Measure & Integration
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Lecture – 01 A
Introduction, Extended Real Numbers

My name is Inder Kumar Rana. I am professor in the department of mathematics IIT Bombay. I will be taking you through this course on measure and integration. This is a course which is normally taught at master's level, MSc in mathematics and sometimes in departments other like physics electrical engineering also. So, let us go through the basic objectives of this course. This course as I said is called measure and integration. This also goes by various names such as real analysis advance real analysis and so on the aim of this course the objectives that will be covering in this course are as follows.

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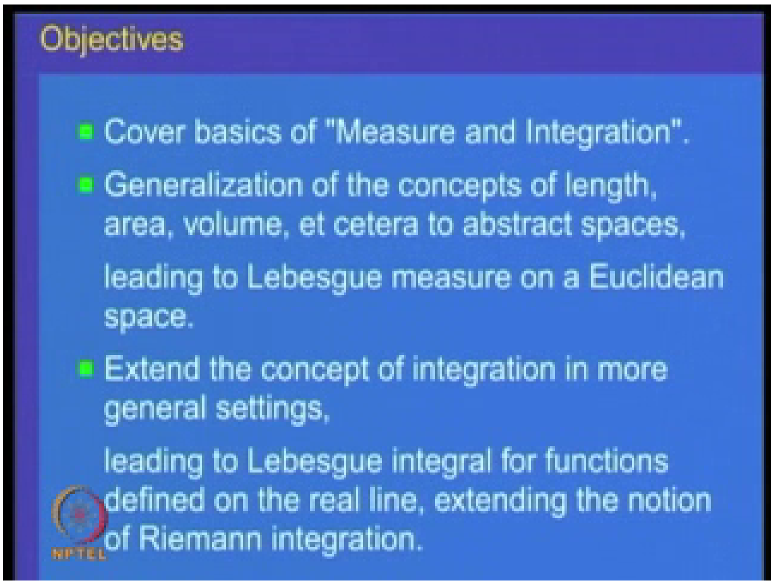

So, the aim is to generalize the concept of length area volume etcetera 2 abstract spaces. And that leads to the notion of Lebasque measure on Euclidean spaces, and general concept of measures on general spaces. And then also we will extend the notion of integration which is normally done in UG levels are called Riemann integration to more general settings and that leads to the notion of Lebasque integral and other notions of abstract integration. So, these are the basic sort of outline of the course that we are going to follow.

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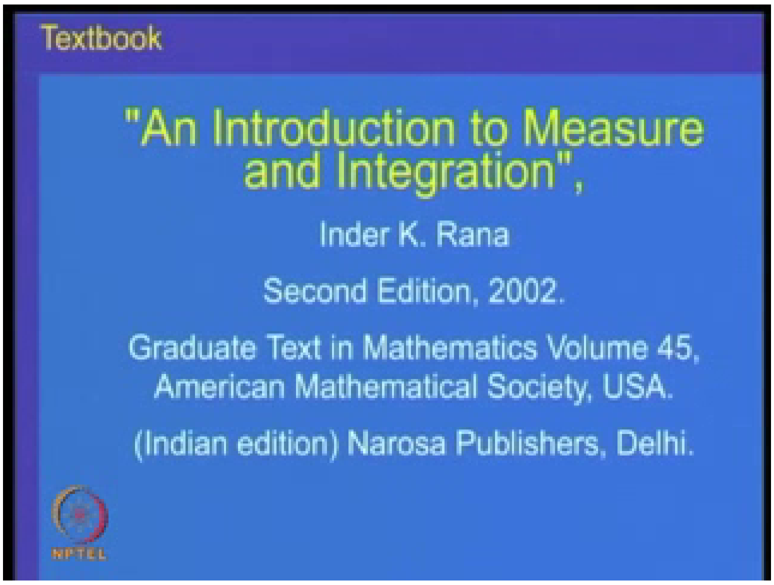

We will be following the text book an introduction to measure and integration written by me. And this is published jointly by graduate text in mathematics know by American mathematical society and a Indian addition of this is available through narosa publishers in new Delhi.

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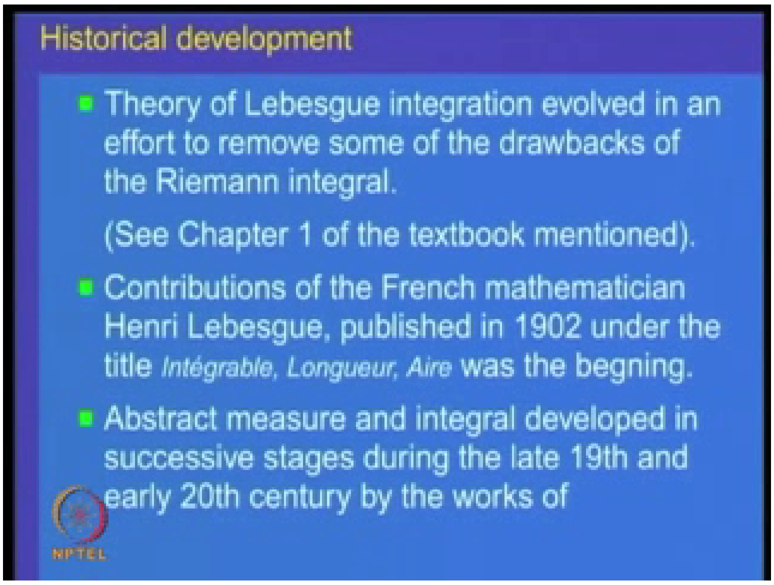

So, why Lebasque integration is needed and why what is the need for extending the notion of Riemann integration. There are some problems drawbacks of Riemann integration and to study about them you should look at chapter one of the text book that I have mentioned. We will not have time to go through this drawbacks of Riemann integration. And now efforts which are made to remove this drawbacks led to the development of labegue measure lebegue integration and so on. So, of were this. So, we refer chapter one of the book. Historically this was developed by the French mathematician Henry Lebasque, who published as a part of his PhD thesis in 1902 integral longuer aire and then this was developed further into the abstract spaces by various mathematicians in nineteenth and twentieth century, some of them being emile borel caratheodory radon and d among others.

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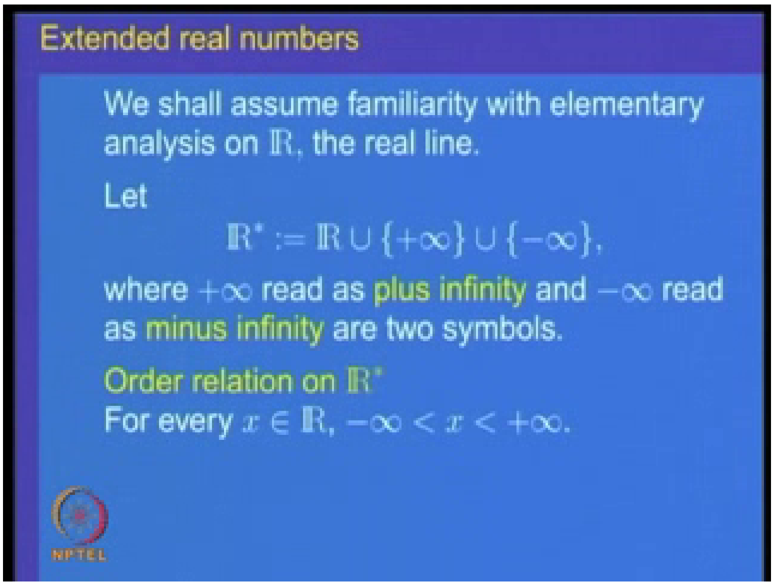

So, we will assume to start with there are some prerequisites for this course, which we shall assume and we will hope that you have gone through in elementary coursing first course in real analysis and you are familiar with the properties of the real line (Refer Time: 03:48) what is real line what are called open intervals closed intervals what is the topology on the real line what are called the compact subsets of the real line.

So, basic course on real analysis is going to be assumed throughout this course. So, if you have difficulty in this I would says look up some elementary book on first course on real analysis and go through these topics. So, that you are you are not left behind and you are able to understand whatever things we are going to discuss concepts. So, we are doing in to discuss there is one the basic space of course, is a real line, but there is a notion of what is called extended real numbers which we are going to use in our course.

And since this is not normally discussed in most of the text books or in courses in real analysis, we will go through some of this ah concept on extended real numbers. So, first of all what is the set of extended real numbers the set of extended real numbers denoted by r star is the set of real numbers to which we had join to new symbols one is called plus infinity and the other is called minus infinity. And now once we adjoin these 2 new symbols to the set r we get the extended set r denoted by r star.

Now, as you all know the set of real numbers has got algebraic operations of addition multiplication there is an order on it. So, when we add this new symbols to them would like to define how does this 2 new symbols this 2 new objects behave with respect to the original order structure the original operation of addition multiplication and so on. So, we are going to define what are called operations of addition multiplication and order on the set of extended real numbers. The first one is the order relation.

So, we are going to assume or we are going to say that for every real number x in r lies between the 2 new symbols minus infinity less than x strictly less than plus infinity. So, this is how the new symbols plus infinity and minus infinity behave with respect to the order section. So, minus infinity in r star is the smallest element and plus infinity is the largest element in r stars as far as the order is concerned for real numbers the same original order stays.

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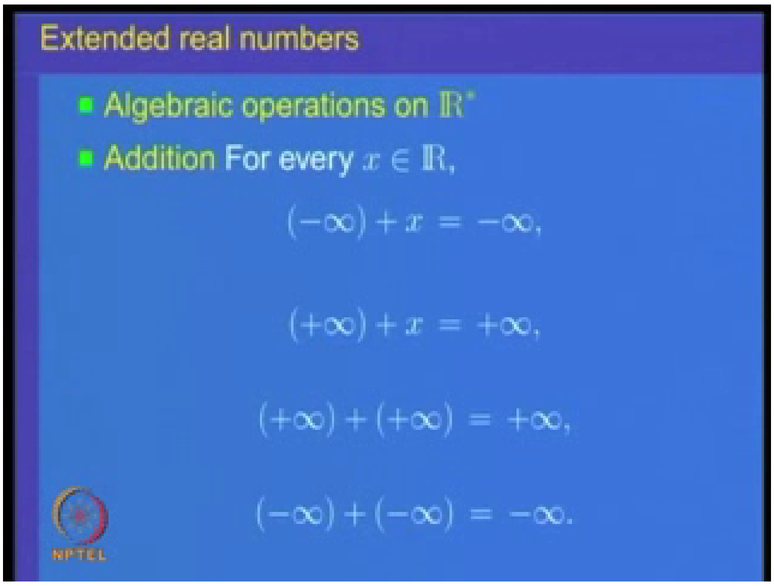

Next let us look at the algebraic operations on r star. So, for real numbers x and y we already know what is x plus y, but for infinity and minus infinity the 2 new symbol, how are these operations defined here are the. So, for every x belong into r if you add minus infinity to x you should get minus infinity. So, that is the rule we are specifying how does minus infinity behave with respect to addition of real numbers and similarly plus infinity plus x is equal to plus infinity.

Whatever be x positive or negative, when added to minus infinity you get minus infinity and when added to plus infinity you will get plus infinity and now how does infinity plus infinity added to itself what is outcome. So, it says plus infinity plus infinity is plus infinity, and minus infinity plus minus infinity is minus infinity. And let us specify that plus infinity plus minus infinity is not defined. So, these are the only 4 relations among addition of plus infinity minus infinity with respect to x plus infinity with itself and minus infinity with itself.

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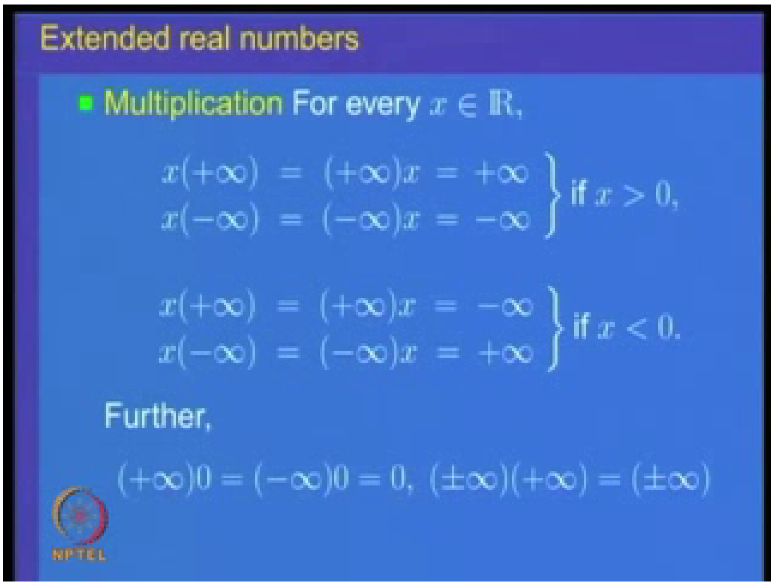

Next comes the rules for multiplication. For every real number x, x into plus infinity is equal to plus infinity into x is plus infinity, if x is non negative and similarly x multiplied by minus infinity is same as minus infinity multiplied by x is equal to minus infinity. Again under the condition that x is bigger than 0. Now more or less we are following the rules of multiplication for real numbers, similarly if x is negative. So, we have x multiplied by plus infinity or plus infinity multiplied by x is equal to minus infinity the sign changes of infinity. And similarly x multiplied by minus infinity is equal to minus infinity multiplied by x is equal to plus infinity if x is less than 0. So, depending upon whether x is bigger than 0 or x less than 0 the rules for multiplication are as specified. Afterwards if x and y are real numbers the multiplication between x and y is same as that of real numbers.

So, this is a rules for multiplication and of course, there is a specific element there is a particular element called 0 in the real numbers. How does that behave with respect to plus infinity and minus infinity here are the rules for plus infinity into 0, is same as minus infinity into 0 is equal to 0 that is the same as for real numbers also x multiplied by 0 whether positive or negative is always equal to 0.

And of course, if I multiplying plus infinity with itself the answer is plus infinity and if minus infinity is multiplied with plus infinity the answer is minus infinity. So, plus minus infinity multiplied by plus infinity is plus minus infinity, and similarly plus minus infinity multiplied by minus infinity is minus plus infinity the sign changes of the outcome. So, these are the rules for addition multiplication and order structure on the set r star which is nothing, but the real numbers along with 2 new symbols plus infinity and minus infinity.

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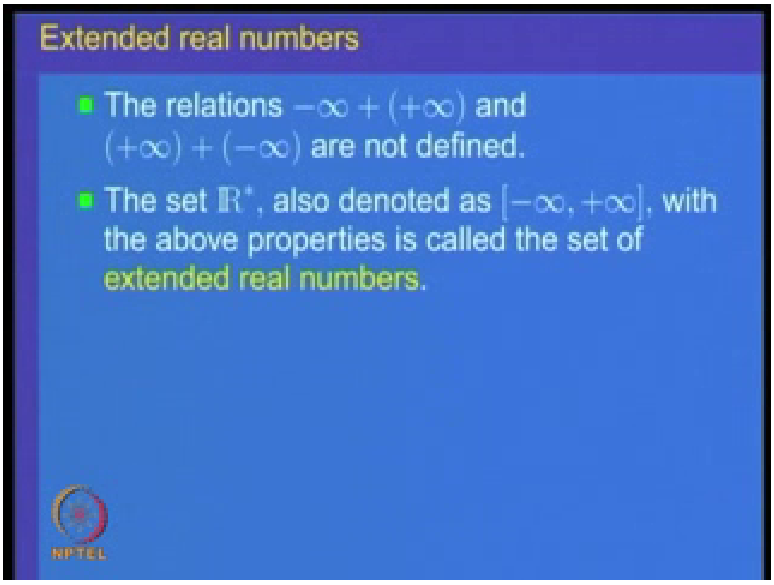

So, once again let us specify that the relations minus infinity plus infinity and plus infinity plus minus infinity are not defined. So, with these rules we get the set r star of extended their numbers which is also denoted by this square bracket minus infinity comma square bracket plus infinity. So, that is essentially something like saying like the real numbers are denoted by the open kind of interval minus infinity to plus infinity if you close it on both sides that is a notation used for extended real numbers.

So, once you are familiar with the order familiar with the addition and multiplication on the extended real numbers we can look at the notion of sequences in real numbers.

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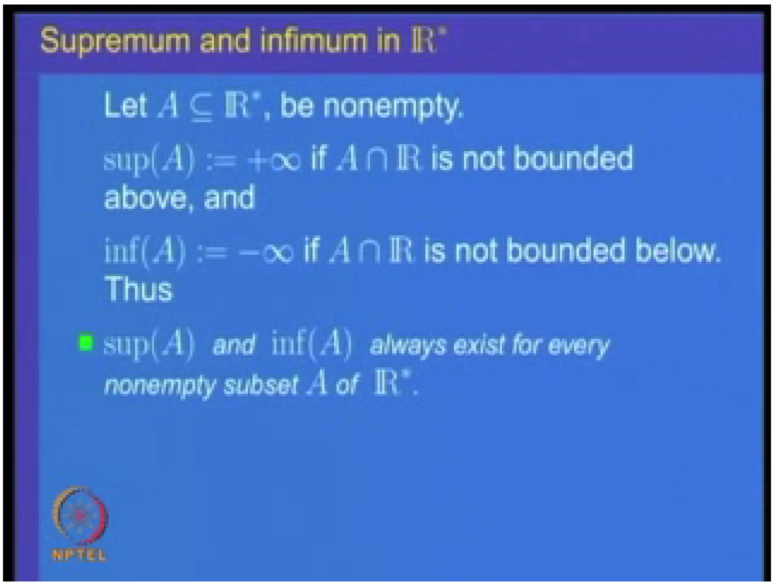

Or and also the notion of supremum and infimum on subsets of a extended real numbers. So, let us first look at a is a subset of a extended real numbers. And let us assume a is a non empty set. Now there is a possibility that a is a subset of real numbers only, then we know that the completeness property of real numbers is if the set a is bounded above it must have least upper bound or namely the supremum.

Now in the case a is a subset of r star is a subset of extended real numbers; that means, there is a possibility of minus infinity or plus infinity being a part of it. And suppose if it is bounded above then it has to be a subset of real numbers and supremum will exist if it is not bounded above; that means, plus infinity is going to be a part of it. So, we will define the supremum of a to be equal to plus infinity if a as a part of r is not bounded above. And similarly we will define the infimum of the set a to be equal to minus infinity if a intersection r is not bounded below.

So, what we are saying is in the subset of extended real numbers a set which is bounded above or bounded below does not we do not have to say that. So, every subset non empty subset of extended real numbers will always have supremum and will always have infimum. Of course, the supremum will be equal to plus infinity if the set a is not bounded above and it will be equal to minus and infimum will be equal to minus infinity if it is not bounded above.

So, it is a very nice situation every subset has supremum as well as infimum.

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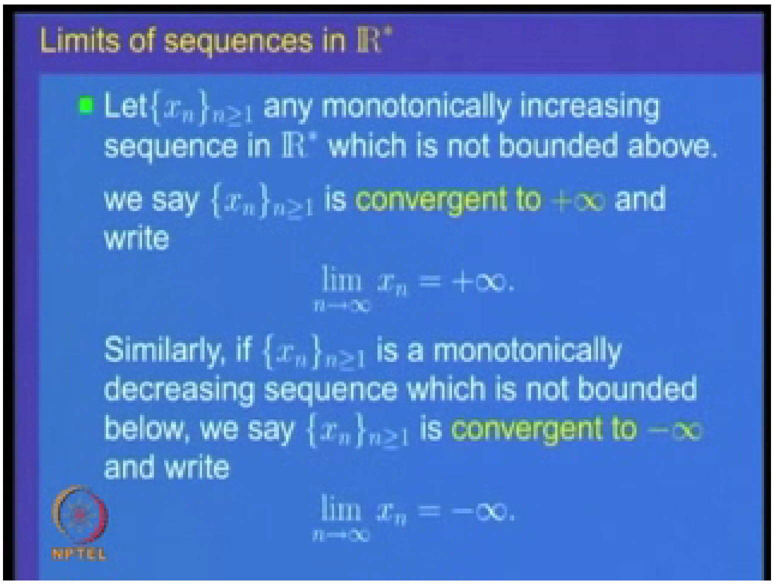

Now a similar condition will hold or similar results will hold for limits of sequences in r star. So, let us look at a sequence x n which is monotonically increasing and which is not bounded above. If you recall a has a subset has a sequence in real numbers if a sequence is monotonically increasing and is bounded above, then it must be convergent. And now if a sequence is monotonically increasing, and it is a it is in r star it is a sequence in of extended real numbers and not bounded above; that means, plus infinity this going to be an element of it.

So, we will say if it is not bounded above we will say a sequence is convergent to plus infinity and write this as equal to plus infinity. This is this is essentially also we say when x n is a sequence of real number which is monotonically increasing and mot bounded above in that case also we write the limit to be equal to plus infinity in so, far a sequence of real numbers it is only a symbolic way of saying that a monotonically increasing sequence not bounded above converges to plus infinity, but as a sequence in r star it convergence to an element of r star namely plus infinity.

Similarly, a sequence x n in r star which is monotonically decreasing and if it is not bounded below we will say it converges to minus infinity, and write this as limit n going to infinity x n is equal to minus infinity. So, this is how we will analyze sequences in r star which are monotonically increasing or monotonically decreasing.

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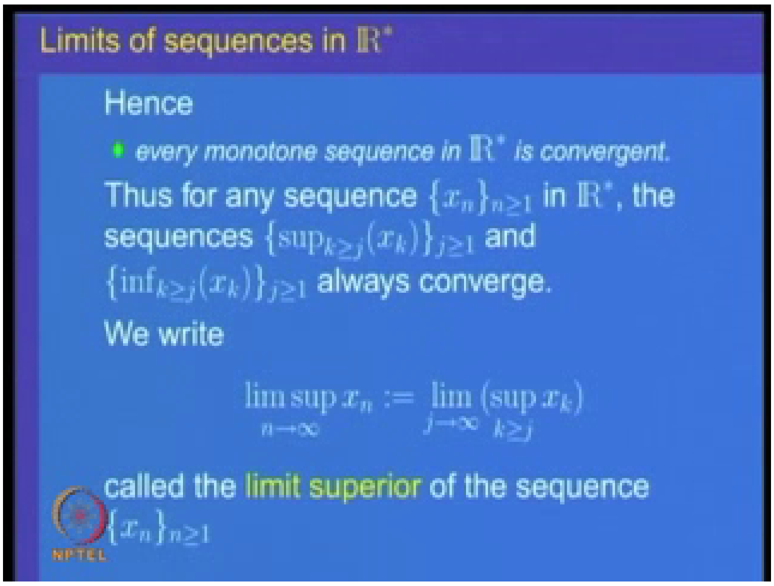

Similar concepts can be developed for series in r. So, let us just say few things about sequences because every monotonic sequence is convergent. So, if I look at the sequence given any sequence look at the supremum's of that sequence from the stage j onwards. So, supremum k bigger than or equal to j of x k. Then that gives me a new sequence and that sequence will always converge and similarly the inflimum from k bigger than or equal to j x k will also converge because these are monotone sequences, and monotone sequences in r star always converge. So, limit of the supremum k bigger than or equal to j for that is denoted by limit superior of x n. And similarly for the inflimum k bigger than or equal to j x k is called the limit inferior of the sequence.

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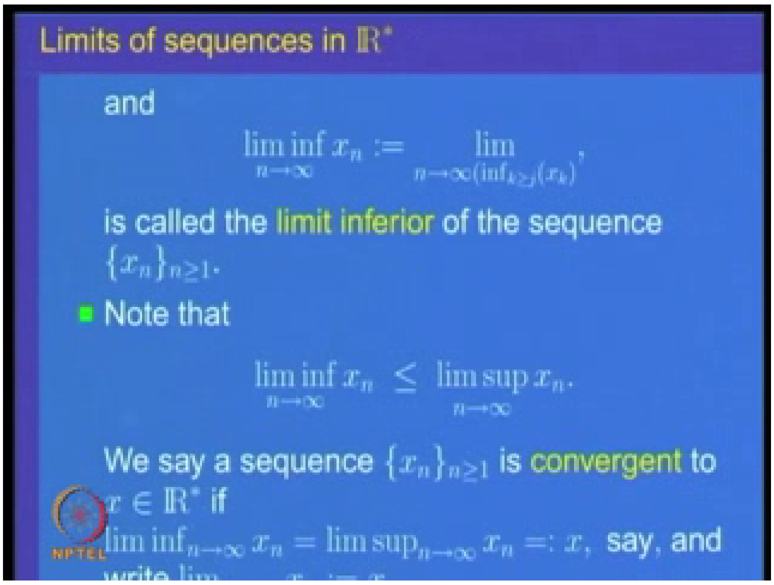

So, in general we already know that limit inferior is always less than or equal to say limit superior and the sequence will converge and the limit inferior is equal to the limit superior even in the case of sequences in r star. So, this is how all sequences behave in.

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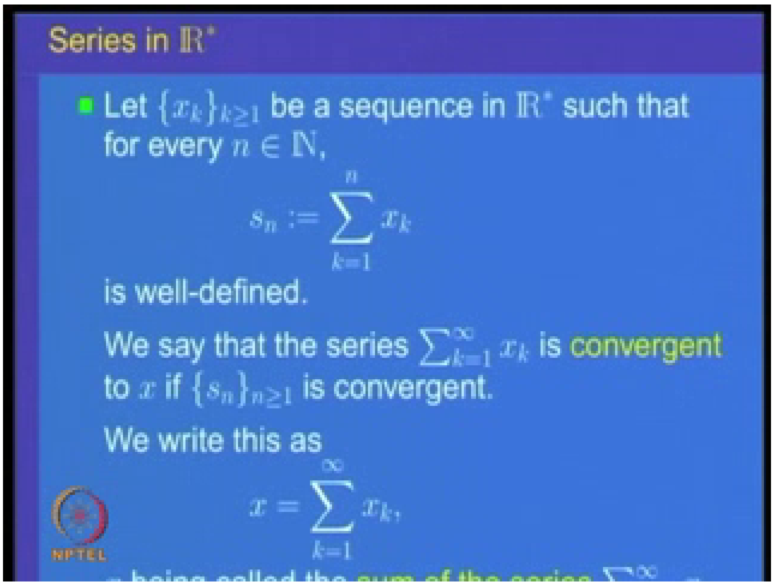

Now, let us look at sequence from sequences let us go to the concept of series suppose x k, k bigger than or equal to 1 is a sequence in R star. Then let us look at the partial sums of the sequence sn that is the sum of the first n terms of the sequence. So, that is denoted by sn which is summation k equal to 1 to n x k.

So, for every n this is well defined and one can ask whether this sequence is convergent or not in r star. So, if the series is convergent in r star then we if the sequence is convergent in r star the sum of partial sums and the sequence of partial sums if it is convergent in r star we say that the series is convergent and the limit is called the sum of the series. So, this is basically same as that of real line only keep in mind how sequence is behave in the real line.

So, with this basic discussion about what is the basic space of extended real numbers we are going to deal with, we will start with a proper concepts in our subject measure and integration. And the first concept are going to be our first few concepts are going to be discussions about class of subsets of a non empty set. So, we are going to look at some collection of subsets of a given nonempty set x with certain properties this collection of a subsets which we are going to call them as semi algebras sigma algebras and monotone classes there are various classes which play an important role later on in our subject. So, let us start with looking at what is called a semi algebra of subsets of a set x. So, let x be a non empty.

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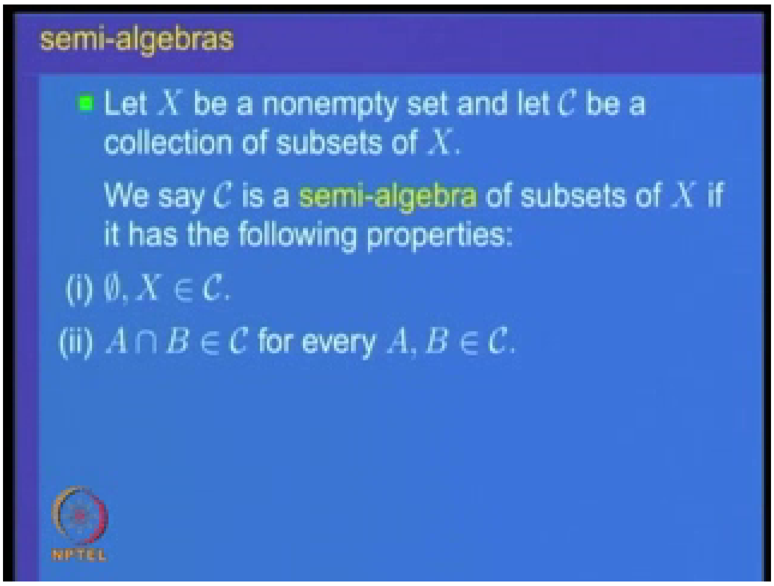

And let c be a collection of subsets of that set x. We say that the class c is a semi algebra of subsets of x, if it has the following properties this collection c has the following properties one the empty set and the whole space are members of this class x.

So, the first property desired of c is that the empty set and the whole space x are members of this class. And the second one is that this class is closed under intersections; that means, if a and b are 2 elements of this collection then the intersection of this sets a and b should also be a member of the class c. So, this class c is closed under intersections that is a second property. There is a third property which will describe soon that is saying that this class need not be closed under compliments, but will require a additional property. So, let me write that property and explain because it is best understood when it is written.

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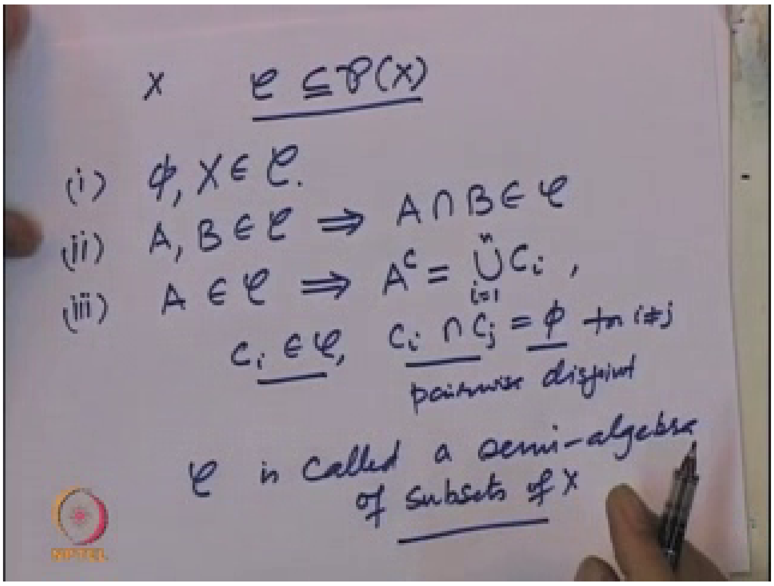

So, x is a non empty set c is a collection of subsets of the set x. So, the first property we said empty set and the whole space belong to c. And second property was if a and b belong to c then that implies a intersection b belong to c. And a third property which is very crucial is that if a belongs to c, then that implies the set look at the set a compliment that need not be in c, but we want to say you can write c as union of element ci finite number of them I equal to someone to n such that, cis are elements of c and they are pair wise disjoint ci intersection c j is empty. So, let us just go through these concepts again a collection c in px having the following properties one, the empty set the whole space are elements of it if a and b belong to it then the intersection of these 2 sets namely a intersection b is also an element of this collection c and the third property is if a is a subset of c then a compliment the complement of the set in x. Of course, should be representable as union of cis I equal to 1 to n where the ci is our elements of c and they are pair wise disjoint.

So, this property that ci intersection c j is non empty for I not equal to j, we just say they are pair wise disjoint. So, such a collection c is called a semi algebra semi algebra of subsets of x. So, let us look at some examples to get familiarized with this notion of semi algebras.

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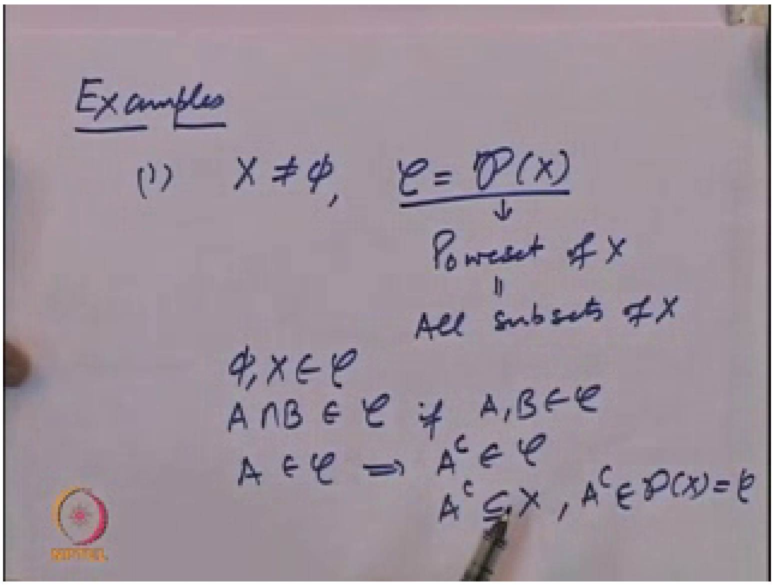

So, let us take a look at some examples. So, let us take x any non empty, set let us take the collection c to be equal to all subsets of x. So, p of x what is p of that is a power set of x. So, normally this is called the power set of x which is same as all subsets of x. So, c is the collection of all subsets of x, they think it is closed under should I think phi and x belong to c; obviously, it is a collection of all subsets. So, phi and x belong; obviously, a intersection b also belongs to it if a and b belong to c. Because if a and b are subsets of it then naturally a intersection b also is a subset and. In fact, if a belongs to c then that implies a compliment also belongs to c because a compliment itself is a subset of x. So, a compliment belongs to px it is sub set of x belongs to p x which is c. So, the collection of all subsets of the set x is a example, which is obvious example of a semi algebra of subsets of a set x. Let us look at some more examples this is an obvious example. So, let us look at the second example.

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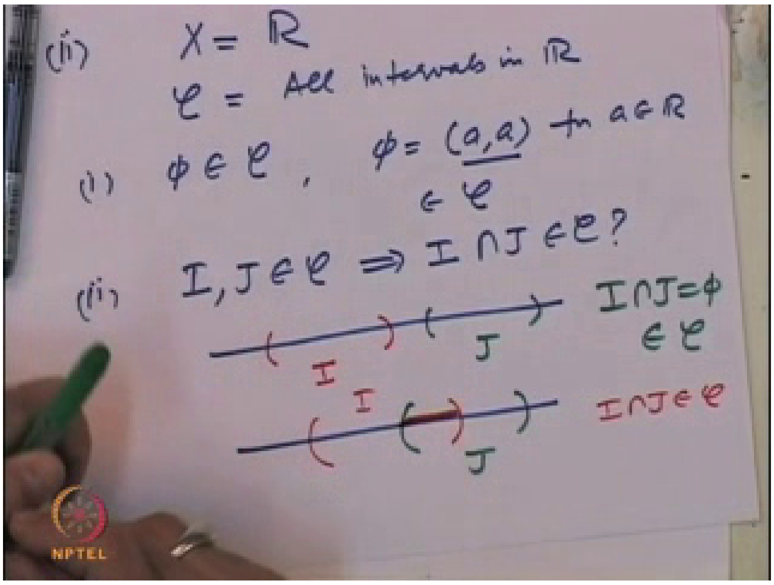

namely let us look at x equal to real line. And let us take the collection c all intervals in r. So, we are looking at the collection of all intervals in r that is a collection c.

So, first property is empty set a member of c. Well here you will to understand of course, yes one way of looking at it is empty set can be written as the open interval a comma a for any point a belonging to r, and that is a interval. So, that belongs to the collection c second property let us take 2 intervals I and j belong to c. So, does this imply the intersection i intersection j belong to c.

So, that is a question means if, i and j are 2 intervals can I say I intersection j is also an interval let us check in the picture. Let us take 2 intervals I and j. So, let us say here is the interval i and the here is the interval j. So, one possibility is that they are disjoint from each other. So, here i intersection j is empty. So, hence belongs to c. What is other possibility? Other possibility is let us take they intersect. So, here is my interval j and here is my interval i then what is the intersection /of these 2. So, this is I and that is j. So, that is my intersection of I and j and clearly that is also. So, I intersection j is also an interval. So, it belongs. So, 2 cases case one when they are disjoint empty set intersection is empty set and belongs to c and if they overlap then the overlap itself is an again an interval and that belongs to c. So, it is quite clear that this collection of all intervals is closed under intersection also.

Let us look at the third property which is crucial and that says if I take an interval i.

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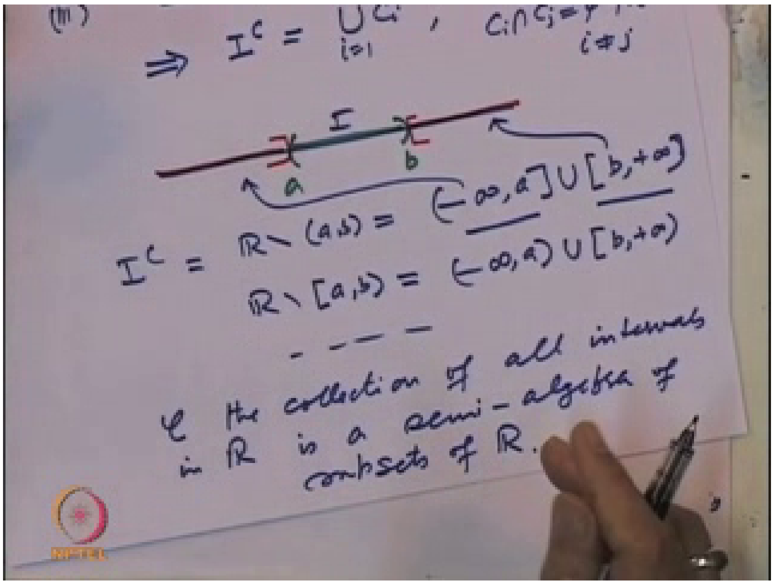

So, the third property that we want to verify is if i is an interval, i should look at the complement of that interval and should be able to write it as union of ci, i equal to 1 to n where ci is belong to c and ci intersection cj is empty for i not equal to j. So, once again let us look at a interval. So, let us look at a interval say open interval a and b. So, that is my interval. So, what is going to be the compliment of this. In fact, the complement of this looks like 2 spaces one is this side other is this side. So, this space and this space.

So, for this I can write that the complements of r minus a b is equal to minus infinity to a in close union b to plus infinity right. So, this is this part and this is this part. So, if I take a interval i. So, that is I complement where i is the interval a to b then it is complement is a disjoint union of 2 elements. Similar case is we will follow if for example, a is left open or right close. So, let us write r if it is of the type this then, I can write this as this point a is in close. So, minus infinity to a the complement will be this open here and union b to plus infinity and similarly the other cases. So, we will I will say yes that you write down yourself. So, we have verified c the collection of all intervals in r is a semi algebra of subsets of real line.