Mathematics in India: From Vedic Period to Modern Times
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Lecture - 36
Proofs in Indian Mathematics I

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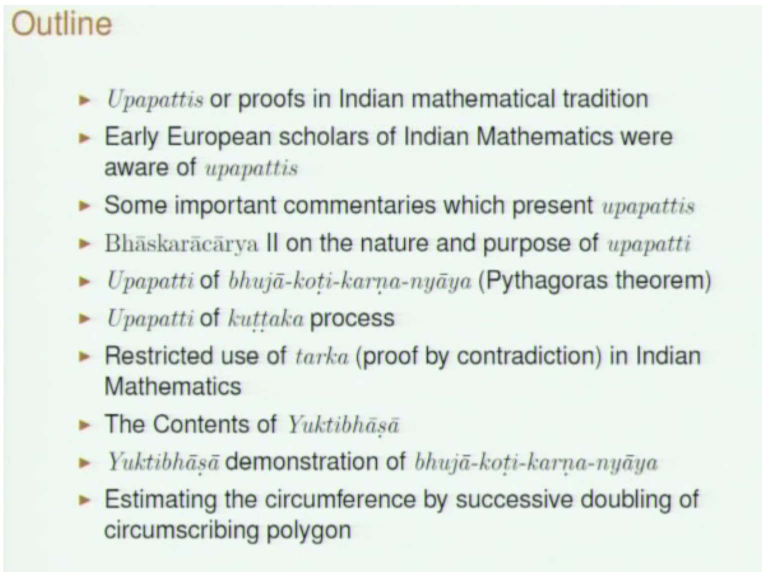

So in this talk and the following 2 talks, we will be discussing the proofs in Indian mathematical tradition. So in this talk we will be talking about the history of proofs and the idea of Upapatti as they are called in Indian mathematical works, where are they present? And something on what is the understanding of Indian mathematicians as to what an Upapatti what is the purpose? What it ought to do? What it purpose to do?

Then we will study few examples of proofs in the earlier Indian text, and come to of course the text Yuktibhasa which is indeed a collection of proofs, many of these proofs of Yuktibhasa will be discussed in the second and third lecture, and at the end of which we will try to analyze what is the notion of proof in Indian mathematics once again towards the end.

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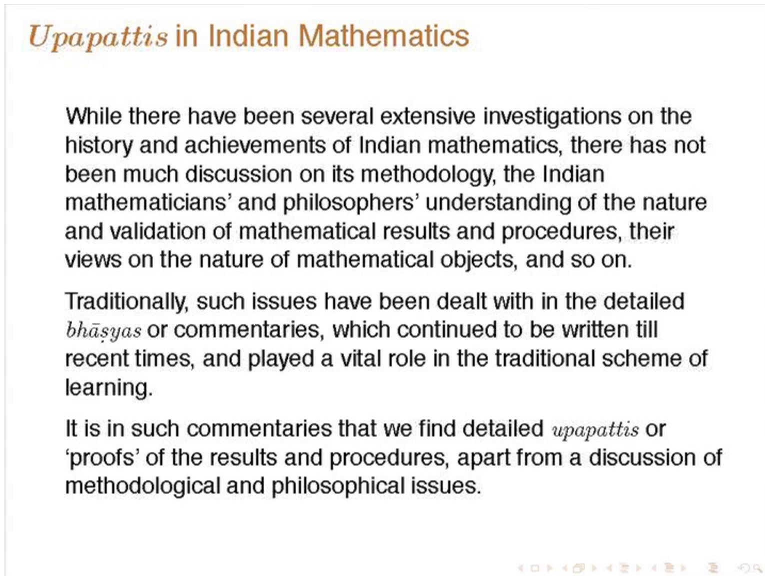

Now while considerable amount of information is known about Indian mathematics, and it has been known for a very long time to modern scholars, there has not been really that much discussion on what is the understanding of Indian mathematicians about the nature of mathematics, what does mathematics deal with? What it is supposed to do? Or having an Indian philosophers discuss these questions.

So these kind of issues have not been studied or analyzed in any detail. The reason is because we have been so sort of impressed by the achievements of the Indian mathematician that they did this very nice results, Aryabhatta at this beautiful approximation for square for pi, or he had this very nice method for square root, cube root. Or Brahmagupta came up with his wonderful formulae for diagonals of a cyclic quadrilateral or we have this Sankara Variyar.

So these results and achievement of Indian mathematician occupied considerable amount of our time and study, and the original works was scanned for the kind of results that they had, and we were not discussing how they were arriving at these results or in what context they were looking at these results, this did not become the subject of serious investigation. One is because the traditionally this issues have been dealt with in the commentaries.

So any text in India has to be understood along with its commentaries and along with the successive oral tradition of teaching of that text, and that is what gives you an orientation towards that text and its position in the overall discipline that is being considered and the way this discipline has evolved. So till very recent times writing of these commentaries was a very important part of Indian Textual Indian scientific tradition, or in all Shastra this was a very important thing.

And in the traditional scheme of learning these commentaries played very, very major role, but since we have been disconnected from that way of studying Indian knowledge systems, we sort of most of the time just look at their results, and then start wondering how did they arrive at it, or did they have any method at all, or did they not have any method at all. And we also start discussing things that maybe Patanjali did not understand Panini correctly, maybe (FL) did not understand Patanjali correctly like that.

I mean obviously nobody would have understood this predecessor so thoroughly that indeed true, but he did not understand there is no reason that we will be able to understand the predecessor much better. So these are the kind of issues in which we are got most of the time. So if we are interested in understanding methodology of Indian mathematics we have to look at the commentaries much more seriously, and this will apply to most of other disciplines most of other traditional Indian disciplines also.

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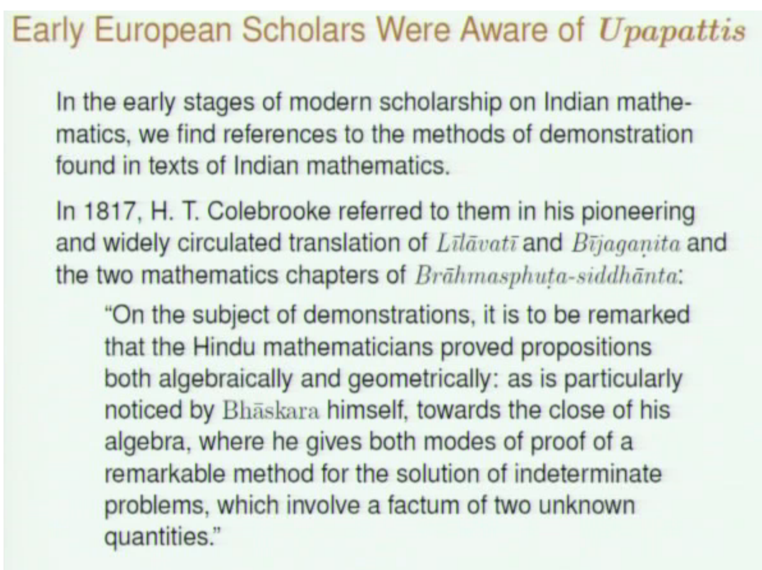

Now when modern scholars started studying Indian mathematics the fact that these books did have various kinds of proofs, the fact that they were commentaries and they contain various proofs was taken out of. Colebrook's work is the pioneering work of in the translation of Indian mathematical text, he translated Lilavati, he translated Bijaganita, he translated the 2 mathematics chapters of Brahmasphuta-siddhanta.

And Colebrook's book is filled with footnotes from various commentaries, and occasionally he refers to the kind of demonstration of Upapattis' which are contained in the commentaries. Colebrook says on the subject of demonstrations it is to be remarked that the Hindu mathematician proved propositions both algebraically and geometrically, as is particularly noticed by Bhaskara himself towards the close of his algebra where he gives both modes of proof of a remarkable method for the solution of the bhavita problem he is referring.

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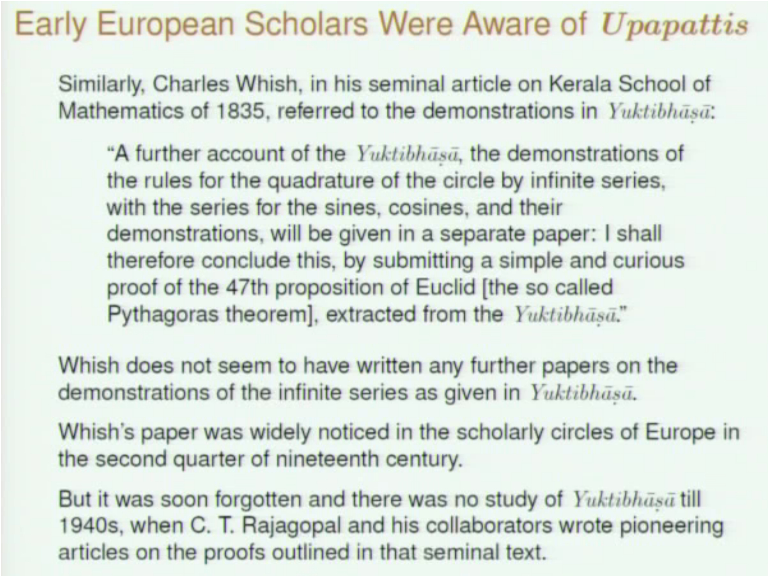

Similarly, Charles Whish, who's article generated interest about the Kerala school of mathematics in recent times, this article was written in 1835, in fact it was submitted as a first a paper to the Madras literary society, and later on it was read as a paper in the it was published in the transaction of Royal Asiatic Society. So Whish says in his article, he talks about 4 texts Tantrasangraha, Yuktibhasa, (FL) except (FL) now all these other text have been fairly good editions and the translations and explanations of these texts are currently available.

So a further account of the Yuktibhasa, the demonstration of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations will be given in a separate paper. I shall therefore conclude this paper that is the paper that is presenting by submitting a simple and curious proof of the 47th proposition of Euclid, what is generally called the Pythagoras theorem extracted from the Yuktibhasa.

But Whish did not write any further paper, in fact there is this detailed manuscript of this Tantrasangraha with Sankara Variyar's Malayalam commentaries is available with his detailed handwritten notes on one side of the book which a professor Ram Subramanian had an occasion to copy and they published some extracts from it on of Sankara Variyar's Malayalam commentary couple of years ago.

Now Whish paper was widely noted at the time when it was written, it was written in 1835, so this was considered an important discovery at that time. However, all this was soon forgotten and there was really no study of Yuktibhasa for almost 120 years, when C. T. Rajagopal and his collaborators started pioneering articles on proofs in Indian mathematics.

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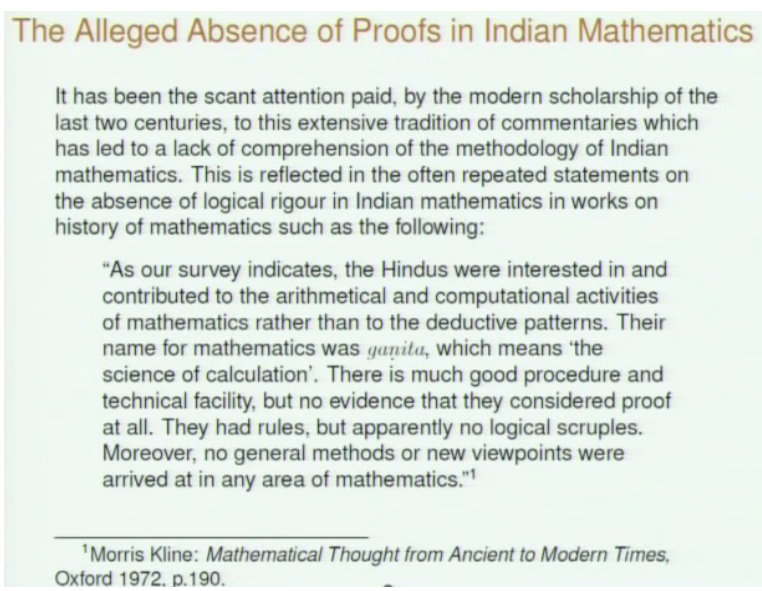

And therefore it is this scant attention paid by modern scholarship of the last 2 centuries, to this extensive tradition of commentaries, it is this which has led to a lack of comprehension of the methodology of Indian mathematics. And this is often reflected in the sort of blank statements that you will find in many of the general history books of mathematics, so if you open any general history of mathematics you are likely to come across a statement like this.

This I have taken from Morris Kline which a book of about 1000 pages Mathematical Thought from Ancient to Modern Times, as our survey indicates the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the detective patterns. Their name for mathematics was ganita which means the science of calculation which is of course correct.

There is much good procedure and technical facility but no evidence that they considered proof at all, they had rules but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics, so this kind of a broad speak generalization view of Indian mathematics you will find this in several books.

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Now the current books will write it with some more sort of caution and some more subordinate clauses added, but the overall thing will be that to a large extent the way these results were obtained by Indian mathematician still remained obscure or something like that will be the general canting. So amongst the published works, there are indeed several books which do contain this Upapattis' so the first is the Bhasya Bhaskara of Aryabhattiya, this contains a few discussions of proofs.

Bhasya Govindasvamin of Mahabhaskariya. The Vasanabhasya of Prthudakasvamin on Brahmasphuta-siddhanta this commentary is still not published in entirety, only some sections are available in print. So the first really book that is serious discussions of Upapattis' is in Vivarana of Bhaskaracharya on Sisyadhivrddhidatantra of Lalla. Vasana of Bhaskaracharya on his own Bijaganita. The Vasana of Bhaskaracharya on his own Siddhantasiromani these are indeed commentaries which contain extensive discussion of proofs.

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Parameswara's commentary on Govindasvamin's Bhasya. Nilakantha (FL), then of course Yuktibhasa of Jyesthadeva. Yuktidipika and Kriyakramakari of Sankara variyar which you referred 2.

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Some Important Commentaries Which Discuss *Upapattis*

12. *Gaṇitayuktayaḥ*, Tracts on Rationale in Mathematical Astronomy by various Kerala Astronomers (c.16th-19th century)
13. *Sūryaprakāśa* of Śūryadāsa (c.1538) on Bhāskarācārya's *Bījagaṇita* (c.1150)
14. *Buddhivilāsinī* of Gaṇeśa Daivajña (c.1545) on *Līlāvatī* of Bhāskarācārya II (c.1150)
15. *Bījanavāṅkura* or *Bījapallavam* of Kṛṣṇa Daivajña (c.1600) on *Bījagaṇita* of Bhāskarācārya II (c.1150)
16. *Vāsanāvārttika*, commentary of Nṛsiṃha Daivajña (c.1621) on *Vāsanābhāṣya* of Bhāskarācārya II on his own *Siddhāntaśiromaṇi* (c.1150).
17. *Marīci* of Munīśvara (c.1630) on *Siddhāntaśiromaṇi* of Bhāskarācārya II (c.1150).

Then about 20 years ago professor K. V. Sharma published a large number of tracks which was small monographs which contain the discussions and proofs of small bits of results on various topics, so he compiled them and published about 20 such tracks 20 to 25 years ago. Buddhivilasini of Ganesha Daivajna we have talked about. Krsna Daivajna of Bijanavarikura we have referred to. The Suryadasa commentary on Bijaganita also a Upapattis'.

Vasanavarttika of Nrsimha Daivajna commentry on Bhaskaracharya's Vasanabhasya is available in print. Marici of Munisvara is another discussion of Siddhantasiromani which contains Upapatti. So there are indeed a large number of books which have been published, so if one wants to find out how a particular result was proved, one should first to go up and look the available commentaries, of course if the proof is not there one has to search more or one has to look around more.

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Kṛṣṇa Daivajña on the Importance of *Upapatti*

The following passage from Kṛṣṇa Daivajña's commentary on *Bījagaṇita* brings out the general understanding of the Indian mathematicians that citing any number of favourable instances (even an infinite number of them) where a result seems to hold, does not amount to establishing it as a valid result in mathematics. Only when the result is supported by an *upapatti* or demonstration can the result be accepted as valid.

ननूपपत्त्या विना वर्गयोगो द्विघ्नघातेन युतो हीनो वा युतिवर्गोऽन्तरवर्गो वा भवतीत्येतदेव कथम्? क्वचिद्दर्शनं त्वप्रयोजकम्। अन्यथा चतुर्गुणो राशिघातो युतिवर्गो भवतीत्यपि सुवचम्। तस्यापि क्वचित्तथा दर्शनात्। तथाहि राशी २, २ अनयोर्घातः ४ चतुर्गुणः १६ अयं जातो युतिः ४ वर्गः १६ वा राशी ३,३ अनयोर्घातश्चतुर्गुणः ३६ अयमेव युति ६ वर्गश्च ३६ वा राशी ४, ४ अनयोर्घातः १६ चतुर्गुणः ६४ अयमेव युति ८ वर्गः ६४ इत्यादिषु। तस्मात् क्वचिद्दर्शनम् अप्रयोजकं क्वचिद्व्यभिचारस्यापि संभवात्। अतो वर्गयोगो द्विघ्नघातयुतोनो युतिवर्गोऽन्तरवर्गश्च भवतीत्यत्र युक्तिर्वक्तव्येति चेत् सत्यम्। इयमुपपत्तिरेकवर्णमध्यमाहरणान्ते।

First because so much has been said about the lack of logical regarding such thing about Indian mathematicians, we will be forced to sort of give quotations where the Indian mathematicians do say that proving some results are indeed important. If you want to take a mathematical results seriously a proof is indeed needed. So Krsna Daivajna who is writing at some beginning of his Bijanavarikura he is saying, he is referring to the result (FL).

How can we say a square+ b square to which 2ab is added or subtracted will lead to a+b whole square or a-b whole square is there a way of understanding this. (FL) if we just show some examples of this that is not really good enough, (FL) otherwise 4 times the product of 2 numbers is equal to the square of the sum of these numbers is also sounds like a proper or a valid result. So he gives 3, 4 examples 2*2=4, 4*4=16, this is also= the square of 2+2.

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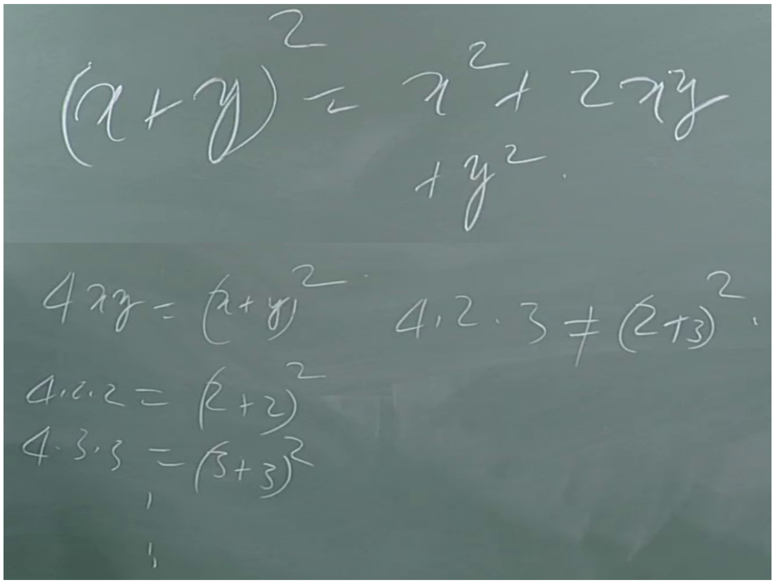

Similarly, 3*3=9 4 times 9=36 this is also the square of 3+3, and again go on 4*4=16 4 times this is 64, so he is saying why do not we say for x y is= x+y whole square even this appears quite alright, because of these examples 4*2*2 is= 2+2 whole square 4*3*3 is= 3+3 whole square, and actually we can give infinite number of examples, and therefore we can conclude that this is correct. (FL) so verifying some result in a certain number of cases is not really of any great use.

Because there might be a contrary since though the moment I go to 2*3 I will find 4*2*3 is != 2+3 whole square, (FL) so this is a counter example or a deviation from what you are, (FL) therefore if one says that one should really give a proof for the fact for the claim that is being made that x+y whole square is= x square + 2xy +y square it simply true that one needs to prove search result. (FL) he will give this proof in the end of the (FL).

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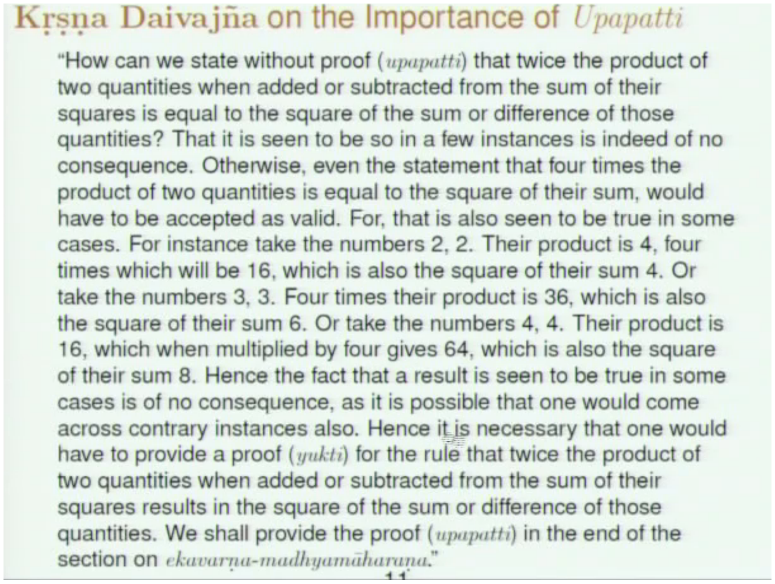

So this is basically the translation of the same passage. So what I am trying to say is that this fact that result in mathematics need to be justified merely giving examples to support them is not enough, in mathematics a certain amount of demonstration is needed is well known think to mathematicians in India.

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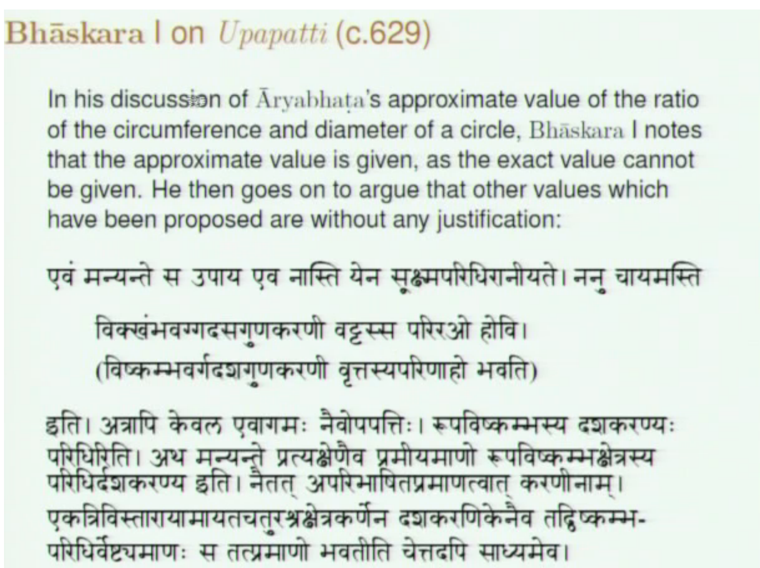

So one of the earliest discussion of nature of Upapattis' occurs in the Bhaskara 1 commentary of Aryabhattiya, he is referring to the discussion of Aryabhatta that the approximate value of ratio of circumference to the diameter is 62832/20000, so then he is saying that approximate value is being given because the exact value cannot be given. Even Bhaskara 1 is saying that exact value of pi cannot be given, then he raises the question (FL).

This is the corresponding prakirta version of that, so the square root of 10 is really the accurate value of the circumference of a circle of diameter 1 is it not true, so why are you saying that exact value of the ratio of circumference to the diameter cannot be given. So then Bhaskara says (FL) so here also what has been given is only a traditional statement no justification has been provided.

(FL) why are you saying that, why do not you measure the circumference and immediately conclude that it is square root of 10 like that this is not possible, (FL) square root of 10 is occurring is and in exact quantity one cannot really determining its exactly, so like that he goes on to give an argument that indeed one has to provide a proof for the fact that certain amount is this ratio of circumference to the diameter.

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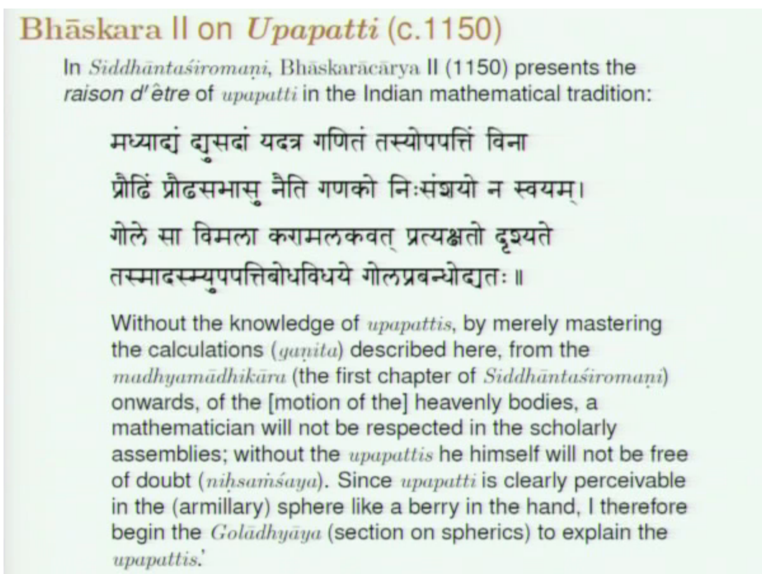

So the fact that the mathematical result is need to be demonstrated is fairly clear in the commentaries, now the book that gives a set of demonstrations how does it view what these demonstrations are supposed to do. So in Siddhantasiromani in the Goladhyaya by the beginning Bhaskara tell us what Upapattis' the word for proofs Ugti, Upapatti are the 2 words used for proofs in Indian mathematics so what this Upapatti are supposed to do.

So this verse (FL) I quoted this in the introductory lecture also that but for whatever that has been discussed in (FL) section of astronomy starting from the madhyamadhikara first chapter of the (FL) section of the astronomy. So whatever is discussed without proofs without Upapatti (FL) a mathematician (FL) he will not be considered as a scholarly mathematician in an assembly of scholarly mathematician, (FL) he will not be free of doubt about the methods and procedures and the results that he is dealing with.

So these are the 2 kinds of handicaps that somebody who does not no proofs will automatically place.

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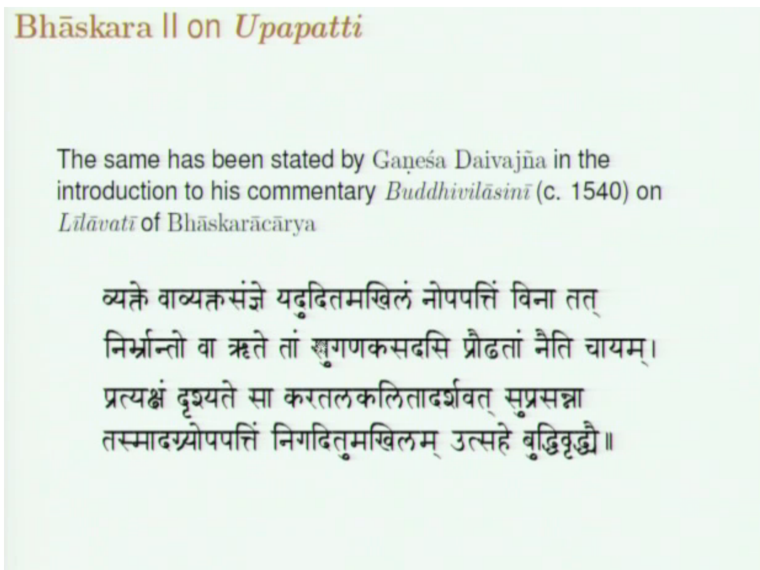

The same thing is repeated by Ganesa Daivajna while starting his commentary on Lilavati the commentary Buddhivilasini, which has a whole lot of Upapattis', (FL) in both (FL) that is both in arithmetic and geometry and in also in algebra whatever he said without Upapatti (FL) that one will not be considered as a well versed mathematician in an assembly of good mathematician (FL) and nor will he be without the confusion about the results and procedure.

So this is the kind of focus that has been put on Upapattis' in Indian mathematics, Upapattis' are important they need to be given.

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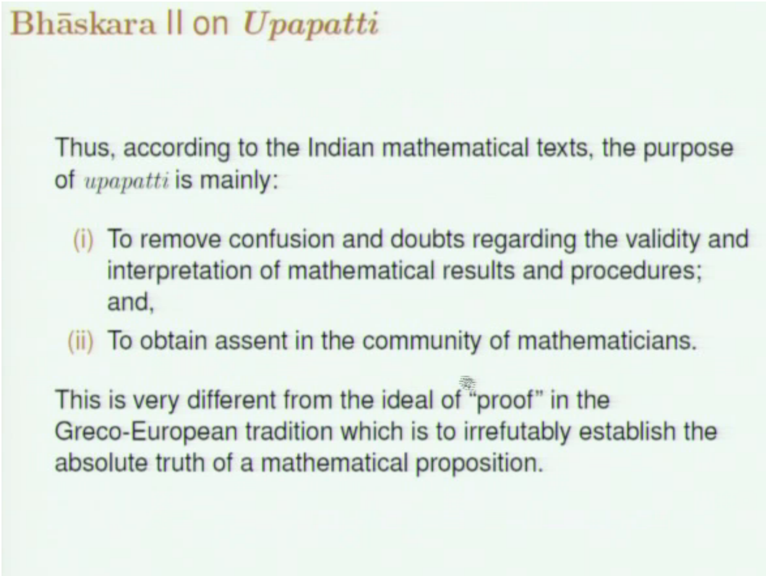

And main purpose is to say that the rest of the mathematicians accept the argument that you are given, so the result becomes a valid result in the community of mathematicians, and it will enable you to be free of how to perform the particular operation or particular result or the particular procedure. So that also a proof will enable without the proof there may be so many conclusions about what that original rule is meaning.

But now this is very, very different I mean if you see the first chapter of the famous commentary of progress on Euclid's elements, you will see what proof is supposed to do, the proof is supposed to do a very unambiguous and irrefutable demonstration of the result stated by the text that it has to once and for all established in an uncontestable manner the truth of the proposition stated in the text. So that is a very different ideal that does not find a central focus in the discussion of the Indian mathematicians.

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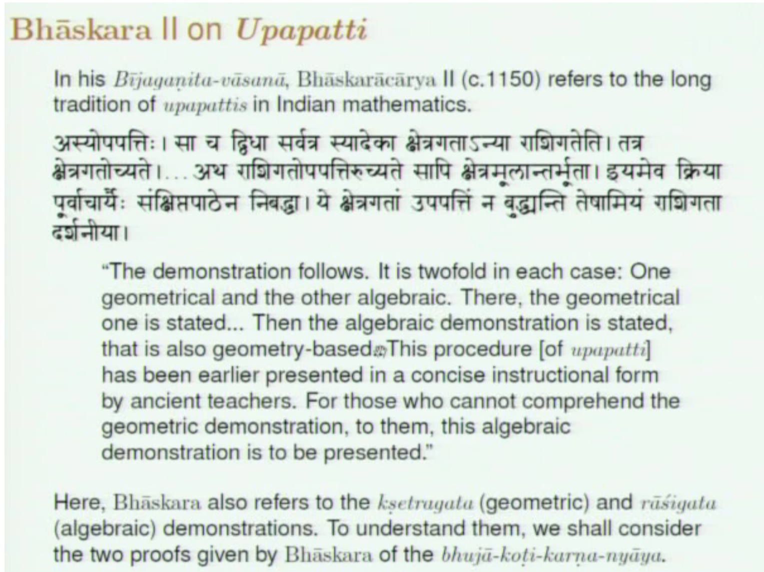

Now how do they actually prove result, so what does Bhaskara say about this proofs, in the Bijaganita-vasana Bhaskara gives a proof of this Pythagoras theorem or the bhuja-koti-karna-nyaya, and he says there are 2 kinds of ways in which you can prove ksetragata and rasigata, ksetragata is geometrical, rashigata is algebraic. (FL) so first I will explain the geometrical, then after some time (FL) so then he will give the algebraic, (FL) that is also based upon geometry only.

(FL) so this Upapatti is something which has been passed down by traditions by the oral tradition of teaching by generation to generation, and of course he says the geometrical proof is somewhat more complex for them the algebraic proof is to be given.

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So let us see what is the geometrical proof and the algebraic proof that the Bhaskara gives for the bhuja-koti-karna-nyaya. So the problem is that in a right angle triangle which sides 15 and 20 what is the hypotenuse? (FL) 15 (FL) is 20, (FL) what is the hypotenuse (FL) so give us also the traditional method of proof or the traditional proof for this traditional method of calculation.

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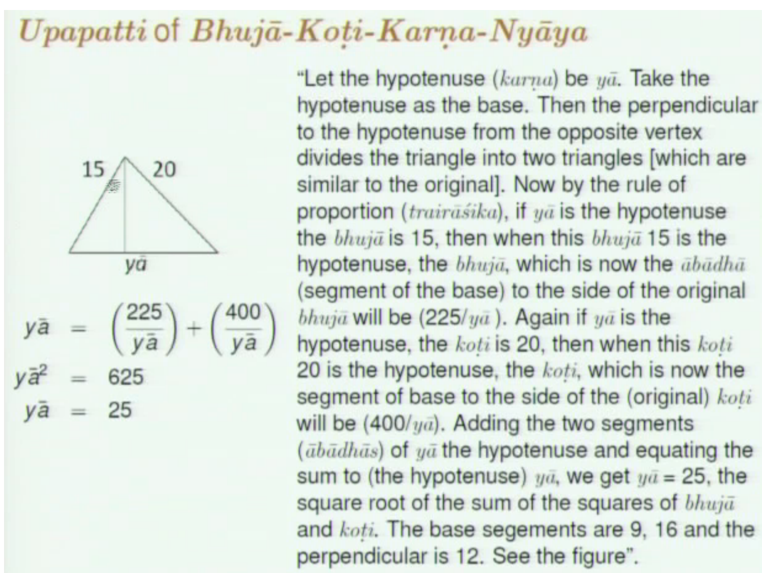

So this is the way Bhaskara is explained let us not going to the details, here it can be read basically all the Bhaskara is doing is drop this perpendicular you want to complete this hypotenuse, the basic this is the right angle of the right angle triangle. So this right angle triangle itself is similar to each of these sub triangles that you obtained, so you use the similar triangles and you obtain both these abadhas side and this side.

So the abadhas this side will turn out to be 225/ya, if this ya is the length of these hypotenuse, the abadhas and this side is 400/ya square of this by this, so the hypotenuse itself is the sum of the 2 and so you will calculate the base to be 25. This is the essence of the geometrical proof given by Bhaskara and the argument goes like this using the trairasika basically the principle of similar triangles.

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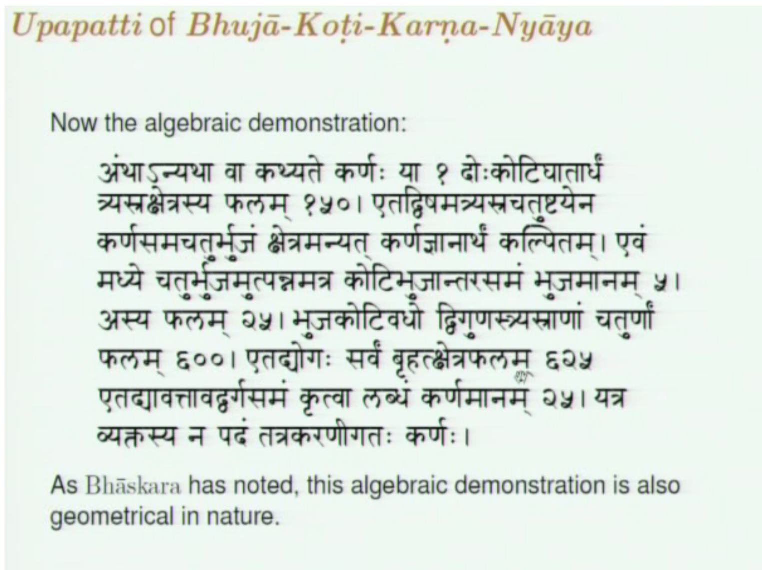

Now then he says let me give the algebraic demonstration, and as he himself said the algebraic demonstration is also based of anything at all you are providing it result in geometry, the algebraic demonstration is also based upon the figures.

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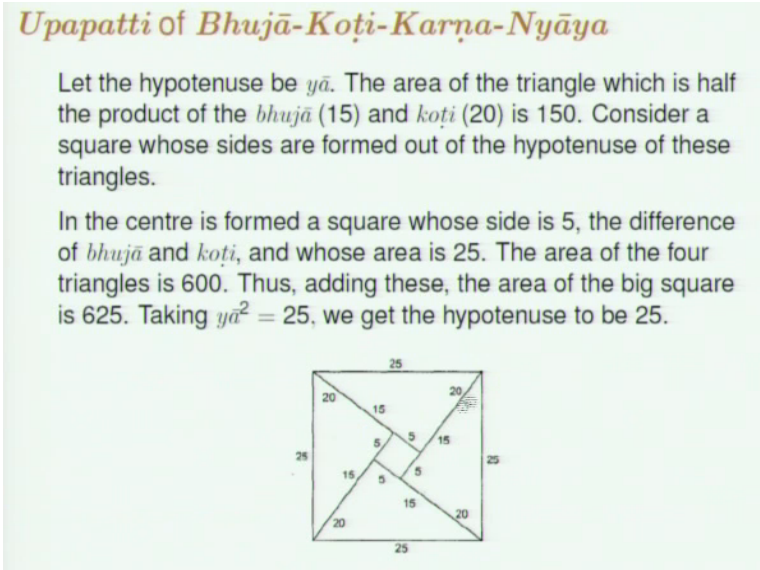

And essentially all that Bhaskara is doing is if this is our right angled triangle, construct a square from the hypotenuse of this right angle triangle, and put the hypotenuse of the right angle triangles in such a way that in the center you are left with another small square.

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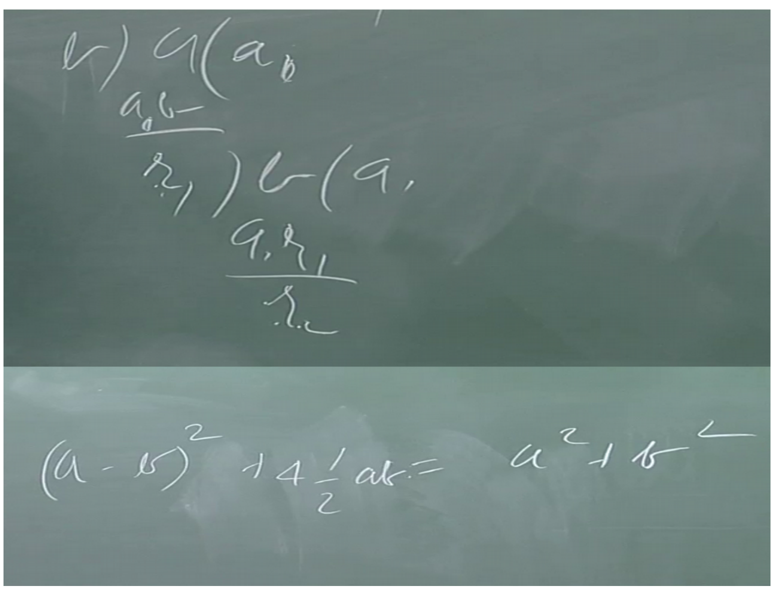

And so essentially what you are doing is you are having this a-b whole square+2 a b is= a square +b square, and this 2ab is actually 4 times of 1/2 ab, so this is essentially the algebra so this is the algebraic relation that is used in proving, here is a-b whole square each of these triangles is 1/2 ab, so you have 4 times 1/2 ab and the sum of a-b whole square +4 times 1/2 ab=a square +b square, therefore this larger square is actually= a square + b square.

So the hypotenuse squared is= sum of the bhuja square and koti square, so it is called algebraic because this algebraic identity is employed in proving this. So from the (FL) time's algebra and geometry have closely intertwined in discovering an algebraic result and even in expressing an algebraic result geometrical result algebra is employed in a very crucial way, and geometrical arguments are also made to discover algebraic identities.

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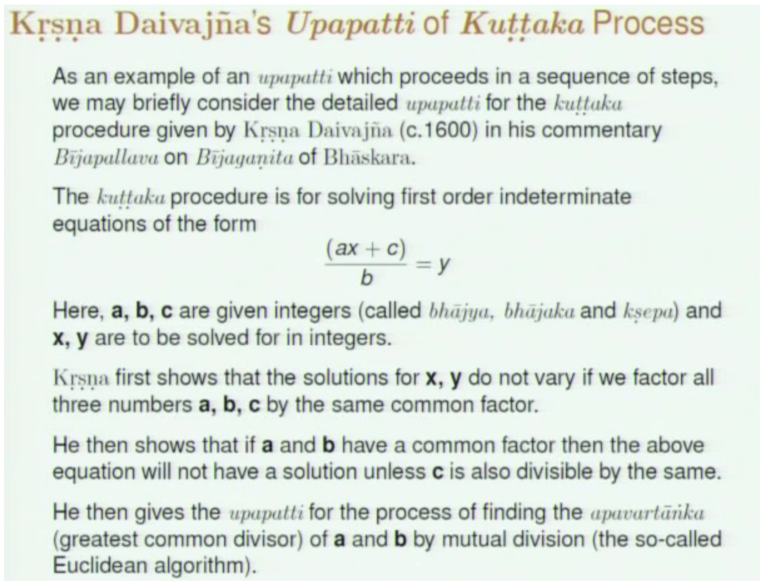

Now let us go to the proof of something else, the in Bhaskara's Bijaganita the kuttaka addhyaya there is a detailed discussion by Krsna Daivajna of the justification of the kuttaka, so how does the Krsna Daivajna justify the kuttaka process. So what is the kuttaka procedure that is for solving the equation a x +c/b= y, a c and b are given integers, x and y are to be determined for integral values that is the kuttaka problem, a is called the bhajya, b is called the bhajaka, c is called the ksepa.

And x is called guna, y is called the labdhi, this is the these are the technical terms for all these quantities in kuttaka. So what does Krsna do? Krsna Daivajna first shows that x and y do not change, if you factor all the three numbers a b c by the same factor, then he shows that if a and b have a common factor then the above equation will not have a solution unless c is also divisible by this same factor, this you already now know that the GCD of the quotient of x and y should divide the ksepa, then only the kuttaka problem has a solution.

Then he gives a proof that the mutual division of a and b that is lead to the greatest common divisor at some stage, so this so-called Euclidean algorithm.

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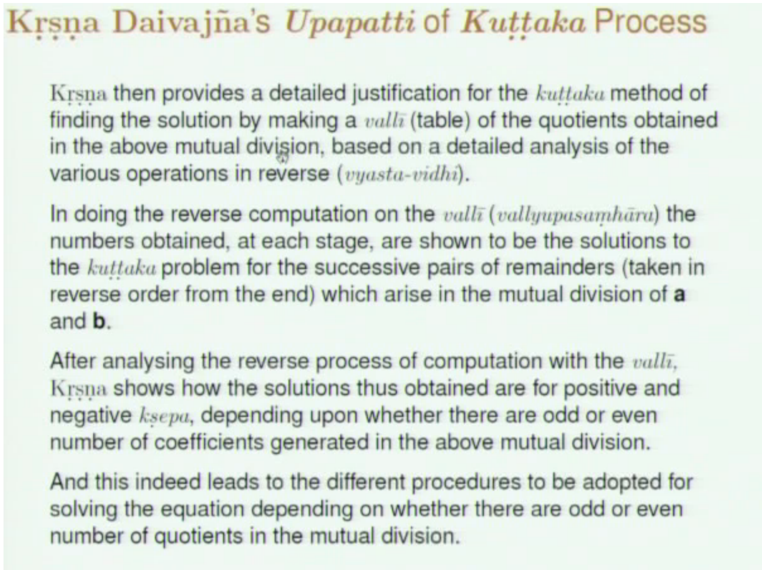

So you can see that there is a sequence of argument going on in Upapatti like the way we do in proving any argument in mathematics that we are familiar with. Krsna then gives a detailed justification of the kuttaka method itself, what is the kuttaka method involved that you make a valli, and that is based upon a detailed analysis of the operations in reverse this is called vyasta-vidhi.

So at each stage what Krsna shows is that the you are obtaining solution for the kuttaka problem for the successive pairs of reminders which appear when you divide a/b, so when you are dividing a/b, so some quotient a0 will appear, a0 b so you will just get the first reminder then you divide r1, so a1 will appear a1 r1, a reminder r2 will appear, so for each of these successive reminders the kuttaka problem that you will appear.

When you go up in the valli you will be solving the problem with the kuttaka problem with the successive reminders that is what is the kuttana, the problem with a, b is converted to problem with r1, r2 then problem with r2, r3 problem with r4, r5, it ultimately comes to very small reminders, so either able to guess this solution or you can work out the solution, so that is what Krsna is explaining.

And then he will also tell you if you have odd number of steps you go in a particular way, if there is even number of steps you go in a particular way. So the proof of each and every one of this is considered in detail.

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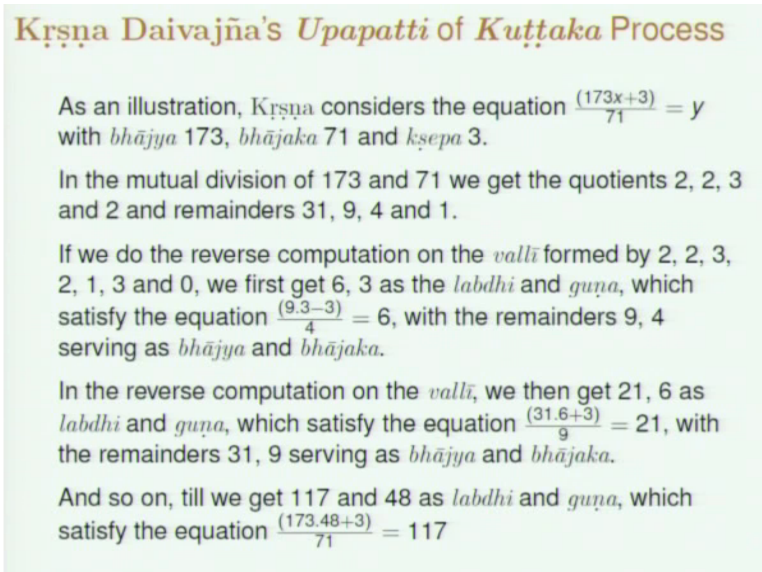

So as an illustration Krsna considers this equation 173 x+3/71, while making the argument the illustration will run parallel, so you will keep an illustration and make the argument. So in the mutual division of 173 and 71 quotients are 2, 2, 3 and 2, the remainders are 31, 9, 4 and 1. So when we do the reverse operation with valli 2, 2, 3, 2, 1 followed by the ksepa is 3, and put a 0 below that right in the end.

So Krsna shows that in the first level you get 6 and 3 as the labdhi and guna in the valli, when you clear the valli and they will satisfy the equation 9*3-3/4= 6, so the quantities involved here 9 and 4 are essentially the 2 reminders which appear here. In the next step you will add a kuttaka problem solution with the 2 reminders 31 and 9, the next guna and labdhi will solve the kuttaka problem with these 2 reminders.

And finally you will get the solution for the kuttaka problem with 173 and 71. So the reverse operation in valli is essentially solutions with the kuttaka problem with the reminders successive reminders that appear here, this is the (FL) of the argument in proving that the kuttaka procedure works. Now he will come to another aspect of proof in Indian mathematics, this is an interesting philosophical issue which seems to be operative here in Indian mathematical tradition.

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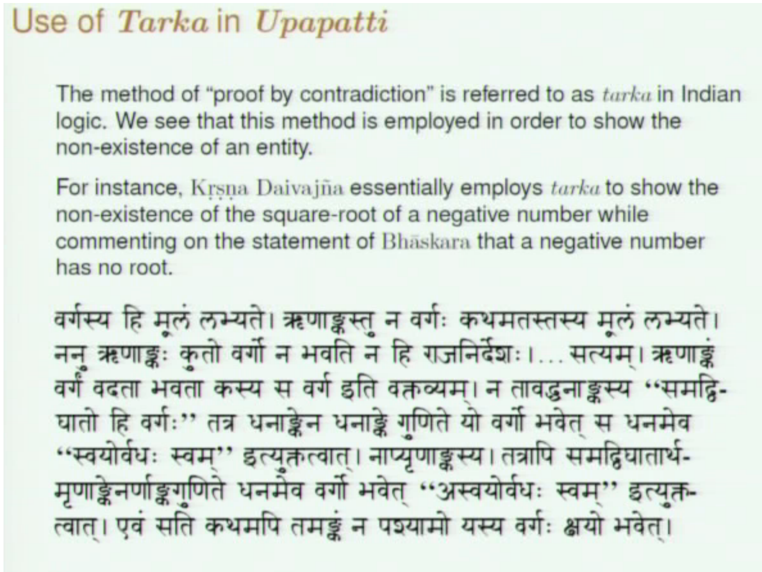

Most of you who would have seen proofs in the standard geometry text might clearly be aware that one of the most commonly met with arguments is what is called the proof by contradiction. What is proof by contradiction? It is also called reduction ad absurdum what is that, so whenever you want to prove a proposition you assume that the opposite of that proposition to be true for the time being.

And then argue and you obtain the result which is in contradiction either with what you have assumed or with any of the results that you have already proved or with anyone of your starting postulate. Therefore, by assuming the contrary of what you want to prove to be true, you arrive at a contradiction and therefore, that the assumption that the contrary of what you wanted to be prove is not justified and therefore, the opposite of that, that is the proposition that you want to be proved has been proved by you.

So this kind of argument is called tarka in Indian logic is nayaya shastra, so this kind of an argument is called tarka, and such arguments do appear in Indian mathematical texts in some simple context. So I have taken the simplest of such arguments to show that a negative number does not have square, so this is the argument of Krsna Daivajna to show that a negative number does not have a square root.

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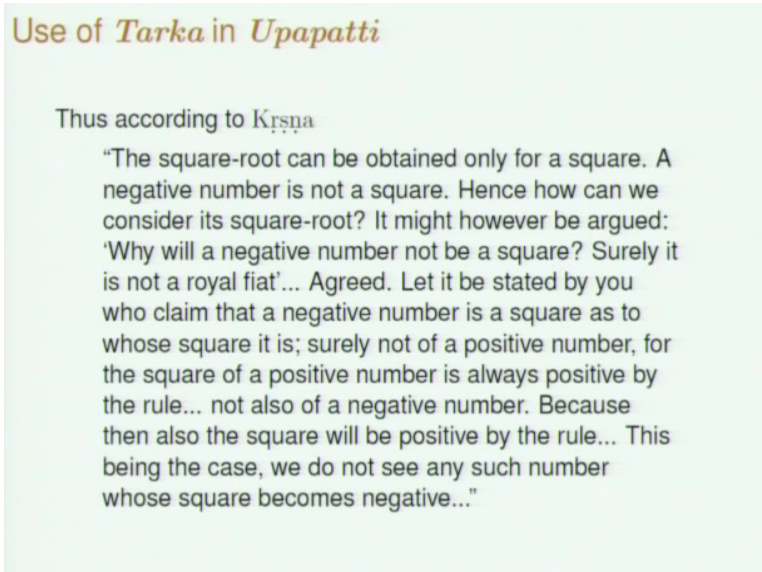

The argument is very simple assume that a positive number is a square root of negative number it cannot be square of that is positive, assume a negative number to be square of a negative number that cannot be its square is positive, so there is any know number which you know which is the square root of a negative number. Therefore, a negative number does not have square, this is the standard argument of proof by contradiction.

And this is employed so (FL) why is it that a negative number is not a square (FL) you were arguing that a negative number is a square you should tell me whose square it is, (FL) it cannot be a square of positive number because of the rule (FL) because square of a positive number is positive, (FL) it cannot be square of a negative number, (FL) they are also negative, negative is positive, (FL) that being the case (FL) so we are not able to see that number whose square is negative, so it is a standard.

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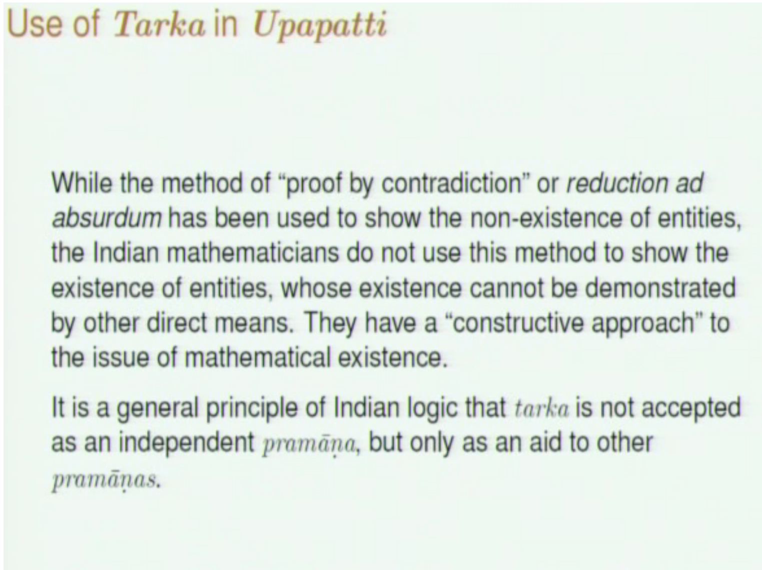

But while this kind of an argument is commonly used you do not see an argument in any of the Indian mathematical texts at least I am not seen much, well you are proving the existence of something (FL) process it would be fairly easy for people to construct a proof by assuming that (FL) process does not lead to a solution, you come up with a contradiction of something that you already know, and in therefore, you say you have proved that the (FL) process leads to a solution.

So to prove the existence of some quantity which is by proof of contradiction is not something that is commonly met with, and I am trying to tell you that this is fairly in tune with these larger principles of Indian logic the way our Indian philosophy. This tarka while it is useful in fact tarka is used in what is called as a sieve in anumana pramana that when you want establish the in variable (FL) you try to refine that (FL) further and further by the use of tarka argument (FL) may not really be giving to that (FL) people who know Indian logic will see that that is the contacts in which tarka is used.

So tarka is used as an auxiliary way of arguing, it is not given as status of an independent pramana what does that mean? That anything that is established by tarka which in principle cannot be established through the other direct means of verification will not be given a state of validly established results. So it is not an independent pramana it is only as an aid to other pramanas.

So basically the Indian mathematician seem to be using reduction ad absurdum kind of arguments if at all they use which are very few for in between mainly to show the non-existence or impossibility of certain things they do not seem to have used it at all, to show the existence of quantities which existence is not possible to be demonstrated by other means, that if you already know what the solution is you do not need reduction ad absurdum.

In the morning question came in to discuss the way Archimedes proved, the argument that the infinite series 1/4+1/4 square etc. adds up to 1/3, this was in specific geometrical contexts in a geometrical construction. So what the Archimedes is showing is let me assume that the sum of all these figures is > that other figure then he will show a contradiction, let me show that the sum of all this figures is< that sum, and then he shows a contradiction.

This is a very, very this is called a double reduction ad absurdum argument that this sum can either be larger nor it be smaller, and therefore, it has to be equal. Whereas you show by successive smaller and difference between the result that you want to prove, and the successive terms of summation becomes smaller and smaller and you are actually going to result that to the result that you are wanting, so that is the kind of proofs that you find in Indian texts.

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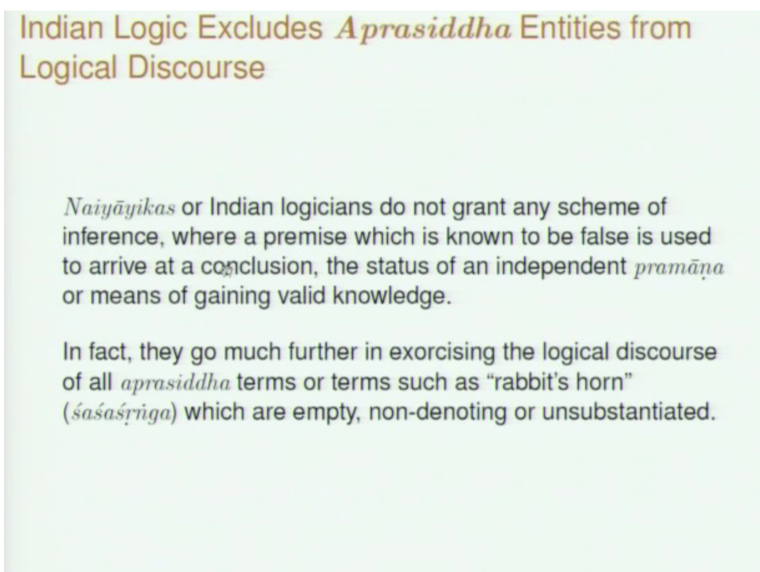

There is in fact an even larger principle in Indian logic it is that you do not grant validity to any scheme of inference, where a premise which is already known to be false is used to arrive at a conclusion, and furthermore you do not allow in any discourse which is logically considered rigorous, a terms which you know are empty, terms which are meaningless like a square circle or a rabbit's horn is the one that is very commonly used sasasrnga.

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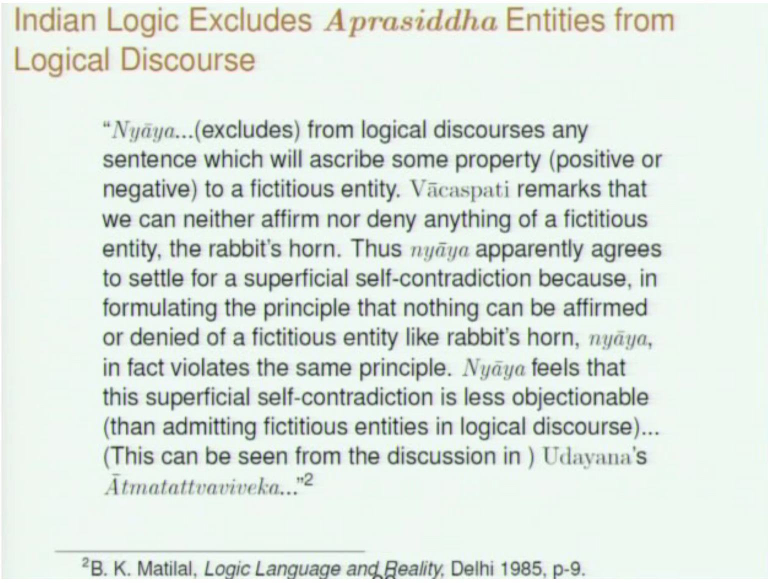

The insistence on not using this undifferentiated or undenotable terms is so deep that one is willing to even accept a contradiction rather than use these terms, so this is the major issue this is a summary, the original work has a detailed discussion which is found in Udayana's Atmatattvaviveka are you can pick up Udayana's Atmatattvaviveka and see this discussion. This this question is very simple.

The nyaya is saying you should not use any statement which uses fictitious entity is not a valid statement, so then the opponent you already made a statement using a fictitious entity, so what are you talking about so he says I am saying something meaningful if you do not want to understand it that is alright but I will live with this contradiction of not using fictitious entities in my discourse, but I will not allow still the fictitious entity to be used.

So we are willing to live with the contradiction that the contradiction is in the statement that rules out the use of these fictitious entities, so that contradiction is alright rather than using these fictitious entities in the argument. Now this issue of contradiction is very special in traditions which base all their arguments on reduction ad absurdum techniques, because what the reduction ad absurdum the moment you find a contradiction the opposite of the premises is true.

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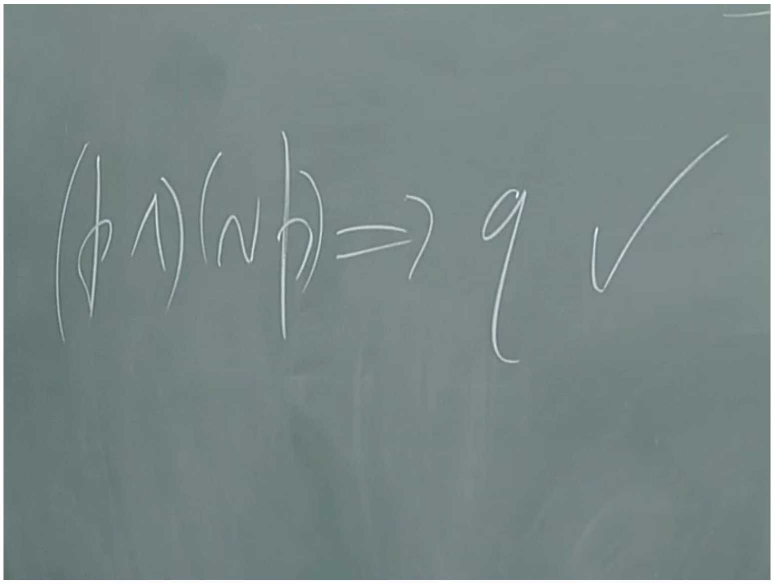

And the moment you are logical system has a contradiction, all the proposition becomes valid in a logical system, where a proposition p and not p are both true, them from p and not p you can imply any q and this is always a universally valid proposition. So this systems of logic which use this kind of reduction ad absurdum to the sort of limit, they are always worried about contradictions in their systems of logic okay.

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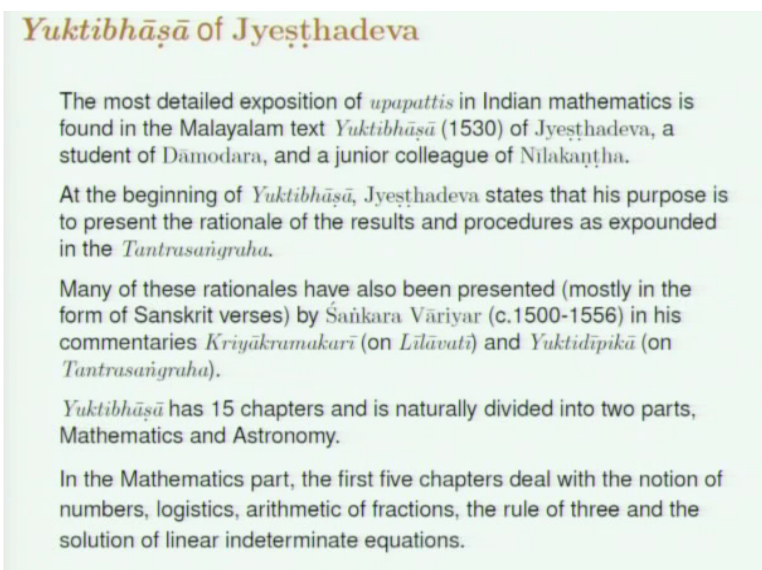

Now with that interlude let us come to Yuktibhasa, the discussion of proofs contained in Yuktibhasa. Today, I will just give **“Professor - student conversation starts”** Yes mam, (()) (38:35) and can you tell us what is a difference between Upapatti and pramana? Pramana is a means of acquiring valid knowledge, so prama is valid knowledge, pramakarana is pramana that what leads you to valid knowledge is pramana.

So the means of valid knowledge are in ordinary understanding observation (FL) argument and generalization from known observation etc. that is called anumana, then some differences then what is called established well established tradition that is agama then different schools of philosophy have their own other definitions of pramana (FL) and (FL) there are other kinds of pramana, this is what is pramana in simple terms.

Now Upapatti is (FL) detailed derivation of that we will leave it to the lecture of professor Ram Subramanian that you are going and arriving at the you have established the result of you are arriving at the result, so that he is I mean a loose sense of QED. **“Professor - student conversation ends.”** So the most detailed exposition over Upapattis’ in Indian mathematics is found in Yuktibhasa by Jyesthadeva.

So Jyesthadeva says that the purpose is to present the rational of the results and procedures outlined in Tantrasangraha, many of these are given in Sanskrit also in the verse of Sankara variyar, Yuktibhasa is 15 chapters, the first 5 mathematics chapters deal with numbers logistics fractions rule of 3 and solutions of indeterminate equation.

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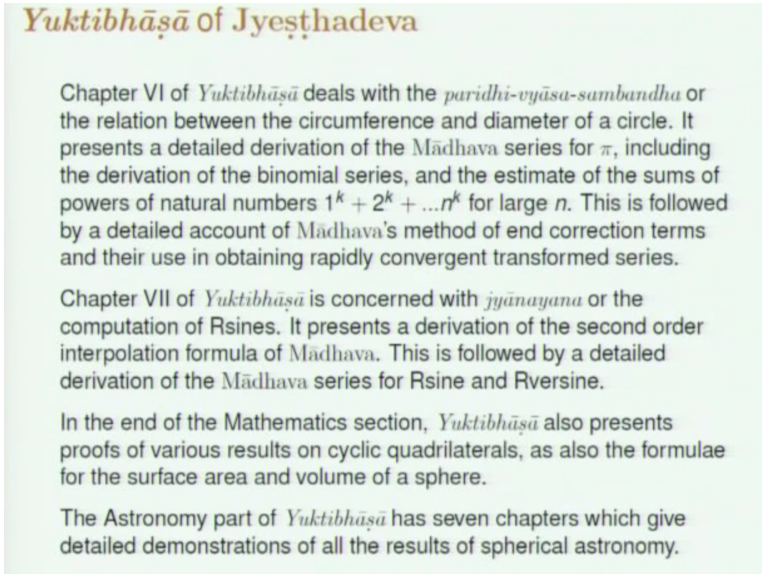

The chapter 6 and 7 of the mathematics part are the crucial ones, chapter 6 deals with the paridhi-vyasa-sambandha, it presents the derivation of Madhava series for pi, derivation of binomial series, the derivation of this estimate of the sum 1 to the power k etc. +n to the power k for large n. And then derivation of the end correction terms, and the transformer series. Chapter 7 deals with jyanayana or computation of Rsines.

It presents a derivation of the second order interpolation formula Madhava I think that will be covered in one of the trigonometry talks, this is followed by the derivation of the Madhava series for Rsine and Rversines. In the end it has a detailed discussion of cyclic quadrilateral and also the proofs of surface area and volume of a sphere. The astronomy part has 7 chapters, which gives detailed demonstration of all the results of spherical astronomy.

Perhaps, you will get a glimpse of the way Indians this sophisticated way the Kerala astronomers work with spherical trigonometry in one of the lectures that will follow.

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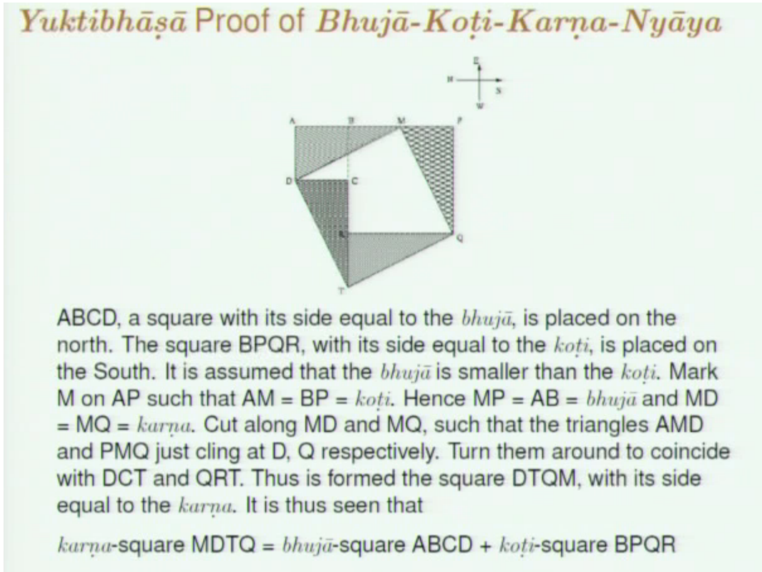

So first is the Yuktibhasa proof of bhuja-koti-karna-nyaya, so we saw 2 proofs of Bhaskara, this is the proof of Yuktibhasa very simple. This is 1 square, BPQR this is the koti square, ABCD is the bhuja square, assume that bhuja square is smaller, now make your AM= koti, so your MP will be= bhuja that is mark this point M such that MP is the same as AB. Then join MP, join MD, construct the square on MD.

Now the constructing the square on MD this procedure is similar to the (FL) procedure for adding 2 squares as it was pointed out in the lecture on (FL). Then how to show that this square this MD square= AM square+ AD square very simple at the point D cut this triangle AMD and place it on DCT, and at the point Q cut this angle MPQ rotate it and place it on QRT, then you have shown that this square= sum of those 2 other squares.

So actually it involves cutting and repasting, so this was another feature of all the Indian mathematical arguments that they could involve moment of the physical objects, cutting of the object. Then even many of the proof say I mentioned Krsna Daivajna proof of -*- =+ by using that let the direction of moving eastwards be taken as positive, direction of moving westwards we taken as negative and with that they will be able to show that +5 -of-3=+8 and things like that.

So even physical arguments of various kinds are acceptable as proofs as long as one is systematic and one is explained. Surprisingly, I should mention that this proof which is available in the available editions of Yuktibhasa is different from the proof Charles Whish has given in his paper saying that this is the proof that I have found in Yuktibhasa, so that still remains a mystery.

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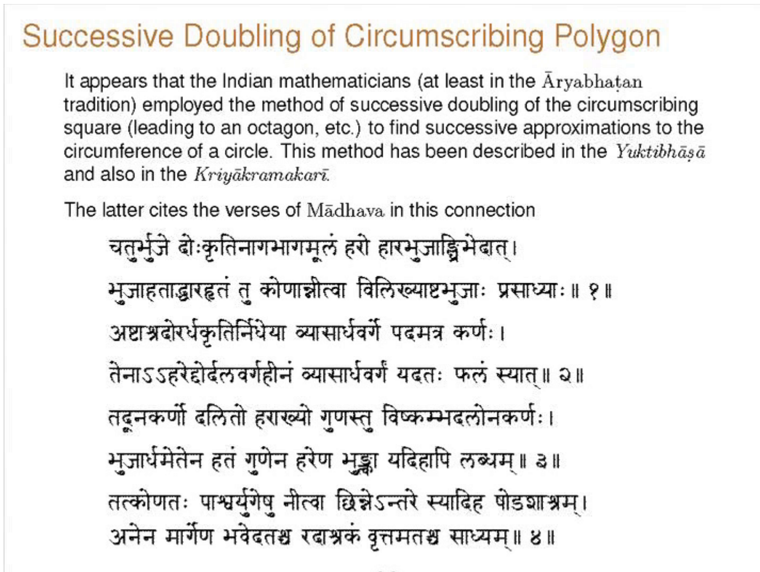

Now there is another complex geometrical proof that you find in Yuktibhasa, this is the method of approximating the circumference of a circle by a square which is circumscribing it, and then dividing the square in such a way such that you have a regular octagon, then dividing it in such a way such that you have a regular polygon of 16 sides. Again Kriyakramakari gives a collection of version due to Madhava which seem to be explaining this method.

So Madhava, who gave the infinite series also has summarized this method which is the classic old method of trying to estimate the circumference of a circle by summing the sides of an inscribed or a circumscribed polygon with large enough number of sides.

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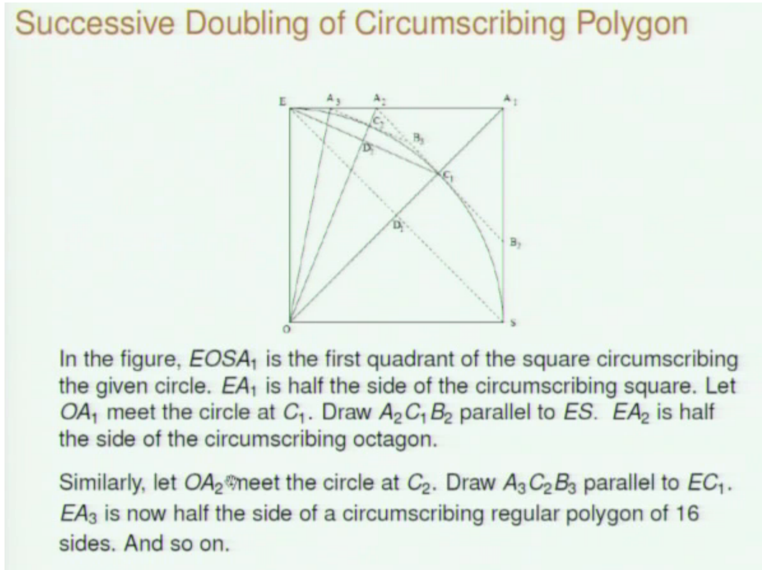

So the method is as given in Yuktibhasa is like this, you take this square which is one fourth of the square which is actually circumscribing the circle, so the side of this particular square is right now we saw this is r, but the bigger square which is circumscribing the circle has side 2r. Now the point at which O A1 meets A1 Yuktibhasa says draw this line A2 C1 B2 which is parallel to ES east south direction, so instead of saying draw the tangent which they do not say this is the way they are explaining.

And then it says okay it goes and meets this side E A1 A2 join O A2, then this O A2 meets at the point C2, now draw A3 C2 B3 which is parallel to C1 E and like that you go on, so this E A2 will be the side of the circumscribing octagon, E A3 will be a side of a circumscribing polygon of 16 sides. So these complex geometrical arguments are indeed dealt with in Yuktibhasa. Now the rest of the argument is using right angle triangles and similar triangles estimating E A2, E A3.

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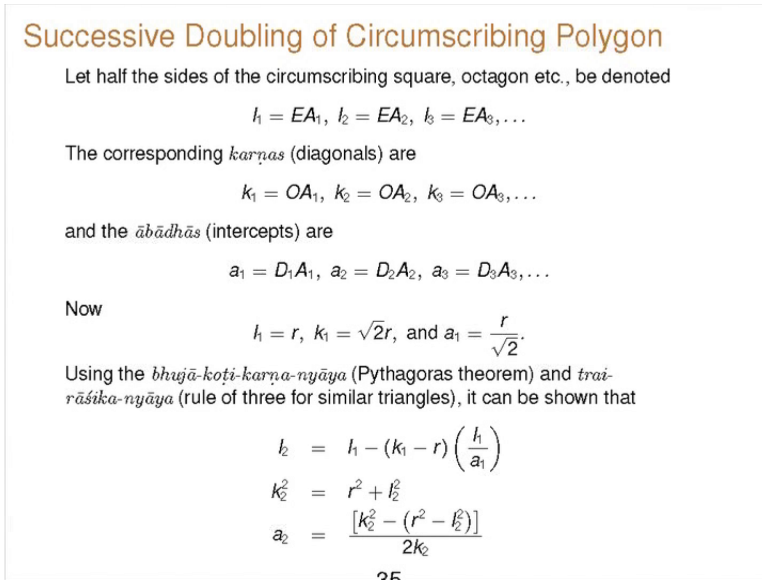

So E A1 is the half of the side of the circumscribing square, E A2 is the half of the side of the circumscribing octagon, octagon will look like this here the octagon this is the part of the octagon it will go like that, so this is half of the side of the circumscribing octagon, E A3 is half the side of the circumscribing 16 sided polygon. The corresponding karna, karna's are O A2, O A1, O A3 so the corresponding karna's are denoted by k1, k2, k3.

The corresponding abadhas are the intercept D1 A1, D2 A2, D3 A3, when you have the first one this D2 A2, D1 A1 will be the abadha that is in this right angle triangle, D2 A2 will be the abadha in this triangle and similarly, D3 is not shown there, so those are the abadhas which will come handy in doing this calculation, they were given by the following. Then you have a serious of recursion relations that you see which are derived from the geometrical argument.

so l2 k2 and a2 can be defined or determined in terms of l1 k1 and first l2 is determined in terms of l1 k, k1 and a1, then k2 is determined then a2 is determined, so l2 k2 a2 is determined, then you go on determine the side of the regular hexagon.

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Successive Doubling of Circumscribing Polygon

In the same way l_{n+1}, k_{n+1} and a_{n+1} can be obtained from l_n, k_n and a_n.

These can be shown equivalent to the recursion relation for l_{n+1} which is the half side of the circumscribing polygon of 2^{n+1} sides:

$$l_{n+1} = \left(\frac{r}{l_n}\right)\left[\left(r^2 + l_n^2\right)^{\frac{1}{2}} - r\right]$$

We of course have the initial value $l_1 = r$.

This leads to $l_2 = (\sqrt{2} - 1)\, r$ and so on.

Kriyākramakarī notes that

> एवं यावदभीष्टं सूक्ष्मतामापादयितुं शक्यम्।
>
> Thus, one can obtain an approximation (to the circumference of the circle) to any desired level of accuracy.

So given ln kn an, ln+1, kn+1, an+1 can be determined, and in fact if you study the process the recursion relation seems to be something like this, and initial value l1 r itself half the side of circumscribing square, l2 will be root 2-1*r and so on. So you keep on using this relation you will obtain half the side of circumscribing polygon of large and larger side, and from that you find out the circumference of the polygon take it as an approximation to the circumference of this square.

And Kriyakramakari says (FL) you can go if you are like (FL) you have patience you can calculate pi 2 16 decimal places there is really no problem.

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References

1. *Līlāvatī* of Bhāskara with comm. *Buddhivilāsinī* of Gaṇeśa Daivajña, Ed Apte, 2 Volumes, Pune 1938.
2. *Bījagaṇita* of Bhāskara with comm. *Bījapallava* of Kṛṣṇa Daivajña, Ed. Radhakrishna Sastri, Saraswati Mahal, Thanjavur 1958.
3. *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (in Malayalam), *Gaṇitādhyāya*, Ed., with Notes in Malayalam, by Ramavarma Thampuran and A. R. Akhileswara Aiyer, Trichur 1948.
4. *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (in Malayalam), Ed. with Tr. by K. V. Sarma with Explanatory Notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, 2 Volumes, Hindustan Book Agency, Delhi 2008.
5. *Kriyākramakarī* of Śaṅkara Vāriyar on *Līlāvatī* of Bhāskarācārya II: Ed. by K. V. Sarma, Hoshiarpur 1975.

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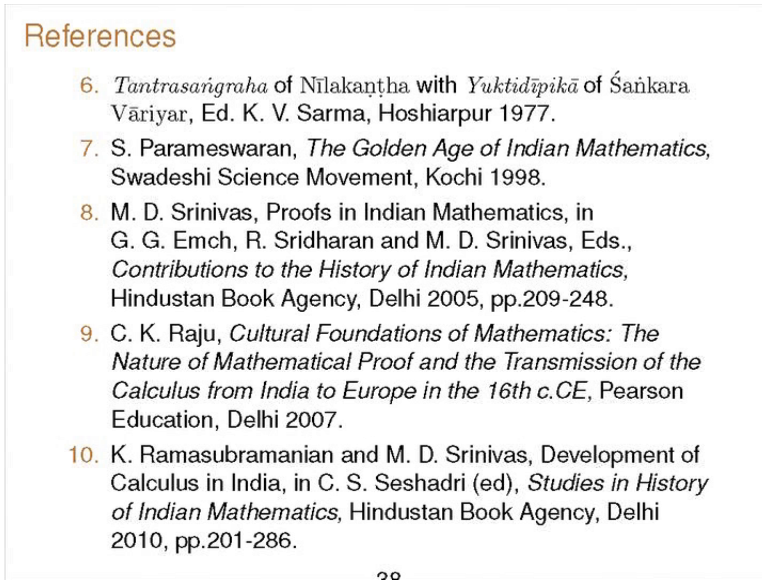

So I think for today, I have covered the tradition of proofs in Indian mathematics from more ancient times. In the next talk, I will discuss the proofs as found in Yuktibhasa of the results of Madhava that will be tomorrow on Monday, thank you.