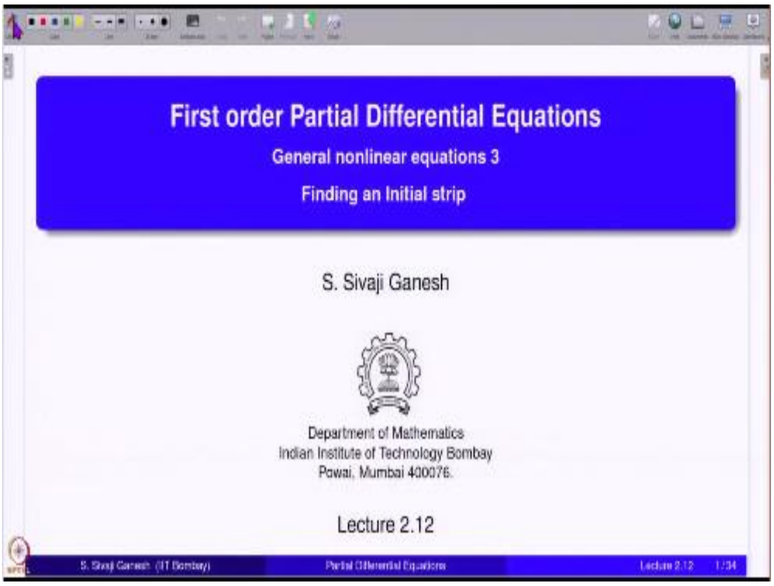
**Partial Differential Equations**
**Prof. Sivaji Ganesh**
**Department of Mathematics**
**Indian Institute of Technology – Bombay**

**Lecture – 2.12**
**General Nonlinear Equations 3**
**Finding an Initial Strip**

While solving Cauchy problem for Quasilinear equations, one of the steps was to pass characteristic curves through points of the datum curve. For that what we do is we solve a system of characteristic ODE with the initial condition, so that the point initial condition point lies on the datum curve.

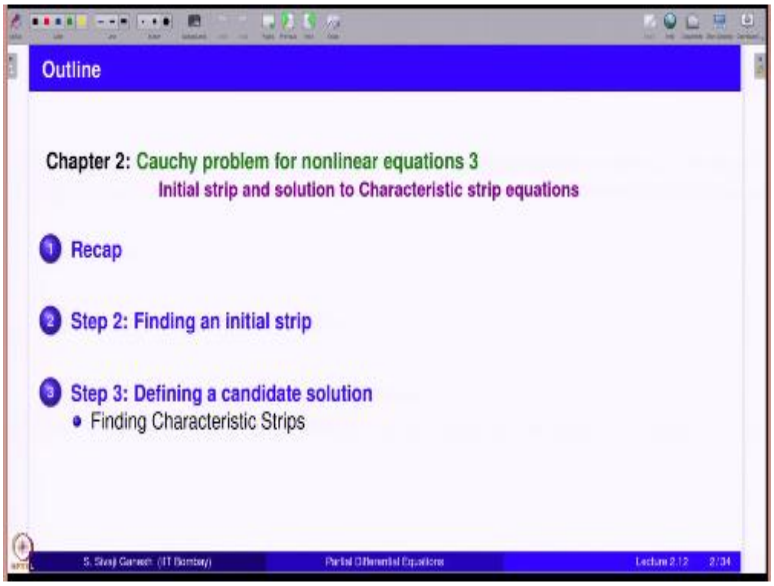
**(Refer Slide Time: 00:47)**




Now, when it comes to the general nonlinear equations, if you want to do the same thing, we did not have a complete system for the characteristic ODE because it involves a p and q which were unknown. Therefore, we extended the system to a system of characteristic strip system of differential equations for a characteristic strip. Now, we would like to solve this. But now, the problem is that characteristic strip is not known on the datum curve.

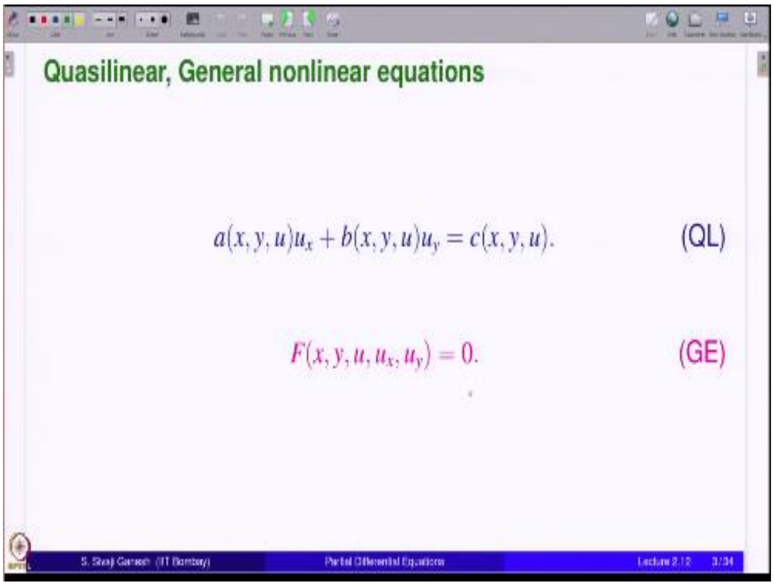
And the datum curve only the values are the initial conditions for x, y, z will be known. Therefore, we need to derive initial strip. That is what we are going to do in this lecture.

**(Refer Slide Time: 01:26)**



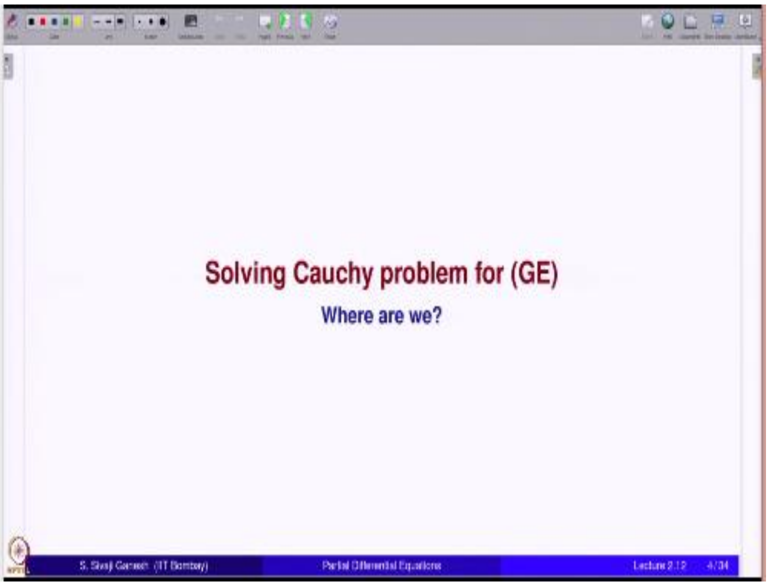

We start with a recap of what we have done so far in the characteristic method for general nonlinear equations. And then we find the initial strip and then we take the third step which is to define a candidate solution and first step in the step 3. In the next lecture, we are going to conclude the entire all the 4 steps.

**(Refer Slide Time: 01:49)**



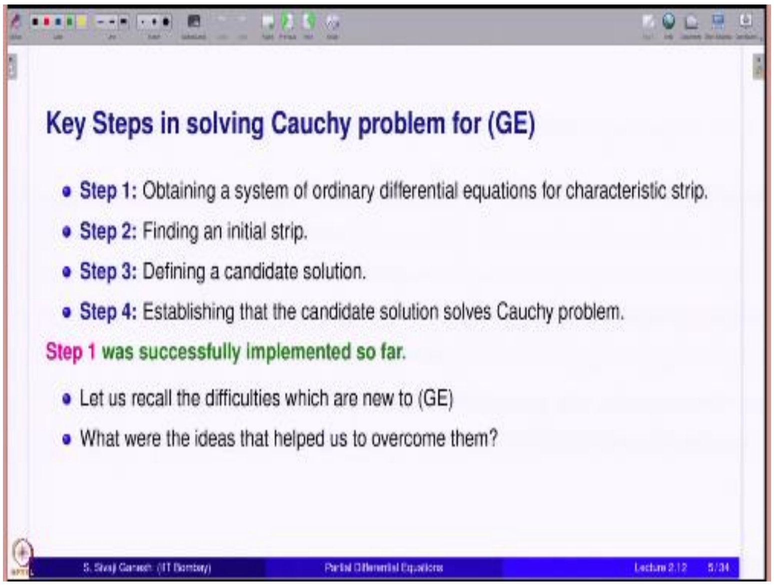

So, this is to just make you recall the notation QL stands for Quasilinear equations a u x + b u y = c. GE we call it general nonlinear equation sometimes people call fully nonlinear equations. It is F of x, y, u, u x, u y = 0.

**(Refer Slide Time: 02:10)**




Now, the solution of the Cauchy problem for general nonlinear equations, where are we?

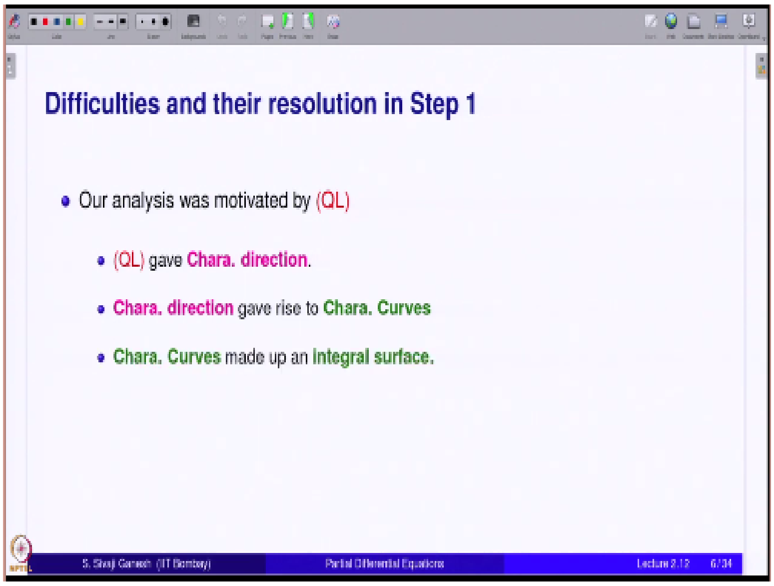
**(Refer Slide Time: 02:15)**




Let us look at that. Recall the key steps involving in the solution of Cauchy problem that we proposed for general nonlinear equations. Step 1 is obtaining a system of ordinary differential equations for characteristic strip, which we have done. Step 2 is finding an initial strip we are going to do today and step 3 defining a candidate solution and establishing that the candidate solution is indeed a solution to the Cauchy problem will be taken up in the next lecture.

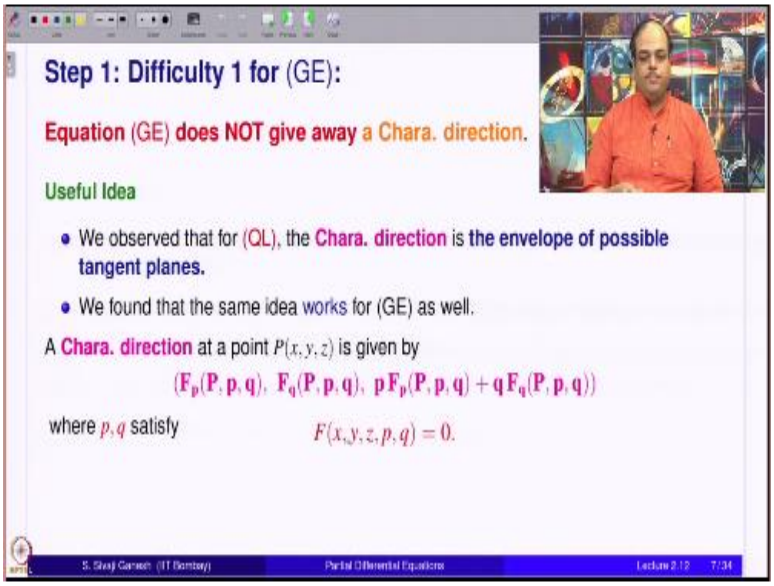
So, step 1 was successfully implemented so far. Let us recall the difficulties which are new to GE when compared to QL. And what were the ideas that helped us to overcome them?

**(Refer Slide Time: 03:02)**




Difficulties under resolution in step 1: Our analysis was motivated by the Quasilinear equations. Quasilinear equation QL gave us a characteristic direction. Characteristic direction gave rise to characteristic curves. Characteristic curves made up an integral surface.

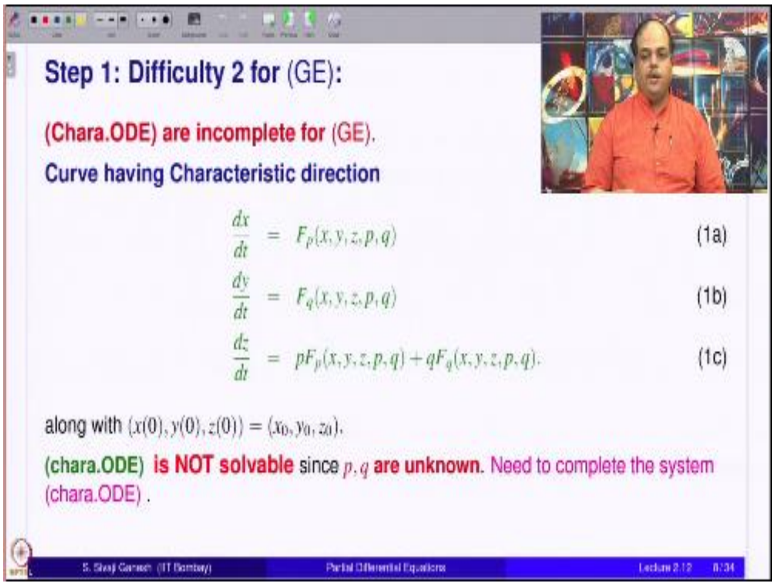
**(Refer Slide Time: 03:24)**




The difficulty 1 for general nonlinear equation is the equation does not give away a characteristic direction. A useful idea was we observed that for Quasilinear equations, the characteristic direction is the envelope of possible tangent planes. Actually, the envelope of possible tangent planes is a straight line whose direction is a characteristic direction. So, we found that the same idea works for GE as well.

A characteristic direction at a point p x, y, z is given by F p, F q, p F p + q F q denote partial derivatives of F with respect to the variable p and q respectively. And the small p, q satisfy F of x, y, z, p, q = 0. Capital P actually is standing for x, y, z.

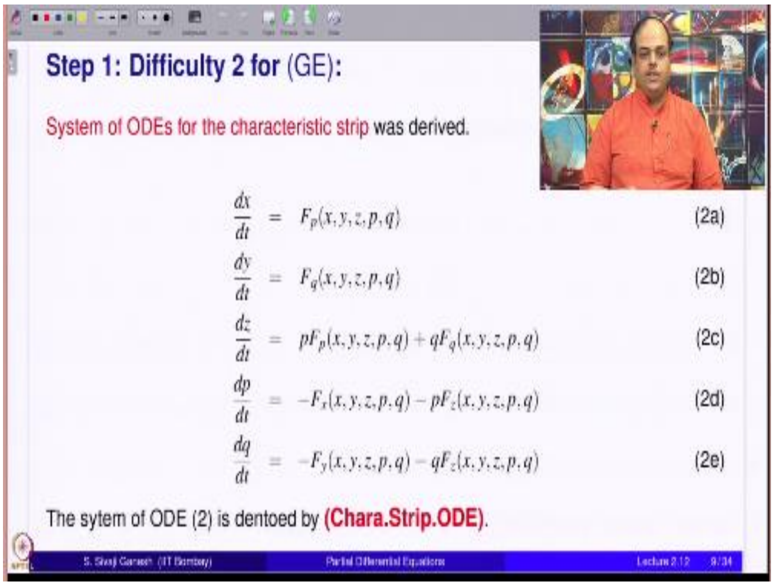
**(Refer Slide Time: 04:21)**




Now, what is the difficulty number 2? Characteristic ODE system is incomplete for GE because we were looking at finding a curve having characteristic direction which was given by this dx by dt, dy by dt, dz by dt = F p, F q, p F p + q F q. We were looking at solutions to this whose image will be a curve. And with this condition x 0 = x 0, y 0 = y 0, z 0 = z 0. Then this will be a characteristic curve passing through the point x 0, y 0, z 0.

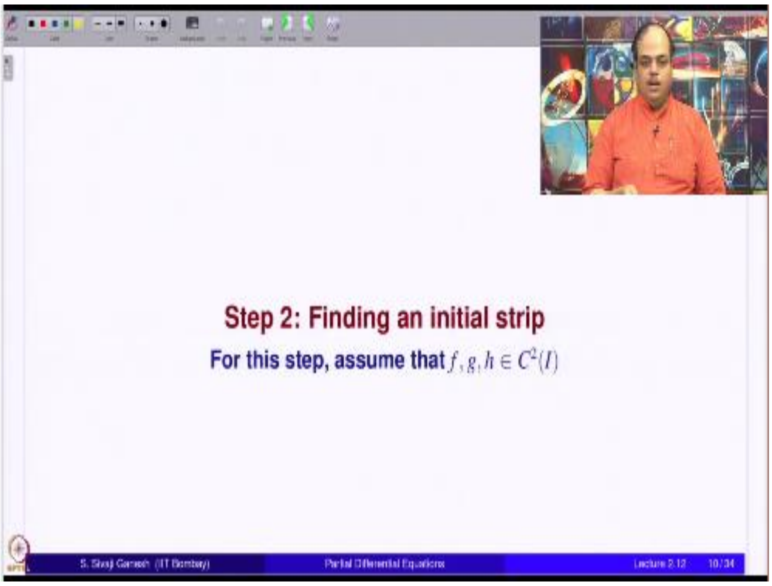
Unfortunately, in this system p, q are involved and p, q also depend on the location of where you are on the characteristic curve that you are trying to find. So, p and q are dependent on x t, y t, z t. That is why Chara.ODE that is this system is called Chara.ODE. It is not solvable. So, we need to complete the system.

**(Refer Slide Time: 05:27)**




That is what we did. We derived the system of ODEs for the characteristic strip, which is a system of 5 equations.

**(Refer Slide Time: 05:36)**



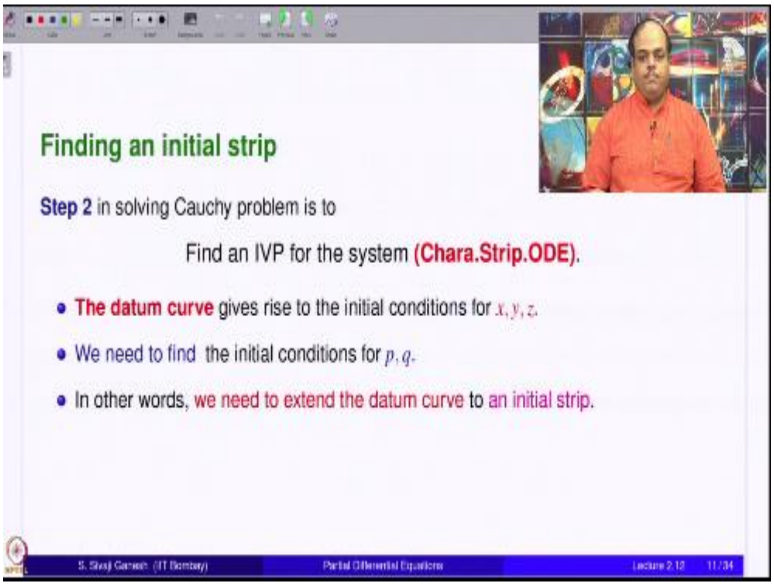

Now, we need to find a solution to this. What is the next step as I told you at the beginning of this lecture, is to find solutions to this so that x 0, y 0, z 0 is a point on the datum curve, let us say f s, g s, h s. But what will be the initial conditions for p and q? Those are not there. So we need to derive that. That is what is called finding an initial strip. Initial curve that is a datum curve is known. f s, g s, h s as s varies in the interval I we are going throughout the curve gamma.

Now, we need to find a p s and q s. And p s and q s are not arbitrary functions. They should be such that p s, q s - 1should be the normal to a possible tangent plane that we want to find.

So, it is something tied with the equation the p s, q s. For this step we are going to assume that the initial data is C 2. f g h are all C 2 functions. This is only a temporary requirement. We will see a comment later on saying that this is not needed at some point.

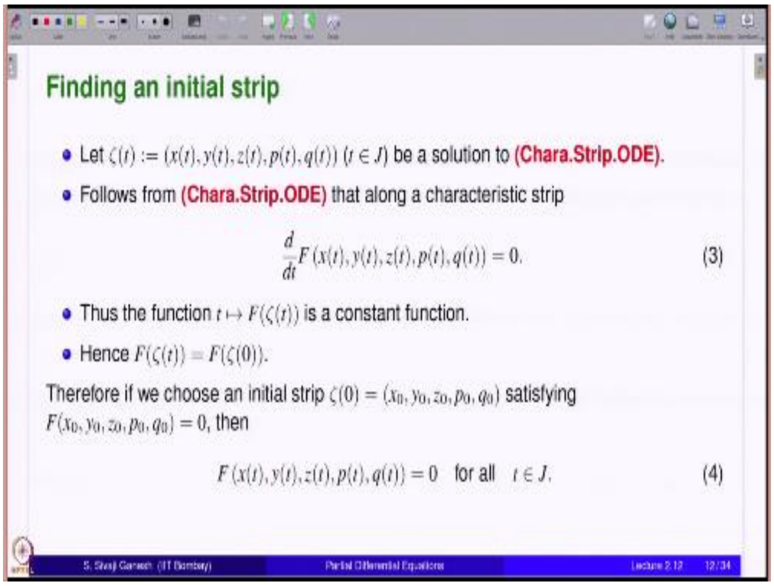
As I told you to derive the equations, we can assume anything that we want. But having got the equations, we have to try to show that things work with the minimal assumptions. Assuming f g h in C 1is somewhat reasonable, C 2 is too much. It is not that much reasonable. But this we are going to do only to derive this initial strip. We will see that. We will also see why or where are we going to need the C 2 s.

**(Refer Slide Time: 07:24)**




So, finding an initial strip. So, step 2, in solving Cauchy problem is to find an initial value problem for Chara.Strip.ODE. The datum curve gives rise to the initial conditions for x, y, z. We need to find the initial conditions for p and q. In other words, we need to extend the datum curve to an initial strip.

**(Refer Slide Time: 07:47)**

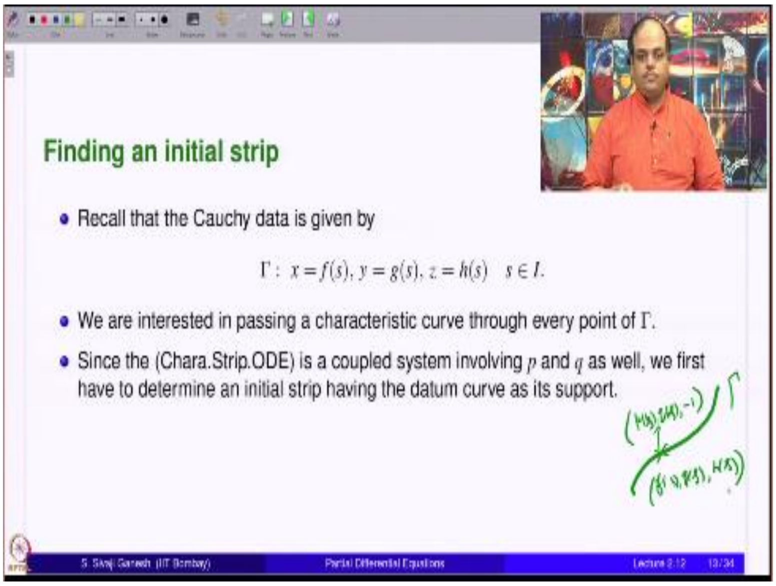



The support of the strip will be the datum card. So, finding an initial strip, how do we do that? So let zeta t be x t, y t, z t, p t, q t be a solution to Chara.Strip.ODE. So, it follows from Chara.Strip.ODE that along a characteristic strip d by dt of this is 0. How does it follow? You do the chain rule. So, it will be F x into x dash t, F y into y dash t, F z into z dash t, F p into p dash t + F q into q dash T.

And if you use these equations, it answered that you end up with 0. Therefore, d by dt of F of x t, y t, z t, p t, q t = 0. So, that means this function t going to F of zeta t is a constant function. Because its derivative is 0, it has to be constant. If it is constant it will be equal to F of zeta of t 0 for any t 0. But I will choose t = 0 because t = 0 is where I am going to stay on the datum curve. Atleast x 0, y 0, z 0 will be on the datum curve.

Therefore, we are going to require that F of zeta 0 is 0. That is what we are going to ask. So, if we choose an initial strip zeta 0 which is x 0,y 0,z 0, p 0, q 0 such that F of that is 0 it means F of zeta t will be 0 for all t in J.

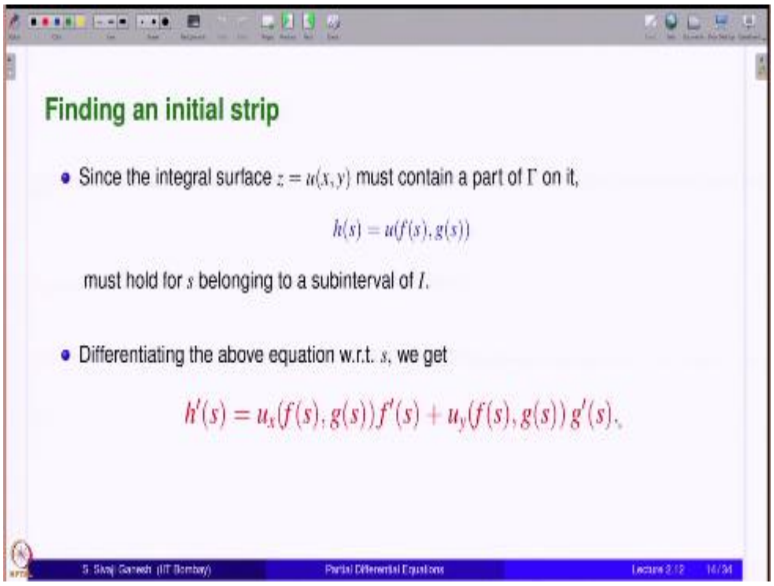
**(Refer Slide Time: 09:29)**




So, recall that the Cauchy data is given by x = f s, y = g s, z = h s. We are interested in passing a characteristic curve through every point of gamma. Since Chara.Strrip.ODE is a coupled system involving p and q also, apart from x, y, z, which only matters to find a characteristic curve. To find a characteristic curve you need x t, y t, z t. But the equations involve p and q. So, we first have to determine an initial strip having the datum curve as its support.

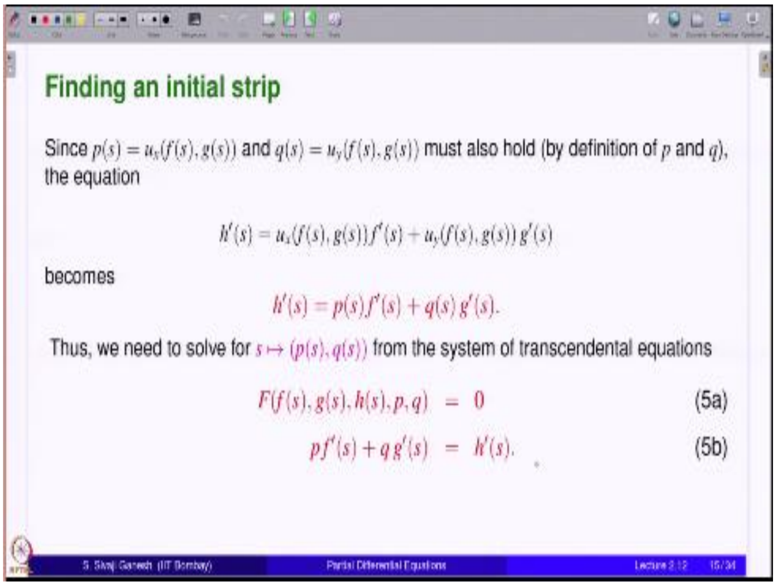
In other words, at every point of the datum curve, this is gamma. At any point, we need to associate a vector I am saying vector because 2 numbers are there and they should be having this finally, this property. That this is p s, the point is s. This is the f s, g s, h s. The strip is f s, g s, h s, p s, q s. The support is f s, g s, h s.

**(Refer Slide Time: 10:45)**



Since the integral surface must contain a part of gamma on it, we should have h s equal to u of f s, g s. It should hold for s belonging to a sub interval of I. Now, once you have that differentiate this equation with respect to s, you get h dash s = u x into f dash plus u y into g dash by chain rule.

**(Refer Slide Time: 11:10)**

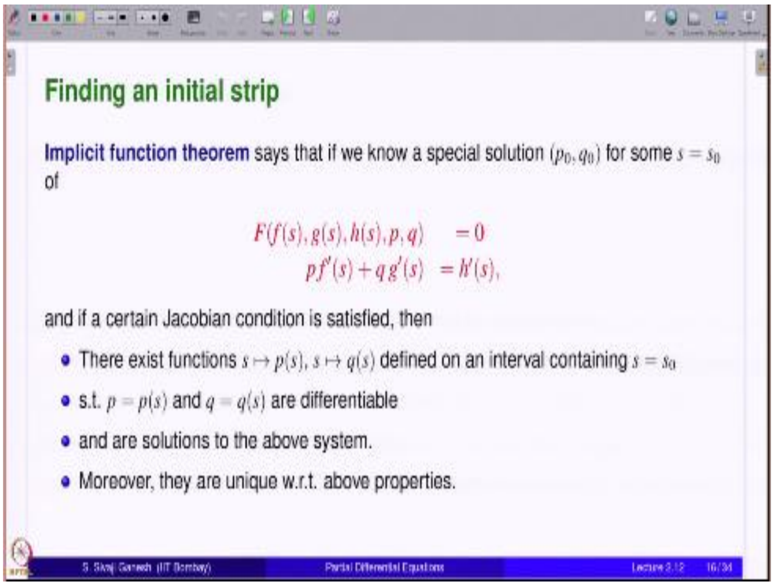



Since p s =, p is supposed to be u x, q is supposed to be u y that should also hold. So, we demand this. p s equal to this should happen q x equals this should happen. Therefore, what does this imply? The equation that we obtained in the previous slide h prime = u x f prime, u y g prime. Now, u x f s, g s must be p and u y must be q. Therefore, we need to solve for p s and q s in terms of s coming from this equation. Capital F of f s, g s, h s, p, q = 0 and p f dash s + q g dash s = h dash s.

So, 5 b is this equation and this is the equation of the partial differential equation. The role p and q are actually for u x and u y. So, that is why we demand these 2 conditions. So, now, if you notice, this is a function of f s, g s, h s are given. Therefore, f dash is known g dash is known h dash is known. So, what it involves is p and q are a function of s, something s.

So, you can think of this as a function of s, p, q = 0. Another function of s, p, q = 0 and you want to solve p, q in terms of s.

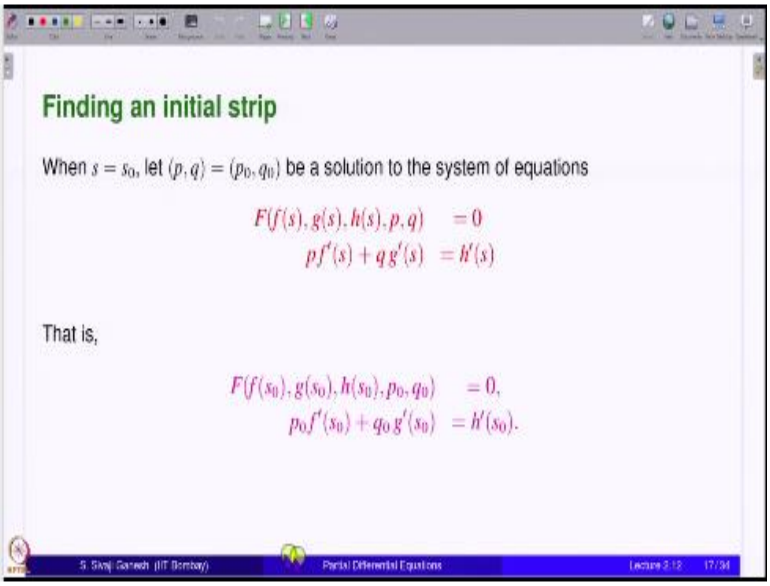
**(Refer Slide Time: 12:39)**




So, implicit function theorem says that if we know a special solution for some s equals s 0, p 0, q 0, which satisfies this as well as this. That means capital F of f of s 0, g of s 0, h of s 0, p 0, q 0 = 0 and p 0 f dash s 0 + q 0 g dash s not = h dash of s 0. If such p 0, q 0 are given and a certain Jacobian condition is satisfied, nonzeroness of certain Jacobian we will do the details on the next slides, then there will exist functions you can express p, q in terms of s essentially that is what we want.

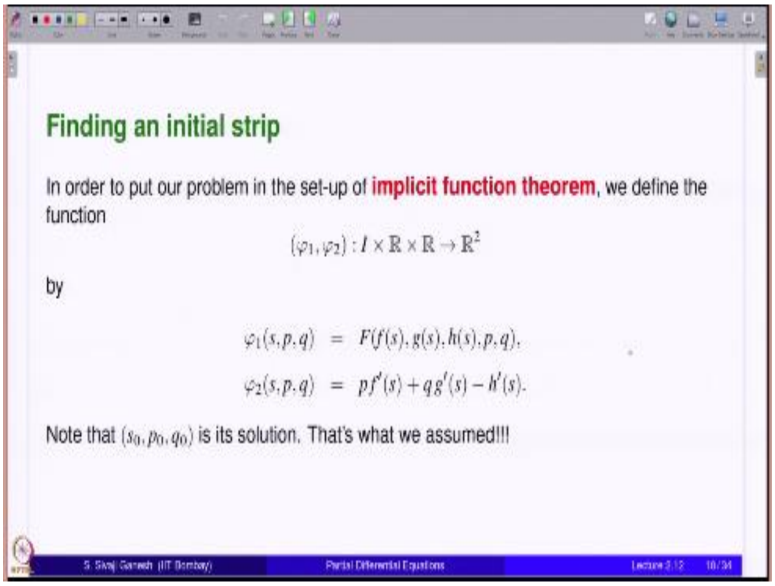
So, there exist functions s going to p s, s going to q s. Of course, the conclusions are local. Therefore, these functions will be defined on an interval containing s 0 where this Jacobian condition is satisfied and where a particular solution p 0, q 0 has been found such that these functions are differentiable and they are solutions to the above system. Moreover, they are unique with respect to the above properties. Implicit function theorem when applicable it gives you a unique solution.

**(Refer Slide Time: 13:52)**




So, when s = s 0, p, q = p 0,q 0. If that is a solution to the system of equations that is exactly this which I have read out earlier.

**(Refer Slide Time: 14: 06)**




So, in order to put our problem in the set of implicit function theorem, we define 2 functions phi 1 and phi 2. As I told you phi 1of s, p, q because f s, g s and h s are known. It is simply a function of s. That is why s. p, q are unknown quantities. I want to solve p, q in terms of s, so, I have 3 variables here. So, s varies in the interval I. p, q vary in an interval R cross R, I have put. But it should be that wherever capital F is defined for this p, q.

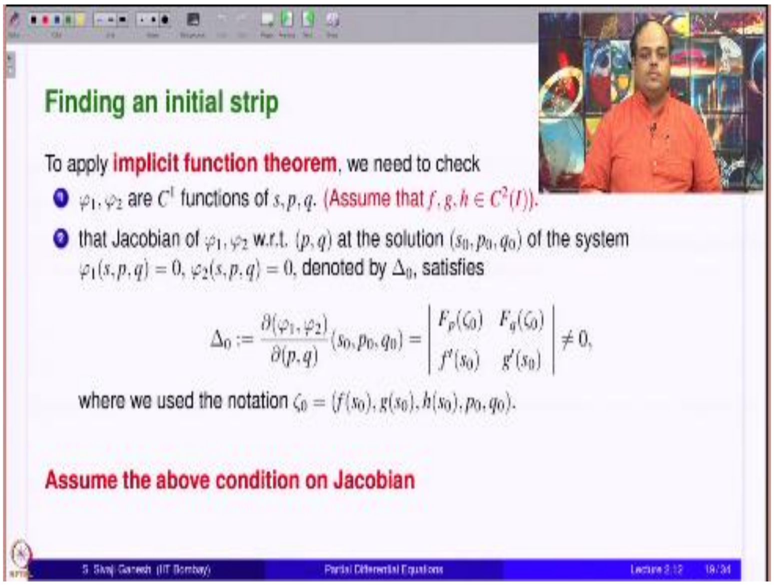
So, I agree with the projection to the last 2 coordinates p, q of the domain of omega 5. That is where it makes sense. So, if you assume that omega 5 is such that the last 2 components is R, then it is R cross R. Otherwise, it is going to be the corresponding projections to the last 2 coordinates that will come in place of R cross R. Because F should be meaningful after all.

Here I am defining this function for p and q belonging to R and R. Of course that makes sense only if F is defined for those p and q's. So, set of all p, q for which F is defined, I will consider this function. Then in that case, I will not write R cross R. I will write something else. So, with this correction phi 2 of s, p, q is the second equation that we had on the previous slide. We are interested in its solution. What is given is s 0, p 0, q 0 is a solution. You have to find the p 0, q 0 such that s 0, p 0, q 0 is a solution. So, that is what we assumed.

Now, to apply implicit function theorem, we need to check that functions are C 1 functions, phi 1, phi 2 are C 1 functions. Are they? Yeah, with respect to s, if f g h are C 1, fine the C one and capital F itself is C 1. Therefore, composition will give you C1, p, q are appearing in this. And we already assumed F is C 1 with respect to all the components. Therefore, p, q also will be C 1. So phi 1 will be C 1, no problem. Phi 2, C 1 ness of phi 2 will involve C 2 ness of f.

So, this is where we need f g h to be C 2. In terms of p and q it is linear. So it is always C infinity. So phi 2 is C 1 provided we assume the initial data that is datum curve f g h are C 2 functions. This is where we need f g h to be C 2. Fine.

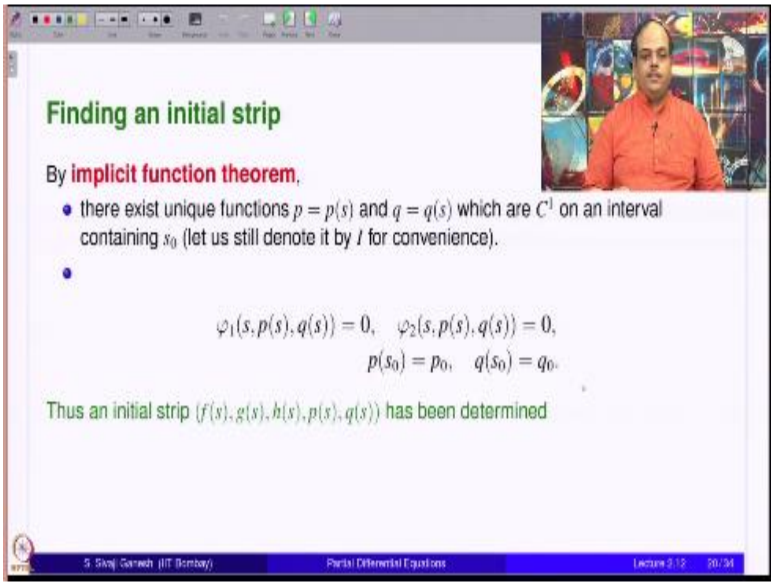
**(Refer Slide Time: 16:48)**




Now, the Jacobian of phi 1, phi 2 with respect to p, q at the solution that we already found s 0, p 0, q 0 of this system that should be nonzero which is this. Though phi 1, phi 2 by dou p, q is simply the Jacobian which is here. Dou phi 1 by dou p, dou phi 1 by dou q, dou phi 2 by dou p dou q will come. Now, at the point s 0, p 0, q 0 that you can compute from the expressions of phi 1, phi 2.

It will turn out to be F p at zeta 0, F q at zeta 0, f prime s 0, g prime s 0. Zeta 0 is this 5 tuple. f s 0, g s 0, h s 0, p 0, q 0. This is nonzero. We have to assume that. Assume the above condition on Jacobian.

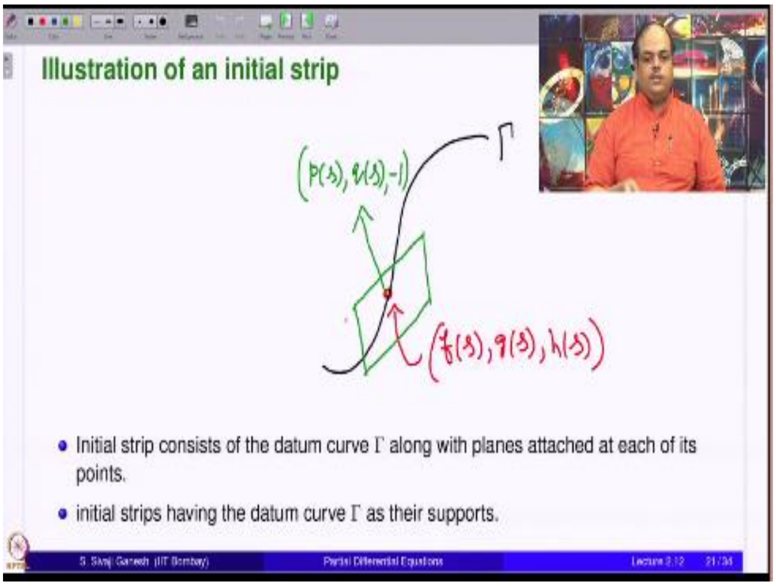
**(Refer Slide Time: 17:41)**




Now we can apply implicit function theorem. It gives us unique functions p = p s and q = q s which are C 1 on an interval containing s = s 0. Let us still denote it by I for convenience if you want you can make it I dash does not make any difference. And what is the property? P s and q s all in the system phi1 and phi 2. Phi 1 s, p s, q s = 0, phi 2 of s, p s, q s = 0 and at s – s 0, it coincides with the p 0, q coincides with q not.

Thus an initial strip has been determined. So, this complete successful implementation of step 2 of our programme.

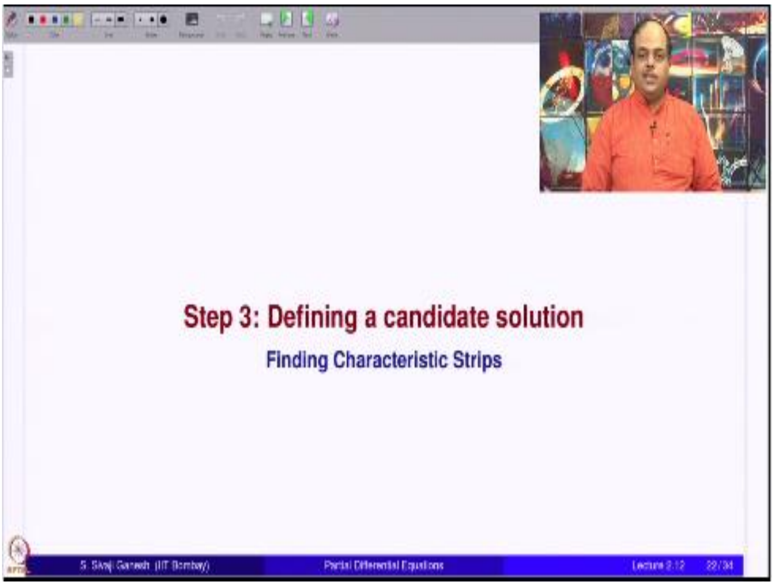
**(Refer Slide Time: 18:27)**




So, initial strip consists of the datum curve gamma which is given to us and along with planes. Planes essentially means the normals to the planes. P s, q s, - 1 attached at each of the points of gamma. What we have shown is that if the Jacobian condition is met at s – s 0 in a piece nearby that is some part of gamma there we can do this. That is what the implicit function theorem said.

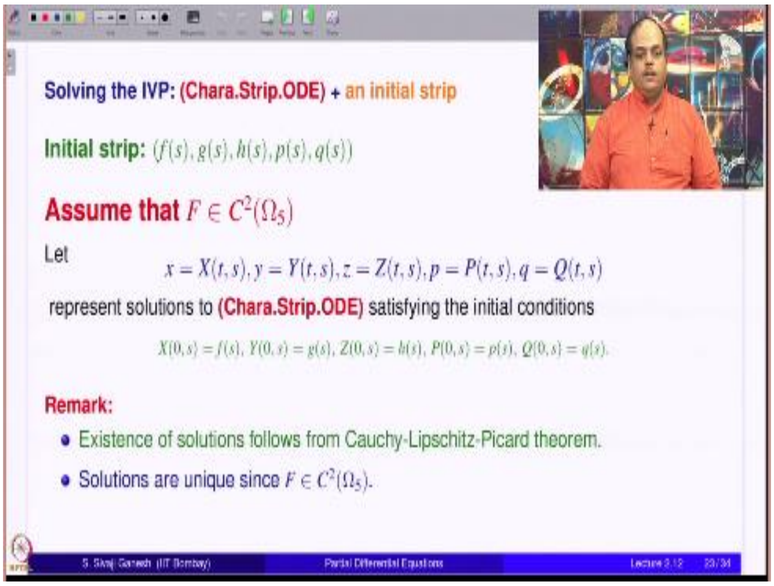
In any case, our theorem is going to be local, we are always going to fix some point and going to assert that near this point p 0 on the datum curve gamma there is an integral surface which contains a piece of gamma. Therefore, this is absolutely fine and initial strips have the datum curve as the support if you can find that throughout all the points.

**(Refer Slide Time: 19:21)**



Now let us go to the step 3, which is defining a candidate solution. To define a candidate solution, first thing is we have to solve characteristic strip equation with the initial step that we have got.

**(Refer Slide Time: 19:34)**

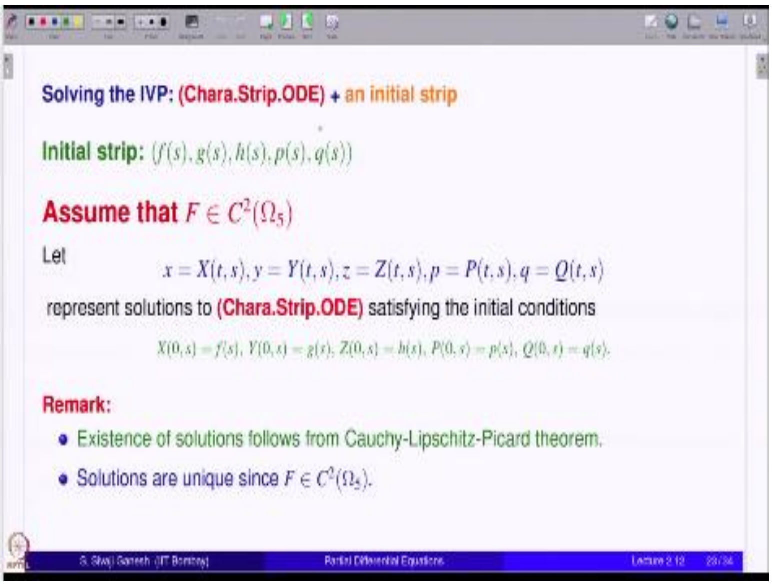



So, here we assume that F E C 2 and so let this represent solutions to Chara.Strip.ODE satisfying the initial conditions. For x, y, z it is f s, g s, h s, for P and Q, it is small P and small q of functions of this which we have determined in step 2. That is the initial step. That means if you look at the trajectory of the solutions, the first 3 coordinates at t = 0 pass through the point f s, g s, h s on the datum curve.

And remaining 2 are supposed to be giving you the components or the normal at that point. That is why the strip. Now existence of solutions, like this follows from Cauchy-Lipschitz-Picard theorem. That is why I assumed F is C 2, because the right hand side will involve f p, f q, etcetera. So to guarantee the right hand side is Lipschitz, one easy way to do is assume that right hand side is C 1, then it will be local elliptious.

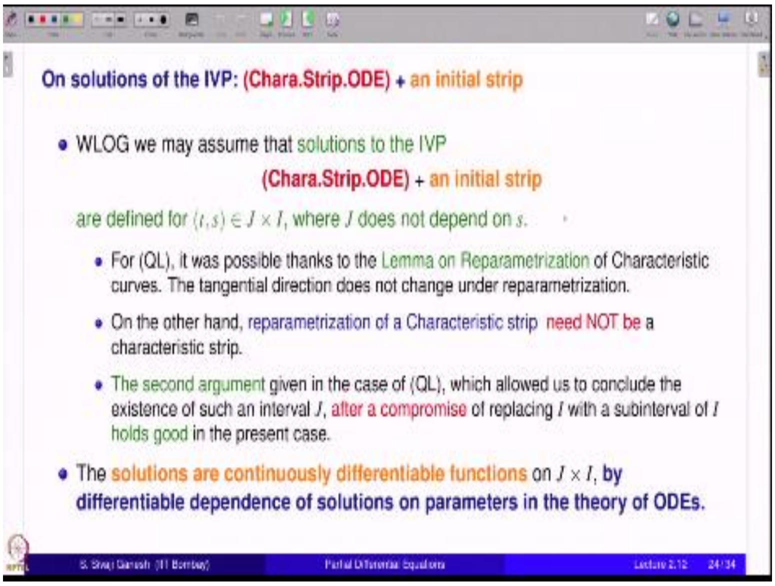
And then we have existence of unique solutions. That would require a few F is C 2. That is why I have added F is C 2. So solutions are unique because of that.

**(Refer Slide Time: 20:52)**




Otherwise, you have existence continuity is enough. We assumed F E C 1. And if you look at the equations of Chara.Strip.ODE the right hand side will involve only derivatives of F at maximum. f p, f q, p f p + p f q – f x, f z and so on f y, they are all assumed to be continuous. Therefore, right hand side is continuous. The moment you assume F is C 1, which means we know Pinos theorem, there exists a solution. If you want uniqueness, you push it C 2. If you make a F to be C 2 uniqueness is guaranteed.

**(Refer Slide Time: 21:27)**



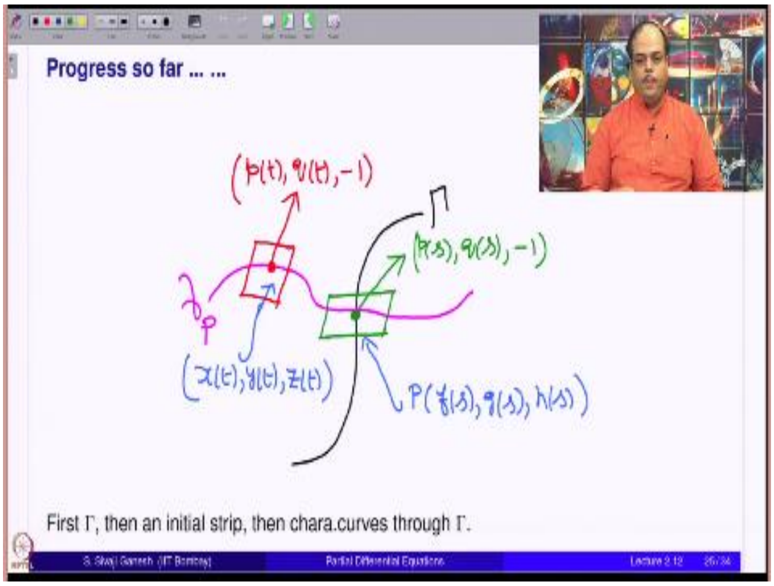

Solutions of the initial value problem that is characteristic strip ODE plus an initial strip, we want to solve this initial value problem. So, without loss of generality, we may assume that solutions to the initial value problem which is given by the ODEs is a characteristic strip ODE 1and the initial conditions are given by the initial strip. They are defined. The solutions are defined for t, s in J cross I, where J does not depend on s.

For Quasilinear equations, it was possible thanks to the lemma on reparameterization of characteristic curves, because the tangential direction does not change under reparameterization. On the other hand, reparameterization of a characteristic strip need not be a characteristic strip. Therefore, this argument does not hold. Luckily, we have another argument. In the case of Quasilinear equations, we had a second argument which we have given that also allows us to conclude the existence of such an interval J which does not depend on s.

But now, there is a compromise that the I the interval I needs to be replaced with a sub interval of I. But that does not matter because later on what we are going to do is we are going to apply inverse function theorem whose conclusions are anyway local. So, it does not matter. So, what is important is that J does not depend on s whether it is s belongs to I or s belongs to some subinterval of I.

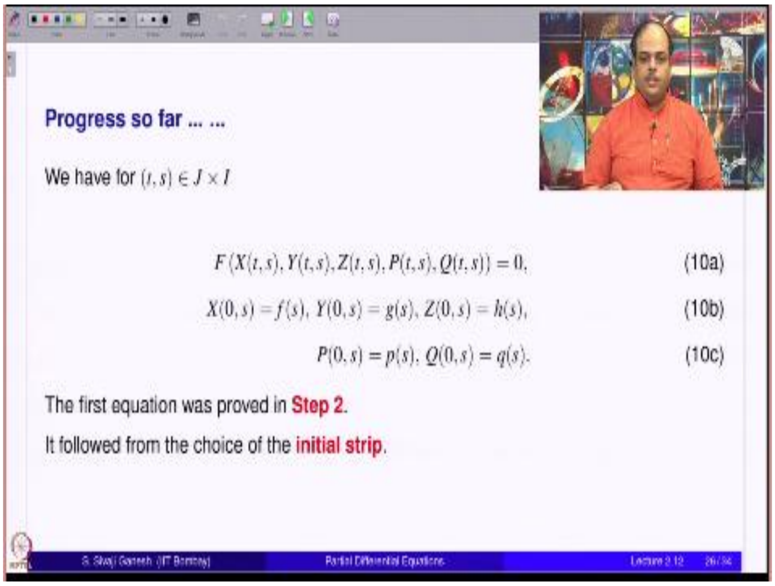
So, the solutions are continuously differentiable functions on J cross I by differentiable dependence of solutions and parameters in the theory of ODEs. In fact, inside the proof of this theorem, one needs to get such catch hold of such an interval which does not depend on s.

**(Refer Slide Time: 23:32)**



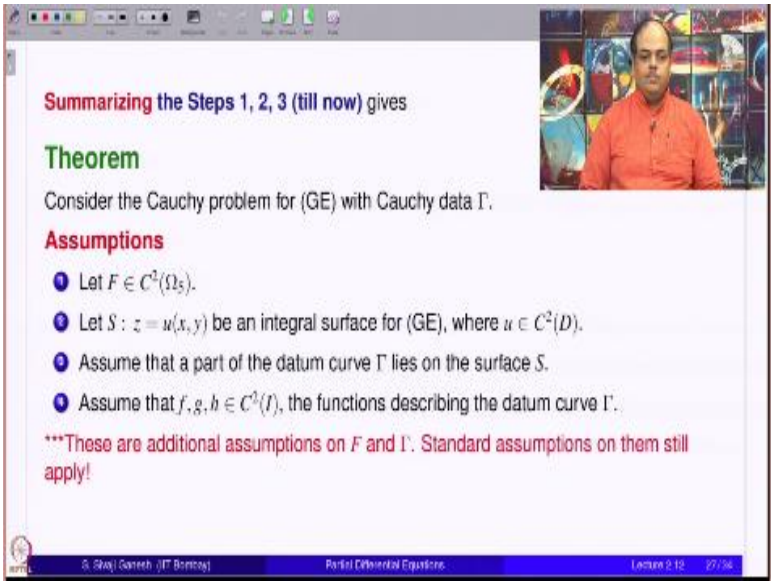

So, the progress so, far has been this first gamma which is given then an initial strip we have obtained that is throughout these points of gamma we have erected these kind of small planes and then characteristic curves going through points of gamma. If this is the point P, these are gamma p, small gamma p and not only characteristic curves throughout at every at any point in the character curve, we also got this particular normal to some plane.

**(Refer Slide Time: 24:08)**




Of course, the plane is expected to be tangent plane later on. So, for t, s in J cross I, we got F of X t, s, Y t, s, Z t, s P t, s, Q t, s = 0 and X 0, s is f s, Y 0,s is g s, Z 0, s is h s. P is p s, Q is q s. So, these are coming from initial conditions. This is the first thing that we did. d by dt F of zeta t is 0. Therefore F of zeta t is constant and we chose zeta 0 such that F of zeta 0 is 0. Therefore f of zeta t is 0. So these are also we approved in step 2. It follows from the way we chose the initial step.

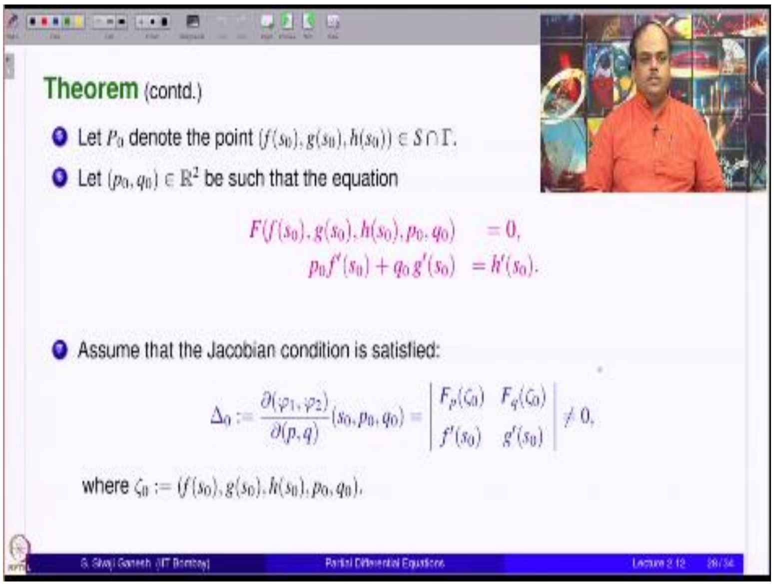
**(Refer Slide Time: 24:47)**




So, summarising the steps 1, 2 and the third one the little bit that we did so far it gives us the following theorem. Consider the Cauchy problem for general nonlinear equation with Cauchy data. Assumptions F is C 2. Let S be an integral surface for this general nonlinear equation, where u is a C 2 function. Assume that a part of the datum curve lies on the surface S, assume that f g h are C 2 functions, which describe the datum curve.

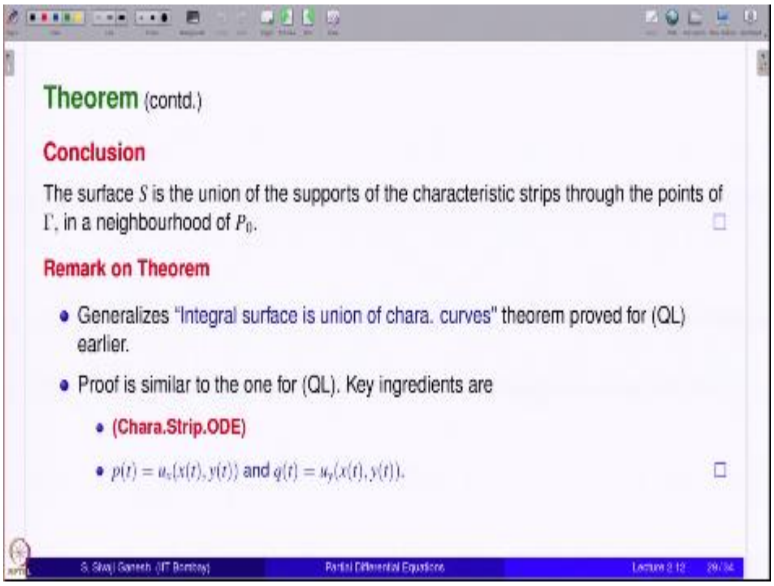
Note these are additional assumptions on F and gamma. The standard assumptions on them still apply namely f p, f q cannot simultaneously vanish and so on.

**(Refer Slide Time: 25:30)**




And f prime, g prime also cannot vanish simultaneously. Now, take a point which is on gamma as well as on S. That means that integral surface which is also there in the datum curve that point. Let p 0, q 0 be such that the equations are satisfied. Assume that the Jacobian condition is satisfied at s 0, p 0, q 0. p 0, q 0 is this s 0 is which we have already fixed and here the zeta 0 stands for this and before.

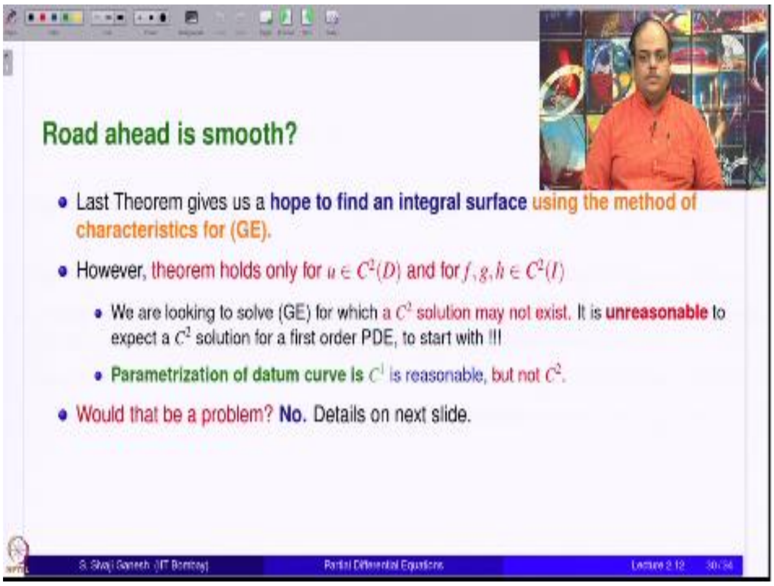
**(Refer Slide Time: 26:06)**




Conclusion: The surface is the union of the supports of the characteristic strips through the points of gamma of course, in a neighborhood of P 0. So, this theorem actually generalises our earlier theorem that we did integral surface is a union of characteristics curves. Integral surface is union of characteristic curves. That theorem we proved for causing the equations.

This is a generalisation of that proof is similar to that same as that. So, key ingredients here are Chara.Strip.ODE and the p t equal to this and q t equal to this.

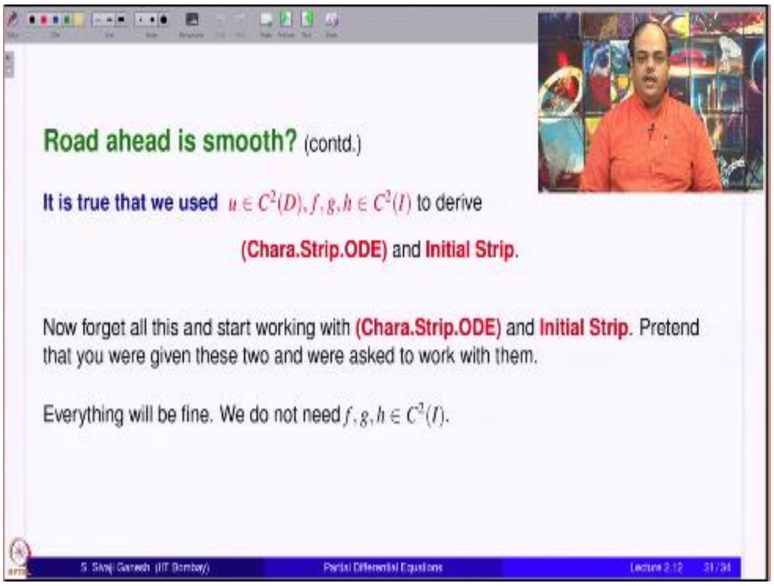
**(Refer Slide Time: 26:45)**




So, now, question is the road I ahead is smooth because last theorem gives us a hope to find an integral surface exactly like the other theorem, which gave us hope in Quasilinear equations using the method of characteristics for General nonlinear equation. However, theorem holds only for u in C 2 and f g h in C 2 of I. We are looking to solve with GE, for which a C 2 solution may not exist, It is unreasonable to expect a CT solution for a first order PDE.

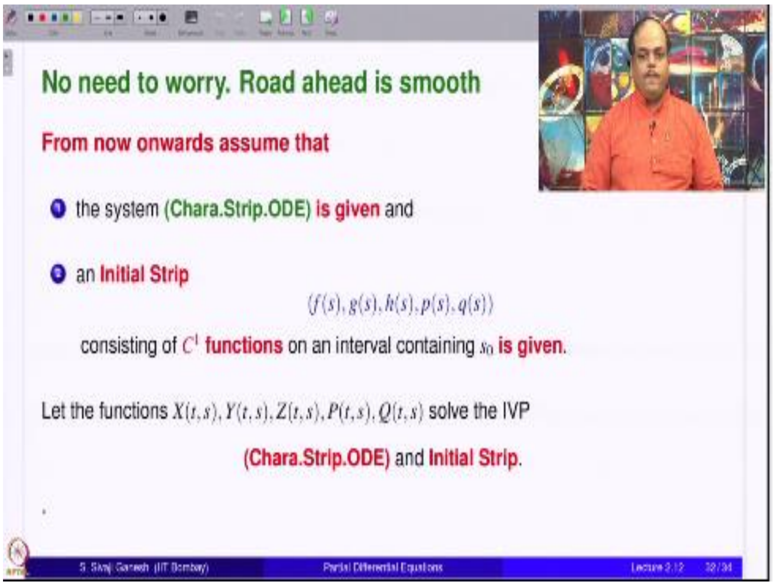
To start with, it may happen that you have C 2 solution or C infinity solution. But to start with you do not expect that. And parameterization of the datum curve C 1 is reasonable, but not C 2. So, would that be a problem? Answer is no that will not be a problem. Details on the next slide.

**(Refer Slide Time: 27:39)**




Yes, it is true that we used u is C 2 of D, f g h is on C 2 of I to derive Chara.Strip.ODE and initial strip. Now, forget all this and start working with Chara.Strip.ODE and initial strip. If we ever used only to derive these 2 things. Pretend that you are given these 2 and you are asked to work with them. Everything will be fine. We do not need f g h in C 2.

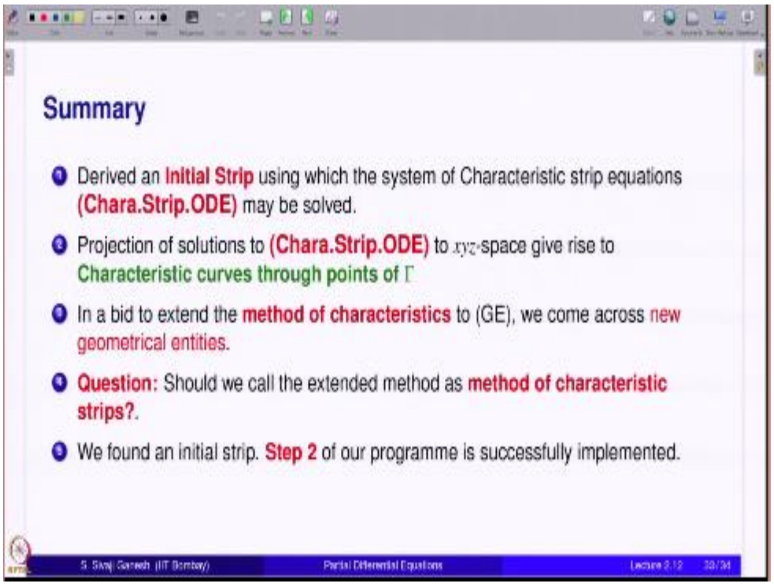
**(Refer Slide Time: 28:14)**




So no need to worry. Road ahead is indeed smooth. From now onwards assume that the system Chara.Strip.ODE is given. And an initial strip f s, g s, h s, p s, q s consisting of C 1 functions on an interval containing s 0 is given. Let the functions X t s, Y t s, Z t s, P t s, Q t s solve the initial value problem and initial strip.

That will be good enough. Of course, you may not have solutions which are unique, but maybe corresponding to each solution you may get one integral surface who knows that. That can happen.

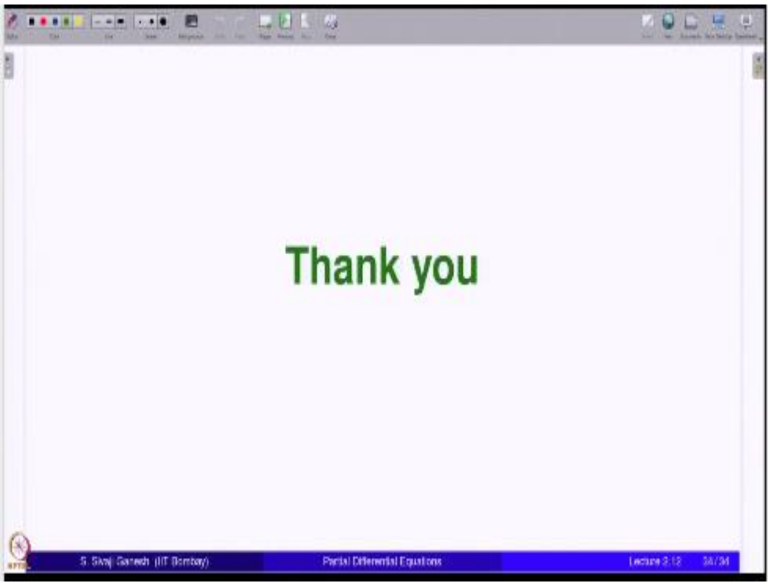
**(Refer Slide Time: 29:01)**




So, we derived an initial strip using which the system of characteristics strip equations may be solved. Projection of solutions to xyz space will give you characteristic curves through points of gamma. In a bit to extend method of characteristics to GE, we come across new geometrical entities. So, a natural question is should we still call it the method of characteristics or should we call method of characteristic strips?

It is okay that does not matter just for you to think ponder about this question. So, we found an initial strip. Step 2 of our programme is successfully implemented.

**(Refer Slide Time: 29:44)**




In the next lecture, we are going to complete the proof of existence and uniqueness, our solutions to Cauchy problem for general nonlinear equations. Thank you.