Decision Making With Spreadsheet
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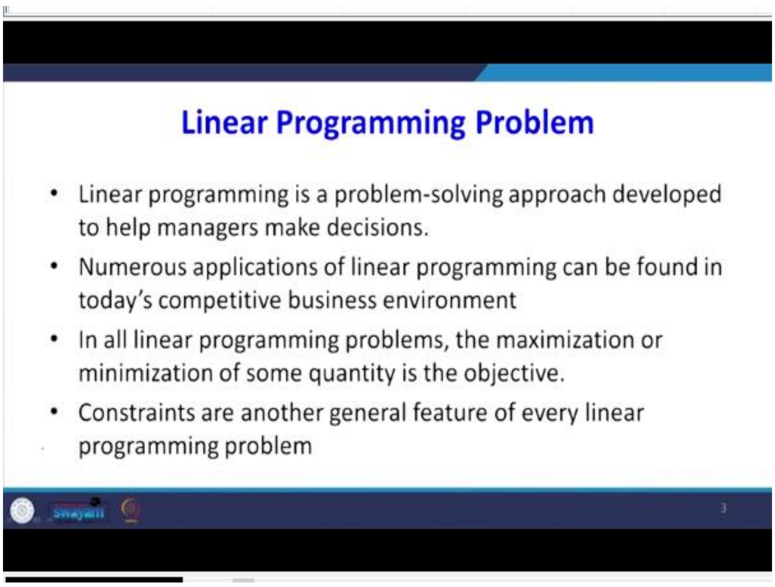
Lecture-03
Linear Programming Problem-Formulations and Assumptions

Dear students, in this lecture, I will explain how to formulate an LP problem because we have seen the classification of that one in the decision-making stages. What is the classification? First, you have to structure the problem, and then you have to analyze the problem. So, structuring the problem is very important once the problem is properly structured. So, analyzing is another stage.

So, we should spend most of the time on how to structure the problem, which is what we are going to see in this lecture. So, the agenda for this lecture is the formulation of linear programming problems. What are the assumptions in linear programming problems?



Linear programming is a problem-solving approach developed to help managers make decisions. Numerous applications of linear programming can be found in today's competitive business environment. In all linear programming problems, the maximization or minimization of some quantities is the objective. It may be the maximization of the profit or minimization of the cost. Another common element in the linear programming problem is a constraint.

Now, I will explain a simple maximization problem, this problem is taken from Anderson et al. That company par incorporation is a small manufacturer of golf equipment and supplies whose management has decided to move into the market for medium-sized and high-priced golf bags; the company's manufacturing is planning to introduce 2 new types of bags. The distributor is enthusiastic about the new product line, and he has agreed to buy all the golf bags that were produced by that company for the next 2 to 3 months.

In the golf bag manufacturing process, there are 4 stages and 4 operations; what are the first operations? Cutting and dyeing the material, the second operation is sewing. The third operation is finishing; the fourth operation is inspection and packaging.

The company is planning to go for 2 types of bags. One is the standard bag, and the other one is a deluxe bag. There are 2 products.

For the manufacturing of the 2 products, we have seen that there are 4 departments. So, there is how much time is consumed for the standard bags? For example, in cutting and dyeing, there are 7/10 hours required for the standard bag. For sewing, it is required for 1/2 hour. For finishing, it is 1 hour. For the inspection packaging, it is 1/ 10 hours for the standard bag.

But the deluxe bag for cutting and dyeing takes 1 hour because the times are in hours. For sewing, it is 5 / 6 hours. For finishing, 2/3 hours, for inspection and packaging, 1/4 hours. These are the time taken to manufacture those 2 bags.

So, there were 4 departments. So, the maximum resources available are the time. So, time is the constraint for you. How can we say the time is constrained? For example, for the cutting and dyeing department, the maximum available hours are 630 hours; for the sewing department, it is 600 hours; for the finishing department, it is 708 hours; for the inspection packaging, it is 135 hours.

Here the company needs to maximize the profit contribution, not the profit. There is a difference between profit contribution and profit. Overhead and other shared costs must be detected before arriving at a profit figure. But here, we are not talking about the profit; we are talking about only the profit contribution because the profit contribution also includes our overhead cost also.

So, the profit contribution for the standard bag is 10 dollars, and for the deluxe bag, it is 9 dollars. So, it is not the profit, it is a profit contribution. Why is it not profitable? We did not subtract the overhead cost.

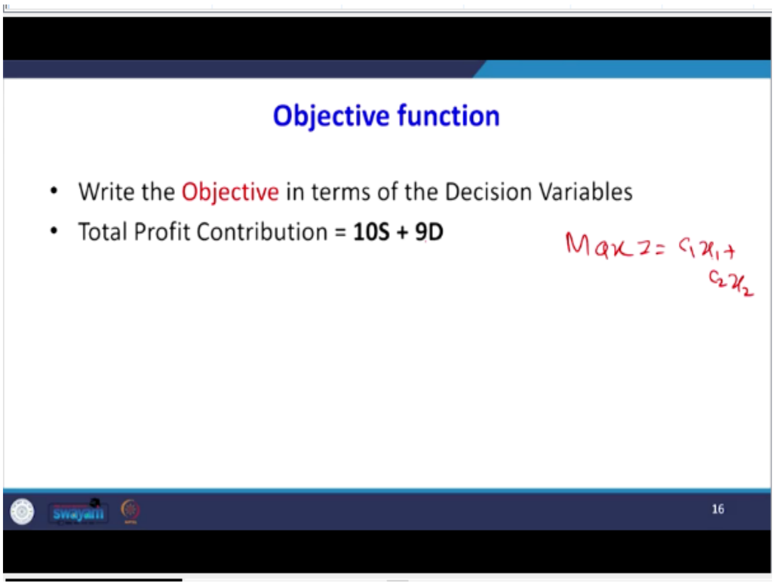
So, the problem is developing a mathematical model for that company that can be used to answer this question. To determine the number of standard bags and the number of deluxe bikes to produce to maximize the total profit. Here, the problem is how many standard bags and how many deluxe bags need to be produced. So, we can maximize the profit without violating a constraint; that is the problem.

The first stage in formulating a problem is described as the objective. The objective is to maximize the total contribution to profit. The next one is to describe each constraint. Four constraints relate to the number of hours of manufacturing time available; they restrict the number of standard bags and the number of deluxe bags that can be produced.

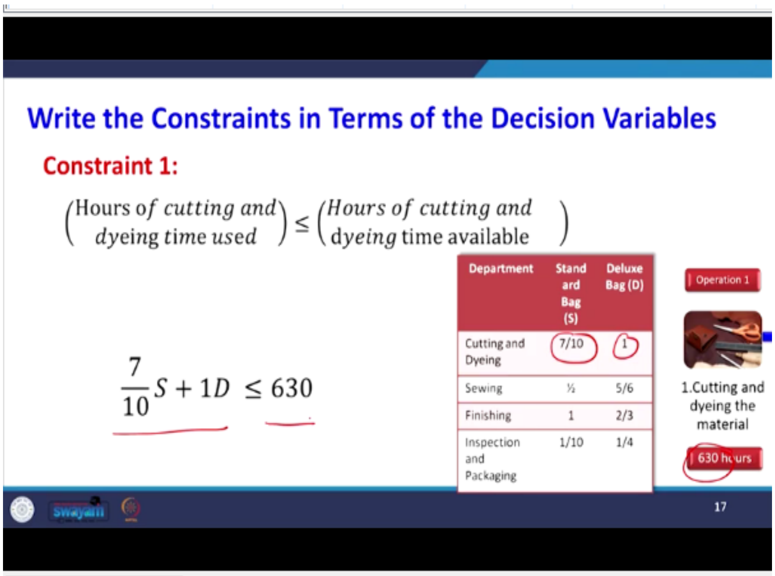
What is constraint number 1 for department 1? The number of hours of cutting and dyeing time used to be less than or equal to the number of hours of cutting and dyeing time available. So, whatever time you are using from department 1 should be less than the number of time available, which is our constraint number 1. Constraint number 2. The number of hours for sewing time used must be less than or equal to the number of hours of sewing time available.

In constraint 3, the number of hours of finishing time used must be less than or equal to the number of hours of finishing time available. Constraint number 4: the number of hours of inspection and packaging time used must be less than or equal to the number of hours of inspection and packaging time available. When you look at the problem, there are 4 departments, and each department has some resource constraints. So, there will be four constraints.

The first one is defining the decision variable while developing a mathematical model. So, in our problem, the controllable inputs are the number of standard bags produced, and the number of deluxe bags produced. Here, the controllable input is nothing but your decision variables. That is our answer. So, S is the number of standard bags, and D is the number of deluxe bags. So, this is what we must find out about the S and D. For that, we have to formulate the problem. In linear programming terminology, these S and D are referred to as the decision variables.

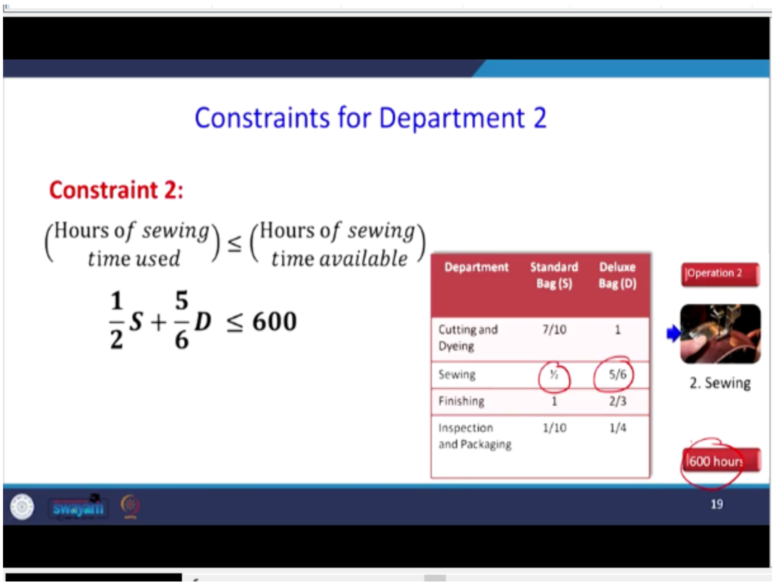


Then, we will come to the objective function. We have to write the objective function in terms of decision variables. So, if I write some objective function maximizing something, say c1*x1, for example, c2*x2, here x1* x2 is called the decision variable, and c1 and c2 are called the coefficient of the decision variables. In our problem, **the total profit contribution = 10S + 90D**. How did we get this 10? By selling 1 unit, the profit contribution is 10 dollars; by selling 1 unit of a deluxe bag, the profit contribution is 9 dollars.



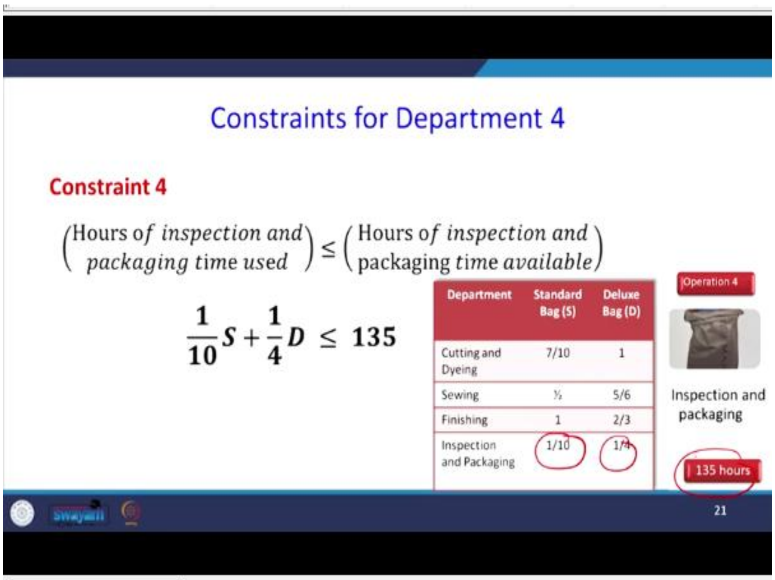
Now, we will come to write the constraint. So, write the constraint in terms of decision variables. What are the decision variables here? S and D are our decision variables. For example, the department's 1 hour of cutting and dyeing time used should be less than or equal to the hours of cutting and dyeing time available. So, (7/10) S + 1 D <= 630.

So, whatever time you are consuming on the left-hand side that should not exceed the capacity of the first department, there is a cutting and dyeing. So, (7/10) S + 1 D <= 630, which is our constraint 1.

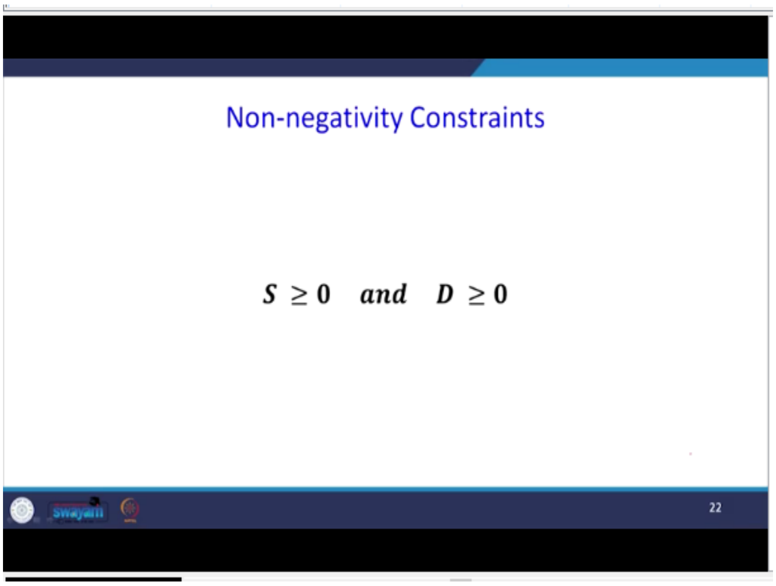


The important point is the unit of measurement on the left-hand side of the constraint must match the units of measurement on the right-hand side. In our problem, the left side unit is hours. So, the right side also should be in terms of hours; you should be very careful on this, then we will go right to the second constraint in the second department. The second department is the sewing department. The hours of sewing time used are less than or equal to the hours of sewing time available.

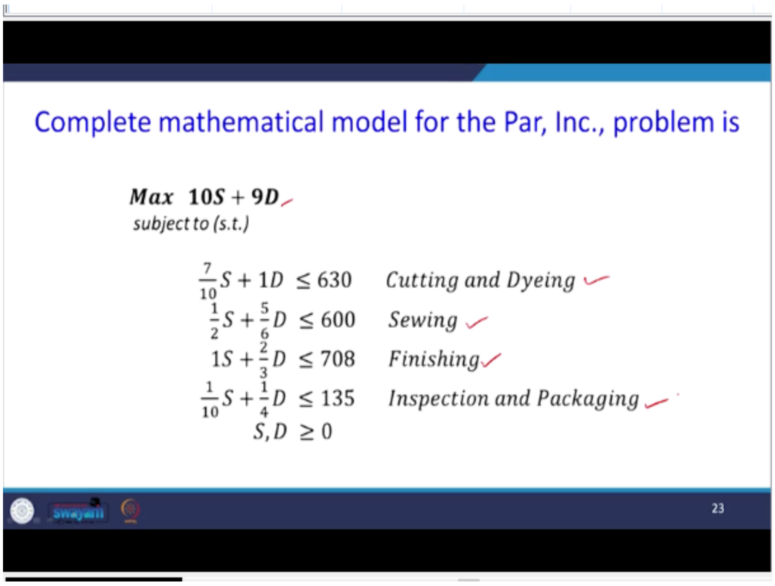
So, for example, $\mathbf{1/2}\ \boldsymbol{S}+\mathbf{5/6}\ \boldsymbol{D} \leq \mathbf{600}$. That is our constraint 2.



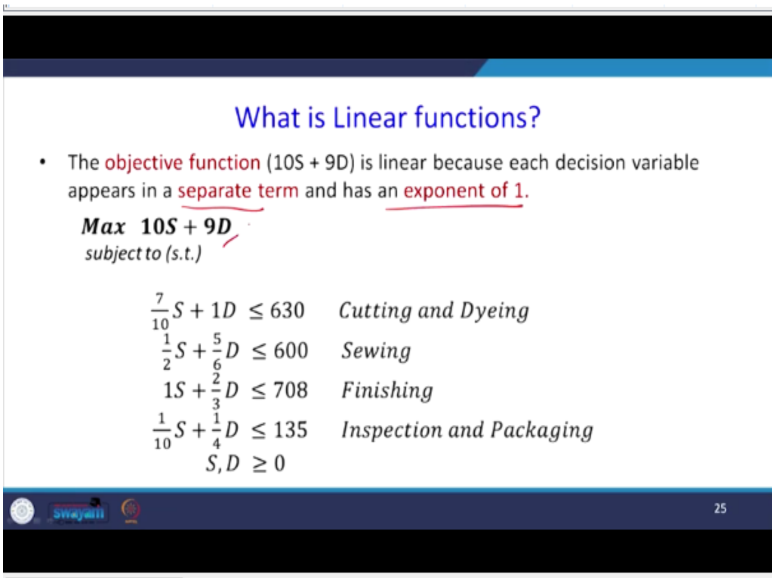
We will go to the third department. The third department is the finishing department. $\mathbf{1S+2/3\ D \leq 708}$ is constraint number 3. Then, the next department inspection and packaging. So, the hours of inspection and packaging time used are less than or equal to the hours of inspection and packaging time available. So, $\mathbf{1/10\ S+1/4\ D \leq 135}$ is the 4th constraint.



Another important constraint is non negativity constraint. So, the number of units produced in standard bags should be greater than or equal to 0. The number of units producing the deluxe bag should be greater than equal to 0 because S and D cannot have negative numbers. $\boldsymbol{S \geq 0 \quad and \quad D \geq 0}$ is the non-negativity constraints. This is an important constraint in the LP problems.

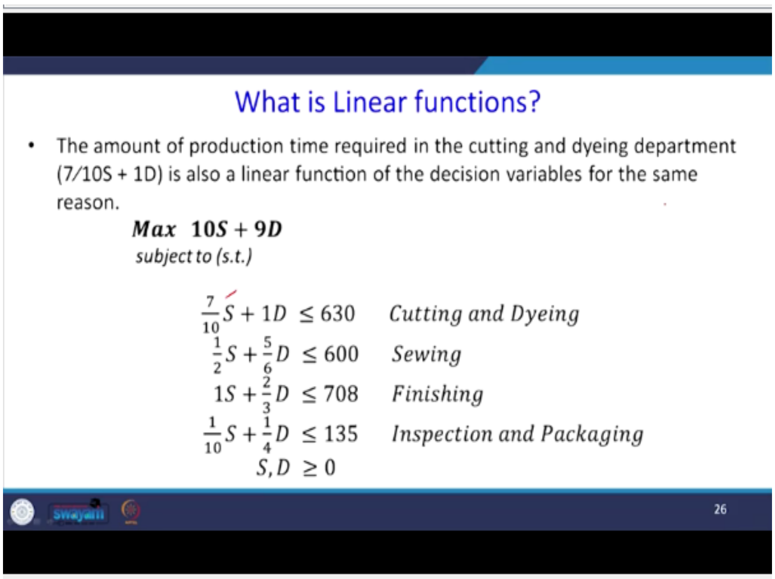


So, the complete mathematical model is like this $\boldsymbol{Max}\ (\mathbf{10S+9D})$ subjected to constrain number 1, constrain number 2, constrain number 3, constrain number 4.

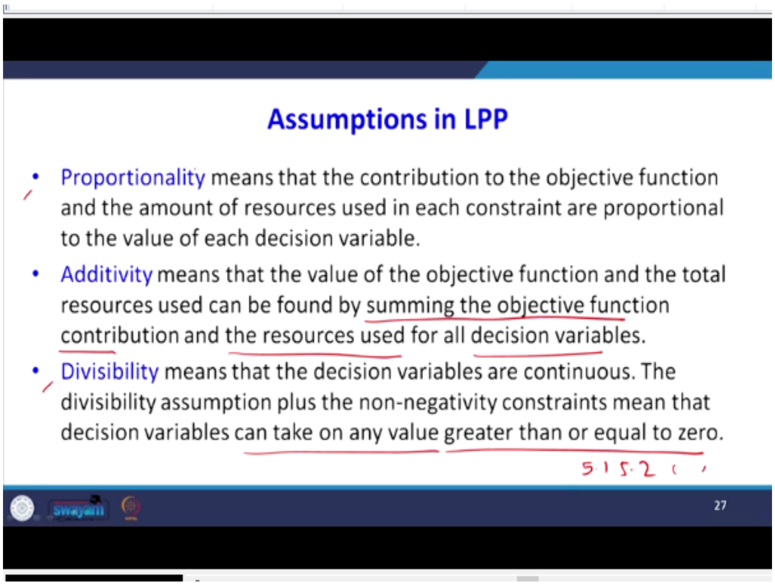


You may ask, what is linear function? Why is it called the linear programming problem? In mathematical functions, each variable appears in a separate term and is raised to the first power, then called a linear function. When you look at this, the power of S is 1, the power of D is 1, and each is in separate terms; it is not S and D. So, we can say this objective function is in the linear form.

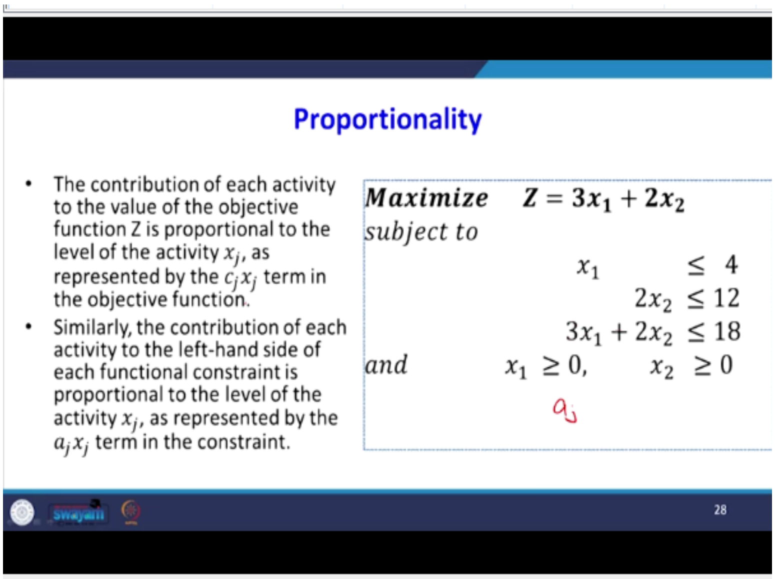
We look at all the constraints; all the constraints, the power is 1, and there is no multiplication of S and D. So, all the constraints are also in the linear form. So this type of model is called your linear programming model. The objective function 10S + 9D is linear because each decision variable appears in a separate term and as an exponent of 1. So, this is our linear.



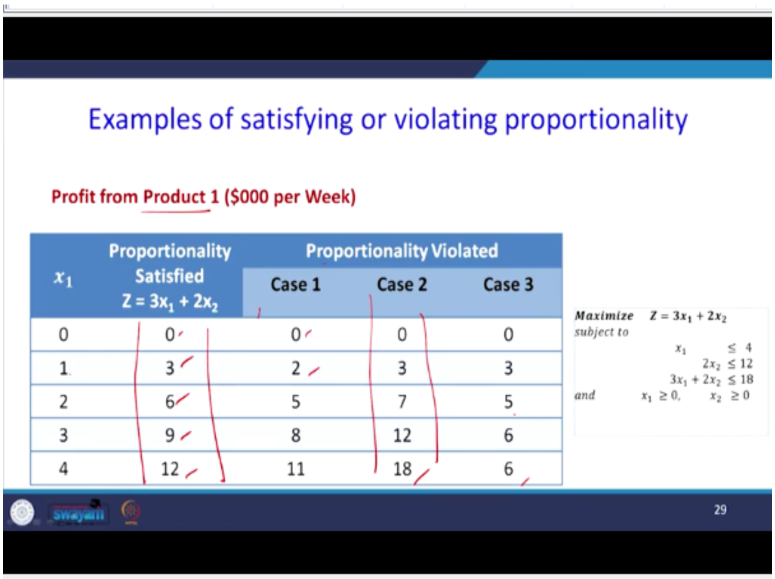
Now, we will come back to the constraint. The amount of production time required in the cutting and dyeing department is 7/10S+1D is also a linear function of the decision variable for the same reason. So, now, the objective function is also linear, and the constraint is also linear, so we are calling it is the linear programming problem.



Then, what are the assumptions in the linear programming problem? The first assumption is proportionality. Proportionality means that the contribution to the objective function and the number of resources used in each constraint is proportional to the value of each decision variable. I have an example in the coming slides; I will explain in detail what proportionality is. Then additivity means that the value of the objective function and the total resources used can be found by summing the objective function contribution and the resources used for all decision variables. That is the additivity assumptions, what the divisibility means, the decision variables are continuous, the divisibility assumption plus the nonnegativity constraint mean that the decision variable can take any value greater than or equal to 0 that means you can have 5.1, 5.2 any number is possible. That is the meaning of this divisibility assumption. Now, we will explain what the meaning of this proportionality is.



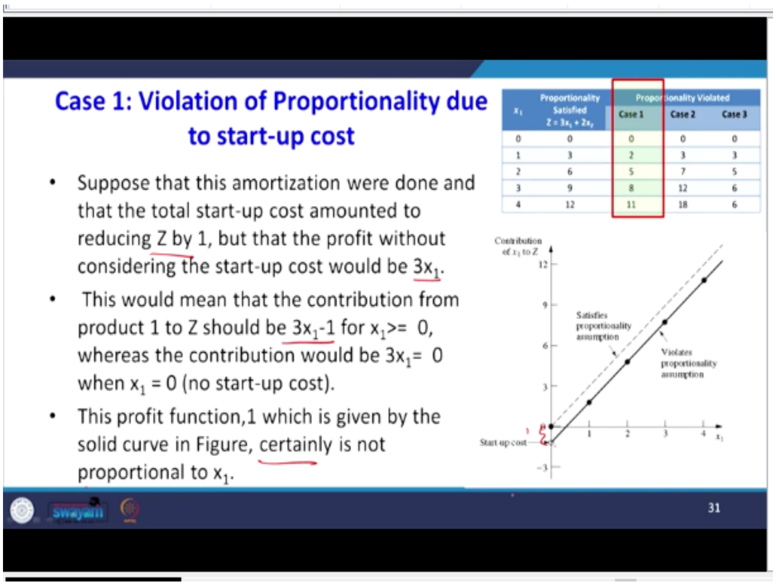
Suppose an objective function is like this: maximize Z = x1 + 2x2. There are constraints is there x1 <= 4, 2x2 <= 12. The contribution of each activity to the value of objective function Z is proportional to the level of activity x as represented by the cj and xj terms in the objective function. Similarly, the contribution of each activity to the left-hand side of each function constraint is proportional to the level of activity xj, which is represented by the ajxj.



Now, look at the example of satisfying or violating the proportionality. Suppose the profit from product 1; say only x1, say 1,000 dollars per week. So, x1 = 0 when you substitute in the objective function, so 3x1 will be 0; this is okay**.** When substitute x1 = 1 so you will get 3. In this 3x1, when you substitute equal to 2, it is 6; when you substitute x1 = 3, it is equal to 9, so when x1 = 4, 12. So, in this portion, there is no problem.

This is the proportionality assumption that is satisfied. What is the meaning when x increases correspondingly, the y also increases by having this assumption that is 3x1? Now, look at case 1. In case 1, what happens when x1 = 0 is okay? When x1 = 1 instead of 3 it is 2. So, one value is decreasing. So, that means it is the proportionality assumption is violated. Here is case 2.

Here is what is happening: when x increases the value of Z, it is increasing beyond 12. So, in that case, also, the proportionality assumption is violated. Look at the last column. When x increases, the value of Z decreases. Here also, the proportionality assumption is required to be violated. I will explain each case.

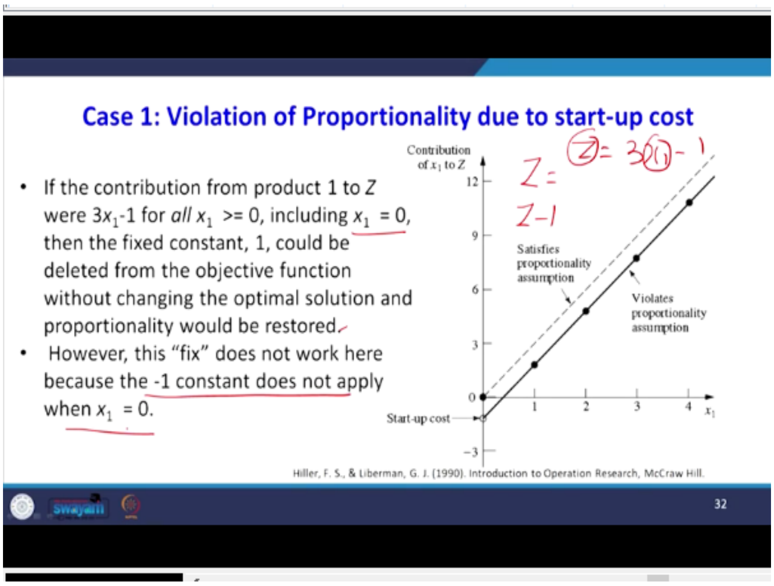


Case 1 violation of proportionality due to start-up cost. This case will arise if there are various startup costs associated with initiating the production of product 1. For example, there might be costs involved with setting up production facilities. There might also be a costs associated with arranging the distribution of the new product because these are one-time costs; they would need to be amortized on a per-week basis to be commensurable with Z.

There is a profit in 1000s of dollars per week. So, what is happening here? The company has spent some money on the start-up. So, every time in every week, so, what is happening every week some money has to be deducted from your objective function; that is why here, you see that every time you see 3, for example, 1 unit is detected, so 3 -1 = 2, 6 – 1= 5. Here, this value is Z, and the value is 9 – 1=8. So, here, 1that is Z - 1, and the -1 is over amortized cost.

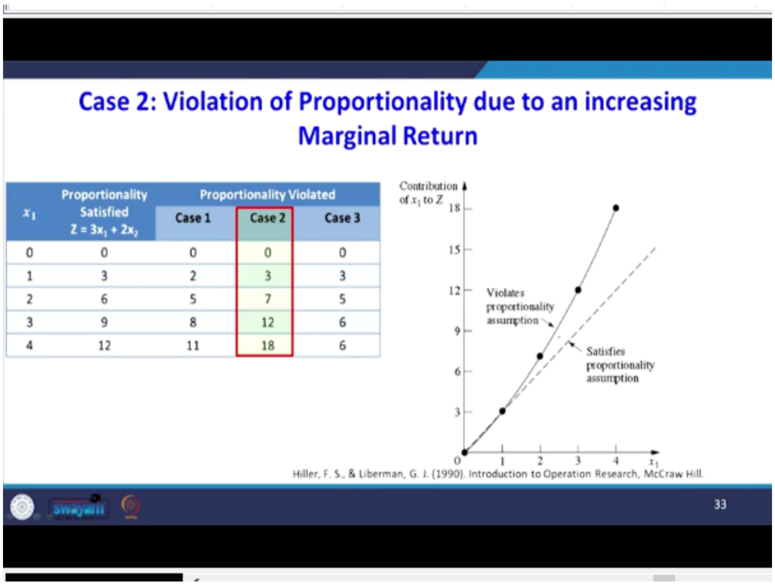
Suppose this amortization was done at the total startup cost, which amounted to reducing Z by 1, but the profit without considering the start-up cost would be 3x1. If there is no startup cost, the profit will be 3x1. But we are considering the start-up cost for the start-up cost every week; some amount is deducted from our profit, that amount is 1 unit. So, this would mean that the contribution from the product 1 to Z should be 3x1 – 1, where x1 is greater than equal to 0.

Because if it is x1 = 0, there is no profit. Whereas the contribution would be 3x1 = 0 when x1 = 0, there is no start-up cost. So, the profit function 1, given by the solid curve in the figure see this one, the bottom, certainly is not proportional to x1. So, the dotted line satisfies the proportionality assumption, and the solid line violates the proportionality assumption why because here, 1 unit is subtracted. This is the violation of proportionality case 1.

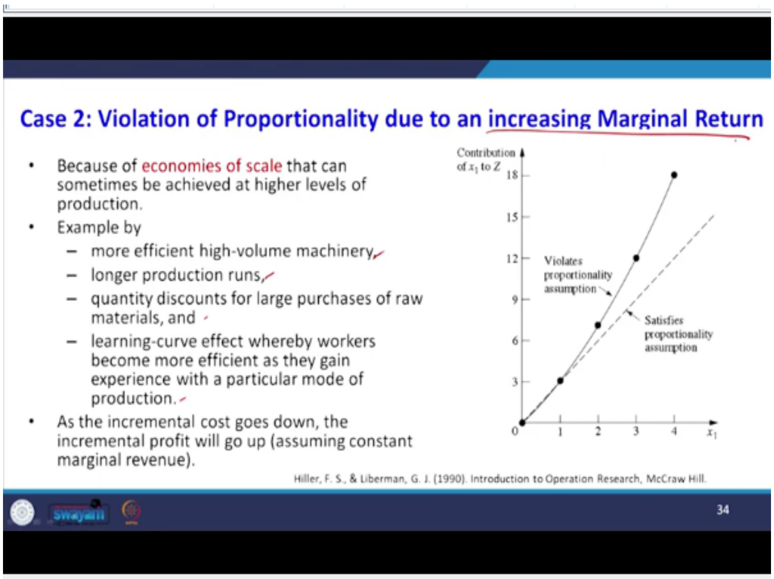


If the contribution from product 1 to Z where 3x1 – 1 for all x1 is greater than 0, including x1= 0, then the fixed constant 1 should be deleted from the objective function without changing the optimal solution and the proportionality will be restored, what we can see we know from the Z to consider this fixed cost, what we can do, we can subtract Z - 1. So, that now, the proportionality assumption will be satisfied, but there will be a problem.

However, this fix does not work here because the -1 constant does not apply when x1 = 0. So, what does that mean? So, Z = 3x1 - 1. Suppose, when you put x1 = 0, you are not at all producing anything, but there is a loss of 1 unit. So, that assumption will not work. So, that fix that is from the objective function, you cannot subtract -1 because that will affect the situation where x1 = 0.

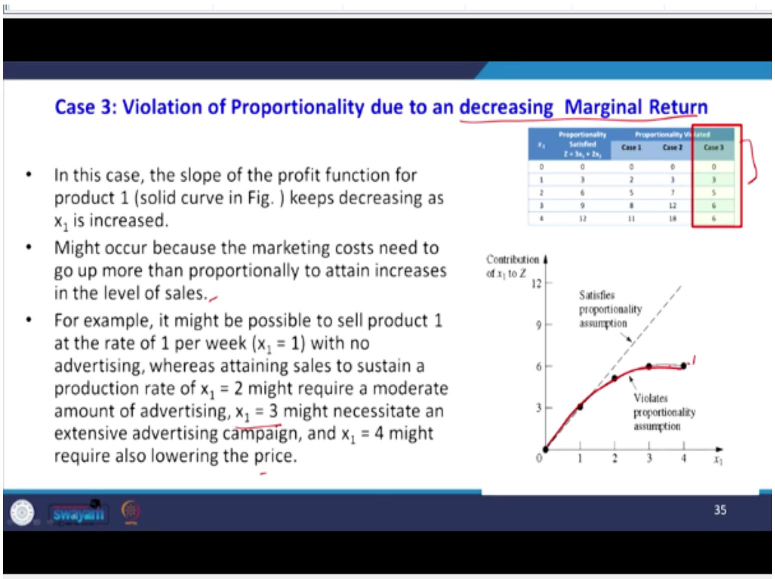


I will look at case 2. This case may occur a violation of proportionality due to increasing marginal return. When you look at this picture and see that the dotted line satisfies the proportionality assumption, you see that here, the Z value is increasing, which is also a violation of your proportionality. This can occur due to an increase in marginal return.



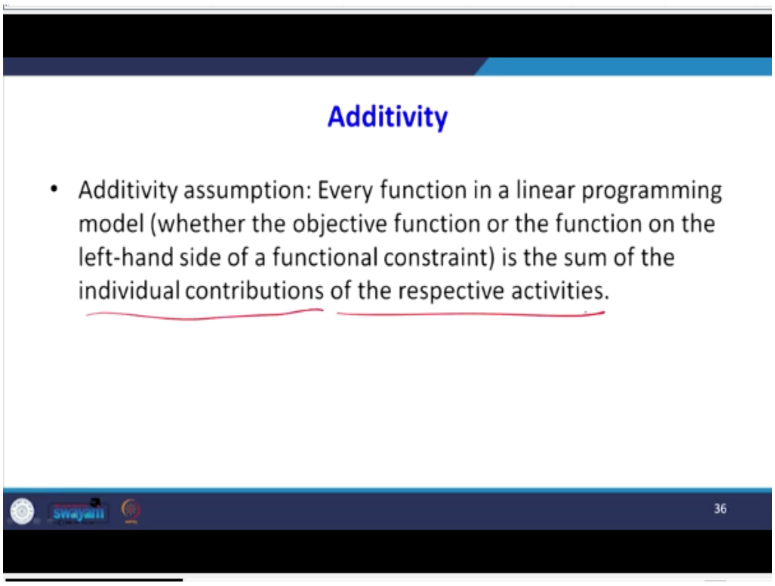
Because of economies of scale, this can sometimes be achieved at a higher level of production. So, because the economics of scale will reduce your costs, your profit will increase. For example, when this kind of situation occurs, with a more efficient volume of missionaries, initially, the profit will be less, but after some time, the profit will be more. Then, longer production runs.

When there is a longer production run, there would not be any setup cost; if the setup cost is less, your profit will increase proportionately. Quantity discounts for large purchases of raw materials, then the learning curve effect whereby the workers become more efficient as they gain experience with a particular mode of production. As the incremental cost goes down, the incremental profit will go up, assuming constant marginal revenue. Due to what is happening here, there is an increasing marginal return. This increasing marginal return is a violation of our proportionality assumption.

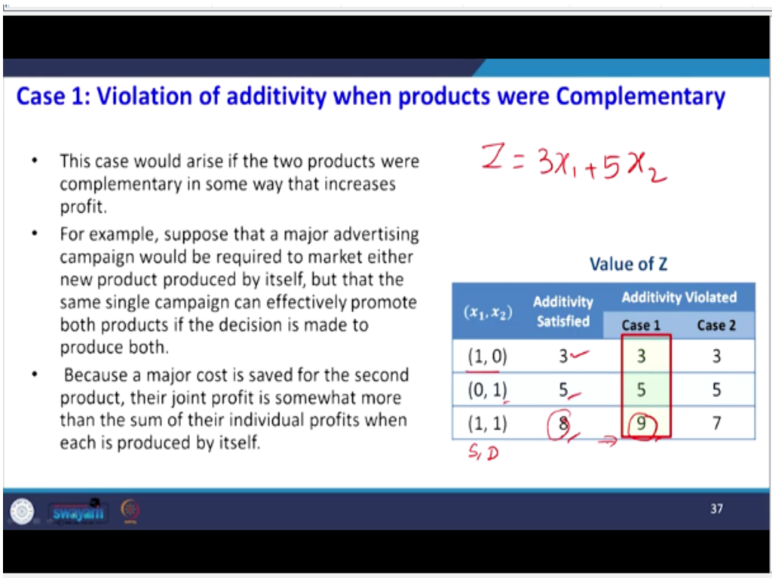


Case 3 is a violation of proportionality due to a decrease in marginal return. In this case, the slope of the profit function for product 1 keeps decreasing as x1 is increased. Look at this last column the corresponding line is this line. This might occur because the marketing costs need to go up more than proportionality to attain an increase in the level of sales. When there are more sales, there are more marketing costs.

For example, it might be possible to sell product 1 at the rate of 1 per week x1 = 1 with no advertising, whereas attaining sales to sustain your production rate at x1 = 2 might require a moderate amount of advertising x1 = 3 might necessitate an extensive advertising campaign and x1 = 4 might require also lowering the price. So, what will happen? When you lower the price, your profit will decrease. That is why you were the solid line is falling like this; this is also a violation of your proportionality assumption. This is due to decreasing marginal return.



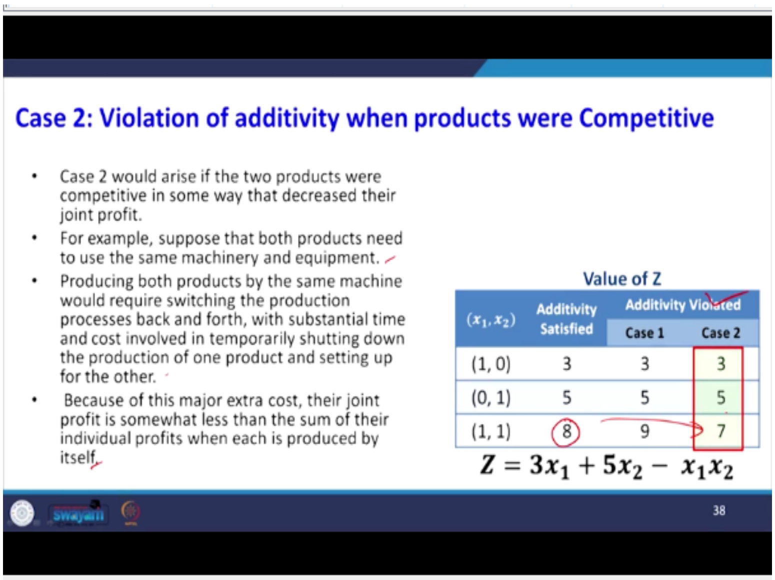
Then another assumption is additivity. Every function in a linear programming model, whether the objective function or the function on the left-hand side of the functional constraint, is the sum of individual contributions of the respective activities.



Now, we will see the violation of additivity when the products are complementary. Assume the objective function is like this: Z = 3x1 + 5x2. Suppose, if I substitute (1, 0), it is 3, (0, 1), it is 5; if I am producing (1, 1), it is 8. Now look at the right-hand side; 3 and 5 are okay. But when I am producing both the products now, instead of the 8, it has increased to 9. This is a situation where both products are complementary.

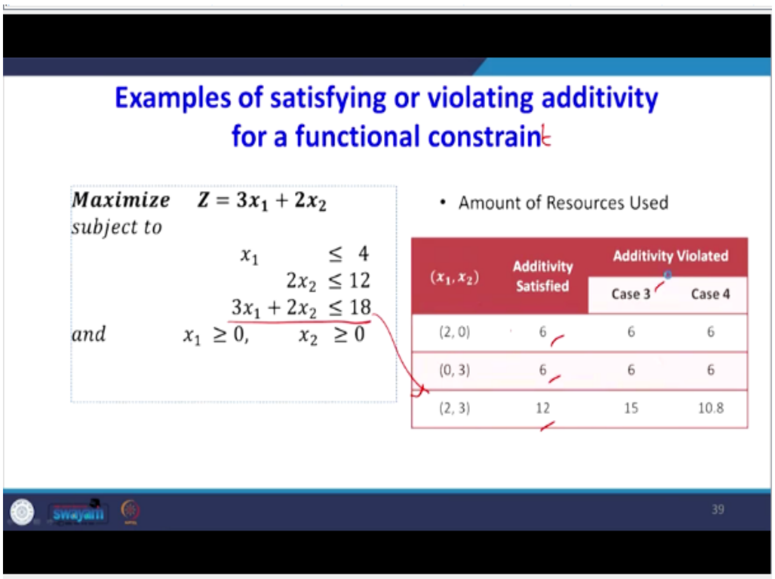
For example, this case would arise if the 2 products were complementary in some way that increases the profit. For example, suppose the major advertising campaign would be required to market either new product produced by itself, but the same single campaign can effectively promote both products if the decision is made to produce both. Because of the major cost to saving for the second product, their joined profit is somewhat more than the sum of their individual profits when each is produced by itself.

So, what is happening is when you produce both standard bags and deluxe bags together, there may be a possibility that it seems both the products are complementary; you were profitable, and instead of 8, it may become 9. So, this is a violation of your additivity.

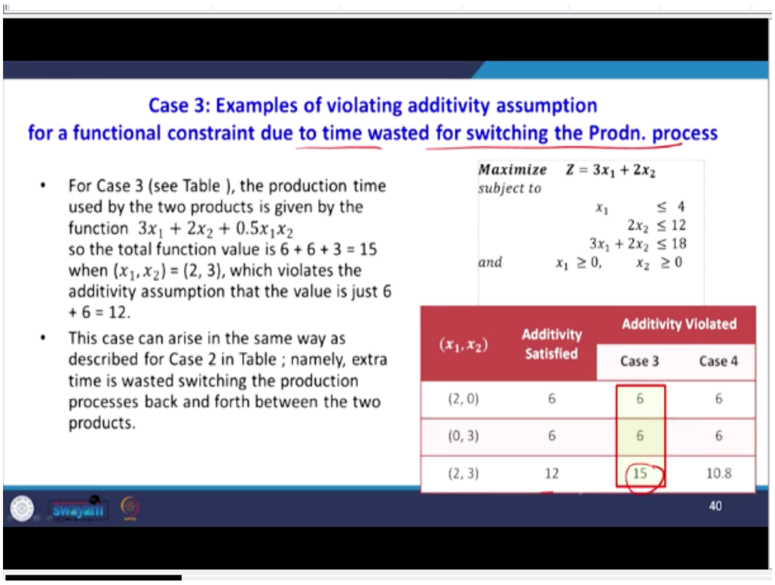


There may be another situation that violates additivity when the products are competitive; in case 2, this situation would arise if the 2 products were competitive in some way, which would decrease their joint profit. For example, suppose that both products must use the same machinery and equipment. So, producing both products by the same machine would require switching the production processes back and forth with substantial time and cost involved in temporarily shutting down the production of one product and setting up for the other.

Because of this major extra cost, their joint profit is somewhat less than some of their individual profits when each is produced by itself. So, what is happening here? Instead of 8, now it becomes 7. If both products are produced on the same machine, there may be a setup cost. So, that setup cost is forcing you to decrease your profit. So, this is a violation of your additivity for the objective function.

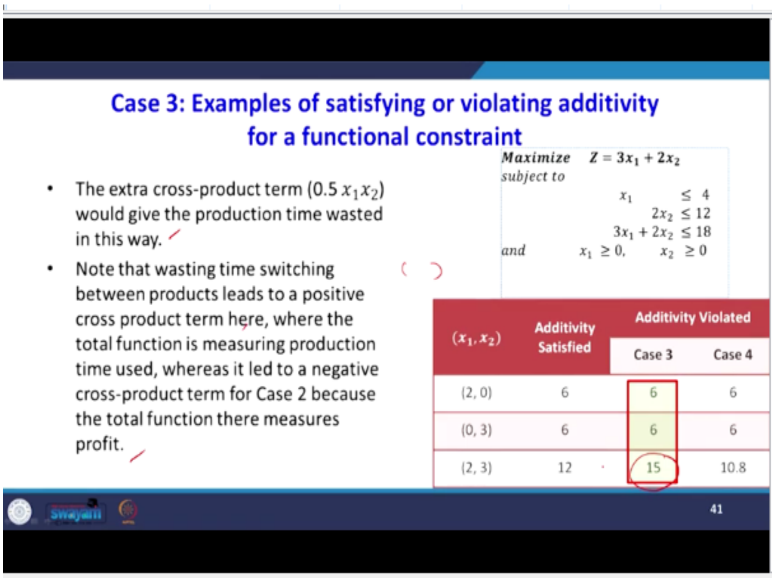


Now, here are examples of satisfying or violating identity for your constraint. Assume that the objective function is 3x1 + 2x2; this is the given problem. Look at the amount of resources used. Suppose x1 = 2 we will take this constraint 3x1 + 2x2 less than or equal to 18. So, it is 6, here are also 6. So, here also, 6 + 6, is 12. Here, the additivity is satisfied for this constraint. Now, look at case 3.

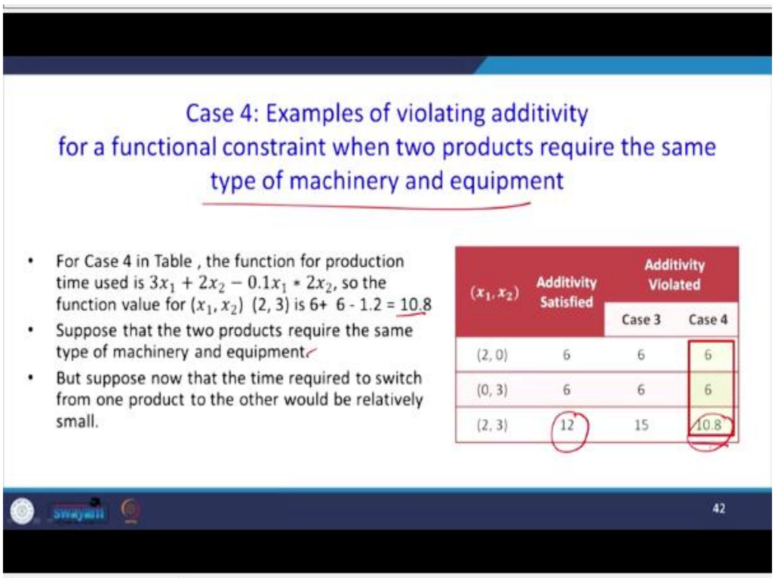


Case 3 may be an example of violating the additivity assumption for the functional constraint due to time wasted switching the production process. In case 3, look at this table. It should be 12, but it is 15. The production time used by the 2 products is given by the function 3x1 + 2x2 + 0.5(x1*x2). So, the total function value is 6 + 6 + 3, which is 15. When x1 and x2 are 2 and 3, respectively, which violates the additivity assumption, the value is just 6 + 6 =12.

It should be 12. But we are getting 15. When this situation may occur, this can arise in the same way as described in case 2 in the table in the previous slides, namely, extra time is wasted switching the production process back and forth between 2 products. So, the time consumed is more.



Another example of satisfying or violating relativity for a functional constraint is the extra cross-production product term 0.5(x1*x2), which would give you the production time wasted in this way. Note that wasting time switching between products leads to a positive cross-product term, whereas the total function measures the production time used. However, it leads to a negative cross-product term for case 2 because the total function there misses the profit. See, in the profit function, when you multiply 2 things, you get the higher value, but here, we are getting the higher value that is not good for you because here it is the time resources.



So, the last case, for example, of violating additivity for a functional constraint when 2 products require the same type of machinery and equipment. In this situation, you see that it has to take 12 units, but now it is consuming only 10.8 units. So, here, in case 4 in the table, the function for the product time used is $3x1 + 2x2 – 0.5(x1*x2)$, so we are getting 10.8. Suppose the 2 products require the same type of machine and equipment; there would not be any extra time for the switching cost or switching time or setup time.

So, his overall time will decrease, and that is why we are getting less time. So, this lecture is very important in formulating the problem. I have taken a sample problem for the 7maximization case; I have formulated the problem. Then, I explained the assumptions in the linear programming problems. All three assumptions are explained with the help of examples. Thank you very much.