Introduction to Algebraic Topology (Part – II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 48
Manifolds with Boundary

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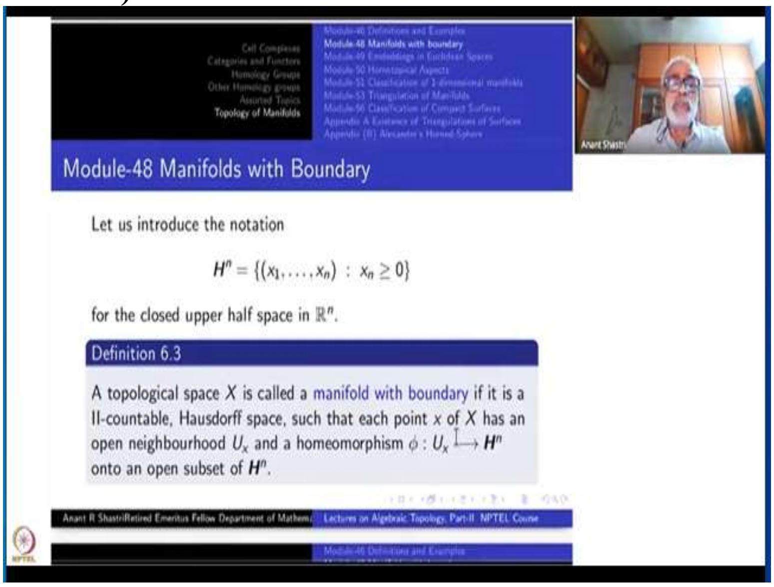

So today we shall continue the study of manifolds with the boundary. Basic thing here is that instead of the model $\mathbb{R}^n$, we shall use the model $\mathbf{H}^n$. $\mathbf{H}^n$ denotes the subspace of all points of $\mathbb{R}^n$ with their n-th coordinate greater than or equal to 0. So if $n = 1$, this is just the closed ray $[0, \infty)$. A topological space X is called a manifold with boundary if it is II-countable, Hausdorff space, (these two conditions are as before but now) but now the atlas consists of charts, which take values inside $\mathbf{H}^n$, viz., at each point $x \in X$, we have an open neighbourhood U_x of X, and a homeomorphism from U_x onto an open subset of $\mathbf{H}^n$. That is the only difference okay?

So if you understand what kind of open subset of $\mathbf{H}^n$ are there, as compared to open subsets of $\mathbb{R}^n$, then you will understand the meaning of this definition as compared to our old definition of a manifold. Since $\mathbf{H}^n$ is a subspace of $\mathbb{R}^n$, a subset of $\mathbf{H}^n$ is open in $\mathbf{H}^n$ iff it is the intersection of an open subset in $\mathbb{R}^n$ with $\mathbf{H}^n$.

Therefore, for examples, suppose I take an open a disc in $\mathbb{R}^n$ and intersect it with $\mathbf{H}^n$, it is a portion of that open disc right? Cut off by the hyperplane, namely $x_n = 0$. All point with their n-th coordinate $x_n < 0$, will be thrown out. So on the other hand, its intersection with

$\mathbb{R}^{n-1} \times 0$ itself will be there inside it. So, if any of those points are present then such a set will not be an open subset of $\mathbb{R}^n$ itself. Indeed, $\mathbf{H}^n$ itself is not an open subset of $\mathbb{R}^n$, okay? So that is the point that you have to pay attention to.

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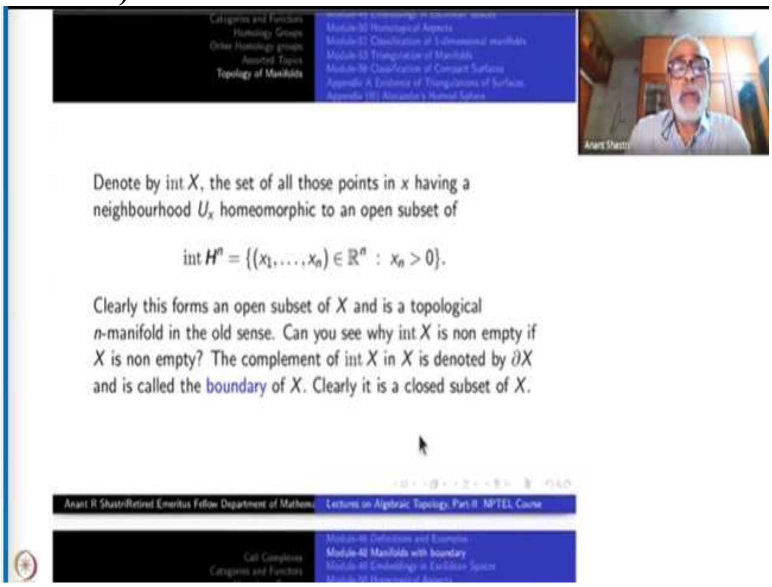

Start with a manifold with boundary. Denote by interior of X the set of all points in X having a neighbourhood U_x homeomorphic to an open subset of interior of $\mathbf{H}^n$, which is the same thing as saying that the n-th coordinate of the image under the homeomorphism is strictly positive. Since this interior of $\mathbf{H}^n$ is an open subset of $\mathbb{R}^n$ also, it follows that $int(X)$ will constitute our manifold with the old definition okay?

So I want $int(X)$ (read it interior of X) to denote the subspace of X consisting of those points which have some open neighbourhood in X which is homeomorphic to some open subset of $int(\mathbf{H}^n)$ and hence already open in $\mathbb{R}^n$ as well. Caution: This should not be confused with the standard notion of the interior of a subset of a topological space.

Clearly, this $int(X)$ is open in X, okay? Once a point is in $int(X)$, all point in an open neighbourhood of it are also in $int(X)$. Therefore, $int(X)$ is a topological n-manifold in the old sense. Of course, it remains to check why $int(X)$ is non empty, if X is non empty. Can you see why? For the moment, given any non empty open subset of $\mathbf{H}^n$, its intersection with $int(\mathbf{H}^n)$ has to be non-empty. Therefore, under the chart, that part of the neighbourhood will be non empty and is contained in $int(X)$. So X non empty implies $int(X)$ is non empty. The complement of $int(X)$ in X is denoted by ∂X and is called the boundary of X. Again, this should not be confused with the with notion of the boundary of a subset of a topological space.

In the old definition in general topology, the boundary of 2-disc in $\mathbb{R}^2$ is a circle. This is true in the present definition also, on no matter where the disc is contained in as a subspace. The circle is the manifold boundary of a 2-disc. That is sacrosanct. Indeed, the general point set topology came late than the study of manifolds, and the definition of the boundary of a subset is adopted from the properties of the boundary of X as a manifold, okay? Rather than the other way round. So this is called the boundary of X so we will have this notation for manifolds only. We will not use this notation for an arbitrary subset A of X and then boundary of X, we will not use that we will use $\bar{A} \setminus A^{\circ}$, where A° denotes the interior of the subset A in X as in general topology. So clearly ∂X is close subset of X okay? Because it is the complement of $int(X)$ which is an open subset of X, alright?

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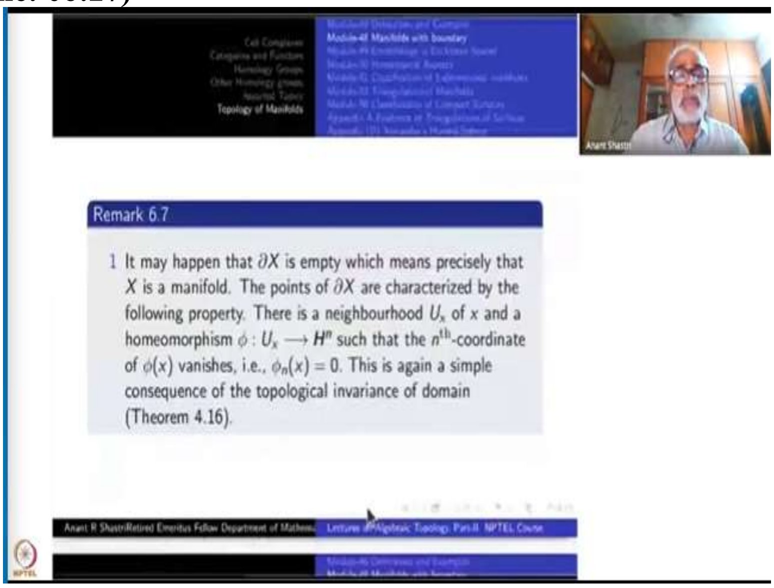

It may happen that ∂X is empty. We cannot say. All the charts may actually land up inside $int(\mathbf{H}^n)$, in which case, the entire of X will be equal to $int(X)$, okay? However, ∂X happens to be the boundary of $int(X)$ inside X in the sense of general topology also, because the closure of $int(X)$ is the whole of X.

All in all, the new definition is actually an extension of the old definition. It would have been better to call this new class of topological spaces simply as `manifolds' and the old class by the name manifolds without boundary. The present practice is justified because often we are studying the old class.

Now comes the crucial thing. Pay attention to this one. The points of ∂X are characterized by the following property. What is it? Each $x \in \partial X$ has an open neighbourhood U_x of $x \in X$ and a homeomorphism ϕ from U_x into $\mathbf{H}^n$ such that the nth-coordinate of $\phi(x)$ is actually 0. The claim is that we can always arrange such a homeomorphism, okay? This is the characterization of the points on the boundary.

If you have one such ϕ such that the n-th coordinate of $\phi(x)$ is positive then x will automatically in $int(X)$. Therefore, remember our definition of $int(X)$ and ∂X are completely non overlapping.

This is again a simple consequence of the topological invariance of domain. I repeat this one, namely, if the n-th coordinate of $\phi_1(x)$ is positive, then the n-th coordinate of $\phi_2(x)$ will be also positive. For we can then choose a smaller open neighbourhood U of $x \in X$ such that $V = \phi_1(U)$ is completely contained in $int(\mathbf{H}^n)$. Since $\phi_2 \circ \phi_1^{-1}$ is a homeomorphism of V to some subset of $\mathbf{H}^n \subset \mathbb{R}^n$. By invariance of domain, it follows that this is an open subset of $\mathbb{R}^n$ contained in $\mathbf{H}^n$ and hence completely contained in $int(\mathbf{H}^n)$. So once a point of X belongs to the boundary, namely, that the n-th coordinate of its image under one chart is zero, then for any other chart at that point, the n-th coordinate of the image of that point will be 0.

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So let us go ahead. Suppose ϕ from U to $\mathbf{H}^n$ is a chart at a point $x \in \partial X$. Take $\hat{U}$ equal to $\phi^{-1}(\mathbb{R}^{n-1} \times \{0\})$. That will be a closed subset of U and contained in ∂X, because under ϕ, the n-th coordinate of each point in $\hat{U}$ is 0. $\hat{U}$ actually an open neighbourhood of $x \in \partial X$ because $\hat{U}$ is equal to $U \cap \partial X$.

So ϕ restricted to ∂X will give you a chart for ∂X which now $(n-1)$-dimensional in the old sense, because $\phi(\hat{U})$ is equal to $\phi(U) \cap (\mathbb{R}^{n-1} \times 0)$ and hence is open in $\mathbb{R}^{n-1} \times 0$. So, we have just proved that the boundary of X itself is a manifold of dimension $n-1$, Hausdorff and II countability being automatically satisfied for subspaces. Therefore, the boundary of X, if it is non empty, then it is topological $(n-1)$-dimension manifold okay? And ∂X will not have any boundary because now every chart is taking values in inside open subsets of $\mathbb{R}^{n-1} \times 0$ which is clearly, homeomorphic to $\mathbb{R}^{n-1}$, okay? So boundary of the boundary of X is always empty.

We shall most often use the word manifold to mean a manifold without boundary okay? Only for that reason we defined the manifold with as the $\mathbb{R}^n$, instead of $\mathbf{H}^n$, okay? Often the results are stated for the so called manifold without boundary, but they are valid for manifolds with boundary also. However sometimes we have to take slightly different version and the proof will also be slightly different. But because of time constraint and to give more time for emphasizing the concepts, we will prove many of these results only for manifolds without boundary, which we keep calling just `manifolds'. If there is a boundary we will specifically mention it because those cases occur fewer times. okay?

So we shall now introduce the notion which comes handy in proving a number of results which are valid for manifolds as well as for manifolds with boundary and once you prove some result for a manifold okay, then automatically or with with a very little effort, it gets proved for manifolds with boundary also. Okay? Of course, there are some properties for which this method will not work, but for many of them it will. So that is why we are introducing it.

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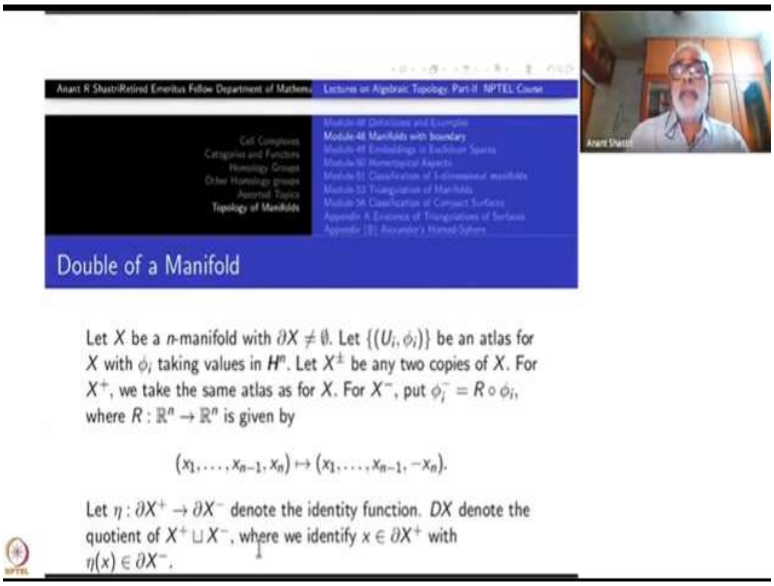

So, that is the concept of `double of a manifold'. Starts with n-manifold X with boundary i.e., ∂X is non empty. That is important. We further assume that X is connected, to get a picture of what exactly is going on okay? Think of an arc A for example, a closed arc inside a circle. Take another copy if it, say $f(A)$ under a homeomorphism. Take the disjoint union of A and $f(A)$ and identify the boundary points of A with that of $f(A)$, x and $f(x)$, corresponding points under f.

What do you get? You get a space homeomorphic to a circle. That is what we mean by a double of a manifold. We should define this one more carefully here. Okay? Let $\{(U_i, \phi_i)_i\}$ be an atlas for X with ϕ_i taking values in $\mathbf{H}^n$. Let $X^{\pm}$ denote any two copies of X, Okay? We take the same atlas on X^+ as for X. For X^-, we will the atlas obtained by the same atlas but followed by the reflection R in the hyperplane $\mathbb{R}^{n-1} \times 0$, and denoted by ϕ_i^-, so the n-th coordinates of $\phi_i^-(x)$ will all non positive. Here $R(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_1, x_2, \ldots, x_{n-1}, -x_n)$.

Now let η denote the identity function from ∂X^+ to ∂X^-. Let DX denote the quotient of X^+ disjoint union X^- where we identify each $x \in \partial X^+$ with $\eta(x) \in \partial X^-$. Alright, that is my quotient space. On the quotient space, we can define an atlas now. If point belongs to the interior of X whether it is in X^+ or X^-, there is no problem you can take an appropriate restriction of ϕ^+ or ϕ^- so we get a chart inside $\mathbb{R}^n$. now okay maybe it is plus or minus ϕ, okay?

The problem is only at points belonging to boundary of X^+ or X^- wherein certain identification is taking place, a point $x \in \partial X^+$ and its copy $\eta(x)$ representing the same point in the quotient space DX. So what do we do there? That also is very easy here. Take the union U_x and its copy on which take ψ equal to ϕ^+ or ϕ^- accordingly. Then ψ will factor down to a homeomorphism of the image in DX of the union these two open sets into an open set in $\mathbb{R}^n$, which is the union of $\phi^+(U_+)$ and $\phi^-(U_-)$ patched up along their intersection which lies inside $\mathbb{R}^{n-1} \times 0$.

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So that is what the double of X is, which is therefore a manifold without boundary. All the boundary points have become interior points in DX. DX has 2 closed subspaces they are manifolds with boundary viz, X and its copy. Indeed, now $\partial X^+ = \partial X^-$ is also the exact boundary of each of them in DX, in the general topological sense as well. Now you see the relation between the manifold boundary and set theoretic boundary. So this DX is called the double of X. You can now see that this concept is defined even if we do not assume that X is connected or for that matter even if ∂X is empty. Just check what is the result in each of these cases.

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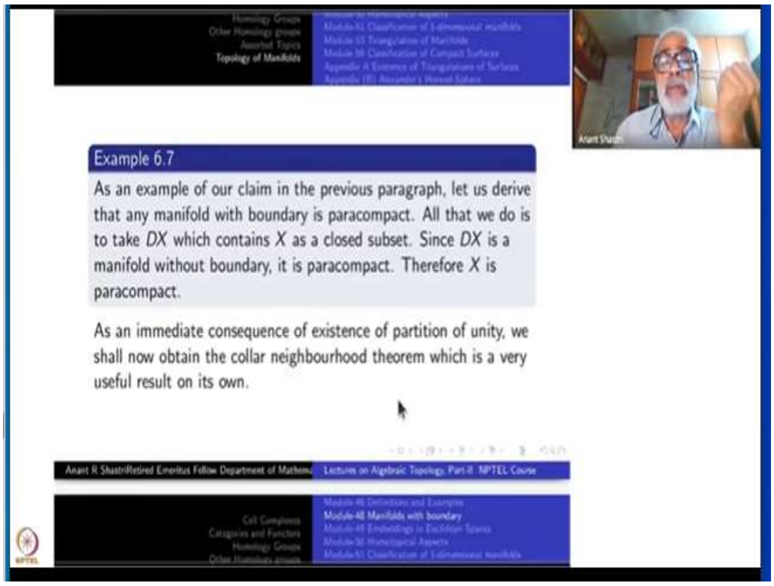

This concept of double of a manifold will be useful in several cases okay? So, as an example of this claim, you can use it to derive that any manifold with boundary is paracompact. See we have proved it for manifolds without boundary. But now X has boundary, but you do not have to prove it again all the way afresh. All that you do is to take DX which contains X as a closed subset. DX is a manifold without boundary and our theorem proved previously, says it is paracompact. Every closed subspace of a paracompact space paracompact okay?

So as an immediate consequence of the existence of partition of unity, we shall now obtain a slightly better topological picture for the boundary of a manifold, namely, the Collar Neighbourhood theorem. You will see that this is a very useful result on its own.

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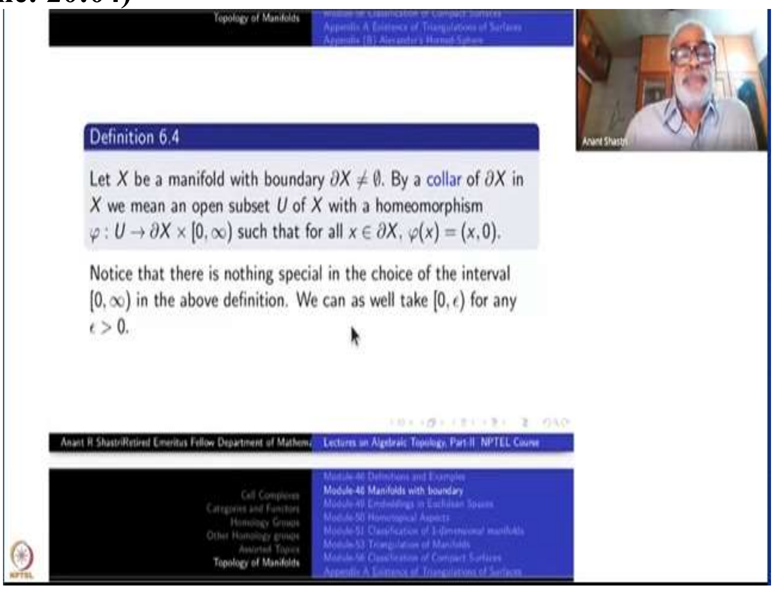

Let X be a manifold with boundary non empty. What is the meaning of a collar of the boundary of X? We mean an open subset U of X which is homeomorphism to boundary of

$X \times [0, \infty)$ such that this homeomorphism ϕ from $\partial X \times [0, \infty)$ should be such that $\phi(x, 0) = x$, okay? So that is the meaning. So this you must be an open subset of X itself, okay?

If you take a disk okay? say the standard unit 2-disk in $\mathbb{R}^2$. The boundary of this, you know is the circle. What will be a collar neighbourhood? All points of the disc which are of the norm bigger than say, $1 - \epsilon$ for some ϵ between 0 and 1, okay? That will be a collar for $\mathbb{S}^1$ inside $\mathbb{D}^2$. More generally this is true for $\mathbb{S}^{n-1}$ inside $\mathbb{D}^n$. To begin with you get a homeomorphism defined on $\partial X \times [0, \epsilon)$ given by (x, t) going to $(1 - \epsilon)x$. You can then use a homeomorphism of $[0, \infty)$ to $[0, \epsilon)$ to get want.

Infact, in the definition itself you can replace $[0, \infty)$ with any half closed interval $[0, \epsilon)$. So there is no problem.

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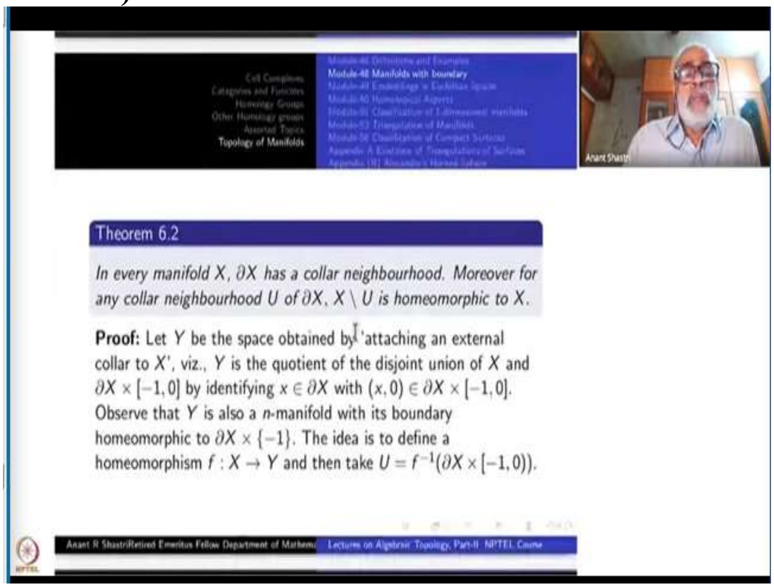

So in every manifold X, boundary of X has a collar neighbour, okay? (A collar neighbourhood which we sometimes just call a collar, is a special kind of neighbourhood. alright?) Moreover for any collar neighbourhood U of boundary of X we have the property that $X \setminus U$, the complement of the collar is homeomorphic to X itself. This is the statement.

If you throw away the $1 - \epsilon$ neighbourhood of the circle from the disk (provided that ϵ is what less than 1, okay 1? You get a smaller disk which is homeomorphic to the old, full disk. So this is easy to see for the disk because you have taken a nice object. However, this is true for all topological manifolds with boundary, Okay? So that is what we want to prove. In fact the second part will also get proved along with the proof of the first part. So, you will see that we do not have to waste so much of time for proving the second part again.

So let Y be the space obtained by attaching an `external collar' to X. So instead of working inside of X, I am going to enlarge X itself, somewhat similar to what we did in the construction of the double of X, but slightly differently.

For instance, consider the subspace A of $\mathbb{R}^n$ consisting of points x whose n-th coordinate x_n is bigger than or equal to $-\epsilon$. You can express A is the union of $\mathbf{H}^n$ with $\mathbb{R}^{n-1} \times [-\epsilon, 0]$. Here we have enlarged the model space $\mathbf{H}^n$ with an extra space which is homeomorphic to $\partial\mathbf{H}^n \times (0, \epsilon]$. We can then visualize A itself to be homeomorphic to $\mathbf{H}^n$. This is the picture of what happens in the model and we want to say that same thing will happen for the general manifold itself. Okay?

So we start with the constructing the space Y obtained by attaching an external collar to X. Y is the quotient of the disjoint union of X (in the double what we did another copy of X we do not bring the whole copy but take something smaller) with boundary of $X \times [-1, 0]$, okay? And identify each $x \in \partial X$ with $(x, 0) \in X \times [-1, 0]$. Observe that, as in the proof of DX is a manifold, Y is also n-manifold with its boundary homeomorphic to $\partial X \times \{-1\}$.

The idea is to define a homeomorphism f from X to Y and then take $U = f^{-1}(\partial X \times [-1, 0))$. So this Y, which is slightly a larger manifold that X, is again homeomorphism to X in such a way that on the boundary $(x, -1)$ corresponds to x. By invariance of domain boundary always goes to the boundary under any homeomorphism. The inverse image of $\partial X \times [-1, 0]$ will then be a collar neighbourhood of ∂X in X.

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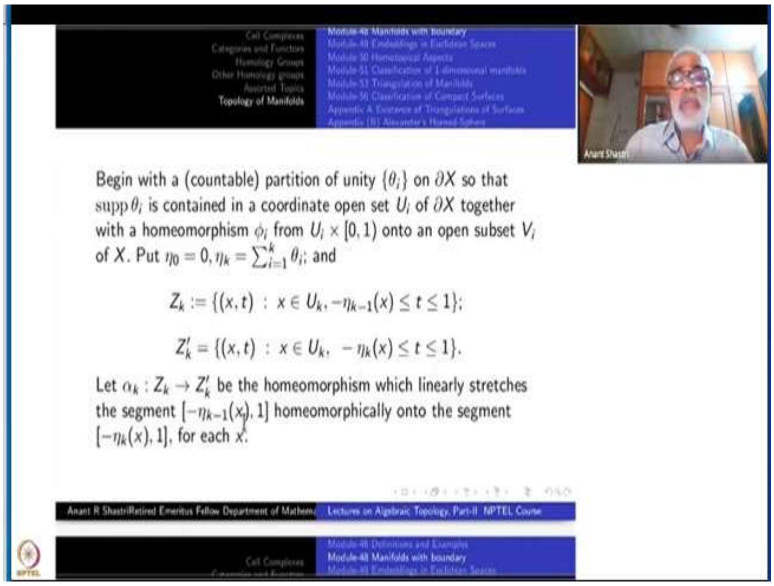

So how to get a homeomorphism like this? Begin with a countable partition of unity $\{\theta_i\}$ on the boundary of X, So that the support of θ_i is contained in a coordinate open set U_i of boundary of X, together with a homeomorphism ϕ_i from $U_i \times [0, 1)$ onto an open subset V_i of X. Okay?

So instead of choosing arbitrary neighbourhoods of a point on the boundary inside X, okay, there are coordinate charts and I am choosing a product expression for them. At each point x of ∂X, first we choose a coordinate neighbourhood W_x of $x \in X$ and a homeomorphism ϕ from W onto an open subset of $\mathbf{H}^n$. We can then choose a nbd U_x and $\epsilon_x > 0$ such that $U_x \times [0, \epsilon_x)$ is contained in $\phi(W_x)$. Put V_x equal to $\phi^{-1}(U_x \times [0, \epsilon_x))$ and take the restriction of ϕ to V_x. Now pass on to a countable subcover $\{V_i\}$ of $\{V_x\}$ and reparametrize all the ϕ_i's appropriately. And then choose the partition of unity $\{\theta_i\}$ as declared.

I start defining the homeomorphism inductively. Take η_0 to be 0 and η_k to be the sum from $i = 1$ to k of θ_i. Okay? Let us take Z_k to be the subset of $U_k \times [-1, 1]$ consisting of all points (x, t), x is in U_k, (U_k is a subset of boundary of X, okay) and t lies between $-\eta_{k-1}(x)$ and 1. Okay?

So, by definition this Z_k is a subspace of $\partial(X) \times [-1, 1]$. Because the entire sum of θ_i's is 1, and I am taking only some partial sum and putting a minus sign, so the lower bound for t is -1. so this will always be less than 1. Let Z'_k be the set of all those (x, t) such that $x \in U_k$ but the lower bound for t is $-\eta_k(x)$, and upper bound 1.

So Z'_k may be slightly bigger that Z_k. Let now α_k from Z_k to Z'_k be the homeomorphism which linearly stretches the line segments $[-\eta_{k-1](x),1]}$ onto $[-\eta_k(x), 1]$ for each x and keeps the first coordinate x at it is. okay? You see the two segments may be equal for some x. Then α on that segment will be identity.

Any two closed intervals (of positive length) are homeomorphic to each other. These homeomorphisms (as well as their domains and codomains) depend on x, right? okay. Luckily for you the endpoints of these intervals are parameterize by x in a continuous fashion. There only one linear homeomorphism each time since we also demand that the upper end point 1 goes to upper end point 1 each time. Therefore, you can write down a formula for the entire α which shows that α is a homeomorphism.

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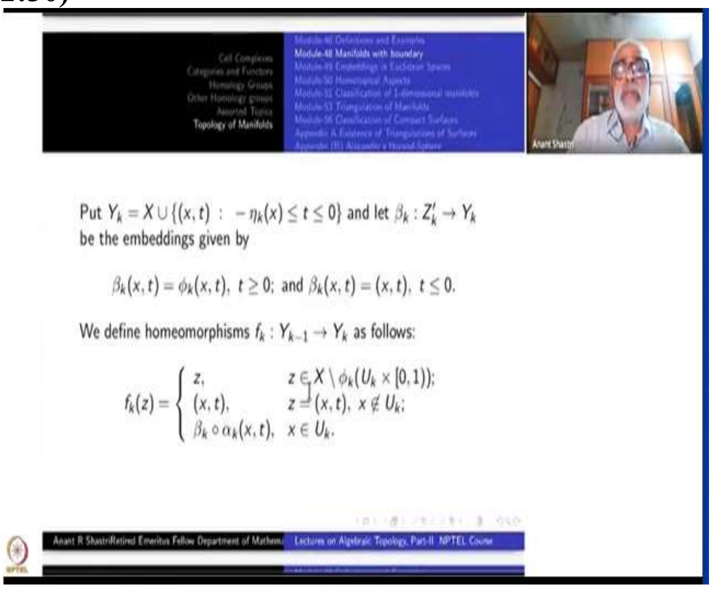

Put $Y_k = X \cup \{(x,t) : x \in \partial X, -\eta_k(x) \leq t \leq 0\}$, okay? With this k-th stage construction is over. Let β_k from Z'_k to Y_k be the embedding given by $\beta_k(x,t) = \phi_k(x,t)$ for $t \geq 0$ and $\beta_k(x,t) = (x,t)$ for $t \leq 0$, okay? So you have to see that these two parts of the definition agree at $t = 0$, namely, with $(x, 0)$ okay, for $x \in \partial X$.

Now define homeomorphisms f_k from Y_{k-1} to Y_k, inductively as follows: f is identity outside $\phi_k(U_k \times [0,1])$, to be (x,t) if $z = (x,t)$ and x is not in U_k,(here also it is identity) and finally, if x is in U_k and $z = (x,t)$, let $f(z) = f(x,t) = \beta_k \circ \alpha_k(x,t)$. Note that Y is the quotient of $\cup Y_k$= the quotient of $X \cup [-1, 0]$.

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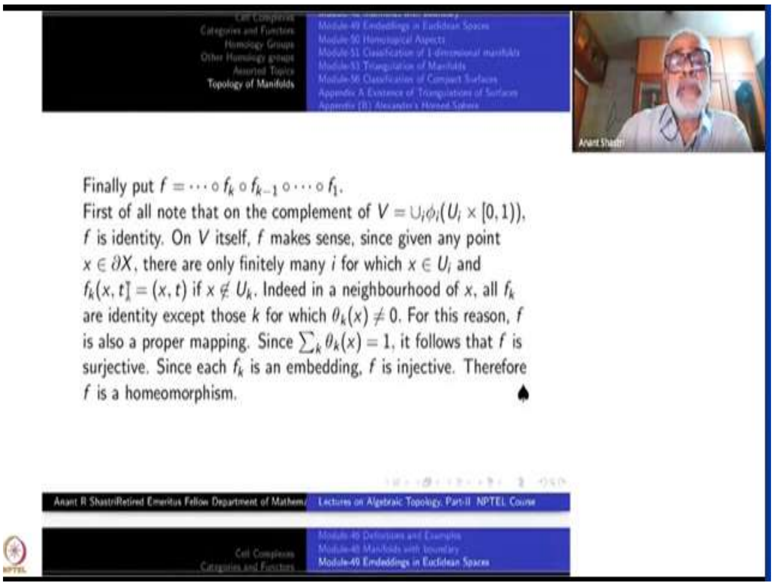

Put $f = \cdots \circ f_k \circ f_{k-1} \circ \cdots \circ f_1$, i.e., f_1 followed by f_2 etc. Though this looks like an infinite composition, for each x, there is some k such that $\eta_\ell(x) = 1$ for all $\ell > k$. It then follows that f_l will become identity and hence this makes sense and is continuous.

First of all note that on the compliment of $V = \cup \phi_i(U_i \times [0,1])$, f is identity. On V itself, given $x \in \partial X$, there are finitely many U_i for which x_i is inside U_i, and if x is not in U_i, then f_i is identity. So all modification is happening inside some finitely many neighbourhoods. Indeed, by local finiteness of the cover $\{U_i\}$, inside a small neighbourhood of x, all f_k are identity except those k for which $\theta_k(x)$ is not 0. Okay? For this reason, f is also a proper mapping. Since $\sum \theta(k) = 1$, it follows that f is surjective. Since each f_k is an embedding f is injective. Therefore f is homeomorphism.

So the partition of unity plays an important role in extending a local topological property into a global one. Today, we will stop here. Next time we consider the problem of embedding topological manifolds inside Euclidean spaces. Thank you.