

Mathematics in India: From Vedic Period to Modern Times
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Lecture - 36
Proofs in Indian Mathematics I

(Refer Slide Time: 00:12)

Outline

- ▶ *Upapattis* or proofs in Indian mathematical tradition
- ▶ Early European scholars of Indian Mathematics were aware of *upapattis*
- ▶ Some important commentaries which present *upapattis*
- ▶ Bhāskarācārya II on the nature and purpose of *upapatti*
- ▶ *Upapatti* of *bhujā-koṭi-karṇa-nyāya* (Pythagoras theorem)
- ▶ *Upapatti* of *kuṭṭaka* process
- ▶ Restricted use of *tarka* (proof by contradiction) in Indian Mathematics
- ▶ The Contents of *Yuktibhāṣā*
- ▶ *Yuktibhāṣā* demonstration of *bhujā-koṭi-karṇa-nyāya*
- ▶ Estimating the circumference by successive doubling of circumscribing polygon

So in this talk and the following 2 talks, we will be discussing the proofs in Indian mathematical tradition. So in this talk we will be talking about the history of proofs and the idea of Upapatti as they are called in Indian mathematical works, where are they present? And something on what is the understanding of Indian mathematicians as to what an Upapatti what is the purpose? What it ought to do? What it purpose to do?

Then we will study few examples of proofs in the earlier Indian text, and come to of course the text *Yuktibhasa* which is indeed a collection of proofs, many of these proofs of *Yuktibhasa* will be discussed in the second and third lecture, and at the end of which we will try to analyze what is the notion of proof in Indian mathematics once again towards the end.

(Refer Slide Time: 01:23)

Upapattis in Indian Mathematics

While there have been several extensive investigations on the history and achievements of Indian mathematics, there has not been much discussion on its methodology, the Indian mathematicians' and philosophers' understanding of the nature and validation of mathematical results and procedures, their views on the nature of mathematical objects, and so on.

Traditionally, such issues have been dealt with in the detailed *bhāṣyas* or commentaries, which continued to be written till recent times, and played a vital role in the traditional scheme of learning.

It is in such commentaries that we find detailed *upapattis* or 'proofs' of the results and procedures, apart from a discussion of methodological and philosophical issues.

Now while considerable amount of information is known about Indian mathematics, and it has been known for a very long time to modern scholars, there has not been really that much discussion on what is the understanding of Indian mathematicians about the nature of mathematics, what does mathematics deal with? What it is supposed to do? Or having an Indian philosophers discuss these questions.

So these kind of issues have not been studied or analyzed in any detail. The reason is because we have been so sort of impressed by the achievements of the Indian mathematician that they did this very nice results, Aryabhata at this beautiful approximation for square for pi, or he had this very nice method for square root, cube root. Or Brahmagupta came up with his wonderful formulae for diagonals of a cyclic quadrilateral or we have this Sankara Variyar.

So these results and achievement of Indian mathematician occupied considerable amount of our time and study, and the original works was scanned for the kind of results that they had, and we were not discussing how they were arriving at these results or in what context they were looking at these results, this did not become the subject of serious investigation. One is because the traditionally this issues have been dealt with in the commentaries.

So any text in India has to be understood along with its commentaries and along with the successive oral tradition of teaching of that text, and that is what gives you an orientation

towards that text and its position in the overall discipline that is being considered and the way this discipline has evolved. So till very recent times writing of these commentaries was a very important part of Indian Textual Indian scientific tradition, or in all Shastra this was a very important thing.

And in the traditional scheme of learning these commentaries played very, very major role, but since we have been disconnected from that way of studying Indian knowledge systems, we sort of most of the time just look at their results, and then start wondering how did they arrive at it, or did they have any method at all, or did they not have any method at all. And we also start discussing things that maybe Patanjali did not understand Panini correctly, maybe (FL) did not understand Patanjali correctly like that.

I mean obviously nobody would have understood this predecessor so thoroughly that indeed true, but he did not understand there is no reason that we will be able to understand the predecessor much better. So these are the kind of issues in which we are got most of the time. So if we are interested in understanding methodology of Indian mathematics we have to look at the commentaries much more seriously, and this will apply to most of other disciplines most of other traditional Indian disciplines also.

(Refer Slide Time: 04:29)

Early European Scholars Were Aware of *Upapattis*

In the early stages of modern scholarship on Indian mathematics, we find references to the methods of demonstration found in texts of Indian mathematics.

In 1817, H. T. Colebrooke referred to them in his pioneering and widely circulated translation of *Līlāvātī* and *Bījagaṇita* and the two mathematics chapters of *Brāhmasphuṭa-siddhānta*:

“On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by *Bhāskara* himself, towards the close of his algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.”

Now when modern scholars started studying Indian mathematics the fact that these books did have various kinds of proofs, the fact that they were commentaries and they contain various proofs was taken out of. Colebrook's work is the pioneering work of in the translation of Indian mathematical text, he translated Lilavati, he translated Bijaganita, he translated the 2 mathematics chapters of Brahmasphuta-siddhanta.

And Colebrook's book is filled with footnotes from various commentaries, and occasionally he refers to the kind of demonstration of Upapattis' which are contained in the commentaries. Colebrook says on the subject of demonstrations it is to be remarked that the Hindu mathematician proved propositions both algebraically and geometrically, as is particularly noticed by Bhaskara himself towards the close of his algebra where he gives both modes of proof of a remarkable method for the solution of the bhavita problem he is referring.

(Refer Slide Time: 05:27)

Early European Scholars Were Aware of *Upapattis*

Similarly, Charles Whish, in his seminal article on Kerala School of Mathematics of 1835, referred to the demonstrations in *Yuktibhāṣā*:

"A further account of the *Yuktibhāṣā*, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper: I shall therefore conclude this, by submitting a simple and curious proof of the 47th proposition of Euclid [the so called Pythagoras theorem], extracted from the *Yuktibhāṣā*."

Whish does not seem to have written any further papers on the demonstrations of the infinite series as given in *Yuktibhāṣā*.

Whish's paper was widely noticed in the scholarly circles of Europe in the second quarter of nineteenth century.

But it was soon forgotten and there was no study of *Yuktibhāṣā* till 1940s, when C. T. Rajagopal and his collaborators wrote pioneering articles on the proofs outlined in that seminal text.

Similarly, Charles Whish, who's article generated interest about the Kerala school of mathematics in recent times, this article was written in 1835, in fact it was submitted as a first paper to the Madras literary society, and later on it was read as a paper in the it was published in the transaction of Royal Asiatic Society. So Whish says in his article, he talks about 4 texts Tantrasangraha, Yuktibhasa, (FL) except (FL) now all these other text have been fairly good editions and the translations and explanations of these texts are currently available.

So a further account of the Yuktibhasa, the demonstration of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations will be given in a separate paper. I shall therefore conclude this paper that is the paper that is presenting by submitting a simple and curious proof of the 47th proposition of Euclid, what is generally called the Pythagoras theorem extracted from the Yuktibhasa.

But Whish did not write any further paper, in fact there is this detailed manuscript of this Tantrasangraha with Sankara Variyar's Malayalam commentaries is available with his detailed handwritten notes on one side of the book which a professor Ram Subramanian had an occasion to copy and they published some extracts from it on of Sankara Variyar's Malayalam commentary couple of years ago.

Now Whish paper was widely noted at the time when it was written, it was written in 1835, so this was considered an important discovery at that time. However, all this was soon forgotten and there was really no study of Yuktibhasa for almost 120 years, when C. T. Rajagopal and his collaborators started pioneering articles on proofs in Indian mathematics.

(Refer Slide Time: 07:31)

The Alleged Absence of Proofs in Indian Mathematics

It has been the scant attention paid, by the modern scholarship of the last two centuries, to this extensive tradition of commentaries which has led to a lack of comprehension of the methodology of Indian mathematics. This is reflected in the often repeated statements on the absence of logical rigour in Indian mathematics in works on history of mathematics such as the following:

"As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their name for mathematics was *ganita*, which means 'the science of calculation'. There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics."¹

¹ Morris Kline: *Mathematical Thought from Ancient to Modern Times*, Oxford 1972, p.190.

And therefore it is this scant attention paid by modern scholarship of the last 2 centuries, to this extensive tradition of commentaries, it is this which has led to a lack of comprehension of the methodology of Indian mathematics. And this is often reflected in the sort of blank statements

that you will find in many of the general history books of mathematics, so if you open any general history of mathematics you are likely to come across a statement like this.

This I have taken from Morris Kline which a book of about 1000 pages Mathematical Thought from Ancient to Modern Times, as our survey indicates the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the detective patterns. Their name for mathematics was ganita which means the science of calculation which is of course correct.

There is much good procedure and technical facility but no evidence that they considered proof at all, they had rules but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics, so this kind of a broad speak generalization view of Indian mathematics you will find this in several books.

(Refer Slide Time: 08:40)

Some Important Commentaries Which Discuss *Upapattis*

1. *Bhāṣya* of Bhāskara I (c.629) on *Āryabhaṭīya* of Āryabhaṭa (c.499)
2. *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629)
3. *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c.860) on *Brāhmasphuṭasiddhānta* of Brahmagupta (c.628)
4. *Vivaraṇa* of Bhāskarācārya II (c.1150) on *Śiṣyadhīvrddhidānta* of Lalla (c.748)
5. *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Bījagaṇita*
6. *Mitākṣarā* or *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Siddhāntaśiromaṇi*

Now the current books will write it with some more sort of caution and some more subordinate clauses added, but the overall thing will be that to a large extent the way these results were obtained by Indian mathematician still remained obscure or something like that will be the general canting. So amongst the published works, there are indeed several books which do contain this Upapattis' so the first is the Bhasya Bhaskara of Aryabhattiya, this contains a few discussions of proofs.

Bhasya Govindasvamin of Mahabhaskariya. The Vasana of Prthudakasvamin on Brahmasphuta-siddhanta this commentary is still not published in entirety, only some sections are available in print. So the first really book that is serious discussions of Upapattis' is in Vivarana of Bhaskaracharya on Sisyaadivarddhidatantra of Lalla. Vasana of Bhaskaracharya on his own Bijaganita. The Vasana of Bhaskaracharya on his own Siddhantasiromani these are indeed commentaries which contain extensive discussion of proofs.

(Refer Slide Time: 09:54)

Some Important Commentaries Which Discuss *Upapattis*

7. *Siddhāntadīpikā* of Parameśvara (c.1431) on the *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629)
8. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somaśutvan (c.1501) on *Āryabhaṭīya* of Āryabhaṭa
9. *Yuktibhāṣā* (in Malayalam) of Jyēṣṭhadeva (c.1530)
10. *Yuktidīpikā* of Śaṅkara Vāriyar (c.1530) on *Tantrasaṅgraha* (c.1500) of Nīlakaṇṭha Somaśutvan
11. *Kriyākramakārī* of Śaṅkara Vāriyar (c.1535) on *Līlāvati* of Bhāskarācārya II (c.1150)

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Parameswara's commentary on Govindasvamin's Bhasya. Nilakantha (FL), then of course Yuktibhasa of Jyesthadeva. Yuktidipika and Kriyakramakari of Sankara variyar which you referred 2.

(Refer Slide Time: 10:09)

Some Important Commentaries Which Discuss *Upapattis*

12. *Garīṭayuktayaḥ*, Tracts on Rationale in Mathematical Astronomy by various Kerala Astronomers (c.16th-19th century)
13. *Sūryaparakāśa* of Śūryadāsa (c.1538) on Bhāskarācārya's *Bījagaṇita* (c.1150)
14. *Buddhivilāsini* of Gaṇeśa Daivajña (c.1545) on *Līlāvatī* of Bhāskarācārya II (c.1150)
15. *Bījanavāṅkura* or *Bījapallavam* of Kṛṣṇa Daivajña (c.1600) on *Bījagaṇita* of Bhāskarācārya II (c.1150)
16. *Vāsanāvārttika*, commentary of Nṛsiṃha Daivajña (c.1621) on *Vāsanābhāṣya* of Bhāskarācārya II on his own *Siddhāntaśiromaṇi* (c.1150).
17. *Marīci* of Munīśvara (c.1630) on *Siddhāntaśiromaṇi* of Bhāskarācārya II (c.1150).

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Then about 20 years ago professor K. V. Sharma published a large number of tracks which was small monographs which contain the discussions and proofs of small bits of results on various topics, so he compiled them and published about 20 such tracks 20 to 25 years ago. Buddhivilasini of Ganesha Daivajna we have talked about. Krsna Daivajna of Bijanavarikura we have referred to. The Suryadasa commentary on Bijaganita also a Upapattis'.

Vasanavarttika of Nrsimha Daivajna commentry on Bhaskaracharya's Vasanabhasya is available in print. Marici of Munisvara is another discussion of Siddhantasiromani which contains Upapatti. So there are indeed a large number of books which have been published, so if one wants to find out how a particular result was proved, one should first to go up and look the available commentaries, of course if the proof is not there one has to search more or one has to look around more.

(Refer Slide Time: 11:09)

Kṛṣṇa Daivajña on the Importance of *Upapatti*

The following passage from Kṛṣṇa Daivajña's commentary on *Bijaganita* brings out the general understanding of the Indian mathematicians that citing any number of favourable instances (even an infinite number of them) where a result seems to hold, does not amount to establishing it as a valid result in mathematics. Only when the result is supported by an *upapatti* or demonstration can the result be accepted as valid.

ननूपपत्त्या विना वर्गयोगो द्विघातेन युतो हीनो वा युतिवर्गोऽन्तरवर्गो वा भवतीत्येतदेव कथम्? क्वचिद्दर्शनं त्वप्रयोजकम्। अन्यथा चतुर्गुणो राशिघातो युतिवर्गो भवतीत्यपि सूच्यम्। तस्यापि क्वचित्था दर्शनात्। तथाहि राशी २, २ अनयोर्घातः ४ चतुर्गुणः १६ अयं जातो युतिः ४ वर्गः १६ वा राशी ३, ३ अनयोर्घातश्चतुर्गुणः ३६ अयमेव युति ६ वर्गश्च ३६ वा राशी ४, ४ अनयोर्घातः १६ चतुर्गुणः ६४ अयमेव युति ८ वर्गः ६४ इत्यादिषु। तस्मात् क्वचिद्दर्शनम् अप्रयोजकं क्वचिद्वाभिचारस्यापि संभवात्। अतो वर्गयोगो द्विघातयुतोनो युतिवर्गोऽन्तरवर्गश्च भवतीत्यत्र युक्तिर्वक्तव्येति चेत् सत्यम्। इयमुपपत्तिरेकवर्णमध्यमाहरणान्ते।

First because so much has been said about the lack of logical regarding such thing about Indian mathematicians, we will be forced to sort of give quotations where the Indian mathematicians do say that proving some results are indeed important. If you want to take a mathematical results seriously a proof is indeed needed. So Kṛṣṇa Daivajña who is writing at some beginning of his *Bijaganita* he is saying, he is referring to the result (FL).

How can we say $a^2 + b^2 + 2ab$ to which $2ab$ is added or subtracted will lead to $(a+b)^2$ whole square or $(a-b)^2$ whole square is there a way of understanding this. (FL) if we just show some examples of this that is not really good enough, (FL) otherwise 4 times the product of 2 numbers is equal to the square of the sum of these numbers is also sounds like a proper or a valid result. So he gives 3, 4 examples $2*2=4$, $4*4=16$, this is also= the square of $2+2$.

(Refer Slide Time: 12:48)

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$4 \cdot 2 \cdot 3 = (2+3)^2$$

$$4 \cdot 2 \cdot 3 \neq (2+3)^2$$

$$4 \cdot 2 \cdot 2 = (2+2)^2$$

$$4 \cdot 3 \cdot 3 = (3+3)^2$$

Similarly, $3 \cdot 3 = 9$ 4 times $9 = 36$ this is also the square of $3+3$, and again go on $4 \cdot 4 = 16$ 4 times this is 64, so he is saying why do not we say for $x \cdot y$ is $= x+y$ whole square even this appears quite alright, because of these examples $4 \cdot 2 \cdot 2$ is $= 2+2$ whole square $4 \cdot 3 \cdot 3$ is $= 3+3$ whole square, and actually we can give infinite number of examples, and therefore we can conclude that this is correct. (FL) so verifying some result in a certain number of cases is not really of any great use.

Because there might be a contrary since though the moment I go to $2 \cdot 3$ I will find $4 \cdot 2 \cdot 3$ is $\neq 2+3$ whole square, (FL) so this is a counter example or a deviation from what you are, (FL) therefore if one says that one should really give a proof for the fact for the claim that is being made that $x+y$ whole square is $= x$ square $+ 2xy + y$ square it simply true that one needs to prove search result. (FL) he will give this proof in the end of the (FL).

(Refer Slide Time: 14:20)

Kṛṣṇa Daivajña on the Importance of *Upapatti*

"How can we state without proof (*upapatti*) that twice the product of two quantities when added or subtracted from the sum of their squares is equal to the square of the sum or difference of those quantities? That it is seen to be so in a few instances is indeed of no consequence. Otherwise, even the statement that four times the product of two quantities is equal to the square of their sum, would have to be accepted as valid. For, that is also seen to be true in some cases. For instance take the numbers 2, 2. Their product is 4, four times which will be 16, which is also the square of their sum 4. Or take the numbers 3, 3. Four times their product is 36, which is also the square of their sum 6. Or take the numbers 4, 4. Their product is 16, which when multiplied by four gives 64, which is also the square of their sum 8. Hence the fact that a result is seen to be true in some cases is of no consequence, as it is possible that one would come across contrary instances also. Hence it is necessary that one would have to provide a proof (*yukti*) for the rule that twice the product of two quantities when added or subtracted from the sum of their squares results in the square of the sum or difference of those quantities. We shall provide the proof (*upapatti*) in the end of the section on *ekavārṇa-madhyamāharaṇa*."

44

So this is basically the translation of the same passage. So what I am trying to say is that this fact that result in mathematics need to be justified merely giving examples to support them is not enough, in mathematics a certain amount of demonstration is needed is well known think to mathematicians in India.

(Refer Slide Time: 14:38)

Bhāskara I on *Upapatti* (c.629)

In his discussion of Āryabhaṭa's approximate value of the ratio of the circumference and diameter of a circle, Bhāskara I notes that the approximate value is given, as the exact value cannot be given. He then goes on to argue that other values which have been proposed are without any justification:

एवं मन्यन्ते स उपाय एव नास्ति येन सूक्ष्मपरिधिगनीयते। ननु चायमस्ति

विस्खंभवग्गदसगुणकरणी वट्टस्स परिओ होवि।

(विष्कम्भवर्गदशगुणकरणी वृत्तस्यपरिणाहो भवति)

इति। अत्रापि केवल एवागमः नैवोपपत्तिः। रूपविष्कम्भस्य दशकरण्यः परिधिरिति। अथ मन्यन्ते प्रत्यक्षेणैव प्रतीयमाणो रूपविष्कम्भक्षेत्रस्य परिधिदशकरण्य इति। नैतत् अपरिभाषितप्रमाणत्वात् करणीनाम्। एकत्रिविस्तारायामायतचतुरश्रक्षेत्रकर्णेन दशकरणिकेनैव तद्विष्कम्भ-परिधिर्विष्टमाणः स तत्प्रमाणो भवतीति चेत्तदपि साध्यमेव।

So one of the earliest discussion of nature of *Upapattis* occurs in the Bhaskara 1 commentary of Aryabhattiya, he is referring to the discussion of Aryabhatta that the approximate value of ratio of circumference to the diameter is 62832/20000, so then he is saying that approximate value is being given because the exact value cannot be given. Even Bhaskara 1 is saying that exact value of pi cannot be given, then he raises the question (FL).

This is the corresponding prakirta version of that, so the square root of 10 is really the accurate value of the circumference of a circle of diameter 1 is it not true, so why are you saying that exact value of the ratio of circumference to the diameter cannot be given. So then Bhaskara says (FL) so here also what has been given is only a traditional statement no justification has been provided.

(FL) why are you saying that, why do not you measure the circumference and immediately conclude that it is square root of 10 like that this is not possible, (FL) square root of 10 is occurring is and in exact quantity one cannot really determining its exactly, so like that he goes on to give an argument that indeed one has to provide a proof for the fact that certain amount is this ratio of circumference to the diameter.

(Refer Slide Time: 16:38)

Bhāskara II on *Upapatti* (c.1150)
 In *Siddhāntaśiromaṇi*, Bhāskarācārya II (1150) presents the *raison d'être* of *upapatti* in the Indian mathematical tradition:

मध्यादां दुसदां यदत्र गणितं तस्योपपत्तिं विना
 प्रौढिं प्रौढसभासु नैति गणको निःसंशयो न स्वयम्।
 गोले सा विमला करामलकवत् प्रत्यक्षतो दृश्यते
 तस्मादस्म्युपपत्तिबोधविधये गोलप्रबन्धोदात्तः ॥

Without the knowledge of *upapattis*, by merely mastering the calculations (*gaṇita*) described here, from the *madhyamādhikāra* (the first chapter of *Siddhāntaśiromaṇi*) onwards, of the [motion of the] heavenly bodies, a mathematician will not be respected in the scholarly assemblies; without the *upapattis* he himself will not be free of doubt (*niḥsaṁśaya*). Since *upapatti* is clearly perceivable in the (armillary) sphere like a berry in the hand, I therefore begin the *Golādhyāya* (section on spherics) to explain the *upapattis*.

So the fact that the mathematical result is need to be demonstrated is fairly clear in the commentaries, now the book that gives a set of demonstrations how does it view what these demonstrations are supposed to do. So in Siddhantasiromani in the Goladhyaya by the beginning Bhaskara tell us what Upapattis' the word for proofs Ugti, Upapatti are the 2 words used for proofs in Indian mathematics so what this Upapatti are supposed to do.

So this verse (FL) I quoted this in the introductory lecture also that but for whatever that has been discussed in (FL) section of astronomy starting from the madhyamadhikara first chapter of the (FL) section of the astronomy. So whatever is discussed without proofs without Upapatti (FL) a mathematician (FL) he will not be considered as a scholarly mathematician in an assembly of scholarly mathematician, (FL) he will not be free of doubt about the methods and procedures and the results that he is dealing with.

So these are the 2 kinds of handicaps that somebody who does not no proofs will automatically place.

(Refer Slide Time: 18:01)

Bhāskara II on *Upapatti*

The same has been stated by Gaṇeśa Daivajña in the introduction to his commentary *Buddhivilāsinī* (c. 1540) on *Līlāvati* of Bhāskarācārya

व्यक्ते वाव्यक्तसंज्ञे यदुदितमखिलं नोपपत्तिं विना तत्
निर्भ्रान्तो वा ऋते तां सुगणकसदसि प्रौढतां नैति चायम्।
प्रत्यक्षं दृश्यते सा करतलकलितादर्शवत् सुप्रसन्ना
तस्मादग्न्योपपत्तिं निगदितुमखिलम् उत्सहे बुद्धिवृद्धौ ॥

The same thing is repeated by Ganesa Daivajna while starting his commentary on Lilavati the commentary Buddhivilasini, which has a whole lot of Upapattis', (FL) in both (FL) that is both in arithmetic and geometry and in also in algebra whatever he said without Upapatti (FL) that one will not be considered as a well versed mathematician in an assembly of good mathematician (FL) and nor will he be without the confusion about the results and procedure.

So this is the kind of focus that has been put on Upapattis' in Indian mathematics, Upapattis' are important they need to be given.

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Bhāskara II on *Upapatti*

Thus, according to the Indian mathematical texts, the purpose of *upapatti* is mainly:

- (i) To remove confusion and doubts regarding the validity and interpretation of mathematical results and procedures; and,
- (ii) To obtain assent in the community of mathematicians.

This is very different from the ideal of “proof” in the Greco-European tradition which is to irrefutably establish the absolute truth of a mathematical proposition.

And main purpose is to say that the rest of the mathematicians accept the argument that you are given, so the result becomes a valid result in the community of mathematicians, and it will enable you to be free of how to perform the particular operation or particular result or the particular procedure. So that also a proof will enable without the proof there may be so many conclusions about what that original rule is meaning.

But now this is very, very different I mean if you see the first chapter of the famous commentary of progress on Euclid's elements, you will see what proof is supposed to do, the proof is supposed to do a very unambiguous and irrefutable demonstration of the result stated by the text that it has to once and for all established in an uncontestable manner the truth of the proposition stated in the text. So that is a very different ideal that does not find a central focus in the discussion of the Indian mathematicians.

(Refer Slide Time: 20:04)

Bhāskara II on *Upapatti*

In his *Bījaganita-vāsanā*, Bhāskarācārya II (c.1150) refers to the long tradition of *upapattis* in Indian mathematics.

अस्योपपत्तिः । सा च द्विधा सर्वत्र स्यादेका क्षेत्रगताऽन्या राशिगतेति । तत्र क्षेत्रगतोच्यते ।... अथ राशिगतोपपत्तिरुच्यते सापि क्षेत्रमूलान्तर्भूता । इयमेव क्रिया पूर्वचार्यैः संक्षिप्तपाठेन निबद्धा । ये क्षेत्रगतां उपपत्तिं न वृद्धान्ति तेषामियं राशिगता दर्शनीया ।

"The demonstration follows. It is twofold in each case: One geometrical and the other algebraic. There, the geometrical one is stated... Then the algebraic demonstration is stated, that is also geometry-based. This procedure [of *upapatti*] has been earlier presented in a concise instructional form by ancient teachers. For those who cannot comprehend the geometric demonstration, to them, this algebraic demonstration is to be presented."

Here, Bhāskara also refers to the *kṣetrāgata* (geometric) and *rāśīgata* (algebraic) demonstrations. To understand them, we shall consider the two proofs given by Bhāskara of the *bhujā-koti-karṇa-nyāya*.

Now how do they actually prove result, so what does Bhaskara say about this proofs, in the Bijaganita-vasana Bhaskara gives a proof of this Pythagoras theorem or the bhuja-koti-karṇa-nyaya, and he says there are 2 kinds of ways in which you can prove ksetragata and rasigata, ksetragata is geometrical, rashigata is algebraic. (FL) so first I will explain the geometrical, then after some time (FL) so then he will give the algebraic, (FL) that is also based upon geometry only.

(FL) so this Upapatti is something which has been passed down by traditions by the oral tradition of teaching by generation to generation, and of course he says the geometrical proof is somewhat more complex for them the algebraic proof is to be given.

(Refer Slide Time: 21:06)

Upapatti of Bhujā-Koṭi-Karṇa-Nyāya

In the *madhyamāharāṇa* section Bhāskara poses the following problem

क्षेत्रे तिथिनखैस्तुल्ये दोः कोटी तत्र का श्रुतिः। उपपत्तिश्च रुढस्य
गणितस्यास्य कथ्यताम्।

"In a right angled triangle with sides 15 and 20 what is the hypotenuse? Also give the demonstration for this traditional method of calculation."

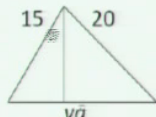
Here Bhāskara gives two proofs. First the geometrical:

अत्र कर्णः या १। एतत् त्र्यस्रं परिवर्त्य यावत्तावत्कर्णो भूः कल्पिता। भुजकोटी तु
भुजौ तत्र यो लम्बस्तदुभयतो ये त्र्यस्रे तयोरपि भुजकोटी पूर्वरूपे भवतः।
अतस्त्रैगुणिकं यदि यावत्तावति कर्णेऽयं १५ भुजस्तदा भुजतुल्ये कर्णे क इति
लब्धो भुजः स्यात्। सा भुजाश्रिताऽऽबाधा २२५/या। पुनर्यदि यावत्तावति कर्ण इयं
२० कोटिस्तदा कोटितुल्ये कर्णे केति जाता कोट्याश्रिताबाधा ४००/या।
आबाधायुतिर्यावत्तावत्कर्णसमा क्रियते तावद्भुजकोटिवर्गयोगस्य पदं
कर्णमानमुपपद्यते। अनेनोत्थापिते जाते आबाधे ९, १६ ततो लम्बः १२।
क्षेत्रदर्शनम्।

So let us see what is the geometrical proof and the algebraic proof that the Bhaskara gives for the bhuja-koti-karna-nyaya. So the problem is that in a right angle triangle which sides 15 and 20 what is the hypotenuse? (FL) 15 (FL) is 20, (FL) what is the hypotenuse (FL) so give us also the traditional method of proof or the traditional proof for this traditional method of calculation.

(Refer Slide Time: 21:47)

Upapatti of Bhujā-Koṭi-Karṇa-Nyāya



$$yā = \left(\frac{225}{yā} \right) + \left(\frac{400}{yā} \right)$$

$$yā^2 = 625$$

$$yā = 25$$

"Let the hypotenuse (*karṇa*) be *yā*. Take the hypotenuse as the base. Then the perpendicular to the hypotenuse from the opposite vertex divides the triangle into two triangles [which are similar to the original]. Now by the rule of proportion (*trairāśika*), if *yā* is the hypotenuse the *bhujā* is 15, then when this *bhujā* 15 is the hypotenuse, the *bhujā*, which is now the *ābādḥā* (segment of the base) to the side of the original *bhujā* will be $(225/yā)$. Again if *yā* is the hypotenuse, the *koṭi* is 20, then when this *koṭi* 20 is the hypotenuse, the *koṭi*, which is now the segment of base to the side of the (original) *koṭi* will be $(400/yā)$. Adding the two segments (*ābādḥās*) of *yā* the hypotenuse and equating the sum to (the hypotenuse) *yā*, we get $yā = 25$, the square root of the sum of the squares of *bhujā* and *koṭi*. The base segments are 9, 16 and the perpendicular is 12. See the figure".

So this is the way Bhaskara is explained let us not going to the details, here it can be read basically all the Bhaskara is doing is drop this perpendicular you want to complete this hypotenuse, the basic this is the right angle of the right angle triangle. So this right angle triangle itself is similar to each of these sub triangles that you obtained, so you use the similar triangles and you obtain both these abadhās side and this side.

So the abadhas this side will turn out to be 225/ya, if this ya is the length of these hypotenuse, the abadhas and this side is 400/ya square of this by this, so the hypotenuse itself is the sum of the 2 and so you will calculate the base to be 25. This is the essence of the geometrical proof given by Bhaskara and the argument goes like this using the trairasika basically the principle of similar triangles.

(Refer Slide Time: 22:38)

Upapatti of Bhujā-Koṭi-Karṇa-Nyāya

Now the algebraic demonstration:

अंथाऽन्यथा वा कथ्यते कर्णः या १ दोःकोटिघातार्थं
त्र्यस्रक्षेत्रस्य फलम् १५०। एतद्विषमत्र्यस्रचतुष्टयेन
कर्णसमचतुर्भुजं क्षेत्रमन्यत् कर्णज्ञानार्थं कल्पितम्। एवं
मध्ये चतुर्भुजमुत्पन्नमत्र कोटिभुजान्तरसमं भुजमानम् ५।
अस्य फलम् २५। भुजकोटिवधो द्विगुणस्त्र्यस्राणां चतुर्णां
फलम् ६००। एतदोगः सर्वं बृहत्क्षेत्रफलम् ६२५
एतदावत्तावद्वर्गसमं कृत्वा लब्धं कर्णमानम् २५। यत्र
व्यक्तस्य न पदं तत्र करणीगतः कर्णः।

As Bhāskara has noted, this algebraic demonstration is also geometrical in nature.

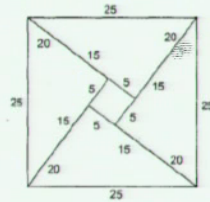
Now then he says let me give the algebraic demonstration, and as he himself said the algebraic demonstration is also based of anything at all you are providing it result in geometry, the algebraic demonstration is also based upon the figures.

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Upapatti of Bhujā-Koṭi-Karṇa-Nyāya

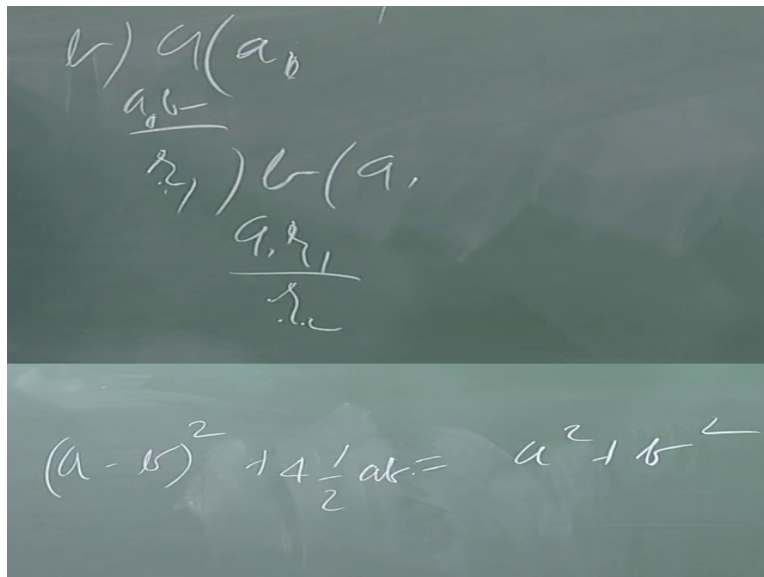
Let the hypotenuse be $yā$. The area of the triangle which is half the product of the *bhujā* (15) and *koṭi* (20) is 150. Consider a square whose sides are formed out of the hypotenuse of these triangles.

In the centre is formed a square whose side is 5, the difference of *bhujā* and *koṭi*, and whose area is 25. The area of the four triangles is 600. Thus, adding these, the area of the big square is 625. Taking $yā^2 = 625$, we get the hypotenuse to be 25.



And essentially all that Bhaskara is doing is if this is our right angled triangle, construct a square from the hypotenuse of this right angle triangle, and put the hypotenuse of the right angle triangles in such a way that in the center you are left with another small square.

(Refer Slide Time: 23:14)



And so essentially what you are doing is you are having this $a-b$ whole square $+ 2ab$ is $= a^2 + b^2$, and this $2ab$ is actually 4 times of $\frac{1}{2}ab$, so this is essentially the algebra so this is the algebraic relation that is used in proving, here is $a-b$ whole square each of these triangles is $\frac{1}{2}ab$, so you have 4 times $\frac{1}{2}ab$ and the sum of $a-b$ whole square $+ 4$ times $\frac{1}{2}ab = a^2 + b^2$ square, therefore this larger square is actually $= a^2 + b^2$.

So the hypotenuse squared is= sum of the bhuja square and koti square, so it is called algebraic because this algebraic identity is employed in proving this. So from the (FL) time's algebra and geometry have closely intertwined in discovering an algebraic result and even in expressing an algebraic result geometrical result algebra is employed in a very crucial way, and geometrical arguments are also made to discover algebraic identities.

(Refer Slide Time: 24:32)

Kṛṣṇa Daivajña's *Upapatti* of *Kuṭṭaka* Process

As an example of an *upapatti* which proceeds in a sequence of steps, we may briefly consider the detailed *upapatti* for the *kuṭṭaka* procedure given by Kṛṣṇa Daivajña (c.1600) in his commentary *Bījapallava* on *Bījagaṇita* of Bhāskara.

The *kuṭṭaka* procedure is for solving first order indeterminate equations of the form

$$\frac{(ax + c)}{b} = y$$

Here, **a, b, c** are given integers (called *bhājya*, *bhājaka* and *kṣepa*) and **x, y** are to be solved for in integers.

Kṛṣṇa first shows that the solutions for **x, y** do not vary if we factor all three numbers **a, b, c** by the same common factor.

He then shows that if **a** and **b** have a common factor then the above equation will not have a solution unless **c** is also divisible by the same.

He then gives the *upapatti* for the process of finding the *apavartānika* (greatest common divisor) of **a** and **b** by mutual division (the so-called Euclidean algorithm).

Now let us go to the proof of something else, the in Bhaskara's Bijaganita the kuttaka addhyaya there is a detailed discussion by Krsna Daivajna of the justification of the kuttaka, so how does the Krsna Daivajna justify the kuttaka process. So what is the kuttaka procedure that is for solving the equation $ax + c/b = y$, a c and b are given integers, x and y are to be determined for integral values that is the kuttaka problem, a is called the bhajya, b is called the bhajaka, c is called the ksepa.

And x is called guna, y is called the labdhi, this is the these are the technical terms for all these quantities in kuttaka. So what does Krsna do? Krsna Daivajna first shows that x and y do not change, if you factor all the three numbers a b c by the same factor, then he shows that if a and b have a common factor then the above equation will not have a solution unless c is also divisible by this same factor, this you already now know that the GCD of the quotient of x and y should divide the ksepa, then only the kuttaka problem has a solution.

Then he gives a proof that the mutual division of a and b that is lead to the greatest common divisor at some stage, so this so-called Euclidean algorithm.

(Refer Slide Time: 26:05)

Kṛṣṇa Daivajña's *Upapatti* of *Kuṭṭaka* Process

Kṛṣṇa then provides a detailed justification for the *kuṭṭaka* method of finding the solution by making a *valli* (table) of the quotients obtained in the above mutual division, based on a detailed analysis of the various operations in reverse (*vyasta-vidhi*).

In doing the reverse computation on the *valli* (*vallyupasaṃhāra*) the numbers obtained, at each stage, are shown to be the solutions to the *kuṭṭaka* problem for the successive pairs of remainders (taken in reverse order from the end) which arise in the mutual division of a and b .

After analysing the reverse process of computation with the *valli*, Kṛṣṇa shows how the solutions thus obtained are for positive and negative *kṣepa*, depending upon whether there are odd or even number of coefficients generated in the above mutual division.

And this indeed leads to the different procedures to be adopted for solving the equation depending on whether there are odd or even number of quotients in the mutual division.

So you can see that there is a sequence of argument going on in *Upapatti* like the way we do in proving any argument in mathematics that we are familiar with. Kṛṣṇa then gives a detailed justification of the *kuttaka* method itself, what is the *kuttaka* method involved that you make a *valli*, and that is based upon a detailed analysis of the operations in reverse this is called *vyasta-vidhi*.

So at each stage what Kṛṣṇa shows is that the you are obtaining solution for the *kuttaka* problem for the successive pairs of reminders which appear when you divide a/b , so when you are dividing a/b , so some quotient a_0 will appear, $a_0 b$ so you will just get the first reminder then you divide r_1 , so a_1 will appear $a_1 r_1$, a reminder r_2 will appear, so for each of these successive reminders the *kuttaka* problem that you will appear.

When you go up in the *valli* you will be solving the problem with the *kuttaka* problem with the successive reminders that is what is the *kuttana*, the problem with a, b is converted to problem with r_1, r_2 then problem with r_2, r_3 problem with r_4, r_5 , it ultimately comes to very small reminders, so either able to guess this solution or you can work out the solution, so that is what Kṛṣṇa is explaining.

And then he will also tell you if you have odd number of steps you go in a particular way, if there is even number of steps you go in a particular way. So the proof of each and every one of this is considered in detail.

(Refer Slide Time: 27:46)

Kṛṣṇa Daivajña's *Upapatti* of *Kuṭṭaka* Process

As an illustration, Kṛṣṇa considers the equation $\frac{(173x+3)}{71} = y$ with *bhājya* 173, *bhājaka* 71 and *kṣepa* 3.

In the mutual division of 173 and 71 we get the quotients 2, 2, 3 and 2 and remainders 31, 9, 4 and 1.

If we do the reverse computation on the *vallī* formed by 2, 2, 3, 2, 1, 3 and 0, we first get 6, 3 as the *labdhi* and *guṇa*, which satisfy the equation $\frac{(9 \cdot 3 - 3)}{4} = 6$, with the remainders 9, 4 serving as *bhājya* and *bhājaka*.

In the reverse computation on the *vallī*, we then get 21, 6 as *labdhi* and *guṇa*, which satisfy the equation $\frac{(31 \cdot 6 + 3)}{9} = 21$, with the remainders 31, 9 serving as *bhājya* and *bhājaka*.

And so on, till we get 117 and 48 as *labdhi* and *guṇa*, which satisfy the equation $\frac{(173 \cdot 48 + 3)}{71} = 117$

So as an illustration Kṛṣṇa considers this equation $173x + 3/71$, while making the argument the illustration will run parallel, so you will keep an illustration and make the argument. So in the mutual division of 173 and 71 quotients are 2, 2, 3 and 2, the remainders are 31, 9, 4 and 1. So when we do the reverse operation with *valli* 2, 2, 3, 2, 1 followed by the *kṣepa* is 3, and put a 0 below that right in the end.

So Kṛṣṇa shows that in the first level you get 6 and 3 as the *labdhi* and *guṇa* in the *valli*, when you clear the *valli* and they will satisfy the equation $9 \cdot 3 - 3/4 = 6$, so the quantities involved here 9 and 4 are essentially the 2 remainders which appear here. In the next step you will add a *kuttaka* problem solution with the 2 remainders 31 and 9, the next *guṇa* and *labdhi* will solve the *kuttaka* problem with these 2 remainders.

And finally you will get the solution for the *kuttaka* problem with 173 and 71. So the reverse operation in *valli* is essentially solutions with the *kuttaka* problem with the reminders successive reminders that appear here, this is the (FL) of the argument in proving that the *kuttaka* procedure

works. Now he will come to another aspect of proof in Indian mathematics, this is an interesting philosophical issue which seems to be operative here in Indian mathematical tradition.

(Refer Slide Time: 29:25)

Use of *Tarka* in *Upapatti*

The method of "proof by contradiction" is referred to as *tarka* in Indian logic. We see that this method is employed in order to show the non-existence of an entity.

For instance, Kṛṣṇa Daivajña essentially employs *tarka* to show the non-existence of the square-root of a negative number while commenting on the statement of Bhāskara that a negative number has no root.

वर्गस्य हि मूलं लभ्यते। ऋणाङ्कस्तु न वर्गः कथमतस्तस्य मूलं लभ्यते।
ननु ऋणाङ्कः कुतो वर्गो न भवति न हि राजनिर्देशः।... सत्यम्। ऋणाङ्कं
वर्गं वदता भवता कस्य स वर्ग इति वक्तव्यम्। न तावदुनाङ्कस्य "समद्वि-
घातो हि वर्गः" तत्र धनाङ्केन धनाङ्के गुणिते यो वर्गो भवेत् स धनमेव
"स्वयोर्वधः स्वम्" इत्युक्तत्वात्। नाप्युनाङ्कस्य। तत्रापि समद्विघातार्थ-
मृणाङ्केनर्णाङ्कगुणिते धनमेव वर्गो भवेत् "अस्वयोर्वधः स्वम्" इत्युक्त-
त्वात्। एवं सति कथमपि तमङ्कं न पश्यामो यस्य वर्गः क्षयो भवेत्।

Most of you who would have seen proofs in the standard geometry text might clearly be aware that one of the most commonly met with arguments is what is called the proof by contradiction. What is proof by contradiction? It is also called reduction ad absurdum what is that, so whenever you want to prove a proposition you assume that the opposite of that proposition to be true for the time being.

And then argue and you obtain the result which is in contradiction either with what you have assumed or with any of the results that you have already proved or with anyone of your starting postulate. Therefore, by assuming the contrary of what you want to prove to be true, you arrive at a contradiction and therefore, that the assumption that the contrary of what you wanted to be prove is not justified and therefore, the opposite of that, that is the proposition that you want to be proved has been proved by you.

So this kind of argument is called *tarka* in Indian logic is *nayaya shastra*, so this kind of an argument is called *tarka*, and such arguments do appear in Indian mathematical texts in some simple context. So I have taken the simplest of such arguments to show that a negative number

does not have square, so this is the argument of Kṛṣṇa Daivajña to show that a negative number does not have a square root.

(Refer Slide Time: 30:57)

Use of Tarka in Upapatti

Thus according to Kṛṣṇa

"The square-root can be obtained only for a square. A negative number is not a square. Hence how can we consider its square-root? It might however be argued: 'Why will a negative number not be a square? Surely it is not a royal fiat'... Agreed. Let it be stated by you who claim that a negative number is a square as to whose square it is; surely not of a positive number, for the square of a positive number is always positive by the rule... not also of a negative number. Because then also the square will be positive by the rule... This being the case, we do not see any such number whose square becomes negative..."

The argument is very simple assume that a positive number is a square root of negative number it cannot be square of that is positive, assume a negative number to be square of a negative number that cannot be its square is positive, so there is any know number which you know which is the square root of a negative number. Therefore, a negative number does not have square, this is the standard argument of proof by contradiction.

And this is employed so (FL) why is it that a negative number is not a square (FL) you were arguing that a negative number is a square you should tell me whose square it is, (FL) it cannot be a square of positive number because of the rule (FL) because square of a positive number is positive, (FL) it cannot be square of a negative number, (FL) they are also negative, negative is positive, (FL) that being the case (FL) so we are not able to see that number whose square is negative, so it is a standard.

(Refer Slide Time: 32:19)

Use of *Tarka* in *Upapatti*

While the method of “proof by contradiction” or *reduction ad absurdum* has been used to show the non-existence of entities, the Indian mathematicians do not use this method to show the existence of entities, whose existence cannot be demonstrated by other direct means. They have a “constructive approach” to the issue of mathematical existence.

It is a general principle of Indian logic that *tarka* is not accepted as an independent *pramāṇa*, but only as an aid to other *pramāṇas*.

But while this kind of an argument is commonly used you do not see an argument in any of the Indian mathematical texts at least I am not seen much, well you are proving the existence of something (FL) process it would be fairly easy for people to construct a proof by assuming that (FL) process does not lead to a solution, you come up with a contradiction of something that you already know, and in therefore, you say you have proved that the (FL) process leads to a solution.

So to prove the existence of some quantity which is by proof of contradiction is not something that is commonly met with, and I am trying to tell you that this is fairly in tune with these larger principles of Indian logic the way our Indian philosophy. This *tarka* while it is useful in fact *tarka* is used in what is called as a sieve in anumana pramana that when you want establish the in variable (FL) you try to refine that (FL) further and further by the use of *tarka* argument (FL) may not really be giving to that (FL) people who know Indian logic will see that that is the contacts in which *tarka* is used.

So *tarka* is used as an auxiliary way of arguing, it is not given as status of an independent pramana what does that mean? That anything that is established by *tarka* which in principle cannot be established through the other direct means of verification will not be given a state of validly established results. So it is not an independent pramana it is only as an aid to other pramanas.

So basically the Indian mathematician seem to be using reduction ad absurdum kind of arguments if at all they use which are very few for in between mainly to show the non-existence or impossibility of certain things they do not seem to have used it at all, to show the existence of quantities which existence is not possible to be demonstrated by other means, that if you already know what the solution is you do not need reduction ad absurdum.

In the morning question came in to discuss the way Archimedes proved, the argument that the infinite series $1/4 + 1/4^2$ etc. adds up to $1/3$, this was in specific geometrical contexts in a geometrical construction. So what the Archimedes is showing is let me assume that the sum of all these figures is $>$ that other figure then he will show a contradiction, let me show that the sum of all this figures is $<$ that sum, and then he shows a contradiction.

This is a very, very this is called a double reduction ad absurdum argument that this sum can either be larger nor it be smaller, and therefore, it has to be equal. Whereas you show by successive smaller and difference between the result that you want to prove, and the successive terms of summation becomes smaller and smaller and you are actually going to result that to the result that you are wanting, so that is the kind of proofs that you find in Indian texts.

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Indian Logic Excludes *Aprasiddha* Entities from Logical Discourse

Naiyāyikas or Indian logicians do not grant any scheme of inference, where a premise which is known to be false is used to arrive at a conclusion, the status of an independent *pramāṇa* or means of gaining valid knowledge.

In fact, they go much further in exorcising the logical discourse of all *aprasiddha* terms or terms such as "rabbit's horn" (*śaśaśṛṅga*) which are empty, non-denoting or unsubstantiated.

There is in fact an even larger principle in Indian logic it is that you do not grant validity to any scheme of inference, where a premise which is already known to be false is used to arrive at a

conclusion, and furthermore you do not allow in any discourse which is logically considered rigorous, a terms which you know are empty, terms which are meaningless like a square circle or a rabbit's horn is the one that is very commonly used sasasrnga.

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Indian Logic Excludes *Aprasiddha* Entities from Logical Discourse

"*Nyāya*...(excludes) from logical discourses any sentence which will ascribe some property (positive or negative) to a fictitious entity. *Vācaspati* remarks that we can neither affirm nor deny anything of a fictitious entity, the rabbit's horn. Thus *nyāya* apparently agrees to settle for a superficial self-contradiction because, in formulating the principle that nothing can be affirmed or denied of a fictitious entity like rabbit's horn, *nyāya*, in fact violates the same principle. *Nyāya* feels that this superficial self-contradiction is less objectionable (than admitting fictitious entities in logical discourse)... (This can be seen from the discussion in) Udayana's *Ātmatattvaviveka*..."²

²B. K. Matilal, *Logic Language and Reality*, Delhi 1985, p-9.

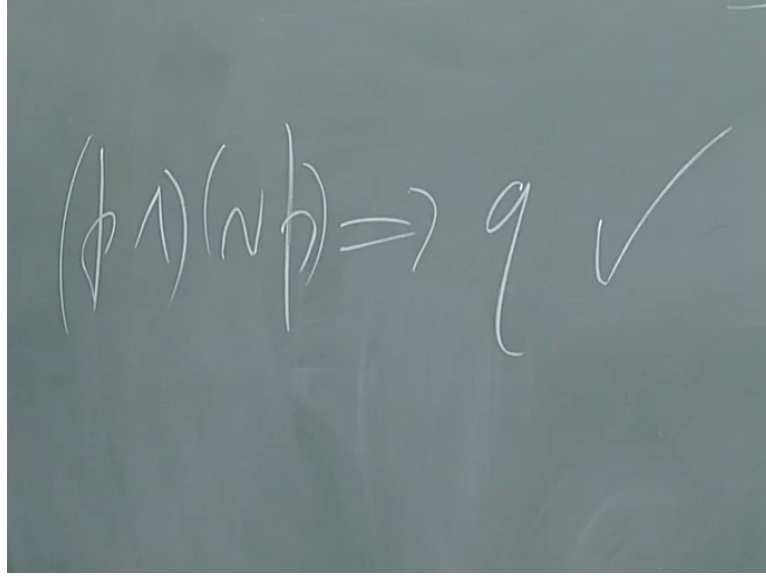
The insistence on not using this undifferentiated or undenotable terms is so deep that one is willing to even accept a contradiction rather than use these terms, so this is the major issue this is a summary, the original work has a detailed discussion which is found in Udayana's *Atmatattvaviveka* are you can pick up Udayana's *Atmatattvaviveka* and see this discussion. This this question is very simple.

The nyaya is saying you should not use any statement which uses fictitious entity is not a valid statement, so then the opponent you already made a statement using a fictitious entity, so what are you talking about so he says I am saying something meaningful if you do not want to understand it that is alright but I will live with this contradiction of not using fictitious entities in my discourse, but I will not allow still the fictitious entity to be used.

So we are willing to live with the contradiction that the contradiction is in the statement that rules out the use of these fictitious entities, so that contradiction is alright rather than using these fictitious entities in the argument. Now this issue of contradiction is very special in traditions

which base all their arguments on reduction ad absurdum techniques, because what the reduction ad absurdum the moment you find a contradiction the opposite of the premises is true.

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And the moment you are logical system has a contradiction, all the proposition becomes valid in a logical system, where a proposition p and not p are both true, then from p and not p you can imply any q and this is always a universally valid proposition. So this systems of logic which use this kind of reduction ad absurdum to the sort of limit, they are always worried about contradictions in their systems of logic okay.

(Refer Slide Time: 38:23)

Yuktibhāṣā of Jyeṣṭhadeva

The most detailed exposition of *upapattis* in Indian mathematics is found in the Malayalam text *Yuktibhāṣā* (1530) of Jyeṣṭhadeva, a student of Dāmodara, and a junior colleague of Nīlakaṇṭha.

At the beginning of *Yuktibhāṣā*, Jyeṣṭhadeva states that his purpose is to present the rationale of the results and procedures as expounded in the *Tantrasaṅgraha*.

Many of these rationales have also been presented (mostly in the form of Sanskrit verses) by Śaṅkara Vāriyar (c.1500-1556) in his commentaries *Kriyākramakārī* (on *Līlāvati*) and *Yuktidīpikā* (on *Tantrasaṅgraha*).

Yuktibhāṣā has 15 chapters and is naturally divided into two parts, Mathematics and Astronomy.

In the Mathematics part, the first five chapters deal with the notion of numbers, logistics, arithmetic of fractions, the rule of three and the solution of linear indeterminate equations.

Now with that interlude let us come to Yuktibhasa, the discussion of proofs contained in Yuktibhasa. Today, I will just give **“Professor - student conversation starts”** Yes mam, (()) (38:35) and can you tell us what is a difference between Upapatti and pramana? Pramana is a means of acquiring valid knowledge, so prama is valid knowledge, pramakarana is pramana that what leads you to valid knowledge is pramana.

So the means of valid knowledge are in ordinary understanding observation (FL) argument and generalization from known observation etc. that is called anumana, then some differences then what is called established well established tradition that is agama then different schools of philosophy have their own other definitions of pramana (FL) and (FL) there are other kinds of pramana, this is what is pramana in simple terms.

Now Upapatti is (FL) detailed derivation of that we will leave it to the lecture of professor Ram Subramanian that you are going and arriving at the you have established the result of you are arriving at the result, so that he is I mean a loose sense of QED. **“Professor - student conversation ends.”** So the most detailed exposition over Upapattis’ in Indian mathematics is found in Yuktibhasa by Jyesthadeva.

So Jyesthadeva says that the purpose is to present the rational of the results and procedures outlined in Tantrasangraha, many of these are given in Sanskrit also in the verse of Sankara variyar, Yuktibhasa is 15 chapters, the first 5 mathematics chapters deal with numbers logistics fractions rule of 3 and solutions of indeterminate equation.

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Yuktibhāṣā of Jyeṣṭhadeva

Chapter VI of *Yuktibhāṣā* deals with the *paridhi-vyāsa-sambandha* or the relation between the circumference and diameter of a circle. It presents a detailed derivation of the Mādhava series for π , including the derivation of the binomial series, and the estimate of the sums of powers of natural numbers $1^k + 2^k + \dots + n^k$ for large n . This is followed by a detailed account of Mādhava's method of end correction terms and their use in obtaining rapidly convergent transformed series.

Chapter VII of *Yuktibhāṣā* is concerned with *jyānāyana* or the computation of Rsines. It presents a derivation of the second order interpolation formula of Mādhava. This is followed by a detailed derivation of the Mādhava series for Rsine and Rversine.

In the end of the Mathematics section, *Yuktibhāṣā* also presents proofs of various results on cyclic quadrilaterals, as also the formulae for the surface area and volume of a sphere.

The Astronomy part of *Yuktibhāṣā* has seven chapters which give detailed demonstrations of all the results of spherical astronomy.

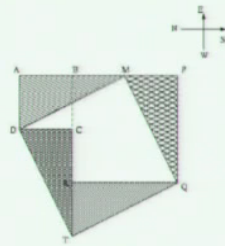
The chapter 6 and 7 of the mathematics part are the crucial ones, chapter 6 deals with the *paridhi-vyasa-sambandha*, it presents the derivation of Madhava series for π , derivation of binomial series, the derivation of this estimate of the sum 1 to the power k etc. $+n$ to the power k for large n . And then derivation of the end correction terms, and the transformer series. Chapter 7 deals with *jyanayana* or computation of Rsines.

It presents a derivation of the second order interpolation formula Madhava I think that will be covered in one of the trigonometry talks, this is followed by the derivation of the Madhava series for Rsine and Rversines. In the end it has a detailed discussion of cyclic quadrilateral and also the proofs of surface area and volume of a sphere. The astronomy part has 7 chapters, which gives detailed demonstration of all the results of spherical astronomy.

Perhaps, you will get a glimpse of the way Indians this sophisticated way the Kerala astronomers work with spherical trigonometry in one of the lectures that will follow.

(Refer Slide Time: 41:30)

Yuktibhāṣā Proof of Bhujā-Koṭi-Karṇa-Nyāya



ABCD, a square with its side equal to the *bhujā*, is placed on the north. The square BPQR, with its side equal to the *koṭi*, is placed on the South. It is assumed that the *bhujā* is smaller than the *koṭi*. Mark M on AP such that $AM = BP = koṭi$. Hence $MP = AB = bhujā$ and $MD = MQ = karṇa$. Cut along MD and MQ, such that the triangles AMD and PMQ just cling at D, Q respectively. Turn them around to coincide with DCT and QRT. Thus is formed the square DTQM, with its side equal to the *karṇa*. It is thus seen that

karṇa-square MDTQ = *bhujā*-square ABCD + *koṭi*-square BPQR

So first is the Yukti-bhāṣa proof of bhujā-koti-karṇa-nyāya, so we saw 2 proofs of Bhaskara, this is the proof of Yukti-bhāṣa very simple. This is 1 square, BPQR this is the koti square, ABCD is the bhujā square, assume that bhujā square is smaller, now make your $AM = koṭi$, so your MP will be = bhujā that is mark this point M such that MP is the same as AB. Then join MP, join MD, construct the square on MD.

Now the constructing the square on MD this procedure is similar to the (FL) procedure for adding 2 squares as it was pointed out in the lecture on (FL). Then how to show that this square this MD square = AM square + AD square very simple at the point D cut this triangle AMD and place it on DCT, and at the point Q cut this angle MPQ rotate it and place it on QRT, then you have shown that this square = sum of those 2 other squares.

So actually it involves cutting and repasting, so this was another feature of all the Indian mathematical arguments that they could involve moment of the physical objects, cutting of the object. Then even many of the proof say I mentioned Kṛṣṇa Daivajña proof of $- \ast - = +$ by using that let the direction of moving eastwards be taken as positive, direction of moving westwards we taken as negative and with that they will be able to show that $+5 - of -3 = +8$ and things like that.

So even physical arguments of various kinds are acceptable as proofs as long as one is systematic and one is explained. Surprisingly, I should mention that this proof which is available in the available editions of Yuktibhasa is different from the proof Charles Whish has given in his paper saying that this is the proof that I have found in Yuktibhasa, so that still remains a mystery.

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Successive Doubling of Circumscribing Polygon

It appears that the Indian mathematicians (at least in the Āryabhaṭan tradition) employed the method of successive doubling of the circumscribing square (leading to an octagon, etc.) to find successive approximations to the circumference of a circle. This method has been described in the *Yuktibhāṣā* and also in the *Kriyākramakārī*.

The latter cites the verses of Mādhava in this connection

चतुर्भुजे दोःकृतिनागभागमूलं हरे हारभुजाद्विभेदात् ।
भुजाहताद्वारहतं तु कोणान्नोत्वा विलिख्याष्टभुजाः प्रसाध्याः ॥ १ ॥
अष्टाश्रदोरर्धकृतिर्निधेया व्यासार्धवर्गे पदमत्र कर्णः ।
तेनाऽऽहरेदोर्दलवर्गहीनं व्यासार्धवर्गं यदतः फलं स्यात् ॥ २ ॥
तदूनकर्णो दलितो हराख्यो गुणस्तु विष्कम्भदलोनकर्णः ।
भुजार्धमेतेन हतं गुणेन हरेण भुज्जा यदिहापि लब्धम् ॥ ३ ॥
तत्कोणतः पार्श्वयुग्मेषु नीत्वा छिन्नेऽन्तरे स्यादिह षोडशाश्रम् ।
अनेन मार्गेण भवेदतश्च रदाश्रकं वृत्तमतश्च साध्यम् ॥ ४ ॥

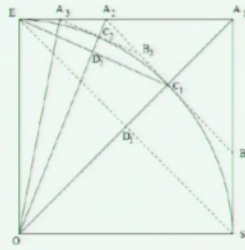
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Now there is another complex geometrical proof that you find in Yuktibhasa, this is the method of approximating the circumference of a circle by a square which is circumscribing it, and then dividing the square in such a way such that you have a regular octagon, then dividing it in such a way such that you have a regular polygon of 16 sides. Again Kriyakramakari gives a collection of version due to Madhava which seem to be explaining this method.

So Madhava, who gave the infinite series also has summarized this method which is the classic old method of trying to estimate the circumference of a circle by summing the sides of an inscribed or a circumscribed polygon with large enough number of sides.

(Refer Slide Time: 44:52)

Successive Doubling of Circumscribing Polygon



In the figure, EOA_1 is the first quadrant of the square circumscribing the given circle. EA_1 is half the side of the circumscribing square. Let OA_1 meet the circle at C_1 . Draw $A_2C_1B_2$ parallel to ES . EA_2 is half the side of the circumscribing octagon.

Similarly, let OA_2 meet the circle at C_2 . Draw $A_3C_2B_3$ parallel to EC_1 . EA_3 is now half the side of a circumscribing regular polygon of 16 sides. And so on.

So the method is as given in Yuktibhasa is like this, you take this square which is one fourth of the square which is actually circumscribing the circle, so the side of this particular square is right now we saw this is r , but the bigger square which is circumscribing the circle has side $2r$. Now the point at which $O A_1$ meets A_1 Yuktibhasa says draw this line $A_2 C_1 B_2$ which is parallel to ES east south direction, so instead of saying draw the tangent which they do not say this is the way they are explaining.

And then it says okay it goes and meets this side $E A_1 A_2$ join $O A_2$, then this $O A_2$ meets at the point C_2 , now draw $A_3 C_2 B_3$ which is parallel to $C_1 E$ and like that you go on, so this $E A_2$ will be the side of the circumscribing octagon, $E A_3$ will be a side of a circumscribing polygon of 16 sides. So these complex geometrical arguments are indeed dealt with in Yuktibhasa. Now the rest of the argument is using right angle triangles and similar triangles estimating $E A_2$, $E A_3$.

(Refer Slide Time: 46:09)

Successive Doubling of Circumscribing Polygon

Let half the sides of the circumscribing square, octagon etc., be denoted

$$l_1 = EA_1, l_2 = EA_2, l_3 = EA_3, \dots$$

The corresponding *karnas* (diagonals) are

$$k_1 = OA_1, k_2 = OA_2, k_3 = OA_3, \dots$$

and the *ābādhās* (intercepts) are

$$a_1 = D_1A_1, a_2 = D_2A_2, a_3 = D_3A_3, \dots$$

Now

$$l_1 = r, k_1 = \sqrt{2}r, \text{ and } a_1 = \frac{r}{\sqrt{2}}.$$

Using the *bhujā-koṭi-karṇa-nyāya* (Pythagoras theorem) and *trai-rāsika-nyāya* (rule of three for similar triangles), it can be shown that

$$l_2 = l_1 - (k_1 - r) \left(\frac{l_1}{a_1} \right)$$

$$k_2^2 = r^2 + l_2^2$$

$$a_2 = \frac{[k_2^2 - (r^2 - l_2^2)]}{2k_2}$$

25

So E A1 is the half of the side of the circumscribing square, E A2 is the half of the side of the circumscribing octagon, octagon will look like this here the octagon this is the part of the octagon it will go like that, so this is half of the side of the circumscribing octagon, E A3 is half the side of the circumscribing 16 sided polygon. The corresponding karna, karna's are O A2, O A1, O A3 so the corresponding karna's are denoted by k1, k2, k3.

The corresponding abadhas are the intercept D1 A1, D2 A2, D3 A3, when you have the first one this D2 A2, D1 A1 will be the abadha that is in this right angle triangle, D2 A2 will be the abadha in this triangle and similarly, D3 is not shown there, so those are the abadhas which will come handy in doing this calculation, they were given by the following. Then you have a series of recursion relations that you see which are derived from the geometrical argument.

so l2 k2 and a2 can be defined or determined in terms of l1 k1 and first l2 is determined in terms of l1 k, k1 and a1, then k2 is determined then a2 is determined, so l2 k2 a2 is determined, then you go on determine the side of the regular hexagon.

(Refer Slide Time: 47:51)

Successive Doubling of Circumscribing Polygon

In the same way l_{n+1} , k_{n+1} and a_{n+1} can be obtained from l_n , k_n and a_n .

These can be shown equivalent to the recursion relation for l_{n+1} which is the half side of the circumscribing polygon of 2^{n+1} sides:

$$l_{n+1} = \left(\frac{r}{l_n}\right) \left[\left(r^2 + l_n^2\right)^{\frac{1}{2}} - r \right]$$

We of course have the initial value $l_1 = r$.

This leads to $l_2 = (\sqrt{2} - 1) r$ and so on.

Kriyākramakarī notes that

एवं यावदभीष्टं सूक्ष्मतामापादयितुं शक्यम्।

Thus, one can obtain an approximation (to the circumference of the circle) to any desired level of accuracy.

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So given l_n, k_n, a_n , $l_{n+1}, k_{n+1}, a_{n+1}$ can be determined, and in fact if you study the process the recursion relation seems to be something like this, and initial value l_1 is itself half the side of circumscribing square, l_2 will be $\sqrt{2}-1$ times r and so on. So you keep on using this relation you will obtain half the side of circumscribing polygon of large and larger side, and from that you find out the circumference of the polygon take it as an approximation to the circumference of this square.

And *Kriyākramakarī* says (FL) you can go if you are like (FL) you have patience you can calculate pi to 16 decimal places there is really no problem.

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So I think for today, I have covered the tradition of proofs in Indian mathematics from more ancient times. In the next talk, I will discuss the proofs as found in *Yuktibhasa* of the results of Madhava that will be tomorrow on Monday, thank you.