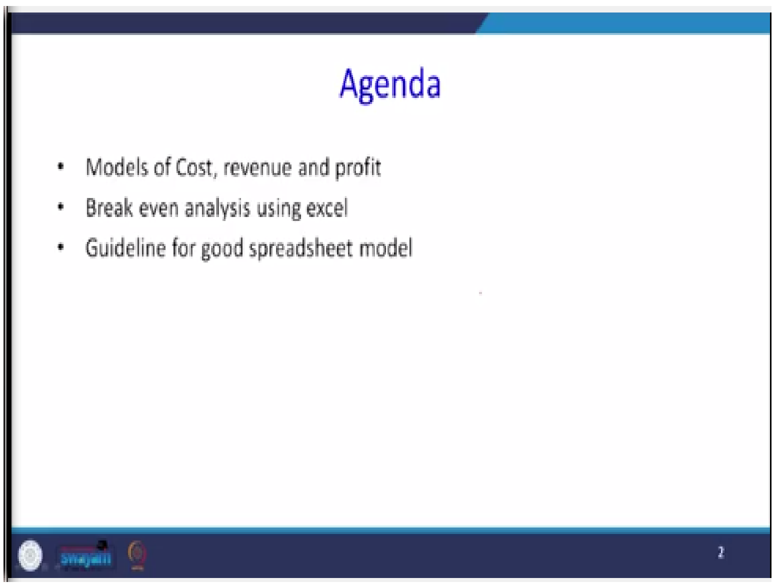
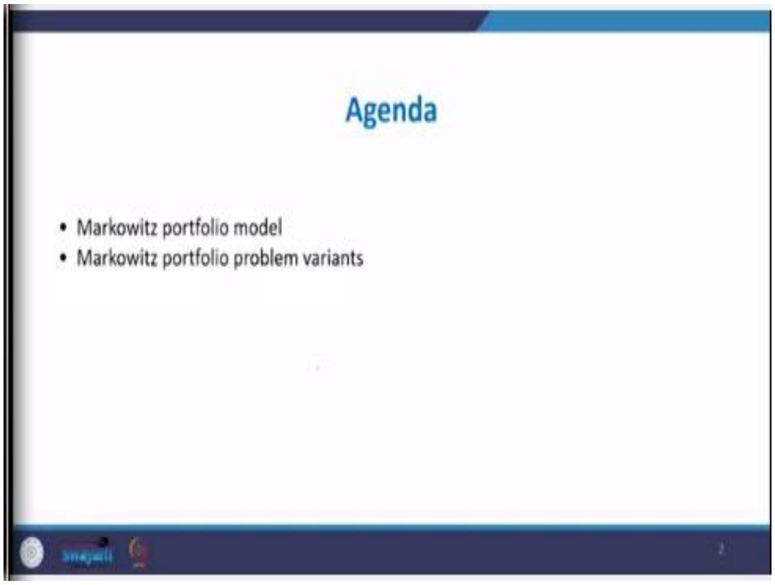
**Decision Making With Spreadsheet**
**Prof. Ramesh Anbanandam**
**Department of Management Studies**
**Indian Institute of Technology-Roorkee**

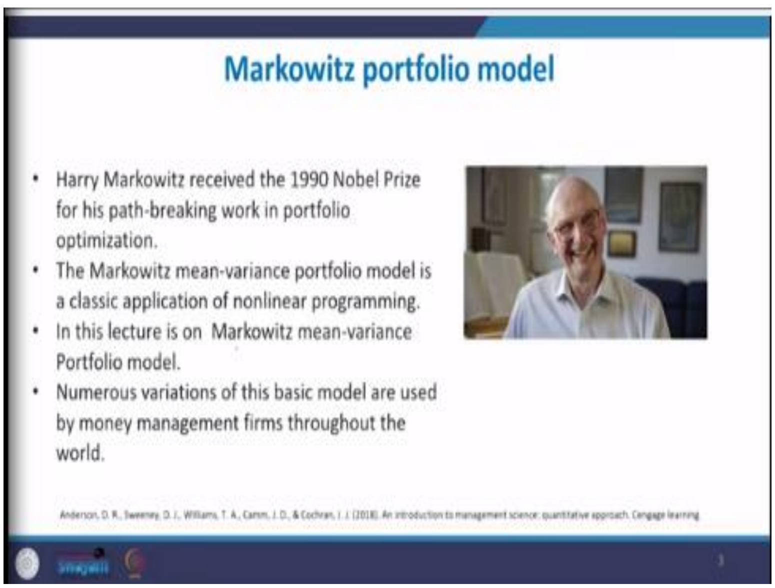
**Lecture-26**
**Non-Linear Optimization Models-III**

Dear students, in the previous class, I discussed index funding using non-linear programming methods. In this class, I am going to explain Markowitz's portfolio model, which also comes under the category of non-linear programming problems. After that, I will explain the variance of these Markowitz portfolio problems.



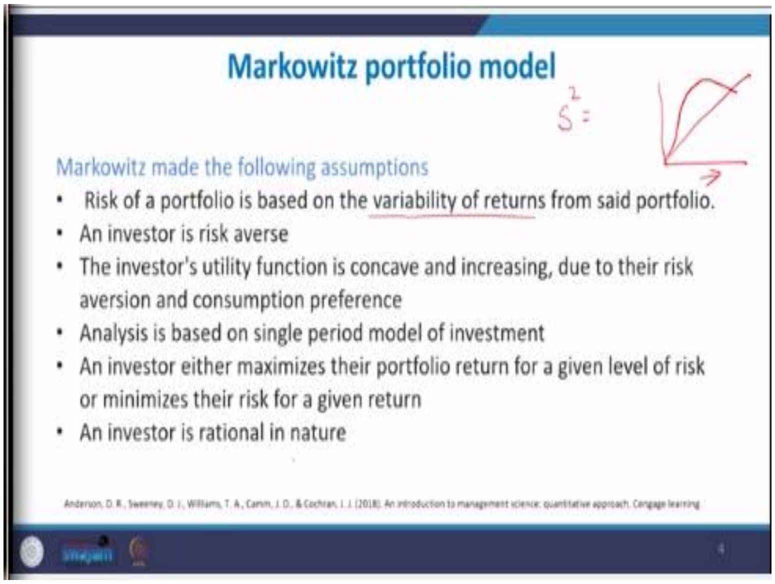

So, the agenda for this lecture is Markowitz's portfolio model and variants.



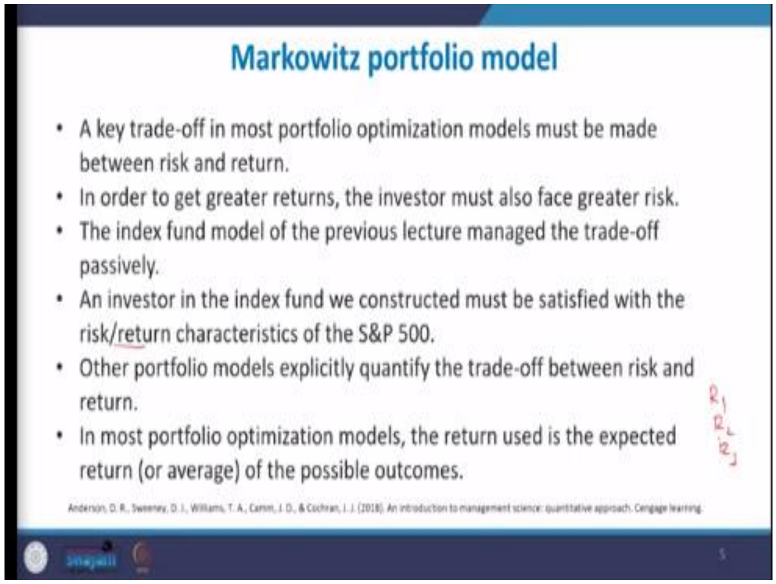

Harry Markowitz received the 1990 Nobel Prize for his path-breaking work in portfolio optimization. This professor of economics, Harry Markowitz, received a Nobel Prize portfolio model, which we will discuss now. So, Markowitz means that the variance portfolio model is a classic application of non-linear programming. In the previous class on index funding, we were concerned about only the mean return rate; we were not bothered about the variance of the portfolio.

However, in the mean-variance portfolio model, we will consider both variances and the mean of the return. So, in this lecture, we will discuss Markowitz's mean-variance portfolio model. Money management firms use numerous variations of this basic model throughout the world.



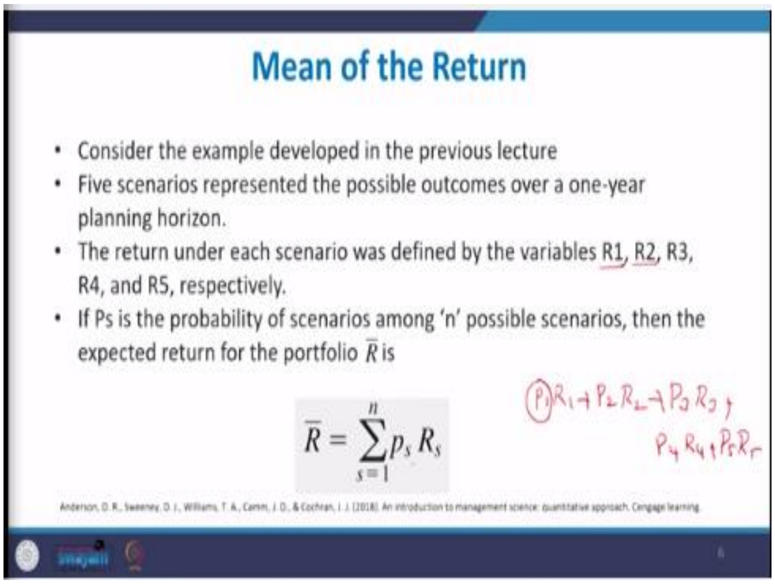

What are the assumptions in Markowitz's model? The assumption is the risk of your portfolio is based on the variability of returns from said portfolio. So, what I mean here is that the variability of the returns measures the risk. What is the variability? It is nothing but the variance of the return. Then, an investor is risk-averse; his assumption is that an investor will try to minimize or avoid risk. The investor’s utility function is concave and increasing. So, the utility function will be in this shape; what is its meaning? It is concave and increasing. So, what is happening? How can we explain this?

See this is the straight model, so what is that? It is utility decreases after when the variable in x is increasing; beyond that, you see that the utility is increasing. After that, the utility is decreasing. The analysis is based on a single-period model of investment; we are considering only one model and one period. The investor either maximizes their portfolio return for a given level of risk or minimizes their risk for a given return; that is why we are calling it the mean-variance theorem. An investor is rational in nature.



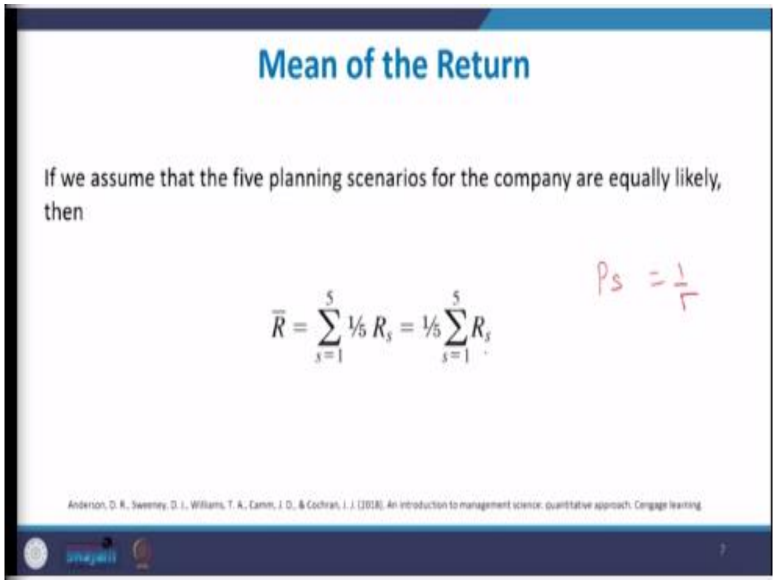

A key trade-off in most portfolio optimization models must be made between risk and return. So, we need to have a trade-off between risk and return. So, to get a greater return, the investor must also face a more significant risk. The index fund model of the previous lecture managed trade-off passively; what is the meaning of this trade-off passively? We did not consider the variance of the return; we have considered only the meaning of the return. An investor in the index fund we constructed must be satisfied with the risk-written characteristics of the S&P 500.

He has to consider the risk at the same time in return; also, previously we have considered only the return. Other portfolio models explicitly quantify the trade-off between risk and return. In most optimization models, the return is used as the expected return or average of possible outcomes. Suppose return 1, 2, return 3. The return means the average of this return; otherwise, the expected value of this return.



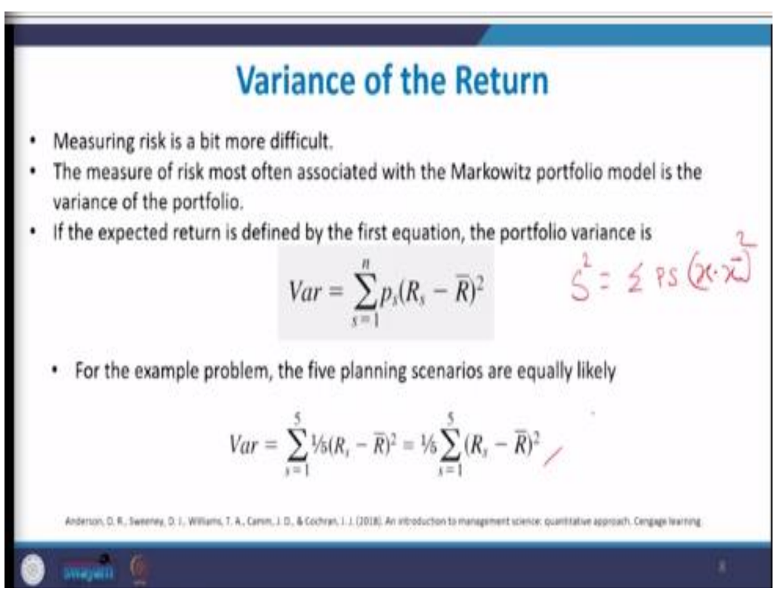

Then we discuss what is the mean of the return. Consider the example developed in the previous lecture; there were 5 scenarios. So, 5 scenarios represented the possible outcomes over a 1-year planning horizon. The return under each scenario was defined by the variables R1, R2, R3, R4, and R5, respectively. That means for year 1 if scenario one is repeated, the return will be R1; the next year, if scenario 2 is repeated, the return will be R2, like this, R3, R4, and R5.

If P is the probability of scenario among 'n' possible scenarios, then the expected return for the portfolio R bar is sigma of ps multiplied by Rs, so s = 1 to n. So, there are five scenarios, so what will happen? This will be P1 R1 + P2 R2 + P3 R3 + P4 R4 + P5 and R5, P1 represents the probability of scenario 1, and P2 represents the probability of scenario 2. So, we are giving the common name Ps probability of scenario, Rs is that corresponding return if that scenario occurs.




So, the meaning of the return, in that expression the Ps, we have to find out the value of Ps; what is happening? If we assume that 5 planning horizons for the company are equally likely, then the probability of each scenario is 1 by 5; that is why we have written 1 by 5 for each scenario. So, the meaning of the return will be one by 5 sigma s = 1 to 5 Rs.

$$\overline{R} = \sum_{s=1}^{5} 1/5\, R_s = 1/5 \sum_{s=1}^{5} R_s$$

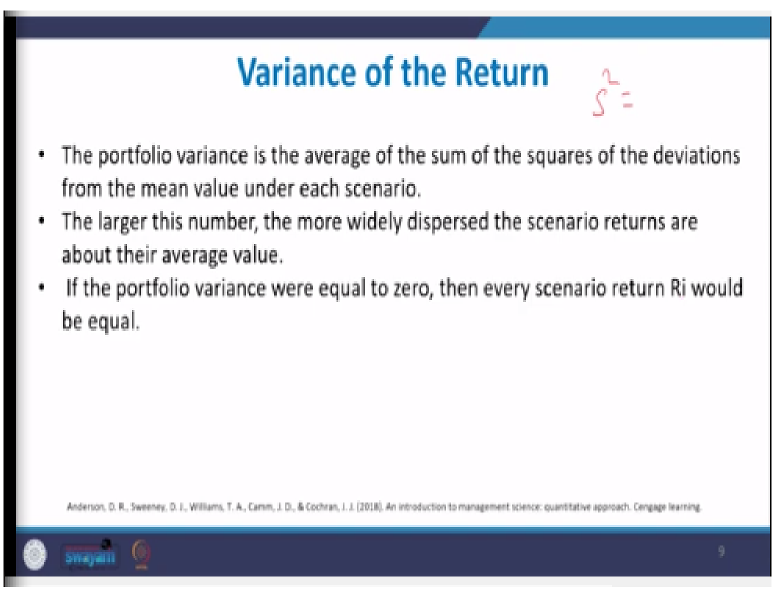


The next one is we must find out the variance of the return. So, measuring risk is more difficult. The measure of risk most often associated with the Markowitz portfolio model is the variance of the portfolio. So, we are capturing, we are measuring the risk by the variance of the portfolio. If the expected return is defined by the first equation that we have discussed here, that is, R bar, the portfolio variance is variance = sigma s = 1 to n, Ps Rs - R bar whole square.

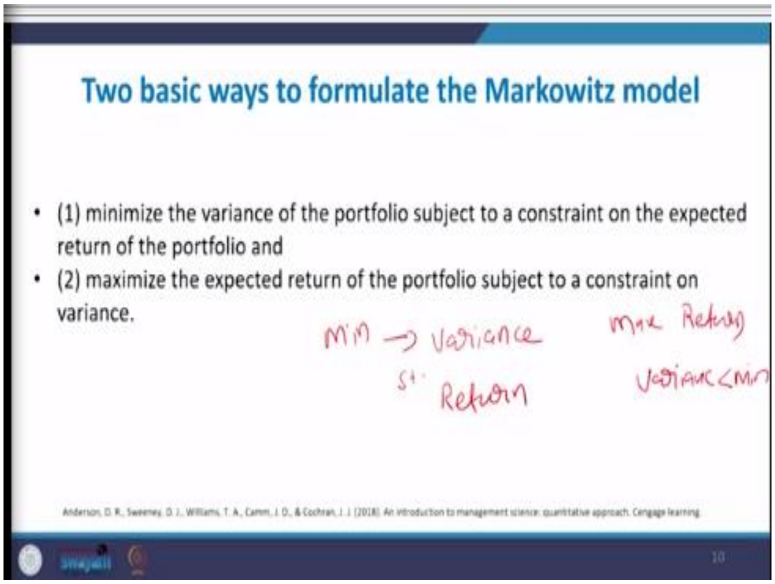
$$Var = \sum_{s=1}^{n} p_s(R_s - \overline{R})^2$$

It is simple to use our variance formula; we know traditionally the sigma of Ps (x - x bar) whole square. Here, the x bar means your R bar, x means the corresponding return and Ps means the probability. For example, for the problem that you have discussed, 5 planning scenarios are equally likely. So, instead of Ps, we can replace 1 by s Rs - R bar whole square, so this is the expression for our variance of the portfolio.

$$Var = \sum_{s=1}^{5} 1/5(R_s - \overline{R})^2 = 1/5\sum_{s=1}^{5}(R_s - \overline{R})^2$$

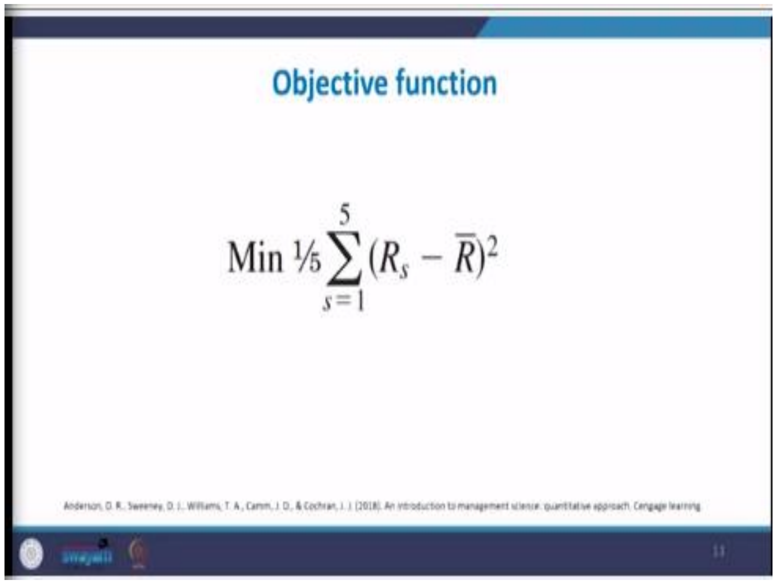



The portfolio variance is the average of the sum of square deviations from the mean value under each scenario. The larger this number, the more widely dispersed the scenario returns are about their average values. If the variance is higher, that means there is more spread in their returns; it is more risk. If the portfolio variance were equal to 0, then every scenario return Ri would be equal. So, that means in every scenario, you are getting an equal return, which means there is no variance, and the variance is 0.

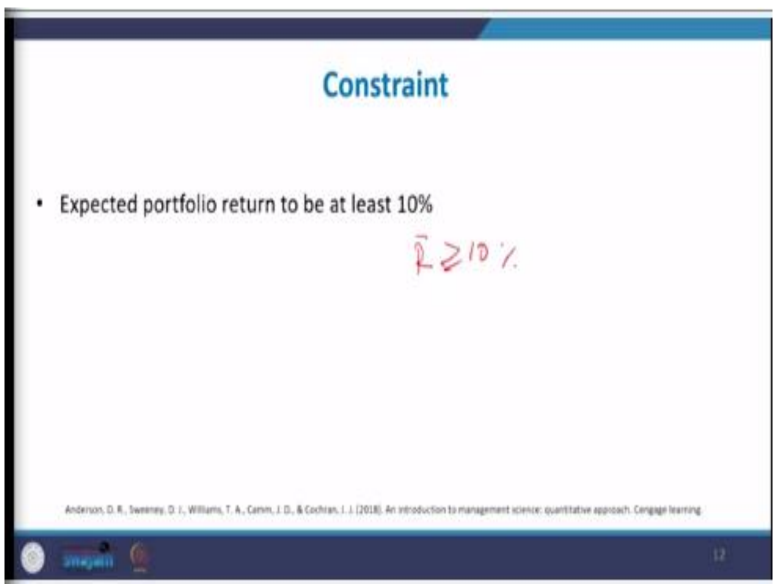



Fundamentally, there are 2 basic ways to formulate the Markowitz model. There are two ways we can formulate the Markowitz model; one way is minimizing the variance of the portfolio subject to a constraint on the expected return of the portfolio. So, we can minimize the variance; what is the constraint? The expected return has to be achieved. Otherwise, we can maximize the expected return of the portfolio if it is subject to a constraint on the variance.

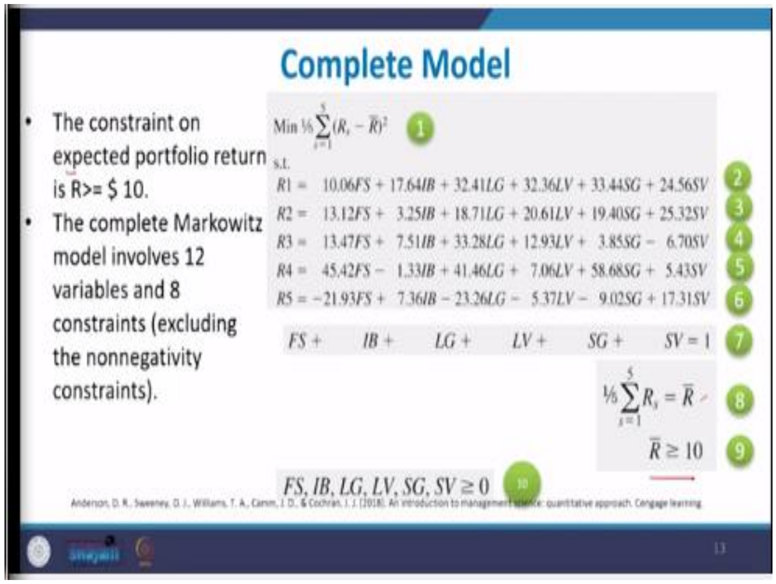
So, there are 2 things: there is variance, and another one is returned; one thing is that we can minimize the variance. The return has to be achieved, and there is a constraint on the return. Otherwise, what can we do? We can maximize the return, but the variance has to be minimized, or it has to meet the constraint. But in this problem, what we are going to say is we are going to consider case 1; what is case 1? We are going to minimize the variance of the portfolio.



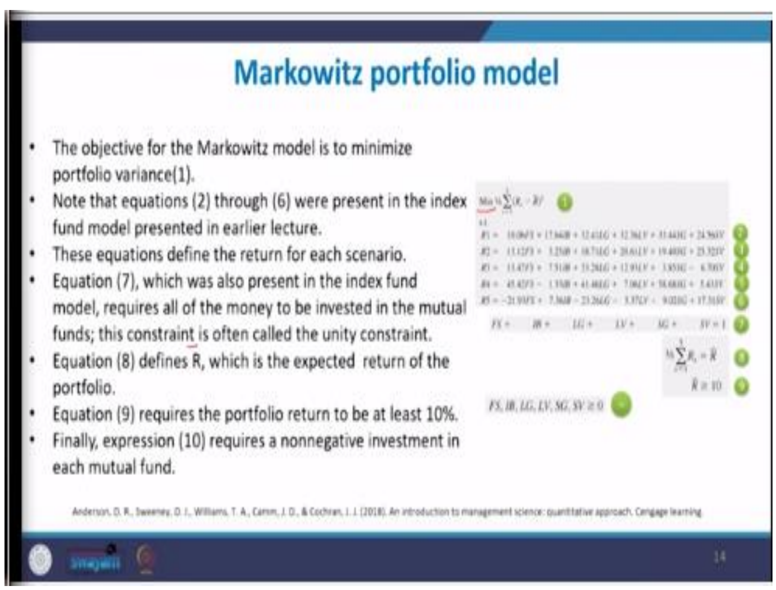

So, the objective function is that we are going to minimize the variance, as we have already seen.




What is the constraint? The constraint is the expected portfolio return to be at least 10%. So, when we say the return is the expected, that means the R bar is greater than 10%, and the variance has to be minimized.

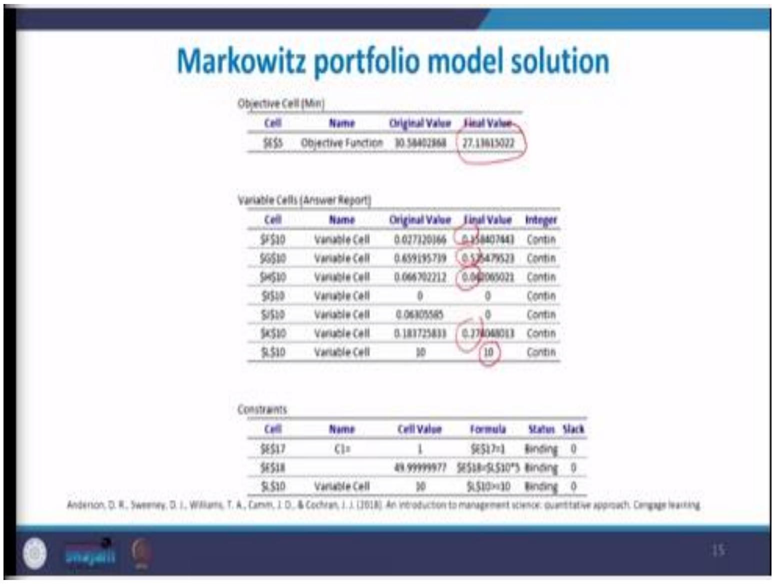



Now, this is a complete model. You see, there are 10 equations; equation 1 represents the objective function, and 2 represents scenario 1. So, 10.06 FS, this is a portfolio 1 IB, LG, LV, SG, SV. The sum of proportions FS, IB, LG, LB, SG, and SV should be equal to 1, and you see that we are finding the mean of the return, so P s sigma of R s. Then the mean should be greater than or equal to 10%. So, the constraint on this expected portfolio return is R should be greater than it is. R bar should be greater than equal to 10. So, the complete Markowitz model involves 12 variables and 8 constraints, excluding the nonnegativity constraint.




Now I will explain the formulation of this Markowitz portfolio model. The objective function of the Markowitz model is to minimize the portfolio variance, which is why we are minimizing it. Equations 2 to 6 were present in the index fund model in the earlier lecture, and 2 to 6 in the previous lecture. We also had the same thing; these equations define the return of each scenario. Equation 7, which is also present in the index fund model, requires all of the money to be invested in the mutual funds. So, this constraint is often called the unity constraint.

That means the amount in which we have all the proportions sum of the proportions the sum of the probability should be 1. Equation 8 defines the average return, expected return R bar, which is the expected return of the portfolio. Equation 9 represents the portfolio return to be at least 10%, which is a constraint. Finally, expression 10 requires a nonnegative investment in each mutual fund.




Now, I am going to solve this problem with the help of a solver; then I will come back to the result. Now, I am going to explain the Excel model of this non-linear problem. The first one is the objective function,

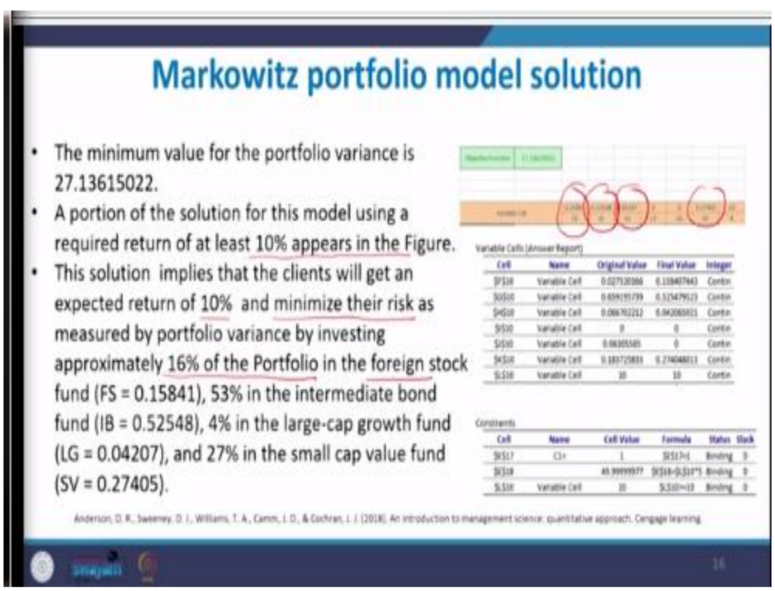
$$\text{Min } 1/5 \sum_{s=1}^{5} (R_s - \overline{R})^2$$

so, the E6 looks at the objective function this is 1 upon 5, so it is a 0.2, R1 - R bar whole square + 1 upon 5 R2 - R bar whole square and R3 - R bar whole square + R4 - R bar whole square and the last one plus R5 - R bar whole square. So, for that, you need to have the mean that is the R bar, so the R bar is written in E18.

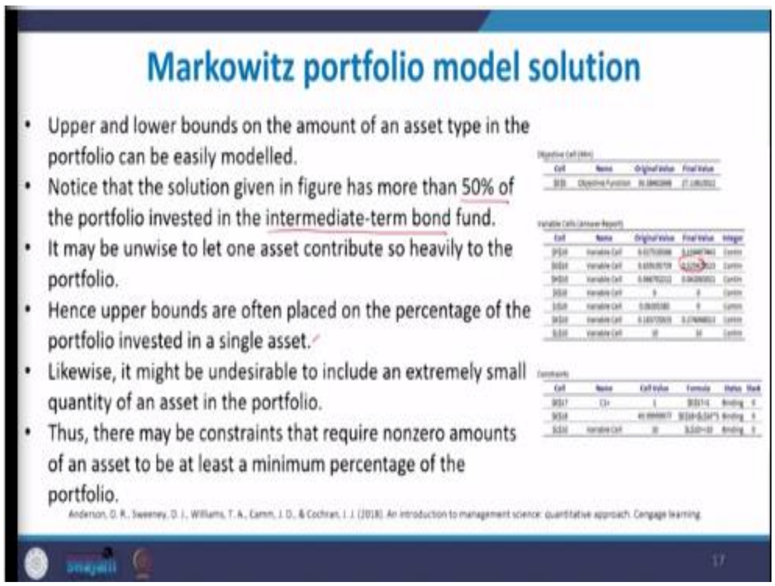
So, see E18; E18 is the mean of the return; what is the mean of the return? R1, R2, R3, R4, R5, so you see E12, E12 is the R1, E12 see that formula E12 sum product of F10 to M10 F12 to M12 that provides the R1 that is the right-hand side of your constraint. You see the L10 when you see L10 and E18 because we are going to write one constraint there that I will explain with the model. So, go to data, solver, and see that this problem is a minimization problem, and one constraint is that the sum of the probability should be 1.

You see that the E10 = L10, so what I am saying is that the expected value of the return is equal to our R-value, that one constraint I have written. Another one is that the L10 should be greater than or equal to 10. That is, the expected return should be at least 10%. Now, I am solving using a non-linear method, so when I solve it, press OK. Now, this 27.13 is my variance, and I got the value for FS; what is the FS?

So, 15% of your amount should be invested in FS, so 52% of your amount should be invested in IB, 4% of your amount should be invested in LG, LB you need not invest, SZ you need not invest, and 27% of your amount should be invested on SV. So, the average return is 10, so this constraint is also achieved. Now will I interpret this result in the lecture? So, the objective function value is 27.13, so F10 is 15%, G10 is 52%, 4%, and 27%, so this says your expected value of our return.

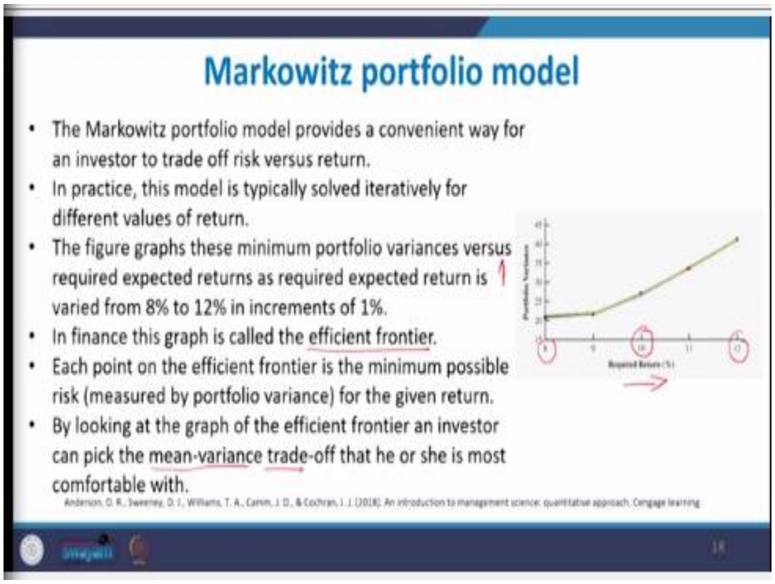


So, further interpretation is the minimum value of the portfolio variance is 27.136. A portion of the solution for this model using a required return of at least 10% appears in the figure, so this 10%. This solution implies that the clients will get an expected return of 10% and minimize their risk as measured by portfolio variance by investing approximately 16% of the portfolio in foreign stock, and 53% in the intermediate bond, this one because the value of IB is 0.52 and 4% in the large-cap growth, this one fund and 27% in small-cap value fund, here 27. So, this is the proportion of the portfolio that we need to invest.




The upper and lower bounds on the amount of an asset type in the portfolio can be easily modeled. Notice that the solution given in the figure has more than 50% of the portfolio invested in the intermediate-term bond; you see this one. It may be unwise because we are investing 52% of your or more than 50% into a single security. It may be unwise to let one asset contribute so heavily to the portfolio. Hence, the upper bounds are often placed in the percentage of portfolio invested in a single asset. We can introduce some more constraints that will make a restriction on investing in a single asset.

Likewise, it might be undesirable to include an extremely small quantity of an asset in the portfolio. So, we can also introduce a new constraint; thus, there may be a constraint that requires non-zero amounts of an asset to be at least a minimum percentage of the portfolio. So, how can we have the upper bound and lower bound of each asset? We can introduce a new constraint that can be incorporated to ensure that the minimum or maximum amount is invested into the portfolio.

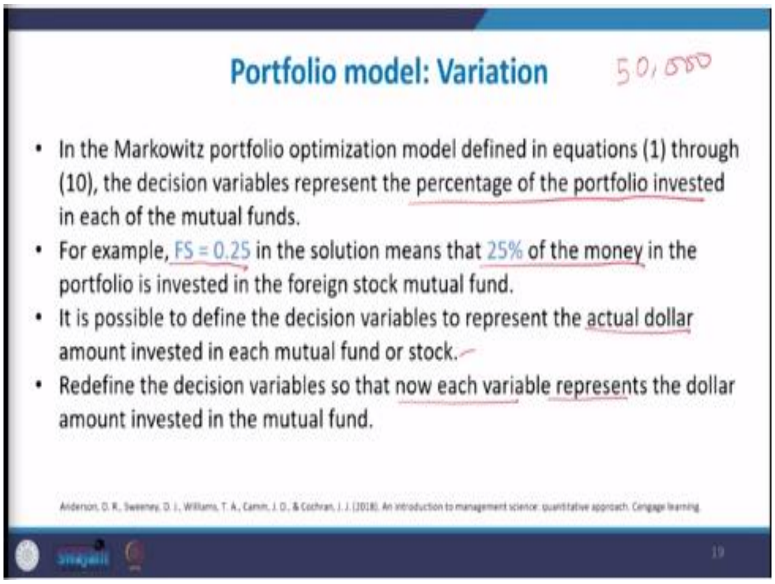



The Markowitz model provides a convenient way for an investor to tradeoff between risk and return. So, on the x-axis, I took the return; on the y-axis, I took the portfolio variance, so we solved this problem by 10%. So, what can we do? We can vary these values. What can we do? We can start from 8 to 12. Suppose the required return is 10%; we can see what the variance is. Suppose the return is 12%, and you see that the variance is increasing.

So, the figure graphs this minimum portfolio variance versus the required expected return as the required expected return varies from 8 to 12%, an increment of 1%. What can we do? We can change the R bar to the problem that we have discussed here in this model. Instead of 10, we can put 11, 12, 13, and 14, and then we can capture the variance. That is what I have done here. So, each point in finance in this graph is called an efficient frontier, so what is an efficient frontier? In the x-axis expected return, the required return is in the y-axis portfolio variance.

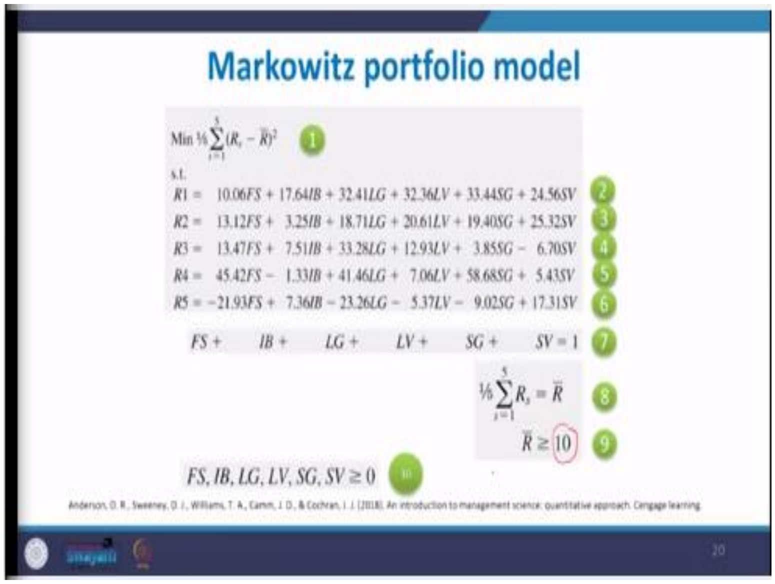
So, each point on the efficient frontier is the minimum possible risk, and we measure the risk by portfolio variance for the given return. So, by looking at the graph of the efficient frontier, an investor can pick the mean-variance trade-off that he or she is most comfortable with. So, this graph efficient frontier is giving the trade-off between return and the variance, it is up to the investor. Suppose I am picking here; my return will be 11%, but my variance will be more than

30. So, this efficient frontier is a helpful graph that will suggest the trade-off between return and the variance.

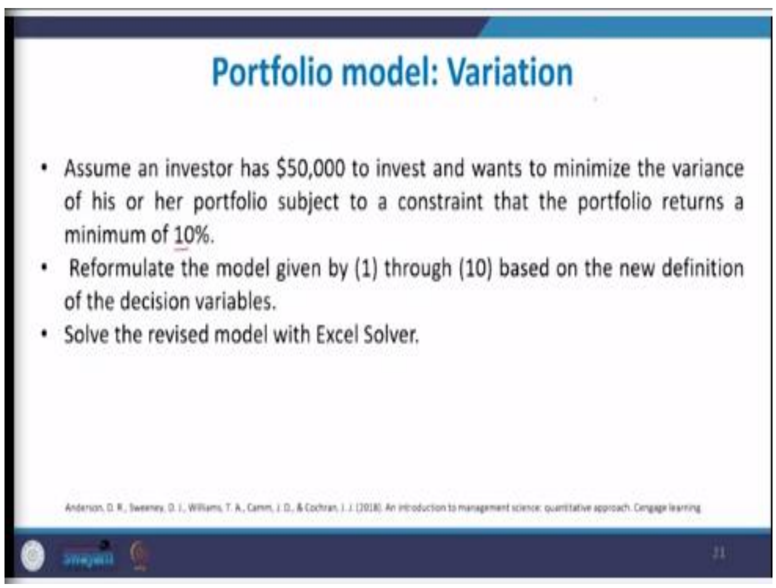



So, we will modify this problem and solve it. So, previously, what have we done? Our result is in terms of percentage into different assets, but assume that now I have some money. Say money is I have assumed that I have 50,000 dollars, so these 50,000 dollars, how much has to be invested in each asset? So, that is the portfolio model variation. So, in the Markowitz portfolio optimization model defined in equations 1 to 10, the decision variables represent the percentage of the portfolio invested in each of the mutual funds.

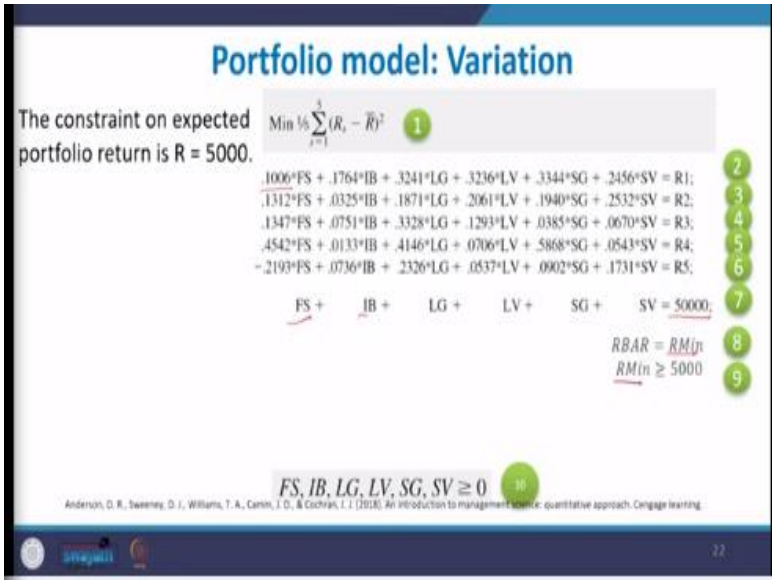
What is the meaning of that? For example, when I say FS = 0.25 in the solution, it means that 25% of the money in the portfolio is invested in the foreign stock mutual fund. It is possible to define the decision variables to represent the actual dollar amount invested in each mutual fund or stock. So, when we say the actual dollar, we are so comfortable instead of saying percentage, so what can we do? We can redefine the decision variables so that now, each variable represents the dollar amount invested in the mutual fund, so this is a little more practical application.




So, this was our own previous problem. Here, FS and IB represent 10.06, which represents the percentage. Here, the return is also more than 10%, but now we are going to consider the actual amount. What is that?




Assume an investor has 50,000 dollars to invest and wants to minimize the variance of his or her portfolio subjected to the constraint that the portfolio returns a minimum of 10%. So, what will we do now? We will reformulate the model, that is, all the equations from 1 to 10, based on the new definition of decision variables. The only thing is the definition of the decision variable has changed; what is that? Previously, it was a proportion; now, it is the actual amount. So, we are going to solve this problem with the help of a solver.




Now you see the formulations. So, here everything is in terms of seeing that I previously put 10.06, but now it is 0.1006. You see, previously it was 1, but now it is 50,000 dollars because I have 50,000 dollars. So how much do I have to invest in FS? How much would I invest in IB and so on? So, the R bar is the expected rate of return, which should be the R mean. So, the R minimum should be greater than or equal to 5000 because the 10% is 50,000, so 10% of 50,000 is 5000 dollars.

$$RBAR = RMin$$

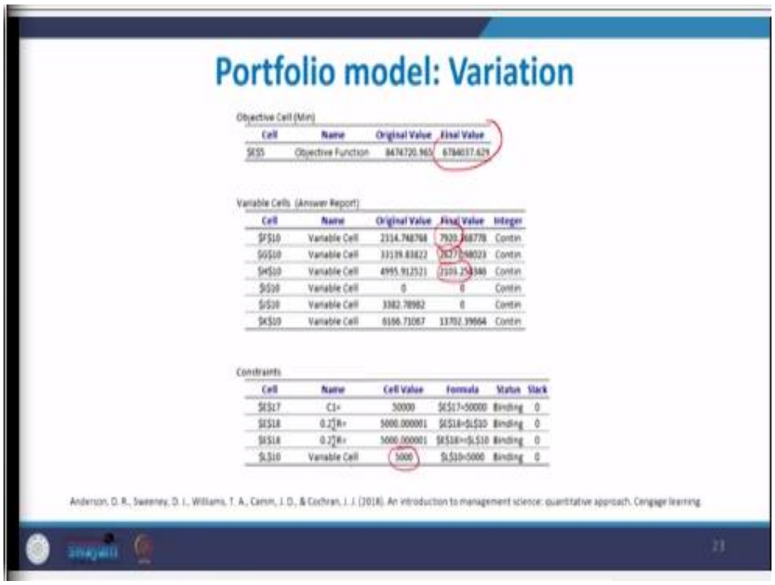
$$RMin \geq 5000$$

So, now we are going to solve this problem, and then we are going to get the result in terms of the actual amount invested in each asset. Now, I will explain the Excel model of this problem. So, when I keep seeing the C5, the answer is that we have to minimize the variance, the minimize the variance that is one upon 5, so that is written 0.2, Rs that is R1 - R bar whole square 1 upon 5 + R2 - R bar whole square, what is the R bar?

$$\text{Min } ⅕ \sum_{s=1}^{5} (R_s - \overline{R})^2$$

R bar I have written C18, so when I keep my cursor on C18, that is a mean value C12 to C16 divided by 5, the mean value of that return. So, that mean value is going to be used in my objective function. When I go to data and solver and look at the constraint, the E17 is 50,000, so E18 should be greater than or equal to L10, but the L10 is 5,000 ok. When I solve it, the answer is you see FS 7920, so out of 50,000 dollars, 70920 dollars must be invested in FS, and 26273 dollars should be invested in IB.

So, 2103 dollars has to be invested in LZ. Otherwise, what can we do? Previously, we got the percentage, and that percentage had to be multiplied by 50,000. Both are the same; you will get the same answer; now, I will go back to my presentation.




This is my variance, so this is the amount that has to be invested in F10. That is that asset, so this much amount has to be invested in another stock. So, the sum of the amount should be 50,000, and the minimum return of 5000 is guaranteed. In this lecture, I explained Markowitz's mean-variance portfolio model. I have taken a sample problem that I have formulated with the help of Excel using my knowledge of non-linear programming. Then, I modified the problem by considering the actual amount that has to be invested in each portfolio. In the next class, I will discuss some more applications of the non-linear programming models; thank you.