Introduction to Algebraic Topology (Part-II)
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Lecture - 08
Product of Cell complexes

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Last time we started the study of product of CW complexes. We began a little bit but today we will continue the study.

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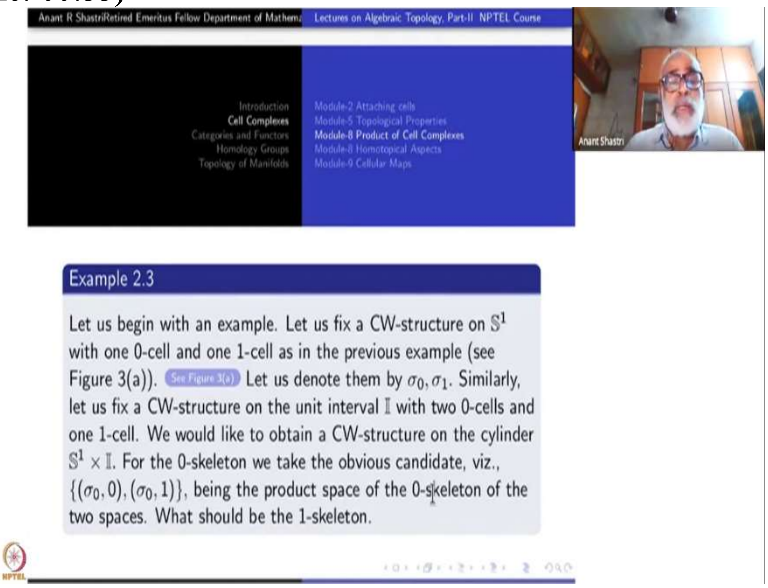

So waht we intend to do is to study just one single typical example, of $\mathbb{S}^1 \times \mathbb{I}$. $\mathbb{S}^1$ is given a CW structure with one 0-cell and 1-cell and the interval $\mathbb{I}$ is given a CW strucutre with two 0-cells and one 1-cell. When we want to think of some CW-structure on a given space, all that we have to do is, as I have told you earlier, to decompose the space into disjoint open cells, of course, in a systematic way. Start with 0-cells, then 1-cells if necessary, then 2-cells and so on.

We have to start with 0-cells, no problem. So here you have $\mathbb{S}^1 \times \mathbb{I}$, the cylinder. Look for a CW-structure this cylinder, which is some way related to the CW-structure on the two factors, $\mathbb{S}^1$ and the CW-structure on the interval $\mathbb{I}$. So look at the 0-cell and the 1-cell in $\mathbb{S}^1$, denote them by σ_0 and σ_1. Similarly I could have denoted the cells of $\mathbb{I}$ also by some notation but I will use the obvious notation here viz., $\mathbb{I} = [0,1]$. I will use the natural notation 0 and 1 as the 0-cells and the interval $\mathbb{I}$ as the 1-cell.

With this notation for individual factors, the 0-skeleton of the product will be the product of the two 0-skeletons, and hence consists of two 0-cells, viz., the points $(\sigma_0, 0)$ and $(\sigma_0, 1)$. Next the 1-skeleton of the product will be the union of the product of the 1-skeleton times the 0 skeleton and the 0-skeleton times the 1-skeleton. And finally the 2-skeleton of the product will be the product of the two 1-skeletons.

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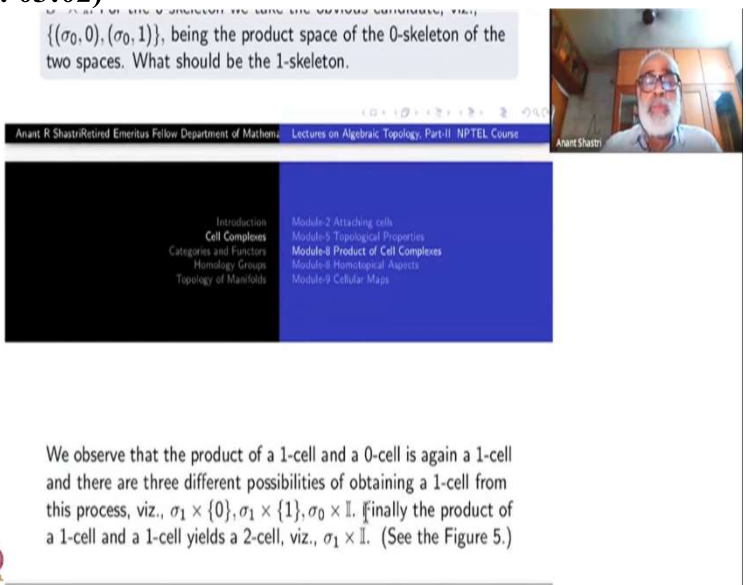

So, $\sigma_1 \times \{0\}, \sigma_1 \times \{1\}$ are two of the 1-cell because σ_1 is the 1-cell in $\mathbb{S}^1$ and the third one is $\sigma_0 \times \mathbb{I}$, because $\mathbb{I}$ is the 1-cell on the other factor.

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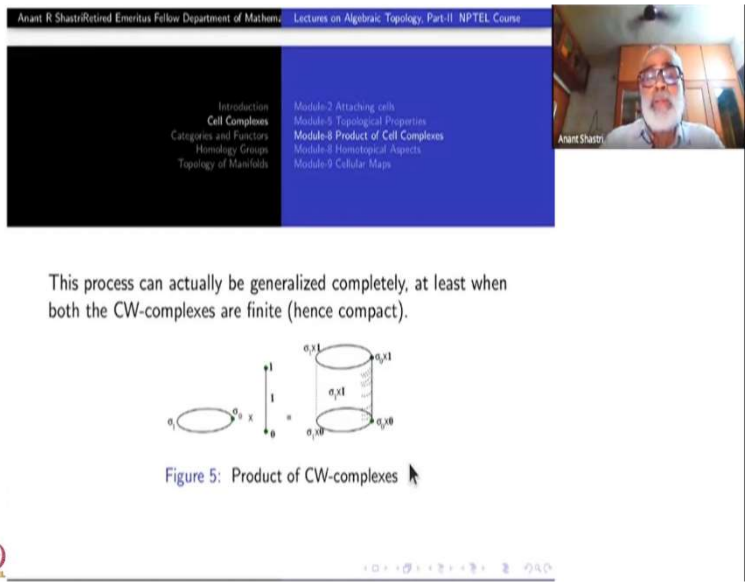

So that is the kind of thing what we have done this is there are two 0-cells here one 0-cell and 1-cell. Two 0-cells here and one 1-cell here. So you start with two 0-cells; $\sigma_0 \times \{0\}$ and $\sigma_0 \times \{1\}$ then this itself will become a 1-cell this to be another 1-cell and that will be another 1-cell corresponding to $\sigma_1 \times \{0\}, \sigma_1 \times \{1\}$ and then the other one is $\sigma_1 \times \sigma_0$ so the remaining cells whatever must be 1 single cell which corresponding to what $\sigma_1 \times \mathbb{I}$.

That will be 1-cell right. So the theme is that this example completely tells you the entire story. No matter what X and Y are. In the general case, all that you have to do is to decompose $X \times Y$ into 0-cells, 1-cells, 2-cells and so on. For any k, write k as a sum $i + j$, in all possible ways, take all i-cells of X cross all j-cells of Y, that will give you all k-cells for $X \times Y$.

Because X and Y are disjoint union of open cells, every point in the product belongs precisely one product cell obtained in the above manner. That is set theoretically completely obvious.

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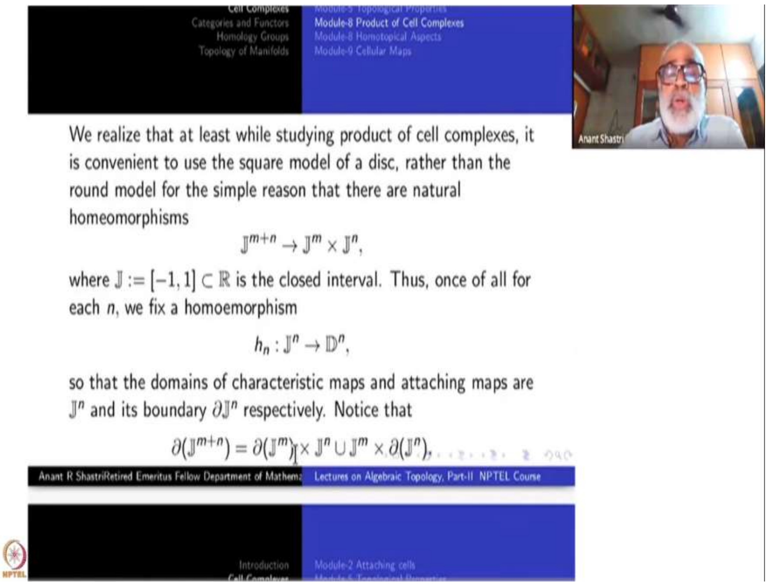

When we come to attaching maps, we have a slight problem here. Because we have modelled our cells on the round mode, viz, on unit discs, the set of all vectors of length less than or equal to 1 inside $\mathbb{R}^n$. So that is the round model. Since we are only interested in things up to homeomorphism anything which is homeomorphic to a disc will be called a-cell. Therefore, we are free to choose any model. Here when you pass to products a disc cross another disc does not look like a round disc. Therefore you have a problem there. So you may change the domains up to homeomorphism type anyway, you have to fix up some homeomorphism from $\mathbb{D}^n \times \mathbb{D}^m$ to $\mathbb{D}^{m+n}$. So that has to be done. Whether you do that every time or you choose a different model each time is left to you. So here is a way of handling it. Right in the beginning we can fix-up some homeomorphisms. For dimension one there is no problem, $\mathbb{D}^1$ is just the interval $[-1, 1]$.

But as soon as the dimension 2, the square and the round disc are not the same spaces, Cubes and $\mathbb{D}^3$ are not the same and so on. So what do you do? Put $\mathbb{J} = [-1, 1]$, the interval. Let us have this notation standard notation, like $\mathbb{I} = [0, 1]$. Fix up a homeomorphism from $\mathbb{J}^m$ to $\mathbb{D}^n$ for each m, once for all. Now the beauty is that $\mathbb{J}^{m+n}$ is cannonically homeomorphic to $\mathbb{J}^m \times \mathbb{J}^n$, just by writing first m coordinates as one m-vector and then the last n-coordinates as one n-vector.

In our old definition of attaching cells, we used the models $\mathbb{D}^n$. We can now use the models $\mathbb{J}^n$. That means the domains of the characteristic maps and the attaching maps will be respectively, $\mathbb{J}^n$ and boundary of $\mathbb{J}^n$ instead of $\mathbb{D}^n$ and boundary of $\mathbb{D}^n$. Note that under the canonical homeomorphism $\mathbb{J}^{m+n}$ to $\mathbb{J}^m \times \mathbb{J}^n$, we have boundary of $\mathbb{J}^{m+n}$ goes to boundary of $\mathbb{J}^m \times \mathbb{J}^n$ union $\mathbb{J}^m \times \partial\mathbb{J}^n$.

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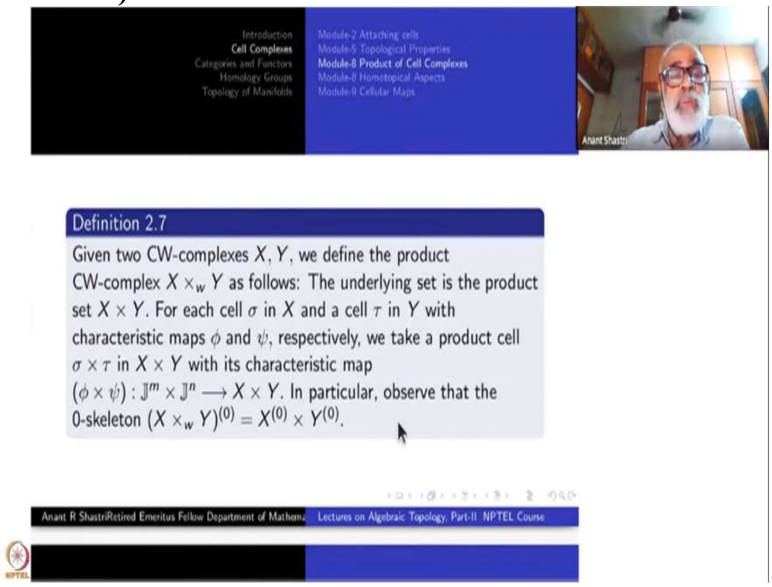

Now let us make a definition here in the most general case. Given two CW complexes X and Y, define the product CW complex $(X \times Y)_w$, this suffix w is to denote the week topology on the underlying set $X \times Y$ which is the product of the two underlying sets.

For each pair of cells (σ, τ), σ in X and τ in Y, with characteristic maps ϕ and ψ respectively, you take a product cell $\sigma \times \tau$ in $X \times Y$, with the characteristic map as the product of the two characteristics maps $\phi \times \psi$.

Observe that the 0-skeleton of $X \times Y$ is nothing but the 0-skeleton of X cross the 0-skeleton of Y. Similarly, the 1-skeleton of $X \times Y$ will be equal to the union of 1-skeleton of X cross the 0-skeleton of Y and 0-skeleton of X cross the 1-skeleton of Y. And so on.

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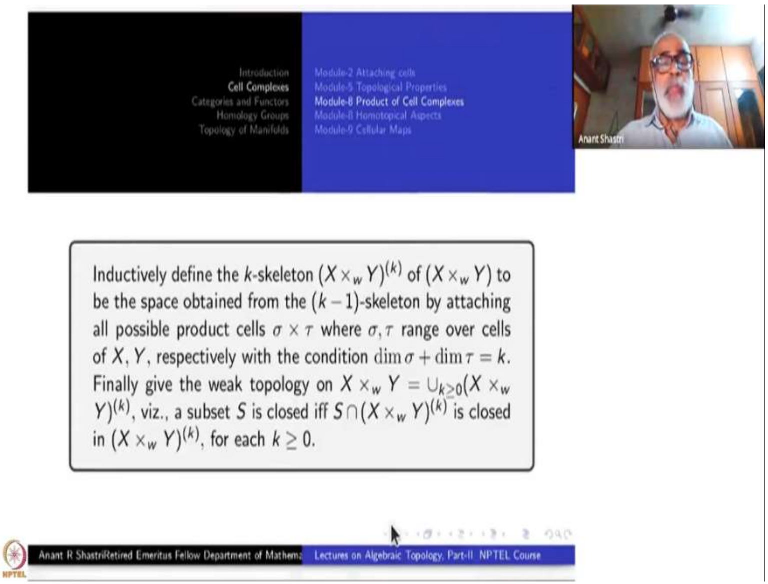

Inductively, the k-skeleton of $X \times Y$ will be the union of $\sigma \times \tau$, where σ ranges over cells of X and τ ranges over cells of Y with the condition that dimension of σ plus dimension of τ should add up to k. Finally we put weak topology on the union of all these k-skeletons. Recall what is the weak topology. A subset is closed if and only if its intersection with the k-skeleton is closed in k-skeleton for all k.

So this completes the definition of $(X \times Y)_w$ as a CW-complex. Once again note that if you are dealing with finite CW-complexes or even finite dimensional CW-complexes then you do not need this extra definition, because if both X and Y are finite dimensional, say m and n respectively, then you will stop at $m + n$ dimensional skeleton. Therefore there is no need to redefine the topology by using this. In other words, the product topology on $X \times Y$ is itself is the weak topology. So this is an easy observation here. Redefining the topology is necessary only when at least one of the factors is infinite dimensional.

And quite often that is the case and therefore we have to study the infinite dimensional case properly. That comes next.

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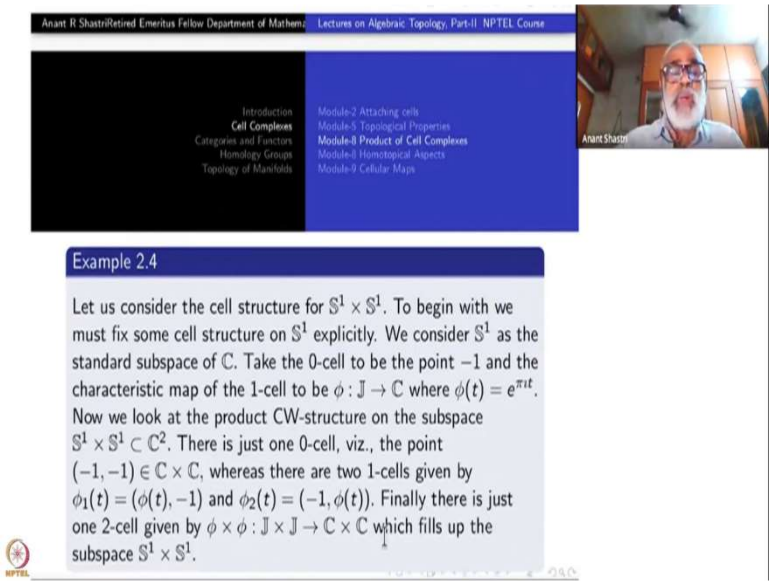

So before proceeding further, let me examine one more interesting example. Instead of $\mathbb{S}^1 \times \mathbb{I}$, now I consider $\mathbb{S}^1 \times \mathbb{S}^1$ and once again take the simplest CW-structure on $\mathbb{S}^1$ on both the factors, namely, one 0-cell and one 1-cell. As usual, of course you can think of $\mathbb{S}^1$ as a subset of $\mathbb{C}$, the set of all unit vectors, and $\mathbb{S}^1 \times \mathbb{S}^1$ as a subset of $\mathbb{C}^2$, just for fixing notations.

So what is the 0-cell? I would like to take the point -1 sitting inside $\mathbb{S}^1$ as the 0-cell and the characteristic map for the 1-cell to be the function ϕ from $\mathbb{J}$ to $\mathbb{C}$ given by t going to $e^{\pi i t}$. Now look at the product structure on $\mathbb{S}^1 \times \mathbb{S}^1$. So there is one just one 0-cell, namely the point $(-1,-1)$.

You see, in the case of $\mathbb{S}^1 \times \mathbb{I}$, you had two 0-cells. Here we have only one 0-cell because on both factors there is only one 0-cell. And that will be $(-1,-1)$. Whereas, there are two 1-cells namely, 1-cell in the first factor cross the 0-cell in the second factor and vice versa. What are the characteristic maps? We have to have, $\phi_1(t) = (\phi(t), -1)$ and $\phi_2(t) = (-1, \phi(t))$. So these are the two characteristic functions from 1-cells. Finally there is just one 2-cell, namely, with the characteristic map from $\mathbb{J} \times \mathbb{J}$ to $\mathbb{C} \times \mathbb{C}$, which is (t,s) going to $(\phi(t), \phi(s))$. So, I have written down this but that is obvious.

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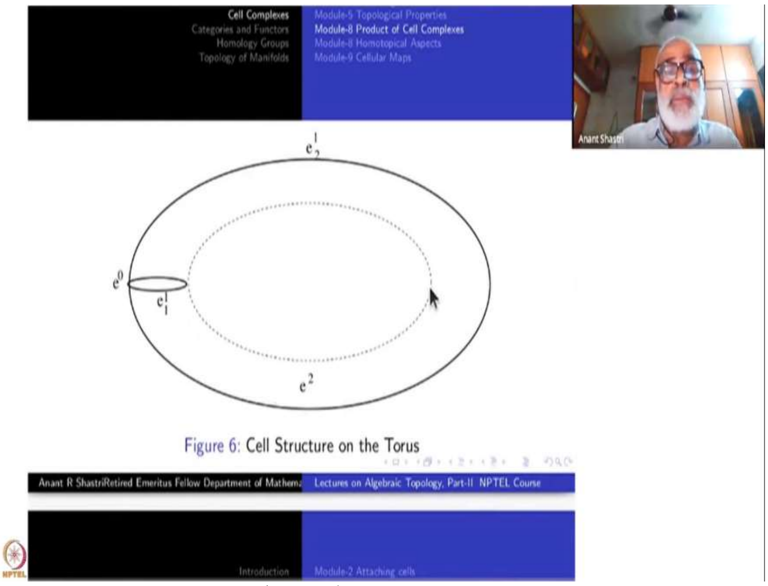

Figure 6: Cell Structure on the Torus

Let us take a look at this picture of $\mathbb{S}^1 \times \mathbb{S}^1$. So here I have used some notation to denote the cells directly in $\mathbb{S}^1 \times \mathbb{S}^1$: e^0, the 0-cell this is e_1^1 and then there is another one e_2^1, these are 1-cells. So this 0-cells is actually is $(-1,-1)$ and e_1^1 the 1-cell equal to $e^1 \times e_0$, and e_2^1 will be $e_0 \times e^1$. Finally the 2-cell e^2 is $e_1^1 \times e_2^1$. So this is e 2-cell.

Note that the inner circle shown by dotted ellipse is not a part of the 1-skeleton. So there are only four cells in all. So this is the way you have to decompose the product space. Automatically the boundaries of each k-cell will be contained in the $(k-1)$-skeleton and the characteristic maps will be product of the corresponding characteristic maps of the two factors.

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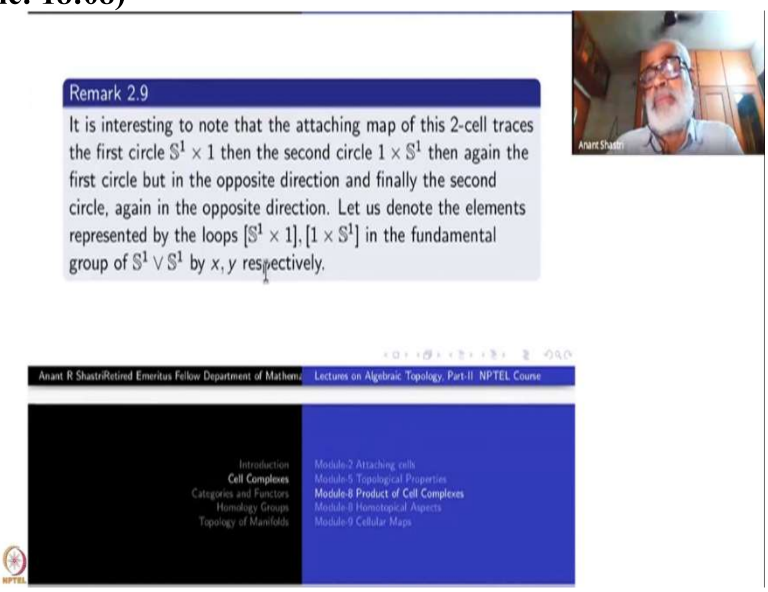

So here let us do that and get a nice output. So, it is interesting to note that the attaching map of this 2-cell here traces the first circle $\mathbb{S}^1 \times \{-1\}$, then the second circle $-1 \times \mathbb{S}^1$ then again the first circle in the opposite direction and again the second circle in the opposite direction. The 1-skeleton is equal to $(\mathbb{S}^1 \times \{-1\}) \cup (\{-1\} \times \mathbb{S}^1)$, with one point in common. Such a space is called wedge of two circles. This is a subspace of the torus. Look at the fundamental group of that space taking $(-1, 1)$ as the base point. Then what is happening is the that the attaching map first goes around the first circle then goes around the second one and then again around the first one but this time in the opposite direction, and finally around the second one again in the opposite direction. The time it comes around the other way round and goes around that way. If you denote the corresponding elements in the fundamental group by x and y respectively, then the characteristic map will represent the element $xyx^{-1}y^{-1}$.

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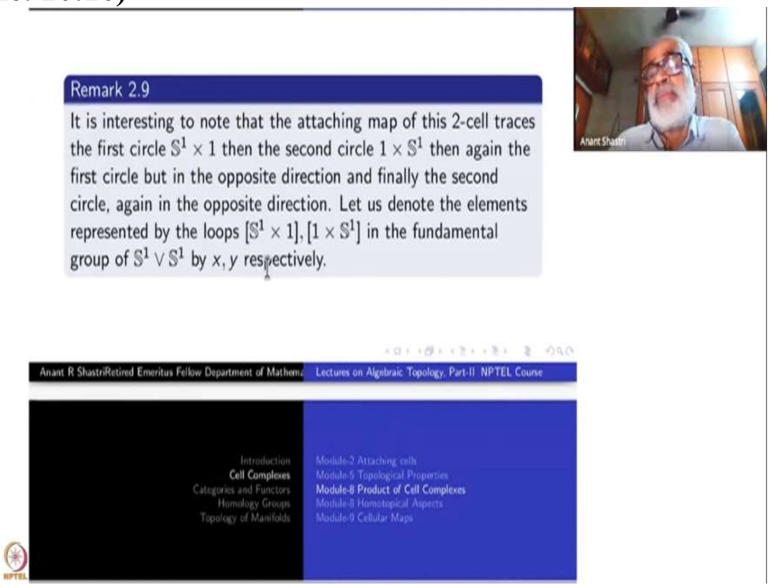

So this is the picture. You all know that the torus can be regard as obtained as a quotient of a square by identifying the opposite sides --- this edge with this edge and this edge, this edge with this, as shown by the arrows placed on these edges. The arrows on each pair of opposite sides are in the same direction. Therefore, the loop formed by the boundary of this square can be traced from here to here to here to here to here. This is going to be the boundary of the 2-cell that I am going to attach, the attaching map first traces this side let us call this x, then it traces this side call it y, but now the side which should be x is being traced the other way, so it is x^{-1} and finally the side y is traces in the opposite direction and is y^{-1}.

So, while studying quotient spaces, you study what happens in the mother space, everything will be very clear. So this is the lesson you have learned.

So this is $xyx^{-1}y^{-1}$. We have done in the first part of the course, that the fundamental group gets quotiented by this element $xyx^{-1}y^{-1}$, which means that you have to take the normal subgroup generated by this element, go modulo that to get the fundamental group of the of the mapping cone.

The mapping cone here is the same as attaching one 2-cell here along the attaching map to the 1-skeleton whose fundamental group is the free group on two generators. Here the generators are x and y. Therefore, we go modulo $xyx^{-1}y^{-1}$. That is the same thing as declaring that x commutes with y. What do you get? The free abelian group on the two generators.

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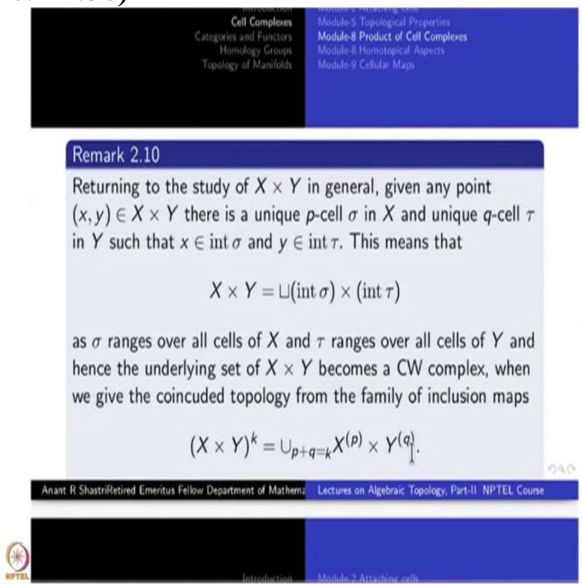

So $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ is a free abelian group over two generators. Of course we know this already by another method, namely, we know that $\pi_1(X \times Y)$ is $\pi_1(X) \times \pi_1(Y)$. Therefore $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ is $\pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1)$, which is $\mathbb{Z} \times \mathbb{Z}$. That much we know. So the point here is when you want to learn a new concept you should check it or you should learn it with some example which you already know very well, so that you will be sure of your ground. So, that is one special example here.

Returning to the general study, let us be sure that whatever you have described as a CW structure on $X \times Y$ is actually a CW-complex as per our definition. Given any point $(x, y) \in X \times Y$, as I have said, there is a unique p-cell σ in X and unique q-cell τ in Y such that x is in the interior of σ and y is interior of τ. Why? Because every CW complex is the disjoint union of its open cells and we have started with X, Y as CW complexes.

So takes σ and τ as required. That would mean that $X \times Y$ is precisely a disjoint union of interior σ cross interior of τ, these are subsets of X and Y respectively. So the product these interiors will be a subspace of $X \times Y$ and different ones will be disjoint from each other, because, the interior of different cells in X and Y are disjoint from each other. When you take all possibilities fro σ and τ, then you get the entire set $X \times Y$.

And not only that what happens to k-skeleton the k-skeleton consists of all cells of dimension less than equal to k. You have to put them together. So, that is why you are taking union over (p, q) with $p + q = k$ of $X^p \times Y^q$.

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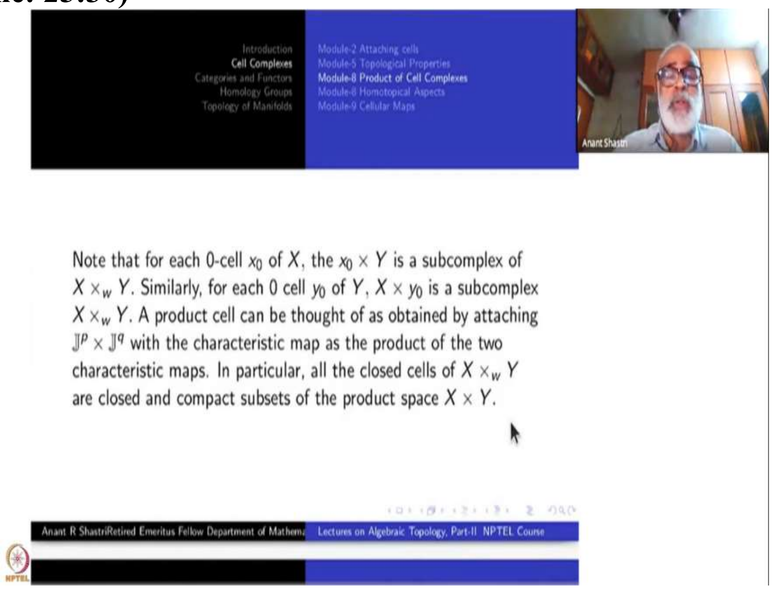

For each 0-cell x_0 of X, if you take $x_0 \times Y$, that will be a subcomplex of $X \times_w Y$, because the attaching maps are the same. Similarly, for every 0-cell $y_0 \in Y, X \times y_0$ is a subcomplex of $X \times_w Y$. However, we cannot take arbitrary points in X or Y here. Remember that. You have to take 0-cell in X (or in Y) and then take product with Y(or with X accordingly) to get a subcomplex.

Each product k-cell can be thought of as obtained by attaching $\mathbb{J}^p \times \mathbb{J}^q$ with characteristic map as the product of the two characteristics maps, where p and q range over all possibilities such that $p + q = k$. In particular, all closed cells of $X \times Y$ are closed and compact subsets of the product space, because they are just products of compact sets.

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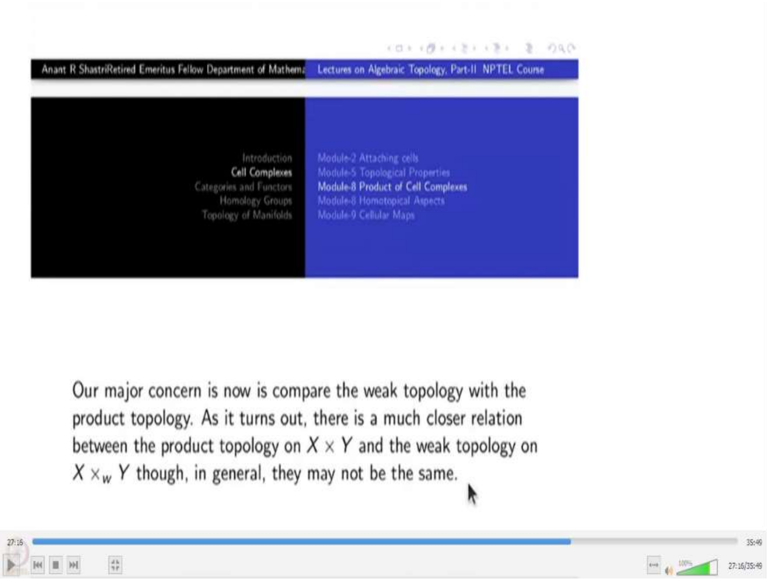

So the only major concern is when X or Y is of dimension infinite. So let us resolve that then we will be very happy with the product of CW complexes.

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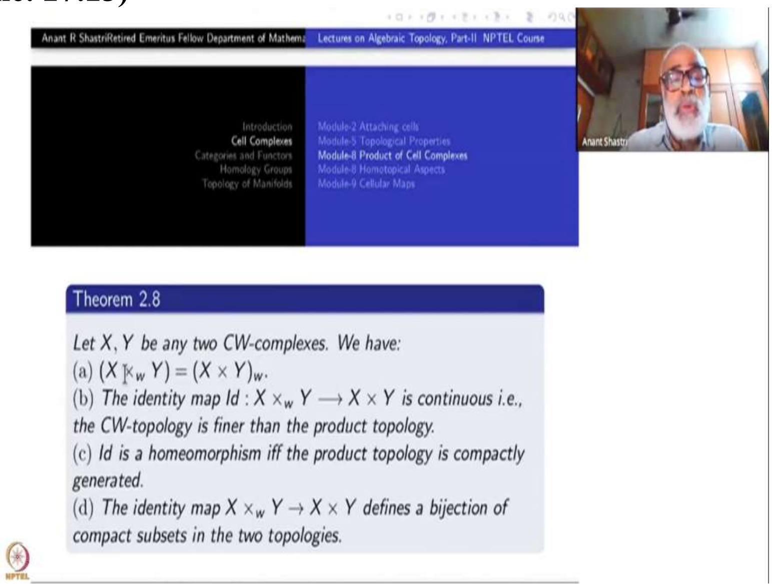

So let X and Y be any two CW-complexes. First thing is to note that the CW-complex structure on $X \times Y$ is nothing as described before, start with the product topology and then retoplogize $X \times Y$ by the compactly generated toopology. That is the meaning of writing the suffix w out the bracket. Remember that for any space Z, Z_w just means the topology coinduced from the family of compact subsets of Z. So you may use the notation $X \times_w Y$ or $(X \times Y)_w$. But do not write $X \times Y_w$ without brackets, that will have different meaning.

Therefore, we know that the identity map from $X \times_w Y$ to $X \times Y$ is always continuous and identity map is a homeomorphism if and only if the product topology is compactly generated. We also know that in any case, the identity map defines a bijection of compact subsets on either side. These general things we have already studied. So we can apply them to the situation of CW-complexes.

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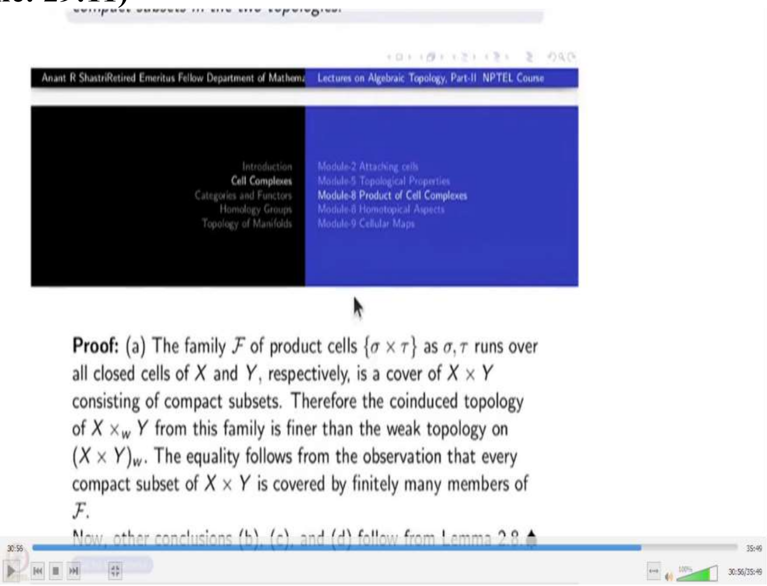

Now look at the family $\mathcal{F}$ of all product cells $\sigma \times \tau, \sigma$ and τ range over all closed cells of X and Y respectively. This $\mathcal{F}$ is a cover for $X \times Y$ and each member of $\mathcal{F}$ is compact. If you have a subfamily of compact sets covering the whole space and give the coinduced topology from such a family, that may not be the same thing as the coinduced topology from the entire family of closed subsets. Indeed, the coinduced topology $X \times Y$ from $\mathcal{F}$ is actually finer than the weak topology on $X \times Y$, viz, $(X \times Y)_w$.

Because you have to verify the same condition for a smaller a smaller family. If you have smaller family, the coinduced topology will be finer. We have perhaps more open sets because there are less number of conditions to be verified. In general, equality of the two topologies is wrong.

But in this case, equality follows from the following observation. Take any compact subset of a CW complex, it is covered by finitely many closed cells of the complex by theorem 1.2, members that one. That will be true for subsets of the product space $X \times Y$ also. Therefore each compact subset K of $X \times Y$ is covered by finitely many members of $\mathcal{F}$.

Therefore these 2 topologies are the same and we have seen that (b), (c), (d) other things are automatically satisfied in the general case. So, this was nice to see.

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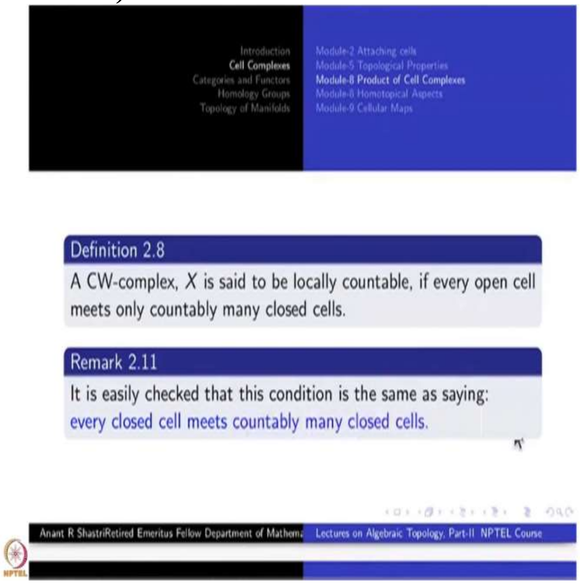

So we still have not come to the case when we have infinite dimensional complexes. We still do not know whether $(X \times Y)_w$ is equal to the product space $X \times Y$. So that is our next concern. It will not happen always. So what are the best conditions under which may happen? That is what we are going to study. So here is a definition which will be a nice condition which will guarantee some kind of equality of the two topologies, $(X \times Y)_w$ and $X \times Y$.

A CW complex is said to be locally countable if every open cell meets only countably many close cells. Let me elaborate on this one. Take a 0-cell, it is both open and closed. So what does the condition mean in this case? It means that the set of all the closed cells which will hit this 0-cell is countable. How does a closed cell hit a 0-cell? Via its attaching maps. In particular, it means that the number of edges or one-cells which are coming out of any fixed 0-cell is countable. Similar conclusion holds for every edge, i.e., for every closed 1-cell and so on.

First remark here is that if I replace open cell by closed cell in the above condition, then this condition will still be true. Suppose I take this condition that every open cell meets only countably many close cells. Take the closure of that open cell, that is the closed cell which a larger set and hence it may intersect too many of other closed cells! No, it will also meet only countably many of them. Why? because the closure of each cell being a compact subset will be covered by finitely many open cells and each of them will meet only countably many closed cells.

So finite union of countable families will be again countable. So, whether I put the word `open' or `closed' here in this definition, it does not matter, they are identical.

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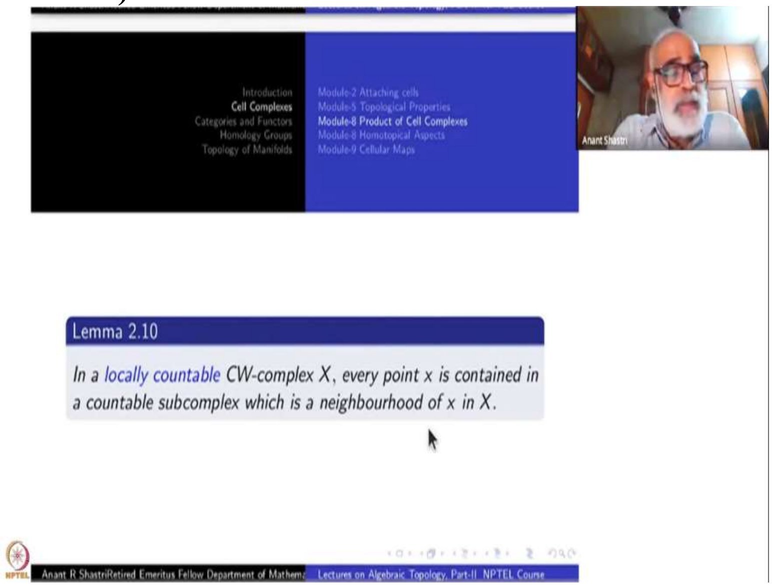

Now here is a lemma. Take any locally countable CW-complex X. Every point x is contained in a countable subcomplex which is a neighbourhood of x in X.

That it is contained in a countable subcomplex is obvious--- you can first take the unique open cell to which x belongs. Then you can try to prove a substatement that every closed cell is contained in a countable subcomplex, which seems to be not so difficult.

However proving that such a subcomplex is also a nbd of x seems be to somewhat non trivial. We shall see this one next time.