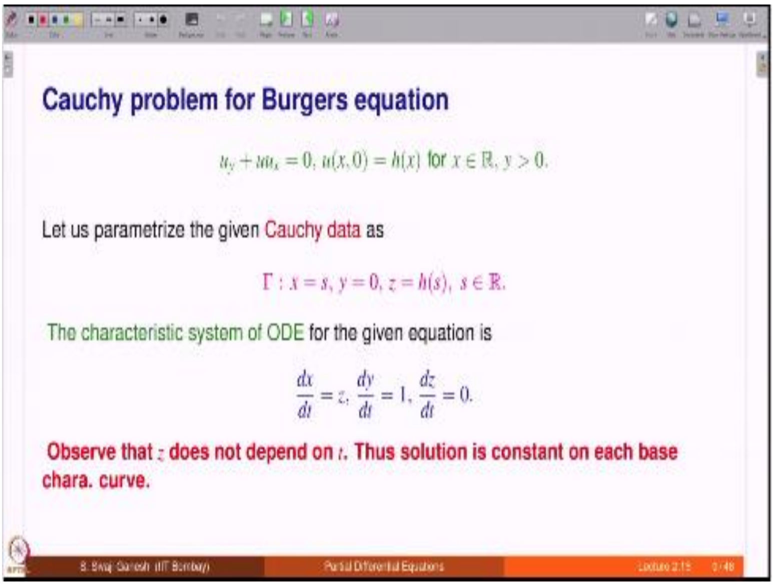

So, this is the Burgers equation. u y + uu x = 0. We would love to write u t + u x = 0, but since we are going to follow a method of characteristics, where t is used as a parameter running on the characteristic curve, we are not using the t here. We still write u y + uu x = 0 and initial condition u x, 0 = h x, x in R, y positive. So, let us parameterize the given Cauchy data. We need to solve the method of characteristics, let us follow that.

So, first is parameterization of the Cauchy data x = s, y = 0, z = h s. Now characteristic system of ODE dx by dt is equal to a. In this case that is au x that is u which is z. dy by dt is 12, dz by dt is 0. We need to solve this set of ODEs, system of ODEs with this initial conditions. Observed that z does not depend on t. What does that mean? It means that solution is constant along each chara, base characteristic curve.

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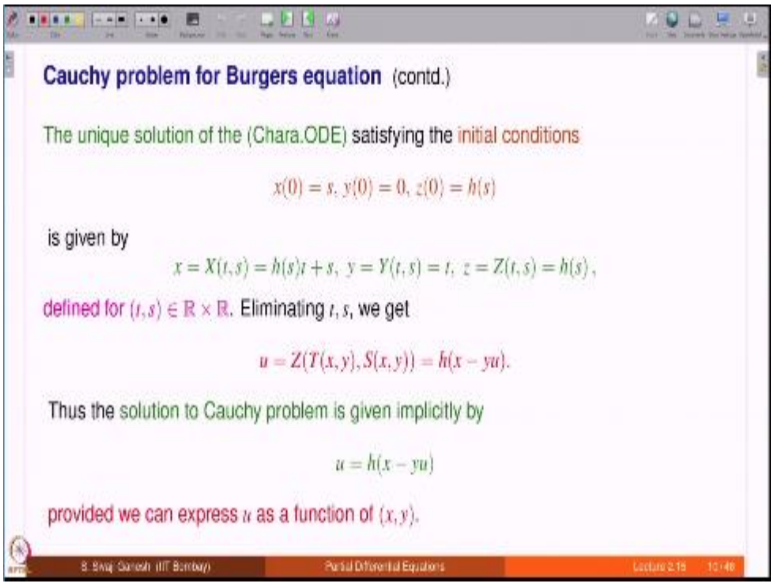


And the solution is satisfying the initial conditions is easily given by this. First is z dz by dt is 0, z is constant at t equals 0. It has to be h of s. So, that is h of s. And dy by dt = 1, therefore y = t + constant. But at 0 it should be 0, it is t. Now when it comes to x dx by dt z. What is z? h s. Therefore dx by dt = h s, if you solve you get this, with this initial condition very easy. Now u equal to h of x – y u. That is what we get using these 3 equations.

Ideally we should solve for t and s in terms of x and y and then go and substitute here. So, h of somebody. So s = x – yu, we are done which comes from here. s = x – h of s is supposed to be a solution u and t is the y. So, that is why you get this implicit expression u = h of x – yu. Of course, when we admit this as an implicit solution, provided you can solve for u as a function of x and y.

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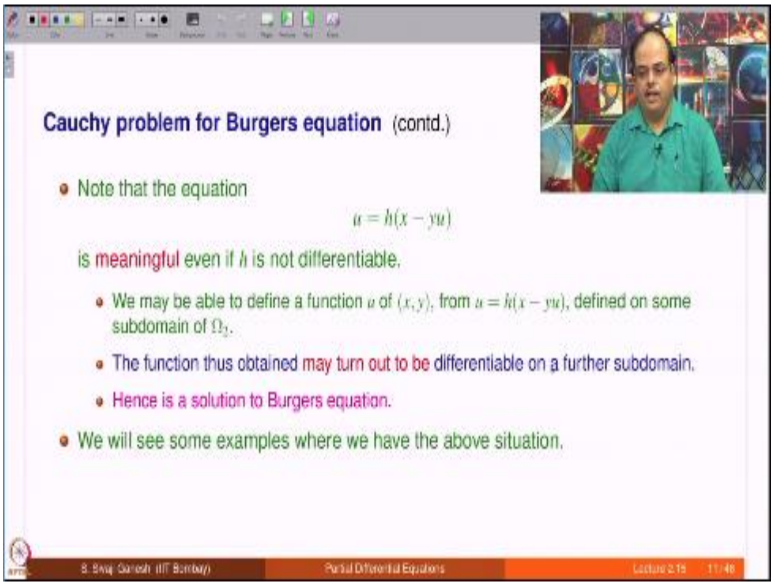


So, note that this equation is meaningful even if it is not differentiable. Really just asking that u = h of x minus yu. Imagine x is continuous equation is meaningful. Now question is whether you can solve u in terms of x and y is another question. We may be able to define a function u of x, y from this equation defined on some subdomain of omega 2. What is omega 2? Recall, it is a production of omega 3.

What is omega 3? Omega 3 is where the equations of the Quasilinear equations are defined. Omega 2 is a projection of omega 3 to xy plane. Solutions will be defined in some sub domain of omega 2. So it is possible. And function thus obtained may turn out to be a solution, may turn out to be differentiable on a further sub domain. Therefore, it will be a solution to Burgers equation.

So, therefore, given this possibilities, what we do in the examples is we arrive at this equation somehow solve for you and then ask you on what domain it is differentiable. Catch hold of that domain and say, this function defined on this domain is a solution or the Burgers equation. That is what we are going to see in the examples later on.

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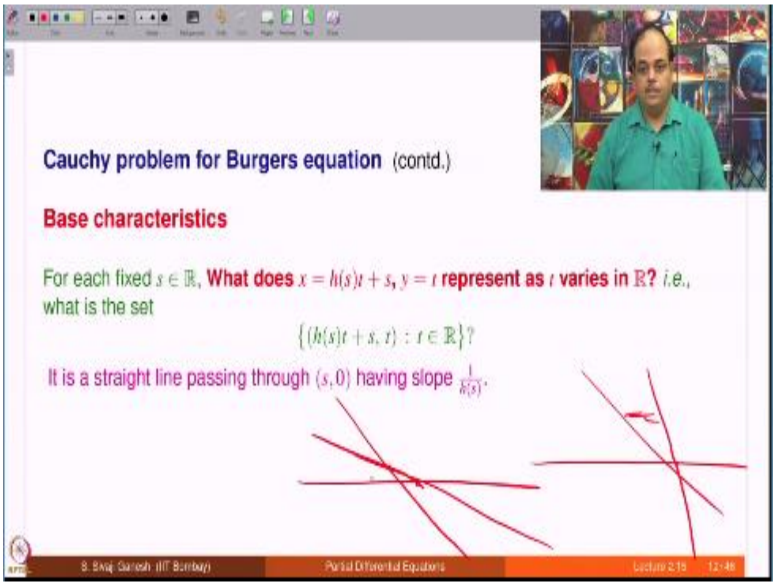


Now, for each fixed s, what does this represent? This is the base characteristics x = h s, t + s, y = t. As t varies, s is fixed. What is the set? It is a straight line passing through the point s, 0. This is the point s, 0. So, it is a straight line passing through s, 0. This point with slope 1 by h s.

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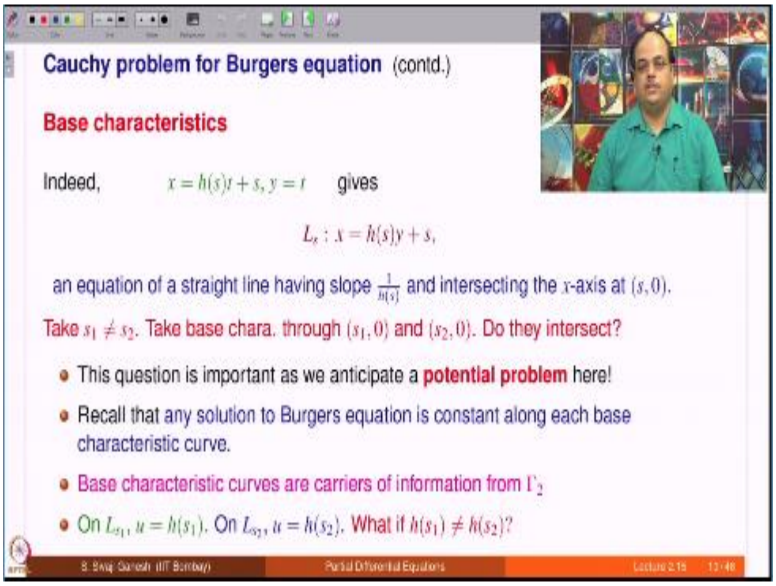


Therefore, what we observed is that it is possible in principle for example, like that. This is some s. At the next point, if the slope is like that, so that the straight line goes like this. Sorry, this is not correct. So, at this point equation goes like that. It is fine, they do not touch here. But on the other hand if you have like this at this point the slope is like that. So, they will intersect.

So, base characters can intersect. It depends on h s. We will see that. So, this base characteristics we use a notation L s. L s is a straight line L for line, s for passing through this point s, 0 and this is the equation. Slope clearly 1 by h s. If you take 2 different s 1and s 2 and take base characteristics through those 2 points, do they intersect? We already saw the picture. This question is important.

Why are we asking this question? This question is important as we anticipate a potential problem. Recall that any solution to Burgers equation is constant along each base characteristic curve. For example, we had this and we had a base characteristic curve like that and we had the base characteristic curve like that. So, this point is s 1, 0. This point is s 2, 0.

What we saw is any solution of the Burgers equation has to be constant on each base characteristic curve and what is the value here? It is h s1 and here sorry, this is h s 2. This is h s 1. So, therefore, on the green line, it is h s 1, on the blue line it is h s 2. What will it be at this point of intersection? Problem. See base characteristics they carry information from the gamma 2. This is gamma 2.

It carries the information which is given namely u must be equal to h x. That is the information given on this x axis. That is being carried forward by these lines which are characteristic curves. So, they are carriers of information from gamma 2. So on L s 1, u is h s 1, on L s 2 u is h s 2. What if they are not same? It means conflicting information is reaching at the points of intersection of the base characteristic curve.

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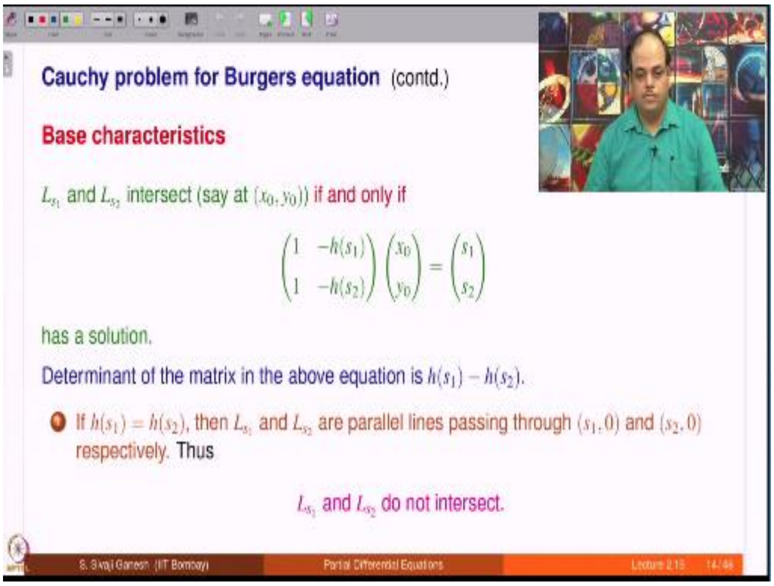


What will happen there? Solution becomes multi-valued? If at all you want to say solution, so there is a trouble at those points. That is why this question is important. Now where do they intersect L s 1 and L s 2? We have 2 equations, straight line equations. When you write down, if they intersect at a point x 0, y 0 that should lie on both L s 1 L s 2. This is the equation for L s 1 first line.

Second one is a equation for L s 2. And it should be satisfied. That means this system should have a solution for x 0, y 0 is a non homogeneous system. So, if the determinant is nonzero, definitely you have a system you have a solution for x 0, y 0. And determinant is h s 1 – h s 2. So, if h s 1 = h s 2, then they are parallel lines, they do not intersect. Therefore no problem.

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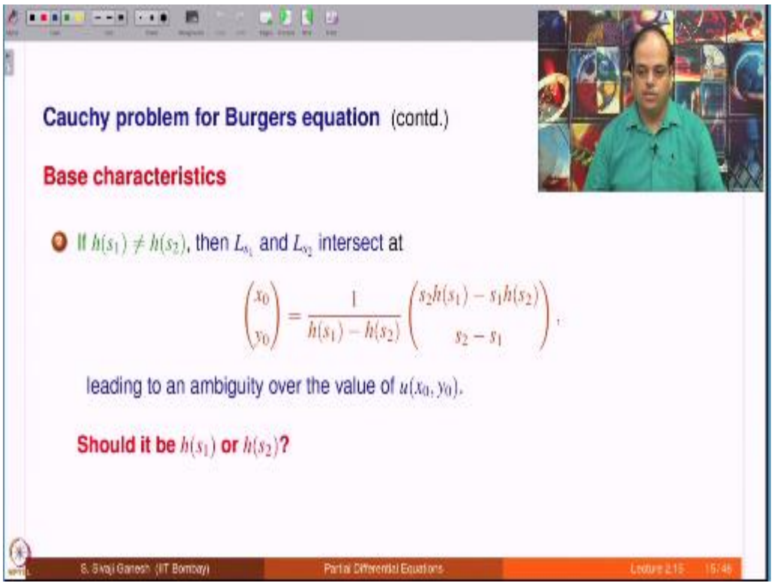

But if they are not equal, then the system has a solution. They intersect at this point leading to an ambiguity in the value of the u at that point x 0 y 0. Because it is both h s 1 and h s 2 and they are not equal. So, there is a problem there is ambiguity. Should it be h s 1or should it be h s 2?

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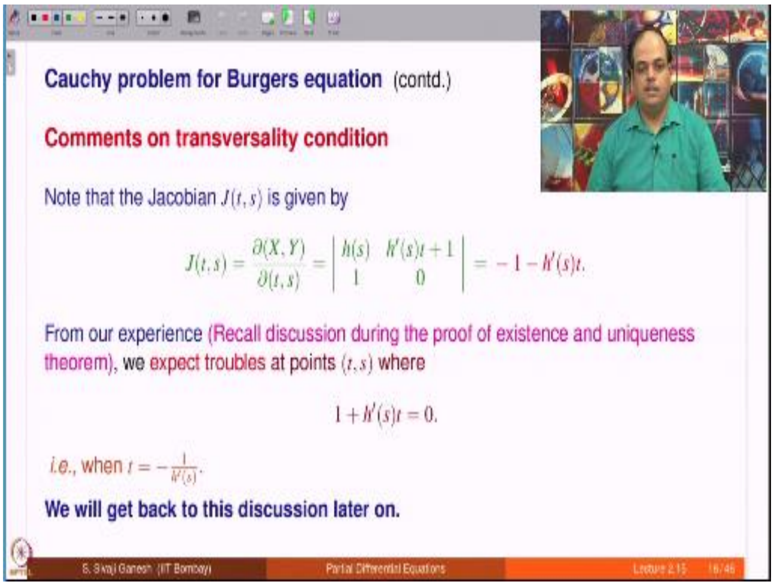


Now, let us look at the transversality condition J t, s, which is do X, Y by do t, s is this. So, this turns out to be minus 0f 1 + h dash s into t. From our existence and uniqueness theorem for Cauchy problems for Quasilinear equations if this is nonzero, no problem. If this is 0 troubles to be expected. Where is it 0? Precisely when h prime s t + 1 = 0. That is from our experience, recall the discussion during the proof of existence uniqueness theorem.

We expect troubles at points where this happens. We will get back to this discussion later, later on.

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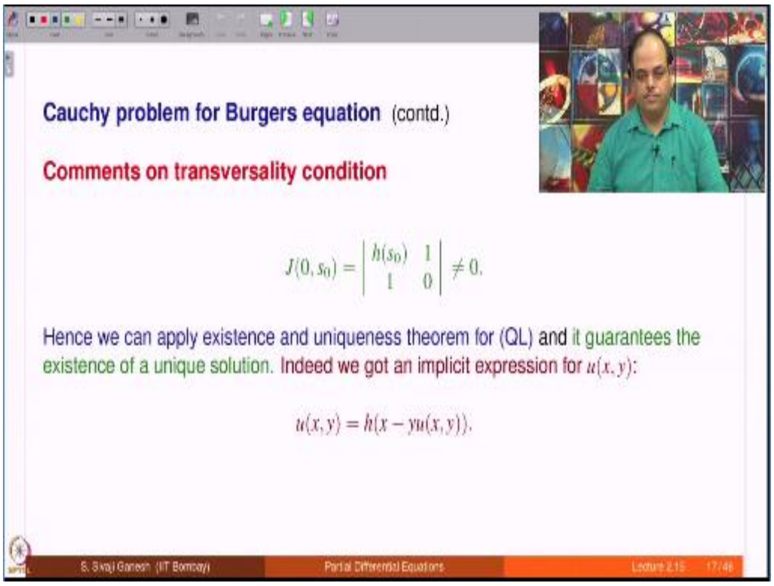


So, J of 0, s 0, this is actually where we used to apply our theorem. Because we do not know at a general t, so t = 0 only we have the information. But in this in the last slide, we have all the information. We have explicitly solved everything. That is why we could write J t, s and we know this expression. But otherwise, we do not know the particular second column at arbitrary t only at t = 0.

So, J of 0, s 0 is –1. It is never 0. So, it just means that there is a existence uniqueness at every point which is nearby at some points nearby the datum curve. This is a solution. Solution exists. Good.

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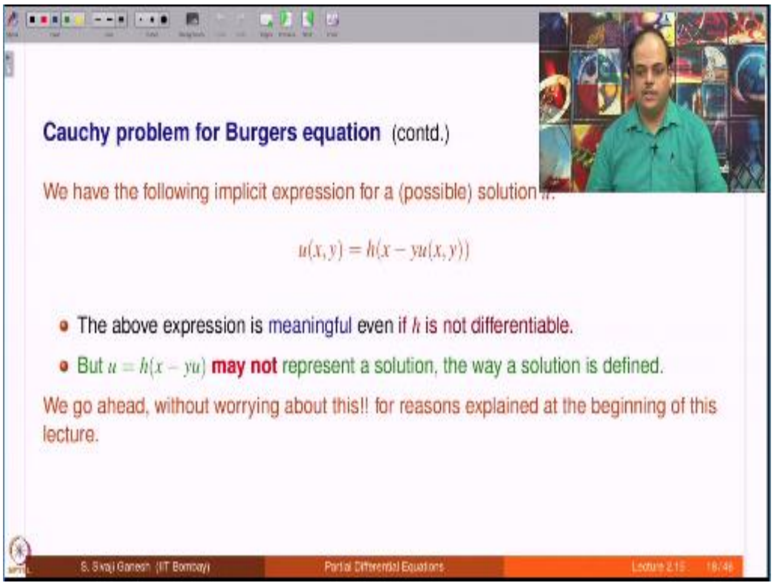


Once again as before the expression is meaningful even if h is not differentiable. But this may not represent a solution, the way a solution is defined, there could be problems. We go ahead without worrying about this, for reasons explained at the beginning of this lecture. Let us look at specific examples.

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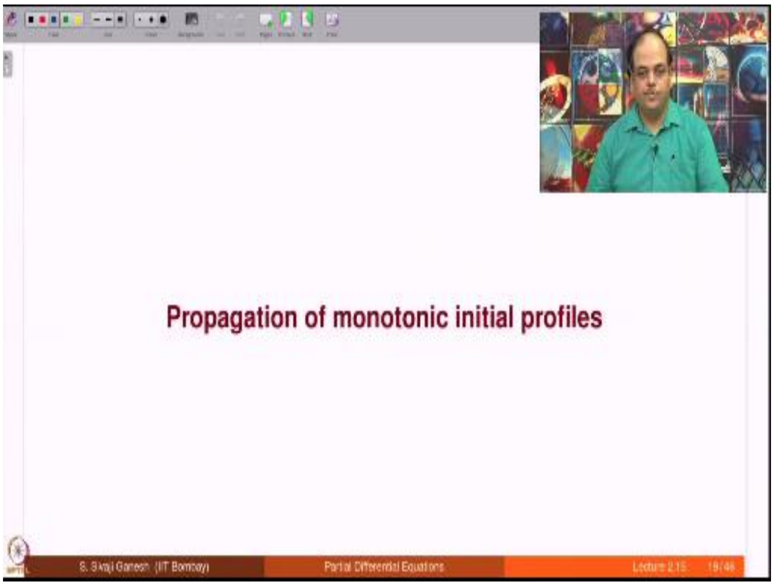


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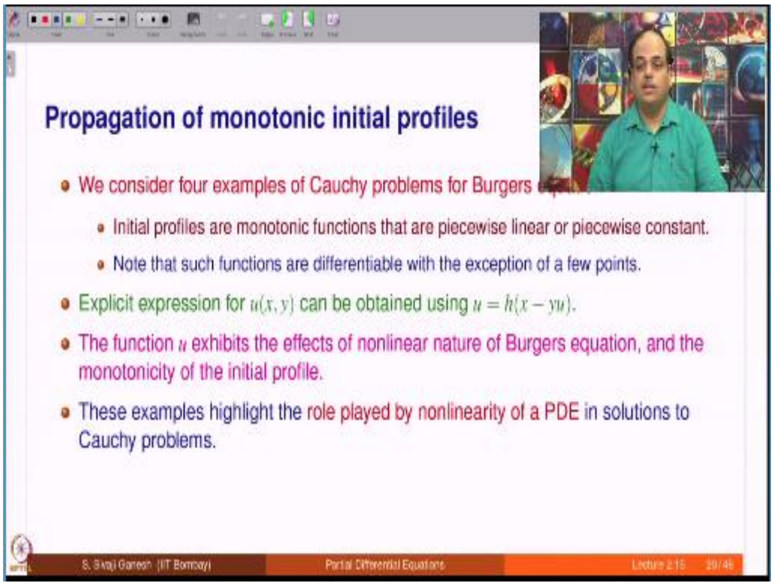


So, we are going to consider 4 examples of Cauchy problems for Burgers equation. Initial profiles, the function h of x. It is graph that can be called as initial profile. Those are monotonic functions that are piecewise linear or piecewise constant. Note that such functions are differentiable. With the exception of a few points, monotonic functions are very close to being differentiable. So, they are differentiable almost everywhere.

An explicit expression for u can be obtained in these examples. The function u exhibits the effects of nonlinear nature of Burgers equation and the monotonicity of the initial profile, both. So, these examples highlight the role played by the non linearity of a partial differential equation in solutions to Cauchy problems.

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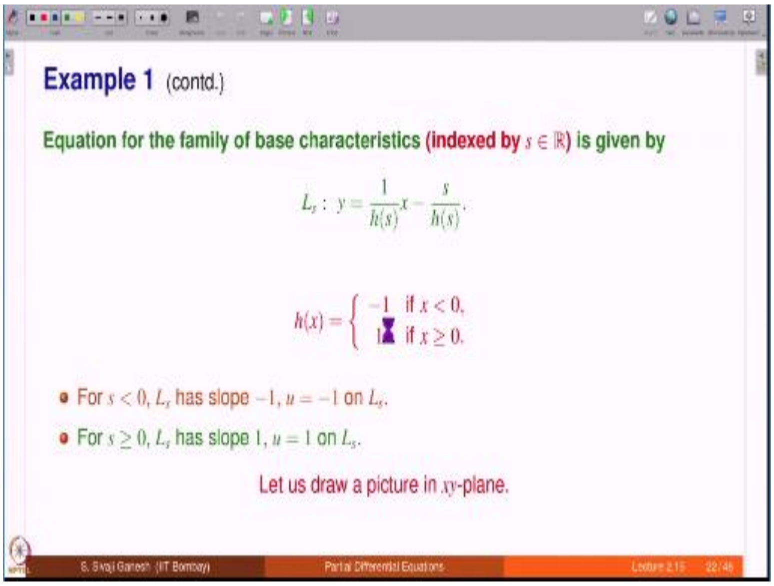


Now, in this example, method of characteristics fails to determine a solution in some region of the upper half plane, which means it is not global with respect to domain. So, consider Burgers equation with the Cauchy data – 1 for negative x and 1 from x = 0 onwards. L sv already observed. The family of base characteristics are equations with slope 1 by h s. So, for s less than 0, L s has slope – 1 and u is – 1 on that because the data carry forward is – 1.

For s greater than equal to 0 slope is 1 because h s is 1. 1 by h s is 1. h s is 1. Therefore, this is 1, slope 1. And information it takes is 1. Solution will be 1. Let us draw a picture.

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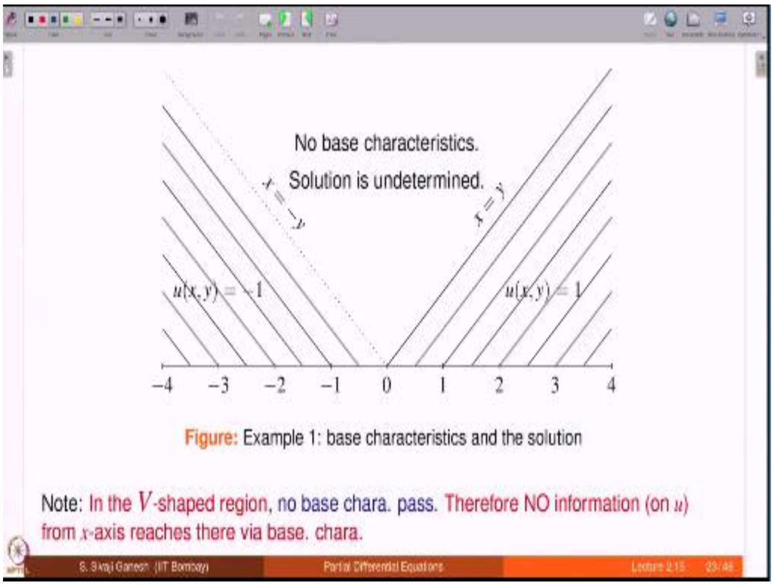

So, this is the picture. So, here we had s negative This is for s greater than or equal to 0. These are the base characteristic curves. These are the base characteristic curves corresponding to positive s. So, these are all lines parallel to y = x, these are lines parallel to y = – x. Now, what happens in the in between this region. In this region no base character curves.

So, no information is being passed from the initial data that is the gamma 2 into this region which is in the V shaped region? That is means this not included because h of x was like that? h of x was – 1 if x is less than 0 that is the reason or reason why the 0 line is not included x = – 1. So, in the V shaped region, no base characteristics pass. Therefore, no information on the solution from x axis reaches there via base characteristic curves.

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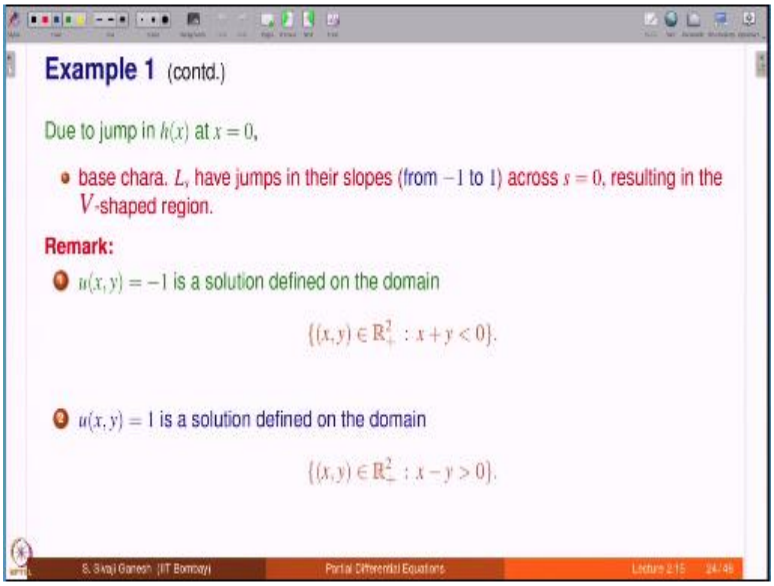


These are happening because the jump in h x, because suddenly shifts from – 1 to 1. Therefore, the slopes also across s = 0. That is what results in the V-shaped region. Now, u = – 1 is a solution defined on this domain. You can see in the picture, 1 is a solution on this domain.

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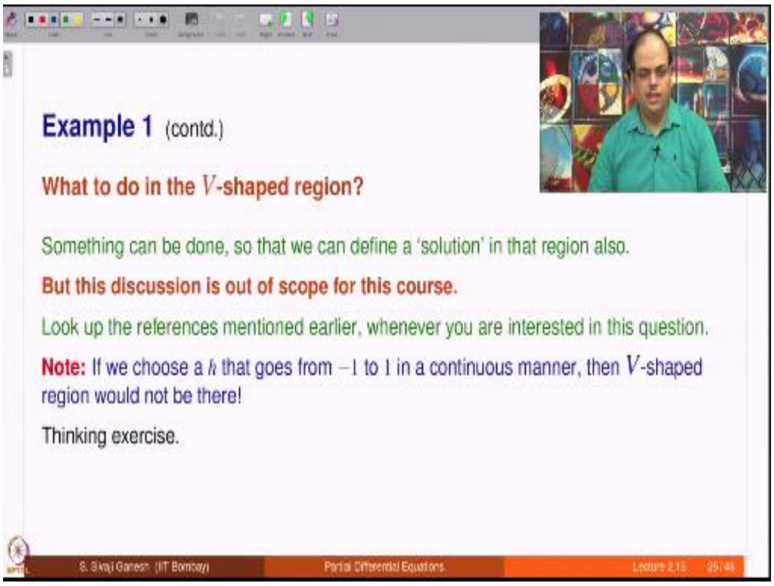


So, what to do in the V-shaped region? That is a natural question. What happens in the V-shaped region we saw were base characteristic curves. But can you define a solution in the V-shaped region? These are questions that we ask. Yes, something can be done, so that we can define a solution. If you observe the solution, I have put it in quotes means some generalised notion or solution. But this discussion is out of scope for this course.

So, look up the references that I mentioned earlier whenever you are interested in this question. So, if we choose a h that goes from – 1 to 1 in a continuous manner, then V-shaped region would not be there. We observed that that is because a jump, sudden jump from – 1 to 1we had this problem. So, think about this.

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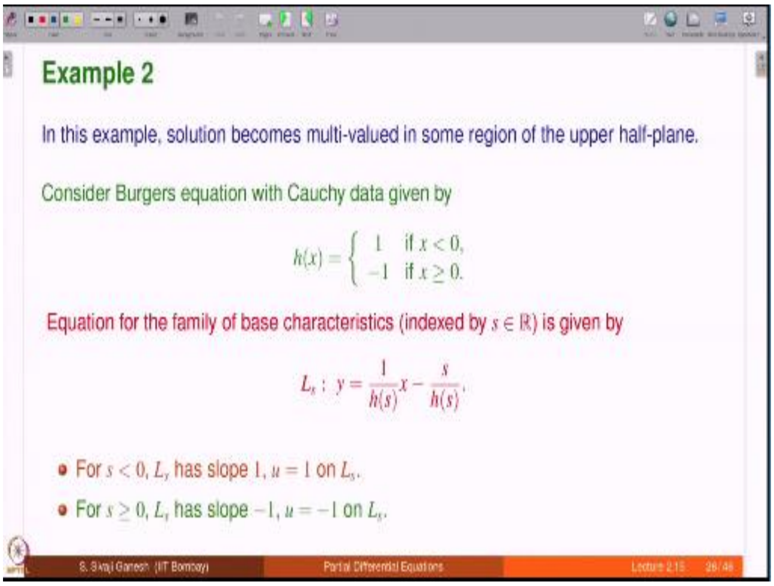

Now let us look at the second example. Here what happens is solution becomes multivalued in some places. So, h is now 1, – 1. It is a decreasing function. It is a monotonic function, but decreasing equation for the base characteristics is the same still. So, s less than 0, L s has slope 1. And it carries, solution will be 1 there. And for s greater than or = 0 h is – 1. Therefore, slope is – 1 and solution will be –1.

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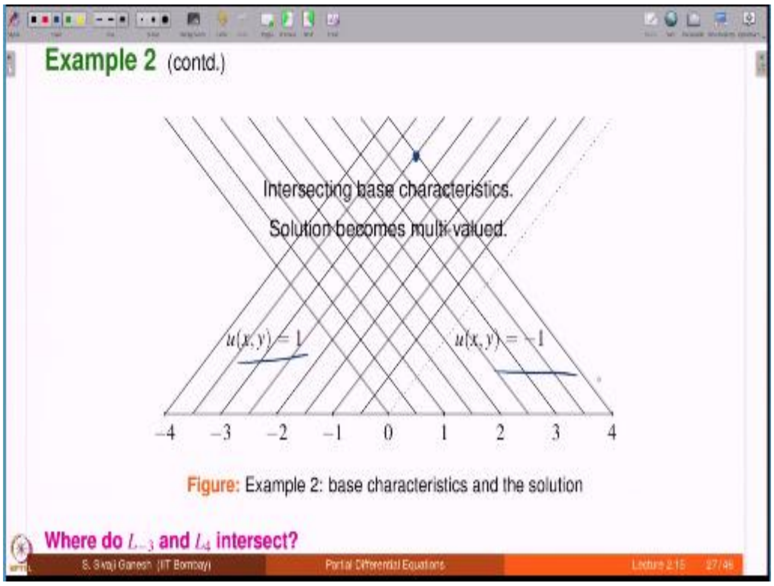


Let us look at the picture here. But these families intersect in this V-shaped region. In this region, maybe this line is included. But here base characteristics are intersecting and they carry 2 different informations. So, if you want to say the solution, at this point, it is both 1 and – 1, if you want to say that. So, you have to change your notion of solution, but we see that conflicting information is reaching there.

So, there is some problem. L – 3 and L 4 for example, L – 3 is this line and L 4 is this line. So, they intersect here. At this point, it is both 1 because of this and – 1 because of this.

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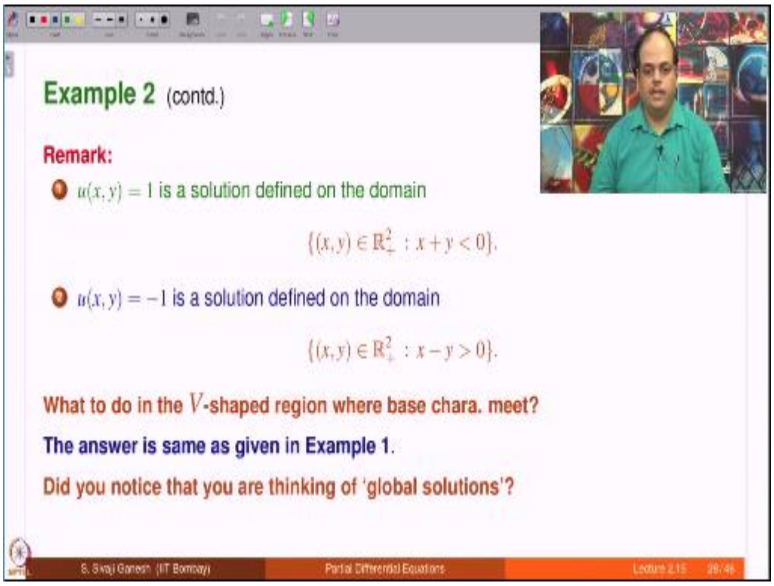


Now, we can write down where 1 is a solution and – 1 is a solution. Once again there is a problem in the V-shaped region. The answer is same as given before something can be done, which is out of scope. You can read the books that I have suggested. Did you notice that you are thinking of global solutions by asking these questions. Once you get some solution in the V-shaped region, you have the solution in the entire upper half plane.

So, it is there the back of our mind.

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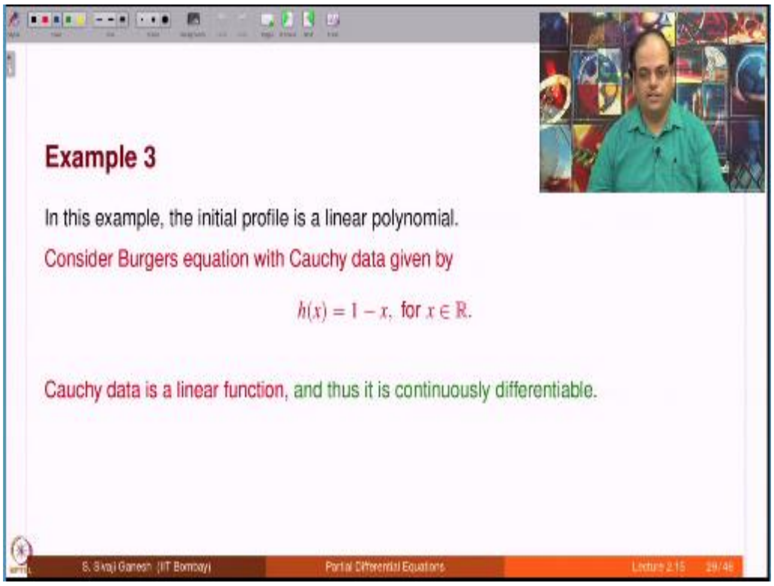


Let us look at the third example. Here initial profile is very nice initial profile. It is a linear polynomial 1 – x. No jumps, nothing. Continuously differentiable C infinity it is.

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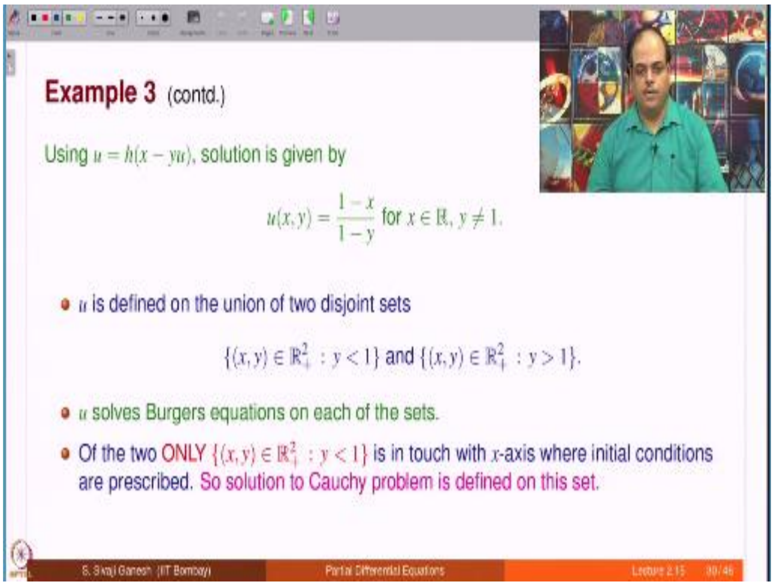


Analytic polynomial afterall. So, let us substitute and get the expressions for you. We get u x, y = 1 – x by 1 – y, obviously y should not be equal to 1. So, it means the line y = 1, u is not defined. So, it is defined on the union of 2 disjoint sets one above the line y = 1, one below the line y = 1. You solve Burgers equation on each of these sets. Of the 2 only this one below the line y = 1 is in touch with x axis which is where that is a gamma 2.

That is where the initial data is prescribed. Therefore that is a solution with this domain.

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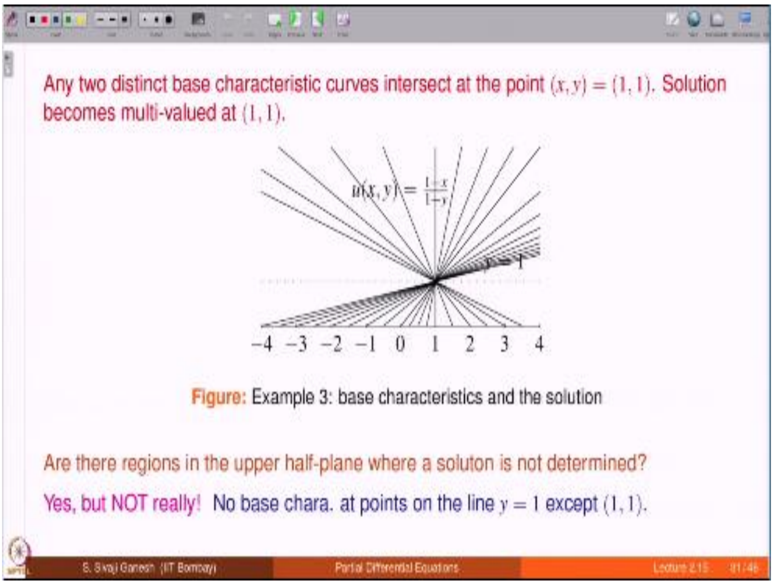


So this is the picture. Any 2 distinct base characteristic curves intersect at this point 1, 1. And solution becomes multivalued at 1, 1. Are there regions in upper half plane where a solution is not determined apart from the line y = 1. It looks like yes because there seems to be no characteristics passing through this. But that is not the problem because as you see the line from here is going like this.

So if you want to go like this, it might have gone out of the picture that is why you are not seeing. So, characteristic do fill up all these points except these dotted points base characteristics go through each and every point.

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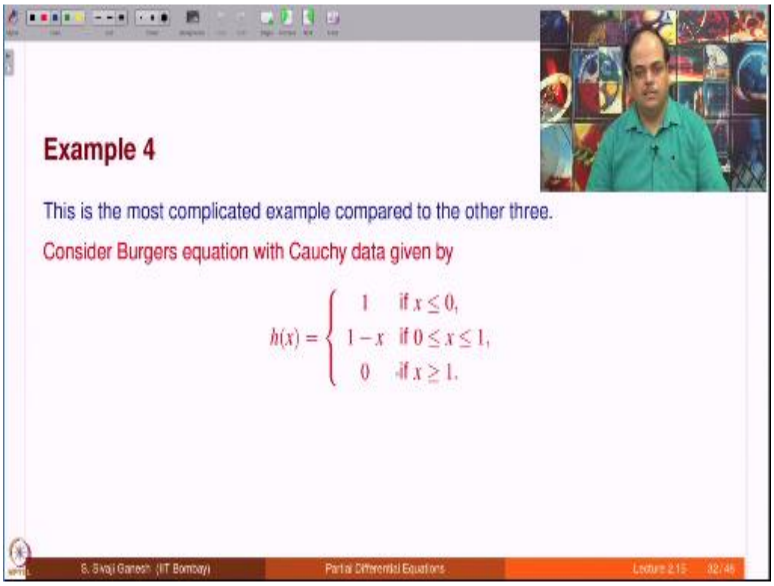


Now this is the most complicated example. Why? Because it is mixing all the 3 things, 1, 1 – x and 0. So it is a piecewise constant and in between linear we will see, what is going to happen.

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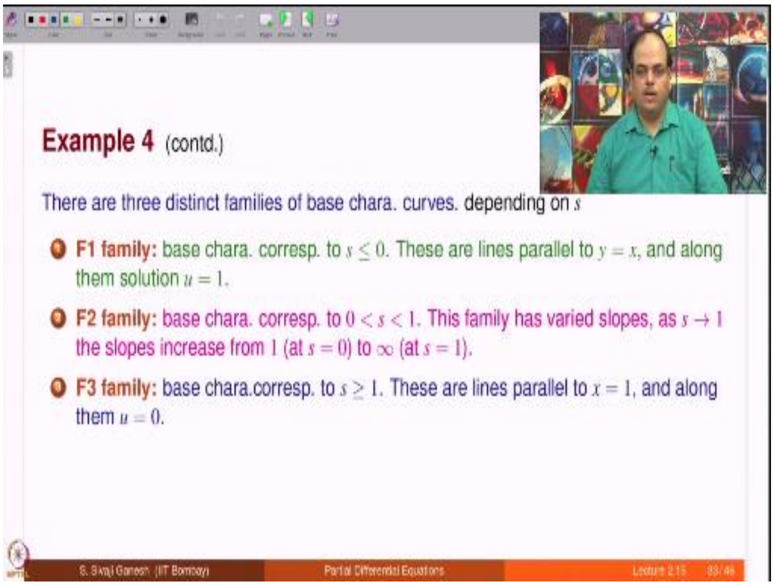

There are 3 distinct families of base characteristic curves depending on s. Earlier we had only s less than 0, s greater than 0. Now what will we have? We will have s less than or equal to 0, one case, between 0 and 1 and bigger than or equal to 1. Let us call give names. So, it will be easy for us to refer to. F1 family corresponds to s less than or equal to 0. These are lines parallel to y = x because the initial condition is 1.

So, therefore, slope is 1 and therefore, the value that solution takes along them will be 1. F2 family that 1 – x. So solution will be like 1 – x by 1 – y. F3 family is where the solution will be 0 and the slope will be like 1 by 0. So, their lines parallel to x = 1.

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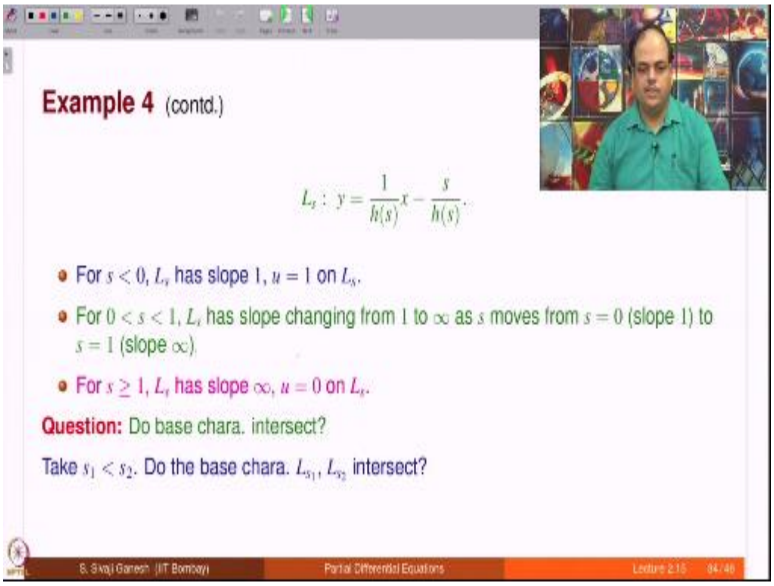


Yes. L s has slope changing from 1 to infinity whenever s is between 0 and 1, as s goes from 0 to 1. For s greater than equal to 1, L s has infinite slope as we saw. These are lines parallel to x = 1. So, question is do base characteristics intersect? So, we are to analyse cases whether family any 2 members of family F1 intersect, F1 and F2 intersect, F1 F3 intersect, any 2 members of family of F2 intersect or some member of F2 intersect some family or some member of F3. These are the various cases we have to analyse. So, take s 1 and s 2, do the base characteristics. L s 1 L s 2 intersect.

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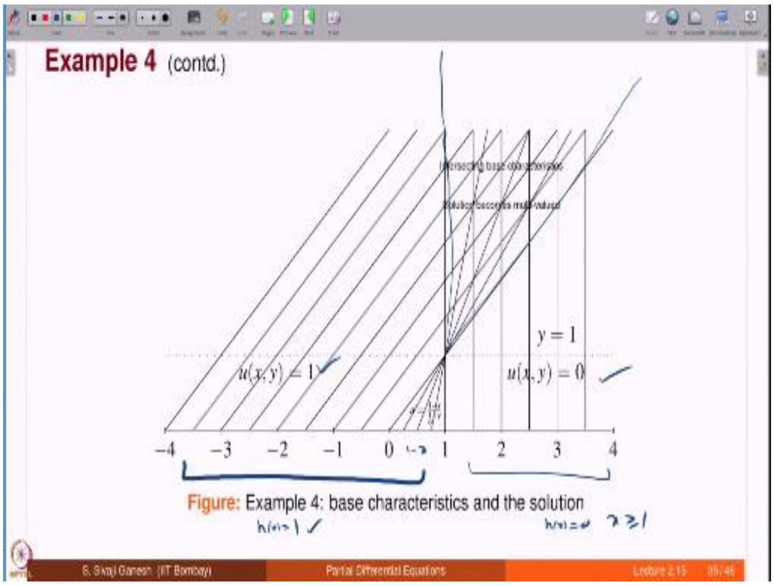


So, this is the picture. Here these are all lines parallel to y = x. Here these are the lines parallel to y axis or x = 1. And in this region which is identified here. In this region which extends base characteristics intersect solution becomes multivalued. So, here u = 1 because my initial condition h of x = 1 here. Here h of x = 0 for x greater than or equal to 1. Therefore, the solution is 0. Here the solution is 1 because of this. Here it was the polynomial 1 – x. Therefore, the solution will turn out to be 1 – x by 1 – y.

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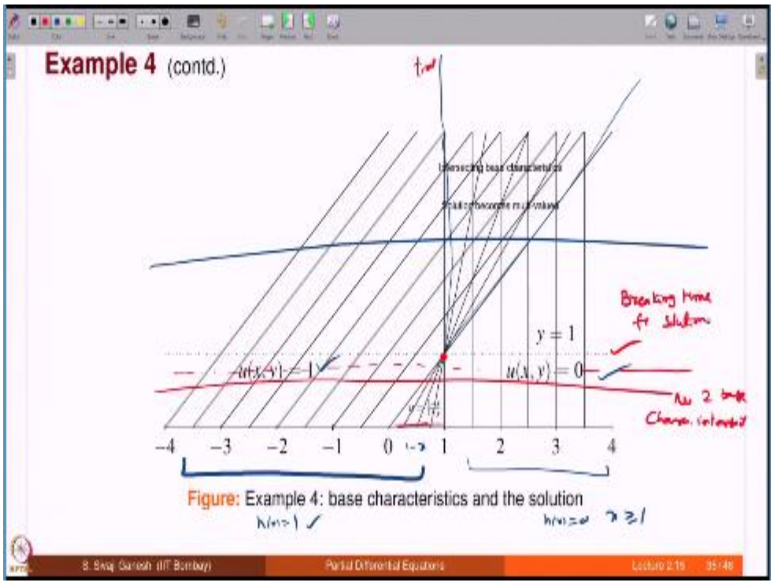


Now, at what point they intersect? It is a algebra. I will skip the algebra and they all intersect here at this point. If you start here again not do not intersect. Now, the question to be asked is at this height do they intersect? Yes they intersect because at these points. At this height do they intersect? I decide no. No 2 base characteristic intersect. Then rise like you know, high jump. I raise the bar a bit and I asked is this the place?

No, it turns out this is the place where base characteristics start meeting. That is basically this family is what is responsible for that. All of them are meeting at this. This is the first time if you, y is called time, time the first instance where some trouble starts brewing is at t equal to 1. So, this is often called breaking time for solution.

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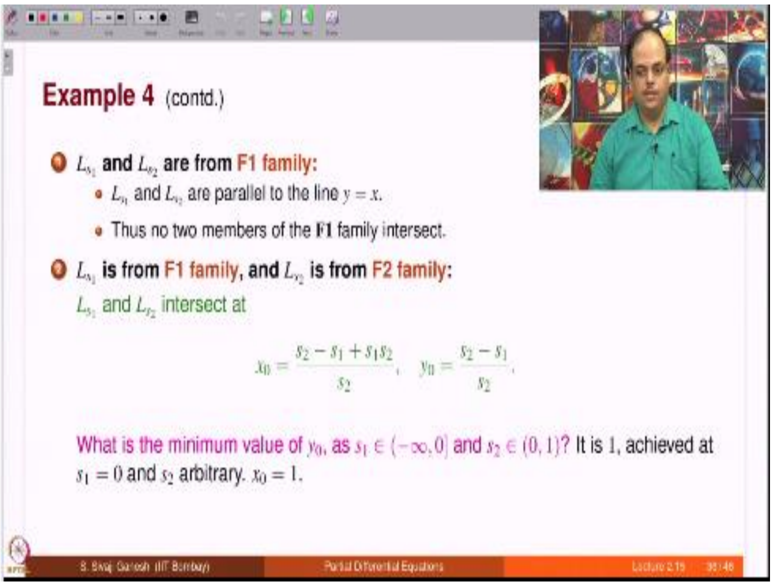


So, 2 members of F1 family clearly do not intersect because all of them are parallel lines, then they never intersect. Similarly 2 members of F3 they do not intersect, but 2 members of F2 they always intersect at this point. And then you can ask when is this family intersect this that is once again at this point. Because this line is going and here the smallest time would be this. So, please compute this.

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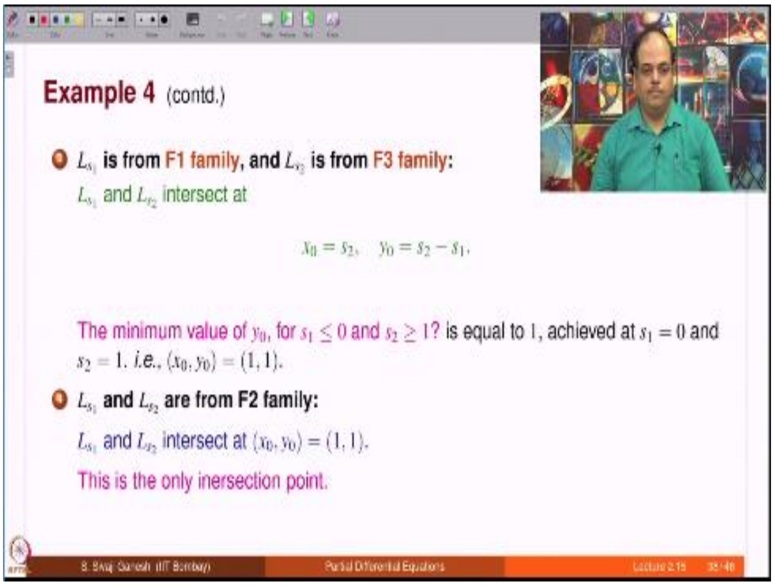

So, whenever base characteristic intersect then the thing of interest is always what is the first time at which they start intersecting until that time they should not be intersecting. So, this is just algebra so I just skipped. But I just keep it on the screen for some time so that you can do the computation. As s 1 varies in minus infinity, 0 means F1 family. This is F2 family. F2 is in 0, 1. This is F 2 family. It is 1. So, I am showing this once again through picture.

Now F1 family and F3 family, where do they intersect? At 1,1. It means some member of the family of F1 meet some member of the family of F3 at the point 1,1. Now, F2, F3. Once again the point 1, 1.

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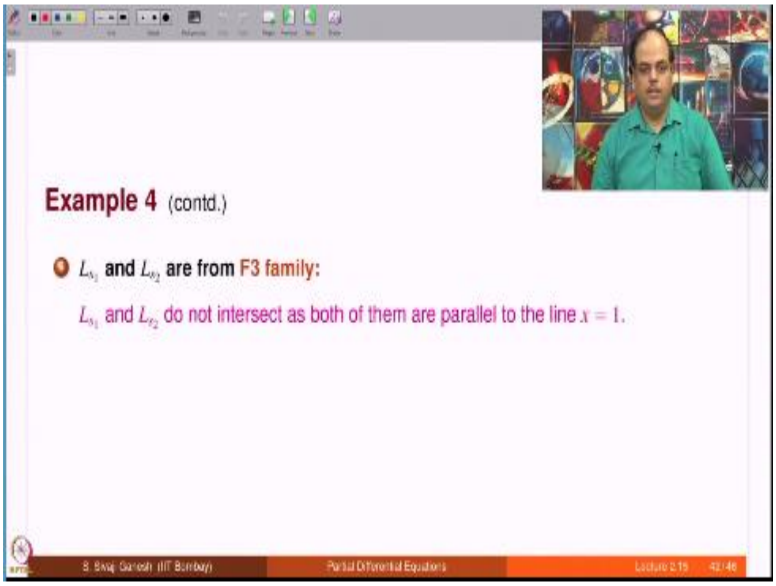


Now, both of them are from F3 family. They do not intersect because they are parallel lines.

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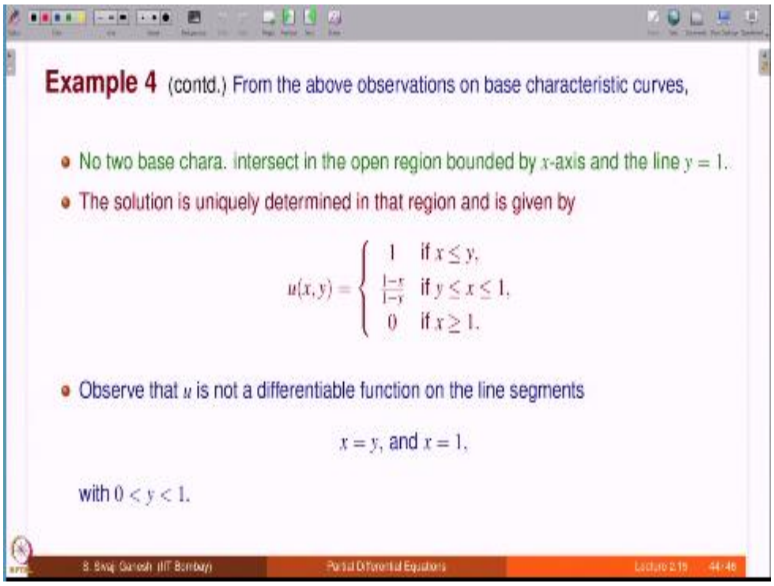

So, no 2 base characteristics intersect in the open region bounded by x axis and the line y = 1. We have observed this. Solution is uniquely determined in that region and is given by this formula. Observe that u is not a differentiable function on the line segment x = y and x = 1.

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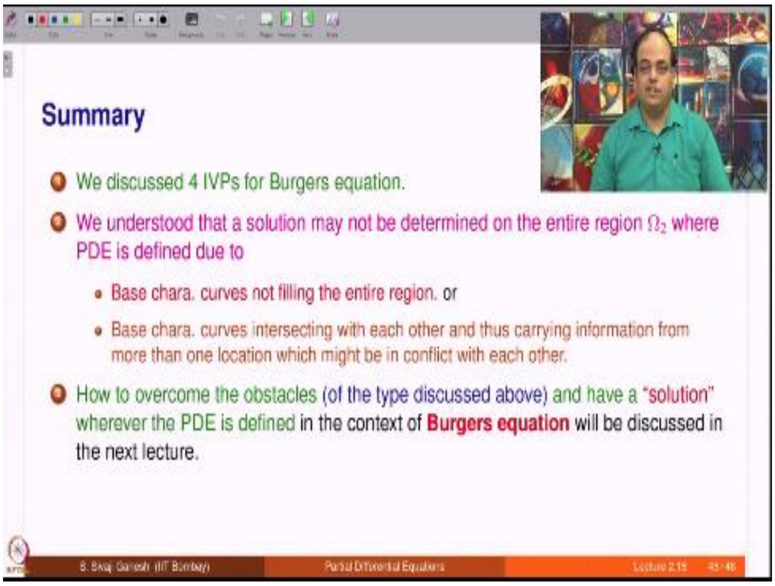


Let us summarize. So, we discussed 4 initial value problems for Burgers equation. We understood that a solution may not be determined on the entire region omega 2 where the PDE is posed due to base characteristics not filling up the entire region. This was the case in one example. Or base characteristic curves intersecting with each other, this is true in the other 2 examples and carrying in conflicting information, possibly. That is a problem.

So, now, question is how to overcome these obstacles of the type that we discussed above and have a solution. That means we have to define a new notion or solution. Wherever the PDE is defined, we want that in the context of Burgers equation. Let us limit ourselves to the context of Burgers equation. That will be discussed in the next lecture, where we will be worried only about how to view a meaningful solution.

Or how to make sense of these functions which you obtain here as solutions in some kind of generalised sense. That will be discussed in the next lecture. Thank you.