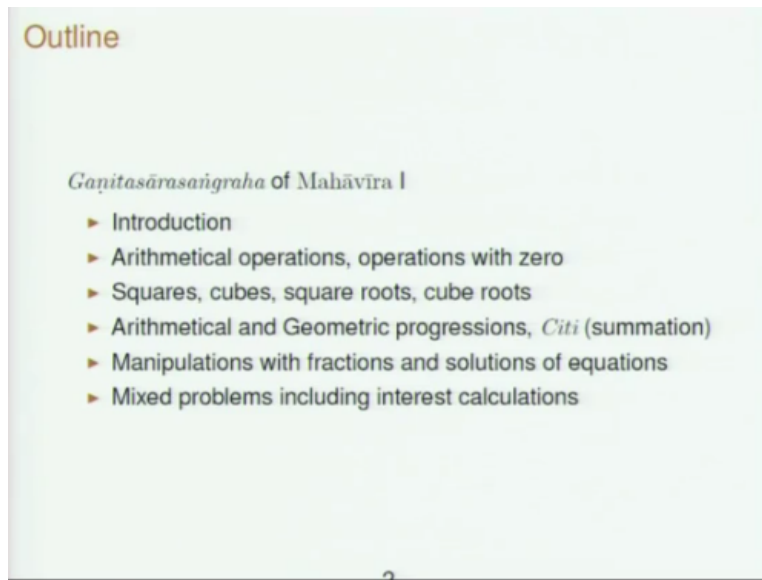


**Mathematics in India: From Vedic Period to Modern Times**  
**Prof. M.S. Sriram**  
**University of Madras-Chennai**

**Lecture-15**  
**Mahaviras Ganitasarasangraha**

Okay, so today I will be talking about Mahaviras Ganitasarasangraha is the outline is something like this.

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First the introduction then the arithmetical operations and operation with zero, squares, cubes, square roots, cube roots, arithmetical and geometric progressions citi summation, manipulations with fractions and solutions of equations and mixed problems including interest calculations. So, this is the outline of the first lecture on Mahaviras Ganitasarasangraha.

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## Mahāvīra's *Gaṇitasāraṅgraha*

Mahāvīra was a Digambara Jain who lived in the later part of the rule of Amoghavarsha Nripatunga (815-877 CE), a great king of the Raṣṭrakūṭa dynasty, ruling over north Karnataka, parts of Andhra, Maharashtra and other parts of India also. Mahāvīra wrote an extensive Sanskrit treatise called '*Gaṇitasāraṅgraha*' (Compendium of the essence of Mathematics) about 850 CE. *Pāṭiganīta* of Śrīdhara (750 CE) is another important work of the same era. I will not be covering that separately, as Mahāvīra's work includes most of the material of *Pāṭiganīta*, as also older works like *Brāhmasphuṭasiddhānta* of Brahmagupta. It also introduces several new topics. It is written in the style of a textbook and provides a rich source of information on ancient Indian mathematics.

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So, mahavira was a Digambar Jain who lived in the later part of the rule of (FL) 815 to 877 CE you know a king who lived and who ruled for a long period a great king of (FL) dynasty ruling over north Karnataka parts of Andhra, Maharashtra and other parts of India also. Mahavira wrote an extensive Sanskrit treatise called (FL), so, whose the I mean the literal translation is compendium of the essence of mathematics about 850 CE.

(FL) which is around 750 Christian era is another important work of the same era but I will not be covering that separately as mahavira's work will include most of the results in (FL) as well as another work by him called (FL) it also introduces several new topics and it is written in the style of a textbook and provides a rich source of information on ancient Indian mathematics what is very significant it is an earlier work we consider Aryabhata.

Aryabhatiya and Brahmapuṭhasiddhānta there mainly works on astronomy and mathematics for the part of it (FL) in Aryabhatiya, so one about 33 verses also and in (FL) though very major things and mathematics are being done it is a small part of the treatise as such. So, that is understandable because much of the development of mathematics in the earlier times both you know in the Indian tradition and Greek tradition and many other traditions went hand in hand with astronomy.

Because astronomy was the exact sciences and they required several tools you know 2 excess things in the mathematical manner to for the precise description, so it is not surprising that you know many of the mathematical developments are very closely associated with astronomy, if for an trigonometry, trigonometry was basically developed in the context of astronomy and later you know it is a concept of instantaneous velocity all these things was developed in the context of astronomy.

So, this astronomy and mathematics went hand in hand and in the in India also lot of works are containing both and typically mathematics will the part of the any some treatise and astronomy. Now Mahavira's (FL) is one of the important works first work to be entirely devoted to mathematics he does not talks about astronomy at all I mean there maybe some application of the result in his book on astronomy.

But he does did if astronomy it not it prime concern mathematics is the prime concern and you know that is what it develops and he develops in a very very you know elaborate manner is a very elaborate work it is can be stay u straight away could have been used must have use as the textbook at that time it is very large work more than 1000 verses and lots of examples of various kinds and any mathematical operation any of this method new method he will have lot of examples you know illustrating various points you know. So, that why it is an extremely important work.

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In Chapter 1 as terminology, Mahāvīra waxes eloquent on the use of mathematics in verses 9-16.

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## Importance of Mathematics

Verses 9-16:

लौकिके वैदिके वापि तथा सामायिकेऽपि यः ।  
व्यापारस्तत्र सर्वत्र सङ्ख्यानमुपयुज्यते ॥ ९ ॥  
कामतन्त्रेऽर्थशास्त्रे च गान्धर्वे नाटकेऽपि वा ।  
सूत्रशास्त्रे तथा वैद्ये वास्तुविद्यादिवस्तुषु ॥ १० ॥  
छन्दोऽलङ्कारकाव्येषु तर्कव्याकरणादिषु ।  
कलागुणेषु सर्वेषु प्रस्तुतं गणितं परम् ॥ ११ ॥  
सूर्यादिग्रहचारेषु ग्रहणे ग्रहसंयुतौ ।  
त्रिप्रश्ने चन्द्रवृत्तौ च सर्वत्राङ्गीकृतं हि तत् ॥ १२ ॥

(FL) he already have this recent extensively about the gives the mathematics in (FL) so, in astronomy naturally it is I have told let the most mathematics developed in this context were not surprisingly lot of mathematics in there (FL) the problem of direction, time and space you know which is important chapter of astronomy in all Indian text (FL).

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## Importance of Mathematics

Verses 9-16:

द्वीपसागरशैलानां सङ्ख्याव्यासपरिक्षिपः ।  
भवनव्यन्तरज्योतिर्लोककल्पाधिवासिनाम् ॥ १३ ॥  
नारकाणां च सर्वेषां श्रेणीबन्धेन्द्रकोत्कराः ।  
प्रकीर्णकप्रमाणाद्या बुध्यन्ते गणितेन ते ॥ १४ ॥  
प्राणिनां तत्र संस्थानमायूरष्टगुणादयः ।  
यात्राद्यास्संहिताद्याश्च सर्वे ते गणिताश्रयाः ॥ १५ ॥  
बहर्भिर्विप्रलापैः किं त्रैलोको सचराचरे ।  
यत्किञ्चिद्वस्तु तत्सर्वं गणितेन विना न हि ॥ १६ ॥

(FL), so nothing happen without it.

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## Importance of Mathematics

"In all those transactions which relate to worldly Vedic or (other) similarly religious affairs, calculation is of use. In the science of love, in the science of wealth, in music and in the drama, in the art of cooking, and similarity in medicine and in things like the knowledge of architecture. In prosody, in poetics and poetry, in logic and grammar and such other things and in relation to all that constitutes the peculiar value of (all) the various arts, the science of computation is held in high esteem. In relation to the movements of the Sun and other heavenly bodies, in connection with eclipses and conjunction of planets and in connection with *tripraśna* (diurnal problems) and the course of the Moon indeed in all these (connections) it is utilised. The number, the diameter and the perimeter of islands, oceans and mountains; the extensive dimensions of the rows of habitation and halls belonging to the inhabitants of the (earthly) world.

You must all have noticed that you know the verses themselves or somewhat you know as have an easy flow compare (FL) and all that he is actually very great poet also and verses are extremely nice and if one more Sanskrit one will be able to appreciate very well. So, in all those transactions which will relate to only vedic or other similarly religious affairs, calculation is of use.

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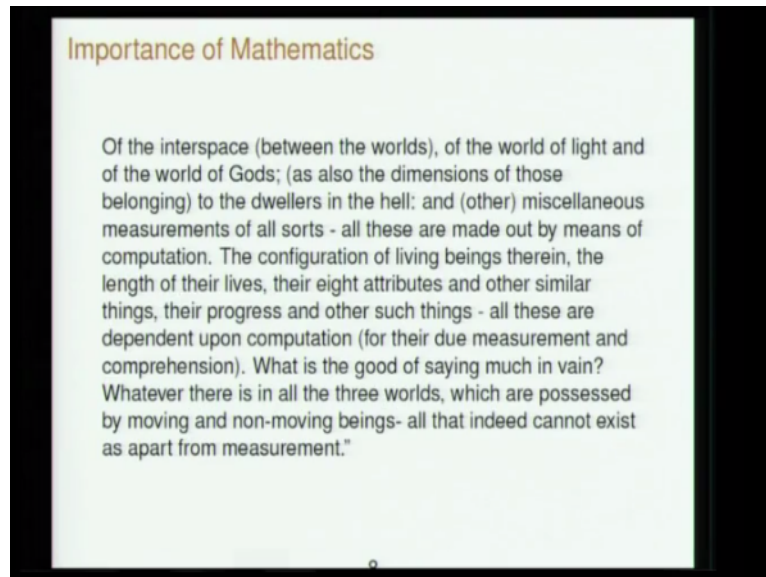
## Importance of Mathematics

"In all those transactions which relate to worldly Vedic or (other) similarly religious affairs, calculation is of use. In the science of love, in the science of wealth, in music and in the drama; in the art of cooking, and similarity in medicine and in things like the knowledge of architecture. In prosody, in poetics and poetry, in logic and grammar and such other things and in relation to all that constitutes the peculiar value of (all) the various arts, the science of computation is held in high esteem. In relation to the movements of the Sun and other heavenly bodies, in connection with eclipses and conjunction of planets and in connection with *tripraśna* (diurnal problems) and the course of the Moon indeed in all these (connections) it is utilised. The number, the diameter and the perimeter of islands, oceans and mountains; the extensive dimensions of the rows of habitation and halls belonging to the inhabitants of the (earthly) world.

In the science of love, in the science of wealth in music, in drama, is a art of cooking, prosody, poetics, poetry all these things the science of computation is held in high esteem relation a movements of sun, moon other heavenly bodies **diurnal** diurnal problems the course of moon the number diameter all the numbers associated with islands oceans, mountains and we already

see that revolution numbers and all that you know over various things can be express. And of course later many more interesting aspects of these mathematics will come in this revolution numbers. So, all these numbers are important and mathematics these to them.

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Of the inter space between the worlds of the world of light and the world of gods to the dwellers in hell it does not for forget the hell dwellers miscellaneous measurements of all sorts. All these are made out by computation the configuration of living beings, length of their lives, eight attributes similar things, their progress and such other things. All these are dependent upon computation what is the good of saying much in vain?. Whatever there is in all the three worlds which are possessed by moving and non-moving beings all that indeed cannot exist as a part from measurement.

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### Arithmetical Operations

He talks of 8 arithmetical operations.

- (i) *Guṇakāra* or *Pratyutpanna*, their multiplication
- (ii) *Bhāgahāra*; Division
- (iii) *Kṛti*; Squaring
- (iv) *Vargamūla*; Square root
- (v) *Ghana*; Cubing
- (vi) *Ghanamūla*; Cube root
- (vii) *Citi* (or *Sanikalita*); summation and
- (viii) *Vyutkalita* (or *Śeṣa*); subtraction of a part of a series, from the whole series.

So, he talks of a 8 arithmetical operations, (FL) then (FL) summation (FL) subtraction of a part of a series from the whole series. It will not be dealing you know with whatever has been done into much detail.

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### Operations with Zero

In particular, consider his understanding of the Operations with zero:  
Verse 49:

ताडितः खेन राशिः खं सोऽविकारी हूतो युतः ।  
हीनोऽपि खवधादिः खं योगे कं योज्यरूपकम् ॥ ४९ ॥

"A number multiplied by zero is zero, and that (number) remains unchanged when it is divided by, combined with (or) diminished by zero. Multiplication and other operations in relation to zero (give rise to) zero; and in the operation of additions the zero becomes the same as what is added to it."

So, for any number

$$a \times 0 = 0 \times a = 0, a \div 0 = 0, 0 \div a = 0,$$

$$a + 0 = 0 + a = a, a - 0 = a.$$

Obviously,  $a \div 0 = 0$  is wrong. Bhāskara call this *khahara* and assigns the value, infinity to it.

So, do not get you know bold expects something straightly more you know **mo** newer things okay see in particular consider is understanding of the operation be 0 one at the first time square the operations at be 0 has been describe (FL) a number multiplied by 0 is 0 and that number remains unchanged when it is divided by combined with (or) diminished by 0 multiplication other operations in relation to 0, give rise to 0. And in the operation of additions the zero becomes the same as what is added to it.



So, for any number  $a \neq 0$  is  $0 \cdot a$  is equal to 0,  $a/0$  is 0,  $0/a$  is 0,  $a+0$  is equal to  $0+a$  is equal to  $a$  and the does not remains unchanged  $a-0$  is  $a$ , so obviously  $a/0$  is the wrong so, when the first time it is happening so, some mistakes are bound to arise because the concepts will become clearer. And very crystallized later only in any mathematical tradition baskara called this khahara operations and assigns the value infinity to it. I think baskara was one of the first to understand the significance of this infinity.

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**Operations with Positive and Negative Quantities**

Verse 50-52:

ऋणयोर्धनयोर्घाति भजने च फलं धनम् ।  
 ऋणं धनर्णयोस्तु स्यात् स्वर्णयोर्विवरं युतौ ॥ ५० ॥  
 ऋणयोर्धनयोर्घाति यथासङ्गमृणं धनम् ।  
 शोध्यं धनमृणं राशेः ऋणं शोध्यं धनं भवेत् ॥ ५१ ॥  
 धनं धनर्णयोर्वर्गौ मूले स्वर्णं तयोः क्रमात् ।  
 ऋणं स्वरूपतोऽवर्गौ यतस्तस्मान्न तत्पदम् ॥ ५२ ॥

"In multiplying as well as dividing two negative (or) two positive (quantities, one by the other), the result is a positive (quantity). But it is a negative quantity in relation to two (quantities), one (of which) is positive and the other negative. In adding a positive and negative quantity, the result is their difference.

And similarly it talks about the operations with positive and negative quantities (FL) in multiplying as well as dividing two negative (or) two positive (quantities, one by the other), the result is a positive (quantity), but it is a negative quantity in relation to two quantities one (of which) is positive and the other negative. In adding a positive and negative quantity the result is difference, so this is understanding you know.

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### Positive and Negative Quantities

The addition of two negative (quantities) or of two positive (quantities give rise to) a negative or positive (quantity) in order. A positive (quantity) which has to be subtracted from a (given) number becomes negative, and a negative (quantity) which has to be (so) subtracted becomes positive.

The square of a positive as well as of a negative (quantity) is positive; and the square root of those (square quantities) are positive and negative in order. As in the nature of things a negative (quantity) is not a square (quantity), it has therefore no square root."

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Already very clear understanding of positive and negative numbers the addition of according to them the addition of two negative (quantities) or of two positive (quantities give rise to ) a negative or positive (quantity) in order. So,  $+$ ,  $+$ ,  $+$  is  $a+$ ,  $-$ ,  $-$  is  $-a$  a positive quantity which has to be subtracted from a given number become negative. So, clearly a okay  $-$  of  $+a$  will become  $a+b$  is  $a-b$ .

And then negative quantity which has to be (so) subtracted becomes positive. So, that is  $a-$  of  $-b$  is  $a+b$  and a square of a positive as well as the negative quantities positive and a square root of those square root of those (square quantities) are positive and negative in order. As in the nature of things a negative(quantity) is not a square (quantity), it has therefore no square root.

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## Positive and Negative Quantities and Notational Places

$$(+a) \times (+b) = (-a) \times (-b) = (+ab); (+a) \times (-b) = (-a) \times (+b) = -(ab).$$

$$a + (-b) = (a \sim b) \text{ [means } a - b \text{ if } a > b \text{ and } -(b - a) \text{ if } b > a].$$

$$(+a) + (+b) = +(a + b), \quad (-a) + (-b) = -(a + b).$$

$$b - a = b + (-a); a - (-b) = a + b$$

$$(+a)^2 = (-a)^2 = +a^2; \sqrt{+a^2} = (+a) \text{ or } (-a).$$

A negative number  $(-b)$  has no square root, because a square is always positive.

In Verses 63-68, names of rotational places:

1-*Eka*, ... 10 - *Daśa*, ...

10<sup>23</sup>: *Mahākṣobha*.

Chapter 2: elaborates on the arithmetical operations (*parikarma*) and simplification of various kinds.

So, what is saying is  $+a \times +b$  is  $-a \times -b$  so, is  $+$  of  $ab$  then you have to multiply naturally and  $a \times -b$  is  $-ab$  and  $a + -b$  is a difference  $b$  which means  $a - b$  if  $a$  is greater than  $b$  and  $-b - a$  is  $b$  greater than  $a$   $+a + +b$  is equal to  $+$  of  $a + b$  like that  $b - a$  is  $b + -a$   $a - \text{of} -b$  is  $a + b$  and  $+a$  whole square is  $-a$  whole square is  $+a$  square and square root of  $+a$  square is  $+a$  or  $-a$  that also he clearly recognizes a negative has  $-b$  have has no square root, because a square is always positive.

See these all look somewhat in a very trivial to us already we all start with that you know in your high school onwards this operations be  $+$  and  $-$  but it took a longtime for this to be digested okay. In fact there one historian of mathematics from the west world come and you know he had given at giving the talk on this use of this negative quantities and all that. So, he was saying that the using of negative quantities in a people were very uncomfortable with it even in 19<sup>th</sup> century.

So, I mean it is not what we take for granted many things you know it is not that trivial you see when it started because something you know some negative dimension see is there and then how do define it how to define operations with it, so that is very important. And of course says you know a negative quantity does not have a square root because if you take a positive quantity it is square is positive.

If you take a negative quantity it is square is again positive, so both positive, negative quantities have positive squares, so how can you have a negative square. So, now of course we are saying

that you know there is a -1 square of -1 we denote it at I and use it in extensively but at depends upon what we are using see what we are dealing with, so we are using only real quantities certainly it is very hard to imagine a quantity with a negative square yeah.

So, and then in a in the history verses later verses see please you know in this this verses means it does not mean 63-68 it is actually is various chapters are there and each of them verses are number like this. So, this is not verse number in the text some chapter only chapter that is all you know, so you should be remember that, so he gives the names of notational it is not notational places up to (FL)like that.

So, I do not have to go to all that you have already heard about in a how various powers of 10 were considered you know from vedic times in India then chapter 2elaborates on the arithmetical operations parikarma and simplification of various kinds.

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**Multiplication**

1. Multiplication: "The multiplicand and the multiplier are placed one below the other in the manner of the hinges of a door" - *Kapāṭasandhi*.

$$\begin{array}{r} (a_n a_{n-1} \dots a_1) \\ (b_m \dots b_1) \leftarrow \text{moved} \end{array}$$

We can use:  $(a \times b) \times (c \times d) = \left(\frac{a \times b}{a}\right) \times (a \times cd)$

or  $= ((a \times b) \times c) \times \left(\frac{c \times d}{c}\right)$  etc.,

Necklace of *Narapāla*:  $12345679 \times 9 = 111111111$

Royal necklace  $142857143 \times 7 = 100000001$

So, multiplication all this things you know he earlier was always deal with various you know the simplest operations they all are you know discuss in great detail. Because multiplication, addition, etc. are very important and multiplication it can be done in different ways obviously people you know who are very, very would do like a shakuthala devi you see who could compute very you know product two huge amount.

She would have some method in you see so, the some very tricky distinction will be methods will be there interesting methods.

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**Multiplication**

1. Multiplication: "The multiplicand and the multiplier are placed one below the other in the manner of the hinges of a door" - *Kapāṭasandhi*.

$$\begin{array}{r} (a_n a_{n-1} \dots a_1) \\ (b_m \dots b_1) \leftarrow \text{moved} \end{array}$$

We can use:  $(a \times b) \times (c \times d) = \left(\frac{a \times b}{a}\right) \times (a \times cd)$

or  $= ((a \times b) \times c) \times \left(\frac{c \times d}{c}\right)$  etc.,

Necklace of *Narapālā*:  $12345679 \times 9 = 111111111$

Royal necklace  $142857143 \times 7 = 1000000001$

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So, there many innumerable method but the common methods are what are discussed in the books. So, it is called (FL) so, this you write the number like this and other the other multiplier and a multiplicand and you keep multiplying each of the digits of the one by the other and keep moving. So, I do not want to explain these things is very simple then  $a*b *c*d +a*b/a*a*cd$  I mean in some applications some multiplications particular examples. It may be convenient to divide by something and then afterwards multiplied by the same thing.

So, that is all it means. So similarly and then he gives some interesting products like  $123456789*9$  111111111 is called (FL) so, similarly  $142857143*7$  is 1 into all the various zeroes into 1 etc... in fact one can see if you read brook work you see the go through the work a really had a very good sense a humor also you know. So, all these things it does things many things through attract students.

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**Division, Squaring, Square root**

2. Division =  $\frac{\text{Dividend}}{\text{Divisor}}$  from left to right. Remove common factors.

3. Squaring:

$$a^2 = (a - b)(a + b) + b^2$$

$$n^2 = 1 + 3 + 5 + \dots + (2n - 1)$$

(Arithmetic Progression with 1 as the first term, 2 as the common difference and  $n$  terms.)

$$(a + b + c + \dots)^2 = a^2 + b^2 + c^2 + \dots + 2ab + 2ac + \dots$$

$$\begin{aligned} (a_n a_{n-1} \dots a_2 a_1)^2 &= (a_n \times 10^{n-1} + a_{n-1} \times 10^{n-2} + \dots + a_2 \times 10 + a_1)^2 \\ &= a_n^2 10^{2(n-1)} + a_{n-1}^2 10^{2(n-2)} + \dots + 2a_n a_{n-1} 10^{n-1+n-2} + \dots \\ &= a_1^2 + a_1 \times 10(a_2 + a_3 \times 10 + \dots + a_n \times 10^{n-2}) + a_2^2 \times 10^2 \\ &\quad + 2a_2 \times 10^3(a_3 + a_4 \times 10 + \dots + a_n \times 10^{n-3}) + \dots \end{aligned}$$

4. Square root: Earlier Indian procedure.

Similarly division is Dividend/ Divisor, some left to right remove common factors. So, all those things are done and squaring see it does not many they get me some implication in you know in splitting into various parts. So, that is what he will discuss the square is  $a-b$  into  $a+b+b$  square. So, in some set up if you want to has some 998 whole square you see is very convenient to write it  $1000-2$  whole square and use this yes like that.

So, similarly  $n$  square if you want to take the  $n$  square it is  $1+3+5$  into  $2$  upto  $2n-1$  arithmetic progression with one as the first term and  $2n-1$  and the last. So, that the sum is  $n$  square and similarly he very clearly say that  $a+b+c$  etc... whole square is  $a$  square  $+b$  square  $+c$  square etc...  $+2ab+2ac$  etc... all the products twice of that you see various products which will come  $n \times n-1/2$  products will come.

So, all these so, in particular if you have this number like this in decimal notation. So, then you have this number like this in decimal notation. So, then you have this finally you get  $a_1$  square units place what a  $a_1$  square in  $10$  lace you get  $a_1 \times 10$  into all these  $a_2+a_3$  into  $10$  etc... so, that is a very I mean he explains all these things in detail you know. Because either I told you nothing is taken for granted from the scratch is building up the whole mathematical vocabulary of those times that is very important.

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## Cube and Cube root

### 5. Cube:

$$(i) a^3 = a(a-b)(a+b) + b^2(a-b) + b^3 \text{ or}$$

$$(ii) a^3 = a + 3a + 5a + \dots + (2a-1)a = a[1 + 3 + \dots + (2a-1)] = a \cdot a^2 \text{ That is, an A.P with } a \text{ as the first term, } 2a \text{ as the common difference and } a \text{ as the number of terms. or,}$$

$$(iii) a^3 = a^2 + (a-1)[1 + 3 + \dots + (2a-1)]. \text{ or,}$$

(iv)

$$a^3 = 3[1 \cdot 2 + 2 \cdot 3 + \dots + (a-1)a] \times a.$$

$$\text{Check. RHS} = 3[2^2 - 2 + 3^2 - 3 + \dots + a^2 - a] \times a$$

$$= 3[(1^2 + \dots + a^2) - (1 + \dots + a)] \times a$$

$$= 3 \left[ \frac{a(a+1)(2a+1)}{6} - \frac{a(a+1)}{2} \right] \times a = a^3$$

And the 100 place it is  $a^2$  square so, like that square root is earlier Indian procedure I do not have to go through that cube is sometimes it is you know convenient to write it like this  $a^3 = a \cdot a^2$  or  $a^3 = a \cdot (a^2)$  square into  $a \cdot (a^2)$  cube or interestingly it cannot also be written as a cube it is equal to  $a + 3a + \dots$ , it is also a sum of an arithmetic expression with  $a$  as the first term and  $2a$  as the common difference.

And similarly a cube can also be written like this  $a^3 = a^2 + a + 1$  into this arithmetic series progression and similarly write a cube it is equal to  $3 \times 1 + 2 \times 2 + 1 \times 3 + \dots$  like this. So, is very, if you see the, a square it is clear that,  $3 \times 2^2 - 2 + 3 \times 3^2 - 3 + \dots$  so, it is very cleverly it has been done. You see because this will be  $1 \text{ square} + \dots + a \text{ square} - 1 + \dots + a \times 2$  progressions okay. So, then we know this we already know this results, so you get the  $a^3$ .

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## Cube and Cube root

$$(a_1 + a_2 + \dots + a_n)^3 = 3 \sum_{i \neq j} a_i^2 \times a_j + \sum a_i^3$$

or

$$(a_1 + a_2 + \dots + a_n)^3 = a_n^3 + 3a_n^2(a_1 + \dots + a_{n-1}) + 3a_n(a_1^2 + \dots + a_{n-1}^2) + a_{n-1}^3 + 3a_{n-1}^2(a_1 + \dots + a_{n-2}) + 3a_{n-1}(a_1^2 + \dots + a_{n-2}^2) + \dots$$

### 6. Cube root: Earlier Indian procedure (Āryabhaṭa, Brahmagupta)

Verse 60: "O mathematician, who are clever in calculation, give out after examination, the root of 859011369945948864, which is a cubic quantity." [Try this as an exercise].

So, similarly it talks about cube so, various the essentially based on the same principle as for the square but of the one more distinguish is there see you got a three so, you will get various terms like this all these things you will explain. And cube root is and I do not want to talk about it all these has been talked about enough whereas same as the earlier Indian procedure the Aryabhata and Brahmagupta.

So, but I think you should try this example O mathematician who are clever in calculation give out after examination the root of 859011369945948864 which is the cubic quantity okay. So, here already given a cubic quantity. So, please do this.

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### Citi Summation

एकविहीनो गच्छः प्रचयगुणो द्विगुणितादिसंयुक्तः ।  
गच्छाम्यस्तो द्विहृतः प्रभवत् सर्वत्र सङ्कलितम् ॥ ६२ ॥

"The number of terms (in the series) as diminished by one and (then) multiplied by the common difference is combined with twice the first term in the series; and when this (combined sum) is multiplied by the number of terms (in the series) and is (then) divided by two, it becomes the sum of the series in all cases."

If  $a$  : First term,  $d$  : Common difference,  $n$ : No. of terms,  $S$  : Sum, of an A.P.,

$$S = a + (a + d) + \dots + [a + (n - 1)d] = \left[ \frac{2a + (n - 1)d}{2} \right] n,$$

So, then a citi or summation it talks about that that is the one of the arithmetical operations so, if your sum is a this a+ etc.  $a+n-d$  is (FL) is the number of term (FL) like that (FL) distinct d the difference (FL) is the first term so, like that so, that is the we already discussed in detail earlier lectures.

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### Example

Verse 67 (Example):

आदिस्त्रयश्चोऽष्टौ द्वादश गच्छस्त्रयोऽपि रूपेण ।  
आसप्तकान्प्रवृद्धास्सर्वेषां गणक भण गणितम् ॥ ६७ ॥

"The first term is 3; the common difference is 8; and the number of terms is 12. All the three (quantities) are (gradually) increased by 1, until (there) are 7 (series). O mathematician, give out the sums of all (those series)."

Expression for number of terms,  $n$  from  $a$ ,  $d$  and  $S$ . Same as in *Brāhmasphuṭasiddhānta* (BSS).

So, he gives an example the first term is 3: the common difference is 8: the number of terms is 12. All the three (quantities) are (gradually) increased by 1, until (there) are 7 (series) okay. So, not 1 7 small sub problems are there in this okay. O mathematician towards the sums of all those series so, this is the verse then expression for number of terms  $n$  from  $a$ ,  $d$  and  $S$  same as in *Brahmasphutasiddhanta*.

We had seen that further and of course Aryabhattachiya also gives that so, this is essentially you got solve a quadratic equation right  $n$  from  $ad$  and  $S$ .

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### Example

Verse 71 (Example):

आदिर्द्वौ प्रचयोऽष्टौ द्वौ रूपेणा त्रयात् क्रमात् वृद्धौ।

खाद्वौ रसाद्विनेत्रं खेन्दुहया वित्तमत्र को गच्छः ॥ ७१ ॥

"The first term is 2, the common difference is 8; these two are increased successively by 1 till three (series are so made up). The sums of the three series are 90, 276 and 1110, in order. What is the number of terms in each series." [Try this as an exercise].

Also gives  $a, d$  in terms of  $d, n, S$  and  $a, n, S$  respectively.

$$a = \frac{S}{n} - \frac{(n-1)d}{2}; \quad d = \frac{\frac{S}{n} - a}{\frac{n-1}{2}}$$

So, and then he gives various examples (FL) okay the first term is 2, the common difference is 8: these two are increased successively by 1 till three (series are so made up) the sums of the three series are 90, 276 and 1110 in order. What is the number of terms in each series so, you have to do this I think you have to copy this and try this exercises. And similarly you can find  $a$  and  $d$  in terms of  $d, n$  and  $S$  and  $d$  in terms of  $a$  and  $n$  and  $S$ . So, there are quite trivial  
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### Mixed Problems

He discusses some "mixed problems". An example:

First series: 1 term =  $a$ , common difference =  $d$ , No. of terms =  $n$ , Sum =  $S$

Second series: 1 term =  $a_1 = d$ ; common difference =  $d_1 = a$ , No. of terms,  $n_1$ , Sum =  $S_1$ .

The first term and common difference of the two series are interchanged. The ratio of the sums is given ( $S/S_1$ ) and the no. of terms  $n$  and  $n_1$  are given. To find  $a, d$  ( $S, a_1, d_1, S_1$  are automatically determined.)

Verse 86 gives the solution:

$$a = n(n-1)p - 2n_1, d = n_1^2 - n_1 - 2pn, \text{ where } p = \frac{S_1}{S}$$

Then it discusses some mixture problems so, you consider 2 series the first series I mean the 2 series is the number of the first term and a common difference they are interchange. In number of the first term and the common difference they are interchange in the first series first term is  $a$ , common difference is  $d$  number of terms is  $n$ , sum is  $S$  in the second series the first term is  $d$  and

the common difference is  $a$  and the number of terms is  $n_1$ , sum maybe  $s_1$ . So, then one can the first term in the common difference of the 2 series of the interchange as I told you the ratio of the sums is given.

And the number of terms  $n$  and  $n_1$  are given you have to find  $a$ ,  $d$  and other things and he has gives a given a solution you understand in the first series the first term is  $a$  common difference is  $d$ , number of terms is  $n$ , sum is  $s$  and the second series the first term is  $d$  the common difference is  $a$ , the number of terms is  $n_1$  this equal to  $n_1$  and sum is  $s_1$ . So, gives a solution like this  $n$  into  $n-1$  by into  $p-2n_1$  and  $d$  is equal to  $n_1^2 - n_1 - 2pn_1$ , where is  $p$  is  $S_1/S$  essentially in fact what is given is only the ratio of the sums okay.

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**Mixed Problems**

or,

$$d = n_1(n_1 - 1) - 2pn_1.$$

Now  $S = n \left[ a + \frac{(n-1)d}{2} \right]$ ,  $S_1 = n_1 \left[ d + \frac{(n_1-1)a}{2} \right]$

There are two equations for  $a$  and  $d$ , in terms of  $n$ ,  $n_1$ ,  $S$  and  $S_1$ . Actually only  $\frac{S}{S_1}$  is given. So it is an indeterminate equations for  $a$  and  $d$ . One can show that for given solutions for  $a$  and  $d$ ,  $\frac{S_1}{S} = p$ . So, it is correct. It is an 'ansatz'.

Exercise: Find the value of  $S$  and  $S_1$  for this ansatz.

So, here you see actually  $S$  is this and  $S_1$  is this right sum of the arithmetic series, so they are 2 equations for unknowns you have to solve for  $a$  and  $d$ , in terms of this but only  $S/S_1$  is given okay, there is only one essentially there will be you have to divide this 2 equations there will be only one equation  $S/S_1$  okay. So, the one equation for 2 quantities  $a$  and  $d$ , so indeterminate equation actually, so you can show that actually if you take  $S_1/S$  is equal to  $p$ .

So, then  $a$  and  $d$  that will be correct, in fact in the modern language is what is known as an ansatz kind of a **non** some kind of a we start with some kind of a you know from some intelligent guess

you assume a form for the solution okay and then carry on because it is actually indeterminate and you do not have a you know unique solution.

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**Mixed Problems**

Example given in GSS.

Verse 88:

द्वादशषोडशपदयोर्व्यस्तप्रभवोत्तरे समानधनम्।  
द्वादशगुणभागधनमपि कथय त्वं गणितशास्त्रज्ञ ॥ ८८ ॥

"In relation to two series (in A.P) having 12 and 16 for their number of terms, the first term and the common difference are interchangeable. The sums (of the series) are equal, or the sum (of one of them) is twice or any such multiple, or half or such fraction (of that of the other). You who are versed in the science of calculation, give out (the value of these sums and the interchangeable first terms and common difference)."

[Try this as an exercise. Here  $n = 12$ ,  $n_1 = 16$ . Take  $p = 1, 2, 1/2, 3/2$ , and  $2/3$  and find  $a, d, S, S_1$ . Check that  $\frac{S_1}{S} = p$  in each case.]

So, he has given some examples (FL), so in relation to 2 series in A.P having 12 and 16 for their number of terms first and common difference are interchangeable. The sum of the series are equal or some of one of them is twice or any such multiple or half or such fractions okay, see is always you know our man is slightly do small problems are embedded you know a small set of problems will involved embedded in each problem okay.

So, here you have to  $n$  is equal to 12  $n_1$  is equal to 16 and take  $p$  is equal to 1, 2, half,  $3/2$  etc., there is not so in this ratio of the sums and you have to find this  $a, d, S, S_1$  like that okay.

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### Geometric Progressions

Next Geometric progressions are considered:  $a, ar, ar^2, \dots, ar^{n-1}$ .

$$S = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{(r - 1)}.$$

[The procedure for finding  $r^n$ , as in BSS is given.]

An important method to find  $r$ , the common ratio, given the first term,  $a$ , number of terms,  $n$  and the sum,  $S$ .

Verse 101 a:

असकृद्विकं मुखद्वितितं येनोद्धृतं भवेत्स चयः ।

"That (quantity) by which the sum of the series divided by the first term and (then) lessened by one is divisible throughout (when this process of division after the subtraction of one is carried on in relation to all the successive quotients) time after time (that quantity) is the common ratio."

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So, now he will come to the geometric progression the important thing as this is the sum whatever is there the earlier thing there is the quote stated but you please notice that there is some improvement in everything there is some small improvement you know some advancement and that is a characteristic of you know Indian mathematics Mahavira you know has some various new things you know from compare to Brahmagupta. Brahmagupta himself had various new results apart from what Aryabhata is stated.

Aryabhata himself of course got some results from maybe earlier tradition and got something see this thing and similarly one can see that you know Bhaskara will give a lot more this things the advance is the mathematical knowledge in various ways and narayapunita much more sophisticated ways you know very important departures and Kerala school of mathematics you know of course the almost a new face you enter you know with some well sophisticated discussion on analysis, infinite series and all that.

So, that is the thing, so though you are toughing with this things whatever was known from the earlier thing is adding up, so that is happening, so this sum of this geometric progressions is given, so this all this is I do not have to explain this result now an important measure to find  $r$  the common ratio is to given the first term number of terms  $n$  and the sum is this is not a you know given  $a$ ,  $r$  and  $n$  you can find  $S$ .

But given S, a and n to find r is it nth degree equation, so it is not easy, so he gives a very interesting method (FL) that quantity by which sum of the series divided by the first term and then lessened by one is divisible throughout when this process of division after the subtraction of one is carried on in relation to all the successive quantities time after time that quantity is the common ratio.

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**Explanation**

$$S' = \frac{S}{a} = \frac{r^n - 1}{r - 1}, \quad S' - 1 = \frac{r^n - 1}{r - 1} - 1 = \frac{r^n - r}{r - 1} = \frac{r(r^{n-1} - 1)}{(r - 1)}$$

$$\therefore S' - 1 \text{ is divisible by } r : \frac{1}{r}(S' - 1) = \frac{r^{n-1} - 1}{r - 1} = S''$$

Subtract 1 from this:  $S'' - 1 = \frac{r^{n-1} - 1}{r - 1} - 1 = \frac{r(r^{n-2} - 1)}{(r - 1)}$  is divisible by r

Again  $\frac{1}{r}(S'' - 1) = \frac{r^{n-2} - 1}{r - 1} = S'''$ .  $S''' - 1$  is divisible by r

One should check r such that at each stage of the process,  $S' - 1$ ,  $S'' - 1$ ,  $S''' - 1$ , ..., etc., is divisible by r, till one gets 1.

**Note:**  $r^n - 1 = (r - 1)(r^{n-1} + r^{n-2} + \dots + 1)$ ;  $r^n - 1$  is divisible by  $r - 1$ .

So, what he is trying to say is, so the S is this S/a is r to the power of n-1 divided by r-1 and S prime-1 is this r to the power of n-1 by this -1, so which is this, so this is divisible by r clearly. So, now take this you take 1/r into S prime-1, so this is call it as S double prime subtract 1 some this again one can read that this is divisible by r, S double prime-1 that is r to the power of n-1-1 you found this ratio then subtract 1, that is also divisible by r.

So, you go on you see and one should check the r said that at each stage of the process S prime-1, S double prime -1 triple prime -1 etc., it is divisible by r till one get 1 okay. So, it is tricky whereas what you sort with you see and sometimes style and error you have to use you see. In a first step it maybe divisible by many numbers and afterwards it subtract 1 and this thing then you know you will be only a few things, so it will get narrowed down you know. Here of course I use the result that r to the power of n-1 is this, so it is always divisible by r-1.

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### Example

Example. Let  $a = 3$ ,  $n = 6$ ,  $S = 4095$ .

$$\frac{4095}{3} = 1365, \quad 1365 - 1 = 1364.$$

Choose by trial  $r = 4$ ,  $\frac{1364}{4} = 341$ .

$$\begin{aligned} 341 - 1 &= 340, \quad \frac{340}{4} = 85; \quad 85 - 1 = 84, \quad \frac{84}{4} = 21; \quad 21 - 1 = 20, \\ &= 20, \quad \frac{20}{4} = 5; \quad 5 - 1 = 4, \quad \frac{4}{4} = 1. \end{aligned}$$

Hence 4 is the common ratio.

Yeah for instance if let  $a$  is 3,  $n$  is 6,  $S$  is 4095 geometric progression first term is 3 and number of terms is 6 and common ratio you have find out, so what you do is it divide the sum by the first get 1365 subtract 1 you get 1364, so now choose by trail  $r$  is equal to 4, so then  $1364/4$  is 341 then subtract 1, 340 is divisible by 4 you get 85, 85 you subtract 1 you get 21 is subtract 1 again that is again divisible by 4,  $5-1$  is 4 that is again divisible by 4, so 4 is the common ratio, so this how you get.

As I told you sometimes it may be tricky so, you have to watch out.

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### Finding $n$ , given $r$ and $S$

Verse 103

एकोनगुणाभ्यस्तं प्रभवद्गतं रूपसंयुतं वितम्।  
यावत्कृत्वो भक्तं गुणेन तद्वारसम्मितिर्गच्छः ॥ १०३ ॥

"Multiply the sum (of the given series in geometrical progression) by the common ratio lessened by one; (then) divided this (product) by the first term and (then) add one to this (quotient). The number of times that this (resulting quantity) is (successively) divisible by the common ratio - that gives the measure of the number of terms (in the series)."

And finding  $n$  given  $r$  and  $S$  that also he gives.

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Finding  $n$ , given  $r$  and  $S$

$$\frac{S(r-1)}{a} + 1 = r^n.$$

Keep dividing  $\frac{S(r-1)}{a} + 1$  by  $r$ . The number of times that this (resulting quantity) is (successively) divisible by  $r$  (till one gets 1), that is ' $n$ '.

Example. In Verse 104:  $a = 3, r = 6, S = 777$ . What is  $n$ ?

$$\frac{S(r-1)}{a} + 1 = \frac{777 \times 5}{3} + 1 = 259 \times 5 + 1 = 1296.$$
$$\frac{1296}{6} = 216, \frac{216}{6} = 36, \frac{36}{6} = 6, \frac{6}{6} = 1.$$
$$\therefore n = 4.$$

So, for instance  $S$  into  $r-1/a+1$  is  $r$  to the power of  $n$ , you can expect that sum like this, so keep dividing by  $r$  this quantity the number of time that is resulting quantity successively divisible by  $r$  till one gets 1 that is  $n$ , so this somewhat little more simpler okay. Suppose  $a$  is 3,  $r$  is 6,  $S$  is 777 what is  $n$ ?. So, this quantity is crucial right, so this is 1296 and 1296 this is more straight forward you know  $1296/6$  is this 6 again is this by 6 again is this, so  $n$  is 4, so it is.

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Example

Exercise from Verse 105.

Verse 105

त्र्यास्ये पञ्चगुणाधिके हतवहोपेन्द्राक्षवद्विद्विप-  
श्वेतांशुद्विरदेभकर्मकरदृष्टानेऽपि गच्छः कियान् ॥ १०५ ॥

What is  $n$ , when the first term is 3, the common ratio is 5 and the sum is  $S = 22888183593$ ? [Try this as an exercise.]

So, exercise is, what is  $n$  in the first term is 3 the common ratio is 5 and the sum is 22888183593, so please try this is an exercise okay.

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### Vyutkalita

$$\underbrace{a, a+b, \dots, a+(d-1)b}_{\text{ista: } d \text{ terms}} + \underbrace{a+db + \dots + a+(n-1)b}_{\text{vyutkalita: } n-d \text{ terms}}$$

$$S_i = \left[ a + \frac{(d-1)b}{2} \right] d$$

$$S_v = \left[ a + db + \frac{(n-d-1)b}{2} \right] (n-d) = \left[ a + \frac{(n+d-1)b}{2} \right] (n-d)$$

So, then he talks about you know as I told you some improvements you know some advancements are made, so consider this series is you know it goes on up to  $n-1$  but it is split into 2 parts where 1 first part of  $d$  term this is  $n-d$  terms, so then this is called (FL) that is the later part of a series where the initial part is called (FL), so this (FL) is actually total sum-ista basically, so this what you get.

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### Chapter 3. Fractions

Multiplication, Division, root, etc.

Number of terms in an A.P can be fractional!

$$S = a, a+b, \dots, a+(n'-1)d = n' \left[ a + \frac{(n'-1)d}{2} \right]$$

where  $n'$  can be a fraction.

Very detailed exposition of manipulations with fractions.

Now multiplication, division, root etc... of fraction she will discuss in rate the various kinds of fractions are there you know. And you know really gives large number of examples to illustrate various kinds of you know manipulations, with fractions, multiplications divisions and so on so both, so that you know things will be fixed firmly in one mind and number of terms in an A.P

very interesting you know that he takes this n prime is this where n prime can be fraction the number of terms can be fraction you see, see this kind of generalization all that.

We are keeping doing only doing the past 2 centuries you know so, he did not mind you know that the number of terms can be a fractional, so it is define like that you see up to sometime and then you know go a little further not the next term you know next integer but fractional thing, as I told you very detailed exploitation of manipulation this fractions.

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### Arithmetic Progressions

Verse 25.

$$a + (a + d) + \cdots + [a + (n - 1)d] = n^2,$$

$$\text{for arbitrary } a, \text{ and } d = \frac{(n - a)}{\left(\frac{n - 1}{2}\right)}.$$

Similarly  $a + (a + d) + \cdots + [a + (n - 1)d] = n^3$ , for  $a = 2n$  and  $b = \frac{(n - 2)n}{(n - 1)/2}$ .

In general if  $a = \frac{x}{4}$ ,  $d = 2a$  and  $n = 2x$ ,

$$S = \left[ \frac{x}{4} + \frac{(2x - 1)}{2} \cdot \frac{x}{2} \right] 2x = x^3$$

Verse 29. If  $a, d, n$  yield  $S$ ,  $a_1 = \frac{S_1}{S} a$ ,  $d_1 = \frac{S_1}{S} d$  yield  $S_1$ .

Then arithmetic progressions I will not all this things are known, so for instance if this if you want this to be square of n, a square of n can be got from an arithmetical progression but arbitrary a and d is this if you want to express this square by a arithmetic series some were of some number integer, so similarly cube also can be express like this in fact any number x is cube can be expressed as a sum of kind arithmetic series, where the first term is x/4, the difference is 2a and n is 2x.

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## Relations between A.P and G.P

An interesting rule. To find the (common) first term of two series of having the same sum, one of them being in arithmetical progression and the other in geometrical progression, their optionally chosen number of terms being equal and similarly, the chosen common difference and the common ratio also being 'equal' in value.

The essential content of Verse 43:

Consider a G.P with  $a$  as the first term,  $r$  as the common ratio,  $n$  terms and sum  $\frac{a(r^n - 1)}{(r - 1)}$ .

Consider an A.P with the same number of terms  $n$ ,  $a$  as the first term, common difference  $d = r$ , and sum

$$S_A = \left[ a + \frac{(n-1)d}{2} \right] n.$$

$$\text{Then } S_G = S_A \text{ if } a = \frac{\frac{n(n-1)r}{2}}{\left[ \frac{r^n - 1}{r - 1} - n \right]}.$$

And he talks about various relations between arithmetical and geometrical progressions for instance you can have an arithmetical progression and geometrical progression suppose you want the sums to be equal, so then you can find the first term and all that. If the first term is taken to or the arithmetical progression is taken to be this then the 2 this sums will be equal.

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## Ingenious Manipulations with Fractions to Solve Problems

A problem of 'śeṣa' variety: Consider a number  $x$  and fractions  $b_1, b_2, \dots, b_n$ . A fraction  $b_1x$  is subtracted from  $x$ . Remainder:  $R_1 = x - b_1x = x(1 - b_1)$ . A fraction  $b_2$  of the remainder is subtracted. Remainder

$R_2 = x - b_1x - b_2(x - b_1x) = (1 - b_1)(1 - b_2)x$ . A fraction  $b_3$  of the remainder is subtracted. Remainder  $R_3$  is given by

$$\begin{aligned} R_3 &= x - b_1x - b_2(x - b_1x) - b_3[x - b_1x - b_2(x - b_1x)] \\ &= (1 - b_1)(1 - b_2)(1 - b_3)x \end{aligned}$$

Then it talks about in a various manipulation with fractions to solve problems. So, a problem was sesa variety I will just go through for an example. Consider a number  $x$  and fractions you know  $b_1, b_2$  etc...  $b$  and suppose a fraction  $b_1x$  I subtracted from  $x$ . So, remainder is  $x - b_1x$  right which is written as  $x$  into  $1 - b_1x$ . A fraction  $b_2$  of the remainder is subtracted so,  $x - b_1x$  a fraction  $b_2$  of this I subtracted from this. So, it will be this quantity which is  $1 - b_1$  into  $1 - b_2x$ .

And a fraction  $b_3$  of the remainder I subtracted so,  $-b_3$  into this whole thing you have to write so, which this.

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**Manipulations with Fractions**

This goes on till  $R_n = y$  a known quantity.  
 To solve for  $x$  in terms of  $y$ .  
 We have  $R_n = (1 - b_1)(1 - b_2)(1 - b_3) \dots (1 - b_n)x = y$ ,

$$\therefore x = \frac{y}{(1 - b_1)(1 - b_2)(1 - b_3) \dots (1 - b_n)}.$$

So, you can go on like that so, this goes on till this  $n$ th remainder is  $y$  a unknown quantity to solve for  $x$  in terms of  $y$ . So,  $R_n$  is this suppose this  $y$  the clearly  $x$  is  $y$  divided by all these.

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**Example**

Example. Verse 32 of Chapter 4.

कोटस्य लेभे नवमांशमेकः परेऽष्टभागादिदलान्तिमांशान्।  
 शेषस्य शेषस्य पुनः पुराणा दृष्टा मया द्वादश तत्प्रमा का॥ ३२ ॥

"Of the contents of a treasury, one man obtained  $\frac{1}{9}$  part; others obtained from  $\frac{1}{9}$  in order to  $\frac{1}{2}$  in the end, by the successive remainders; and (at last) 12 *purāṇas* were seen by me (to remain). What is the (numerical) measure (of the *purāṇas*) contained in the treasury?"

Solution: Here  $b_1 = \frac{1}{9}$ ,  $b_2 = \frac{1}{9}$ ,  $b_3 = \frac{1}{8}$ ,  $b_4 = \frac{1}{7}$ ,  $b_5 = \frac{1}{6}$ ,  $b_6 = \frac{1}{5}$ ,  $b_7 = \frac{1}{4}$ ,  $b_8 = \frac{1}{3}$ ,  $b_9 = \frac{1}{2}$ ,  $y = 12$ .

$$\begin{aligned} \therefore x &= \frac{12}{(1 - \frac{1}{9})(1 - \frac{1}{9})(1 - \frac{1}{8})(1 - \frac{1}{7})(1 - \frac{1}{6})(1 - \frac{1}{5})(1 - \frac{1}{4})(1 - \frac{1}{3})(1 - \frac{1}{2})} \\ &= \frac{12}{\frac{8}{9} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}} \\ &= \frac{12 \cdot 81}{8} = \frac{243}{2} = 121\frac{1}{2} \text{ purāṇas.} \end{aligned}$$

The furnishing gives the example (FL) name of the monetary unit (FL) 12 puranas for seen and the contents of the treasury one man obtained one nine part others obtained from  $1/9$  in order to  $1/2$  in the  $n$  by successive remainders. And at least 12 puranas were seen. What is the numerical

measure of the puranas contained in the treasury?. So, here straight away you just method b1 is 1/9, b2 is 1/9, 1/8 etc etc. and then well is the, what is found so, x is 12 divided by all these so, this will be equal to 243/2 are 121 and a half puranas is solution.

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Problems of the *Śeṣamūla* variety

Suppose we have problems of the form:  $x - bx - c\sqrt{x} - a = 0$ .  
Then

$$\sqrt{x} = \frac{c/2}{1-b} + \sqrt{\left(\frac{c/2}{1-b}\right)^2 + \frac{a}{1-b}}; \quad x = (\sqrt{x})^2$$

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So, this of course will be (FL) so, moola will also come root x kind of a thing suppose your problems is form  $x-bx-c$ , root  $x-a$  is equal to 0. So, then you have the quadratic equation to write this solution like this one of the roots and then x is root x by this quadratic equation solution.

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Example

Verse 49 (Example):

सिंहाश्चत्वारोऽद्रौ प्रतिशेषपडंशकादिमार्थान्ताः ।  
मूले चत्वारोऽपि च विपिने दृष्टा कियन्तस्ते ॥ ४९ ॥

"Four (out of a collection of) lions were seen on a mountain; and fractional parts commencing with  $\frac{1}{6}$  and ending with  $\frac{1}{2}$  of the successive remainders (of the collection), and (lions equivalent in number to) twice the square root (of the numerical value of the collection) as also (the finally remaining) four (lions) were seen in a forest. How many are those (lions in the collections) ?"

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So, he gives an example (FL) four (out of a collection) of lions were seen on a mountain: okay(FL) mountain okay and fractional parts commencing with 1/6 (FL) right (FL) starting from



that commencing with that and (FL) ending with half as you remind as and (lions equivalent in number to) twice is square root (FL) twice the (of the numerical value of the collection) as also finally remaining four lions were seen (FL). How many are those (lions in the collections) ok.

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**Solution**

Solution:

$$\left(1 - \frac{1}{6}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{2}\right) (x-4) - 2\sqrt{x} - 4 = 0.$$

or,  $\frac{1}{6}(x-4) - 2\sqrt{x} - 4 = 0$

$$\frac{1}{6}x - 2\sqrt{x} - 4 \cdot \frac{7}{6} = 0$$

Here  $1 - b = \frac{1}{6}, c = 2, a = 4 \cdot \frac{7}{6}, \frac{c}{2} = 1.$

$$\therefore \sqrt{x} = \frac{1}{6} + \sqrt{\left(\frac{1}{6}\right)^2 + \frac{4 \cdot \frac{7}{6}}{6}} = 6 + \sqrt{36 + 28} = 6 + 8 = 14.$$

$$\therefore x = 196$$

So, then the essentially is the problem is this so, four of them were seen on the mountains you subtract that of the remaining you know 1/6 went away like that. So, it is 1-1/6 of thus for remaining out of that 1/5 went so, like that to this x-4 and then twice the moola okay – so, that is equal to twice the moola. So, then if you subtract that all also 4 was finally remaining if you subtract twice the moola of the original number. So, what to get is a solution sorry equation like this x-4 into this so, this is this thing so, one can and x is equal to 196.

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## Rule of Proportions

Chapter 5 is on the Rule of three, five, ...

Example. Verse 31:

द्वात्रिंशद्वस्तदौर्यः प्रविशति विवरे पृथुभिस्सप्तमार्धः  
कृष्णाहीन्द्रो दिनस्यासुखपुरजितः सार्धसप्ताङ्गुलानि ।  
पादेनाहोऽङ्गुले द्वे त्रिचरणसहिते वर्धते तस्य पुच्छं  
रन्त्रं कालेन केन प्रविशति गणकोत्तंस मे ब्रूहि सोऽयम् ॥३१॥

"A powerful unvanquished excellent black snake, which is 32 *hastas* in length, enters into a hole (at the rate of)  $7\frac{1}{2}$  *angulas* in  $\frac{5}{14}$  of a day; and in the course of  $\frac{1}{4}$  a day its tail grows by  $2\frac{3}{4}$  of an *angula*. O ornament of mathematicians, tell me by what time this same (serpent) enters into the hole" [1 *hasta* = 24 *angulas*. Try this as exercise].

So, the chapter 5 in a rule of three, five etc... again in order going to detail but just mention in interesting problem (FL) A powerful unvanquished excellent black snake which is 32 hastas in length okay (FL) enters into a hole okay enters into a hole at the rate of  $7\frac{1}{2}$  angula okay (FL) in  $\frac{5}{14}$  of a day; and in the course of  $\frac{1}{4}$  of a day it is tail grows by  $2\frac{3}{4}$  of an angula okay (FL) okay (FL) so,  $2\frac{3}{4}$  right (FL) I  $1\frac{1}{4}$  the of the day (FL) okay tail. So, (FL) tell me by what time this same (serpent) enters into the hole (FL)) okay expert mathematician please tell the solution.

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## Mixed Problems

Chapter 6 is on "mixed problems". Mahāvīra shows ingenuity in solving determinate and indeterminate equations.

Principal (capital), Interest, Mixed amount (A), etc.

Now, if  $I$  is the interest on the principal,  $P$  (rate principal) in time  $T$  months and  $i$  is the interest on a capital  $p$  in  $t$  months,

$$i = \frac{t \times p \times I}{T \times P}.$$

Suppose we are given  $m = p + t$  along with  $P$ ,  $T$ ,  $I$  and  $i$ . How to find  $p$  and  $t$ ? The solution is given in Verse 29 :

So, that is the (FL) little careful it is direct okay so, then interactive discusses various varieties have mixed problems okay. So, I will come into that in fact that is the lot of much as Varghese on the so, called mixed problems. So, for instance typically if capital is a interest on principal

capital  $p$  and time capital  $T$  months and  $I$  is a interest on a capital small  $p$  in small  $t$  months okay then  $I$  is this  $p \cdot p \cdot I / T \cdot p$  okay.

I may if the rate of interest is the same I am saying if the rate of interest is the same so, then one can you know you can easily see that  $I/I$  will be  $t \cdot p / T \cdot p$  so, is the rate of interest is the same. So, this equation will be true but suppose there given  $m + p + t$  along with  $P, T, I$  and  $i$  okay. So, then how to find outs small  $p$  and  $t$  so, these a slightly invert problem. So, using this small principal and the time okay only the sum is given.

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**Mixed Problems**

Verse 29:

स्वफलोद्धृतप्रमाणं कालचतुर्वृद्धितादितं शोध्यम् ।  
मिश्रकृतेस्तन्मूलं मिश्रे क्रियते तु सङ्क्रमणम् ॥ २९ ॥

"From the square of the given mixed sum (of the capital and the time), the rate-capital divided by its rate-interest and multiplied by the rate-time and by four times the given interest is to be subtracted. The square root of this (resulting remainder) is then used in relation to the given mixed sum so as to carry out the process of *sankramaṇa*."

Then one show that

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**Solutions**

Now 
$$t = \frac{i \times T \times P}{p \times I}$$

Now 
$$m = p + t = p + \frac{i \times T \times P}{p \times I}$$

$$\therefore p^2 - mp + \frac{i \times T \times P}{p \times I} = 0$$

$$p = \frac{m \pm \sqrt{m^2 - \frac{P \times T}{I} \times 4i}}{2}, \quad t = m - p.$$

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I will not so, t is this so, m is equal to p+t is you know this so, p squared-mp you will get a quadratic equation so, p will be this actually and t is equal to m-p essentially somewhat similar to brahma what Brahmagupta at consider as a one of the problems that I told so, told you straightly different kind of a problem. But same technique is used.

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### Example

Example. Verse 31:

त्रिकषष्ट्या दत्तैकः किं मूलं केन कालेन।  
प्राप्तेऽष्टादशवृद्धिं षट्षष्टिः कालमूलमिश्रं हि ॥ ३१ ॥

"By lending out what capital for what time at the rate of 3 per 60 (per month) would a man obtain 18 as interest, 66 being the mixed sum of that time and that capital?"

And finally for instance by lending out what capital for what time at the rate of 3 per 60 (per month) would a man obtain 18 as interest, 66 being the mixed sum of that time and that capital?

Okay

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### Solutions

Solution: Here  $m = 66$ ,  $P = 60$ ,  $I = 3$ ,  $T = 1$ ,  $i = 18$ .

$$\begin{aligned} p &= \frac{66 \pm \sqrt{66^2 - \frac{60 \times 1}{3} \times 4 \times 18}}{2} \\ &= \frac{66 \pm 6\sqrt{11^2 - 40}}{2} = \frac{66 \pm 6 \times 9}{2} = 60 \text{ or } 6. \end{aligned}$$

Correspondingly  $t = 6$  or  $60$ .

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So, here what is given is  $m$  is 66 see for a capital  $p$  is equal to 60 for one month the interest  $I$  is 3 okay. So, no, what is the interest suppose the interest is given to be 18 okay. It is some percent some other lending. The interest generated is given to be 18 and the time+the principal is given to be 66 so, is in what is that principal and what is that time. So, for that you have to use the quadratic equation you get the solution 60 or 6 okay.

So, is there you know  $t$  is equal to 6 or 60 so, principal is 60 okay so, the interest will be 18 if it is 6 months. But if the principal is 6 the interest is 18 if the time is 60 months so, this is the.

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### Mixed Problems

There are many more mixed problems involving money lent out at various rates involving the contributions of several persons to the principal and so on, as also proportionate division in other situations.

Example: Suppose there are various quantities purchased at specified different rates and the total amount of the purchase is specified. To find the amounts of quantities purchased.

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So, there are many more mixed problems involving money lent out at various rates involving contributions the several to the principal and so on as also proportionate division in other situations okay. Suppose there are various quantities purchased at specified different rates and the total amount is the purchase is specified to find the amounts of quantities purchased here all very simple. But you know (FL) more general you know a readily usable form kind of everything.

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**The Rule for Mixed Quantities**

The rule:  
Verse 87 a, Chapter 6:

भक्तं शेषैर्मूलं गुणगुणितं तेन योजितं प्रक्षेपम्।  
तद्व्यं मूल्यं प्रक्षेपविभक्तं हि मूल्यं स्यात् ॥ ८७ १/२ ॥

"The (number representing the) rate - price is divided by (the number representing) the thing purchasable herewith; (it) is (then) multiplied by the (given) proportional number; by means of this, (we get at) the sum of the proportionate parts (through) the process of addition. Then the given amount multiplied by the (respective) proportionate parts and then divided by (this sum of) the proportionate parts gives rise to the value (of the various things in the required proportion)."

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For instance okay I will tell what it is and then may be.

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**Mixed Quantities**

Rate price,  $R'_i$  (per unit quantity of  $i$ ) =  $\frac{R_i}{n_i}$  (where  $R_i$  is the price for quantity,  $n_i$ ).

Let the quantities of 1, 2, 3, ...,  $n$  be  $x_1, x_2, \dots, x_n$  ( $x_i$  for  $i$ ).

∴ Total amount of money for purchase,

$$A = \sum x_i R'_i = x_1 R'_1 + x_2 R'_2 + \dots + x_n R'_n.$$

If  $x_1 : x_2 : x_3 \dots$  is given as  $\alpha_1 : \alpha_2 : \alpha_3 \dots$ , that is,  $x_i = \alpha_i X$ ,

$$\left( \sum \alpha_i R'_i \right) X = A, \text{ given total amount.}$$

$$\therefore \left( \sum \alpha_i \frac{R_i}{n_i} \right) X = A$$

$$\therefore x_i = \alpha_i X = \frac{\alpha_i A}{\sum \alpha_i \frac{R_i}{n_i}}, \text{ which is the rule.}$$

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You know because you do not have a too much time see suppose various quantities are bought okay. The rate price is  $R$  prime  $i$ , I will edge a you will know why I am writing  $R$  prime here  $R_i / n_i$   $R_i$  is the price of the quantity  $n_i$  okay. Some quantity is there some item is there the quantity of that is  $n_i$ . The price is  $R_i$  for that quantity that is the rate is  $R_i / n_i$  and a write that the quantity is the 1, 2, 3,  $n$  be  $x_1$  to  $x_n$ .

So, the total amount for purchase is clearly the quantity of each and then multiplied by the rate price. So, this will be the this now suppose you are given the ratios of this quantities you know  $x_1 : x_2$  etc... is equal to  $\alpha_1 : \alpha_2$  etc. So that is essentially  $x_i$  is equal to  $\alpha_i$  I into  $x$  okay. One quantity is there so, you get if you substitute this here you get this so, you get capital  $X$  to be you know this.

So,  $x_i$  is  $\alpha_i / \sum \alpha_i R_i / n_i$  okay so, you can directly work with  $R_i / n_i$  you do not have to use a rate and do all these things.

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### Example

Example. Verse 90 b and 91:

द्वाभ्यां त्रीणि त्रिभिः पञ्चभिस्तप्त मानकैः।

दाडिमाम्रकपित्थानां फलानि गणितार्थवित् ॥ ९० १/२ ॥

कपित्थात् त्रिगुणं ह्याम्रं दाडिमं षड्गुणं भवेत्।

क्रीत्वानय सखे शीघ्रं त्वं षट्सप्ततिभिः पणैः ॥ ९१ १/२ ॥

"Pomegranates, mangoes and wood apples are obtainable at the (respective) rates of 3 for 2, 5 for 3 and 7 for 5 respectively. O you friend, who know the principles of computation, come quickly having purchased fruits for 76 *panas*, so that mangoes may be three times as the wood-apples, and pomegranates six times as much."

The for a example you have one example is given (FL) is pomegranate (FL) mango (FL) wood apple (FL) okay the number of a mangoes are three times as (FL) wood apples okay. And pomegranate (FL) 6 times this is things. So, and the rates are given (FL) you see for pomegranate the rate is 3 for 2 and similarly for the other mangoes it is 5 for 3 okay (FL) like that you know.



So, and for the wood apples it is 7 for 5 so, you have to find out how many quantities so, then here.

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**Solutions**

1 (Pomegranates), $n_1 = 3, R_1 = 2;$	2 (Mangoes), $n_2 = 5, R_2 = 3;$	3 (Wood apples) $n_3 = 7, R_3 = 5$
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$\alpha_1 = 6, \alpha_2 = 3, \alpha_3 = 1$

Amount  $A = 76$ .

$$\sum \alpha_i \frac{R_i}{n_i} = 6 \times \frac{2}{3} + 3 \times \frac{3}{5} + 1 \times \frac{5}{7} = 4 + \frac{9}{5} + \frac{5}{7}$$

$$= \frac{140 + 63 + 25}{35} = \frac{228}{35}$$

$$\frac{A}{\sum \alpha_i \frac{R_i}{n_i}} = \frac{76}{\frac{228}{35}} = \frac{35}{3}$$

$\therefore x_1(\text{No. of pomegranates}) = 6 \times \frac{35}{3} = 70,$

$x_2(\text{No. of mangoes}) = 3 \times \frac{35}{3} = 35,$

$x_3(\text{No. of wood-apples}) = 1 \times \frac{35}{3} = \frac{35}{3}.$

One has to just straight away use this formula so, here for a pomegranates the rate is I mean you get 2 pomegranates per 3 pomegranates for two 2 units of money per mangoes so, the 3 for 5 you get 3 and similarly for wood apple you 5 you get for 7. And the ratios are given to be this 6:3:1. So that is the pomegranates, mangoes and wood apples. So, then amount is given to be 76 so, total amount is this.

What each individual thing you will get so, you will edit to be you have to calculate all these things and finally you get the number of pomegranates to be 70, number of mangoes to be 35, and number of wood apples is equal to 35 units. So, these what you will get for 76 (FL) okay 76 (FL) when it is mention that the number of mangoes is thrice the number of wood apples and number of pomegranates is 6 times the number of wood apples. So, this is one typical problem.

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The references are given here so, thank you so, we will discuss some more problems of this solve you know various fixed quantities you know that which are of direct interest in daily life. I will stop here.