Decision Making With Spreadsheet
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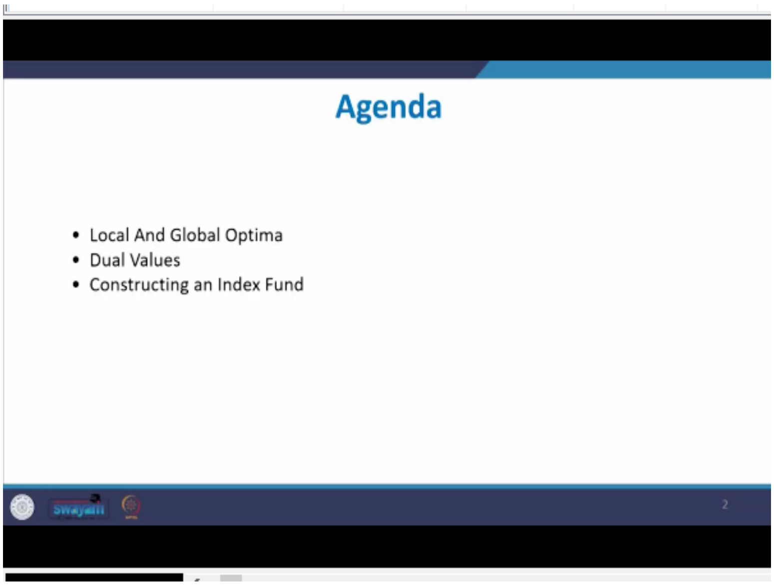
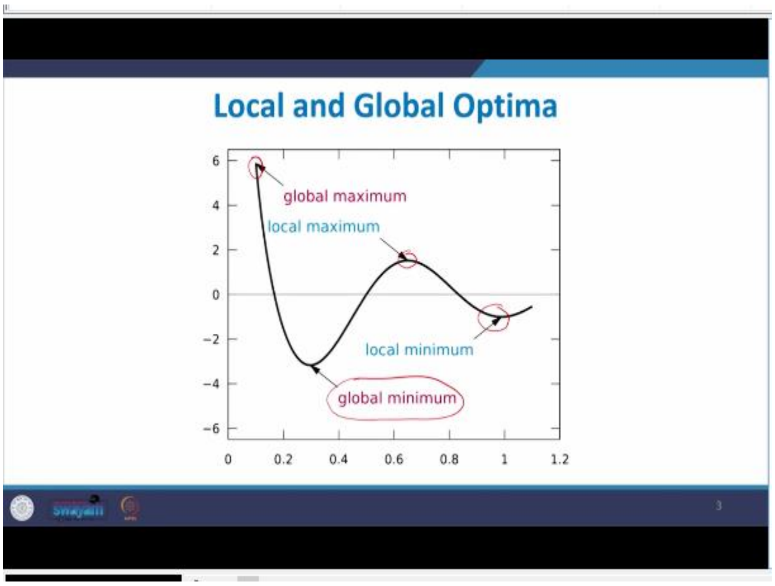
Lecture-25
Non-linear Optimization Models-II

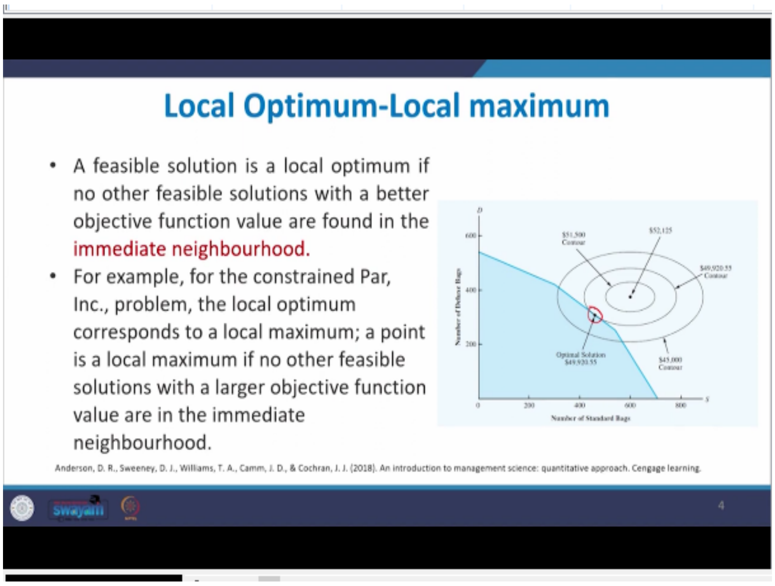
Dear students, we will continue with our previous discussion on non-linear optimization models in this lecture.



So, the agenda for this lecture is I will explain what is local and global optima then I will interpret the meaning of dual values for non-linear programming. After that, I took a sample problem on non-linear programming. The problem name is an index fund. So, for this problem, I am going to formulate it in the form of a non-linear problem, and then I am going to solve it.

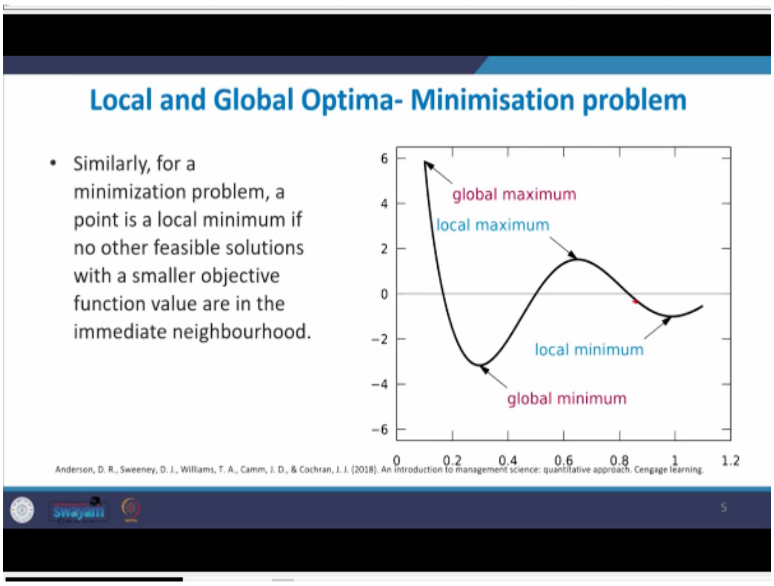


Before going to our problem, first, we will understand a little bit about concepts of local and global optima. Look at this picture; here, at the bottom, there is a point called local minimum. So, among this minimum, which is the global minimum? This point is the global minimum, and the same thing you see: there is a local maximum, then there is a global maximum. So, this is an example of the problem of what the local and global optima are.

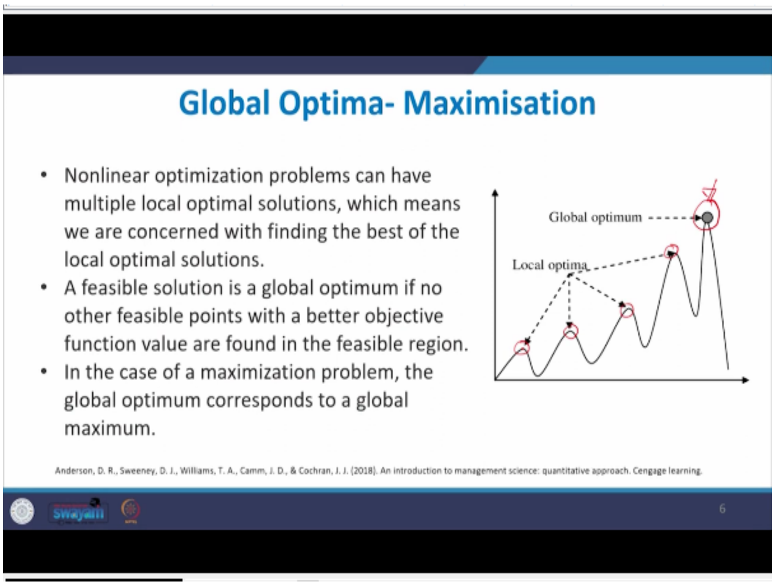


First, I will discuss the local optimum in the context of the local maximum. A feasible solution is a local optimum if no other feasible solutions with a better objective function value are found in the immediate neighborhood. This is the meaning of your local optimum. So, when we say a problem is local optimum, a solution is local optimum, no other value is better than this value in the neighborhood.

For example, in the problem that you have discussed, the local optimum corresponds to the local maximum. So, this point was our local optimum, and the same point was also the local maximum. So, a point is a local maximum if no other feasible solutions with a larger objective function value are in the immediate neighborhood. That is the meaning of your local maximum.

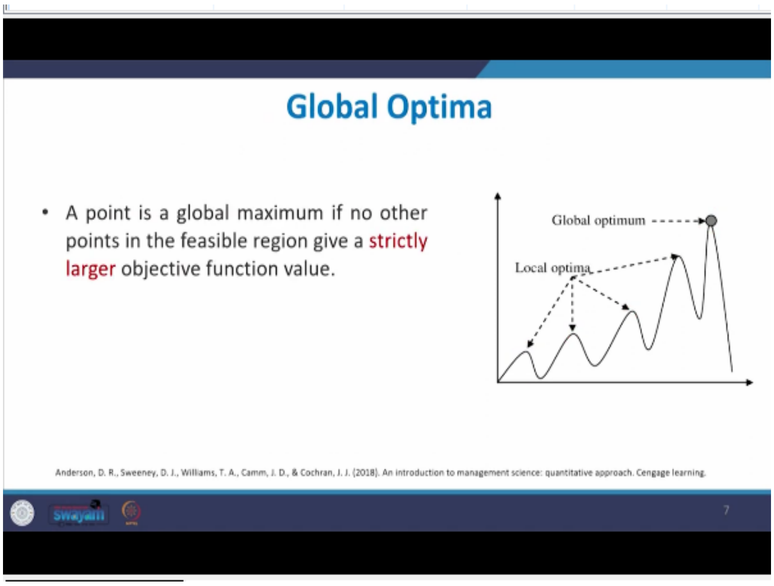


Similarly, for a minimization problem context, a point is a local minimum if no other feasible solutions with a smaller objective function value are in the immediate neighborhood. Look at this point; suppose there may be some other point. This is a local minimum, but when you compare it to this, this is also a local minimum; this is also a local minimum because no other values are in the immediate neighborhood, which minimum, so it is a local minimum.

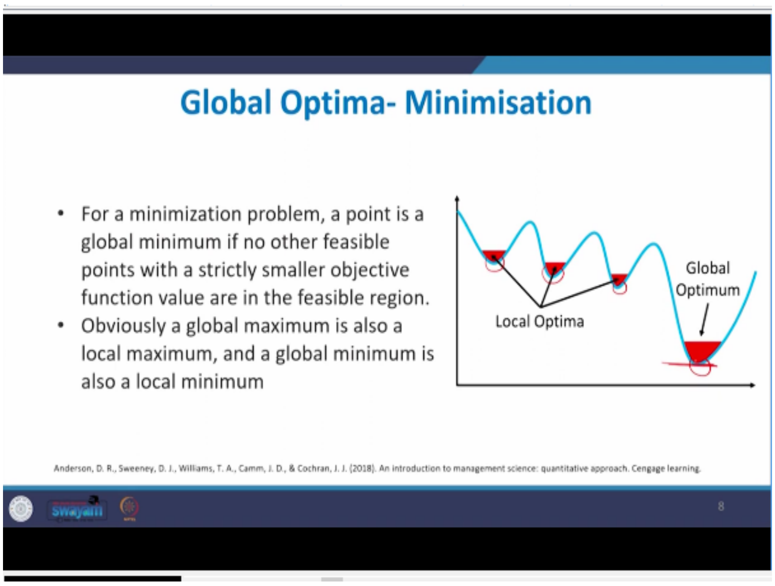


Then, what is the meaning of global optimum in the maximization context? Look at these, these points are maximum local. This is one local maximum, local maximum, local maximum, local maximum. So, among this local optimum, which is maximum, this point is the highest point. So, this is called global optimum. Previously, we discussed the local optimum, but now I am discussing the global optimum. Non-linear optimization problems can have multiple local optimal solutions which means we are concerned with finding the best of the local optimal solutions.

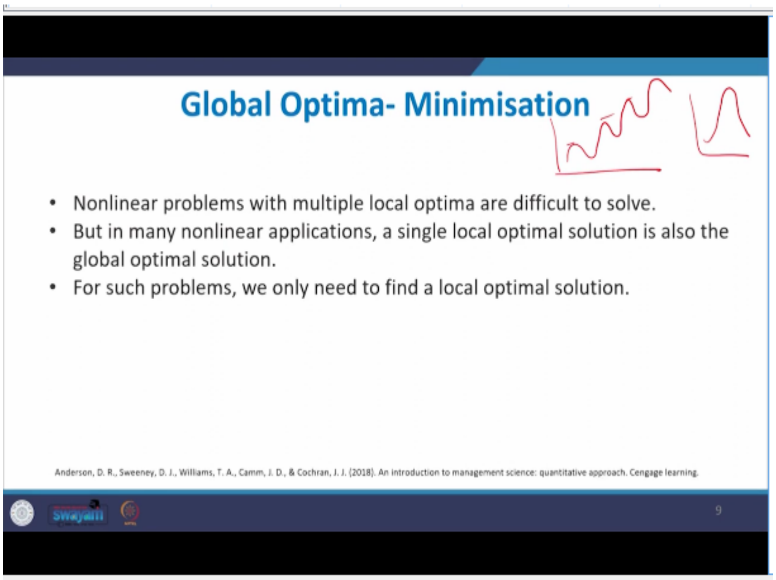
Look at this picture. There are multiple local optimums; among these multiple local optimums, we are going to find out which is the best one. A feasible solution is a global optimum if no other feasible point with a better objective function value is found in the feasible region. So, you see this is our global optimum because no other value is better than this. In the case of the maximization problem, the global optimum corresponds to a global maximum.



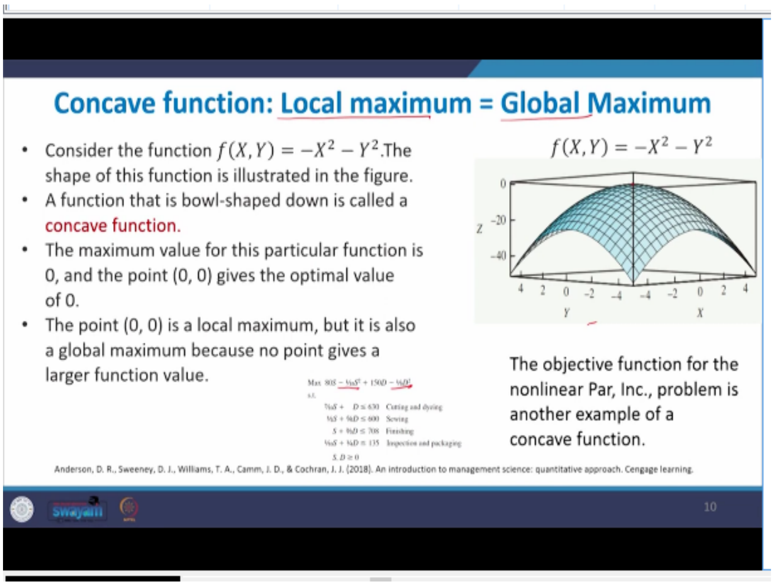
A point is a global maximum if no other point in the feasible region gives a strictly larger objective function value. See among these different local optimum values, so this is called global optimum because it provides a strictly larger objective function value.



Then what are the global optima for a minimization problem? Look at this picture. There are different local minimums available; among these local minimums, which is the best solution among these minimums? So, this point is the minimum among these minima. So, this point is called your global optimum. So, for a minimization problem, a point is a global minimum if no other feasible points with a strictly smaller objective function value are in the feasible region. This point obviously, a global maximum is also a local maximum, and a global minimum is also a local minimum. This is also one of your local minimums, but at the same time, it is the global minimum.



Non-linear problems with multiple local optima are difficult to solve if there are different local optima is there. For example like what we discussed in the previous lecture, this is local maximum, local maximum, local maximum. In that situation, if this kind of problem is difficult to solve, but in many non-linear applications, a single local optimal solution is also the global optimal solution. Sometimes, there may be one single solution that is also a global optimal solution. For such a problem, we need to only find the local optimal solution because that local optimal solution is equal to your global optimal solution.



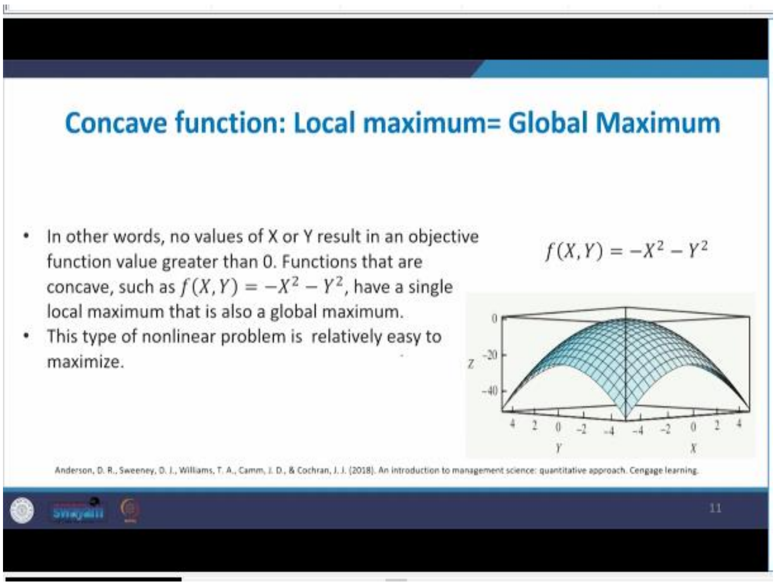
Now, we are going to discuss some important concepts in non-linear programming. So, we are going to classify the function into two categories: concave and convex.

Consider if the f(X, Y), = - $X^2 - Y^2$.

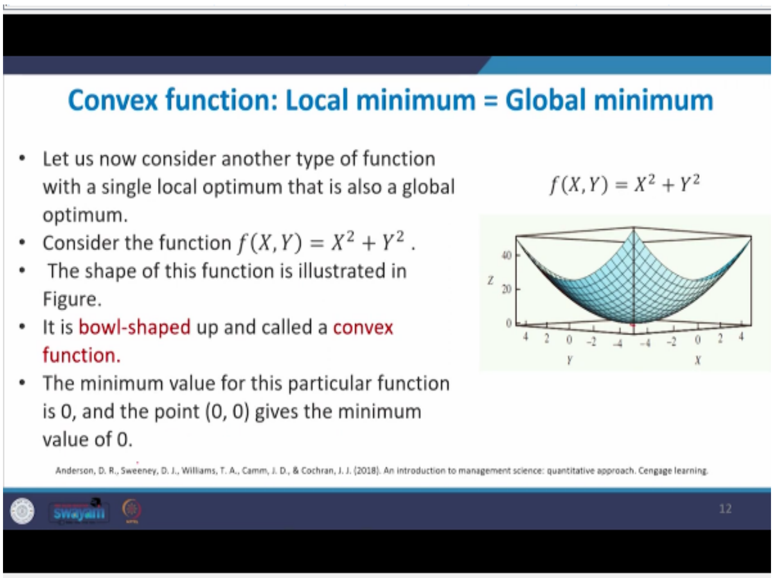
The shape of this function is illustrated in this figure; look at the right-hand side. This example is taken from Anderson et al. A function that is bowl-shaped down is called a concave function. It is ball-shaped, but it has a down concave function.

The maximum value for this function is 0, so this point is the maximum value, and the point (0, 0) gives the optimal value of 0. So, point (0, 0) is a local maximum, but it is also a global maximum because no point gives us a larger function value. So, if a function is a concave function there, the local maximum is equal to the global maximum. That is the point at which the local maximum is equal to the global maximum. So, the objective function for the non-linear problem that you have discussed is another example of a concave function.

How can we call it a concave function? look at this shape – S^2 and – D^2; if the objective function is this form so, then we can call it a concave function. In a concave function context, your local maximum is equal to the global maximum.

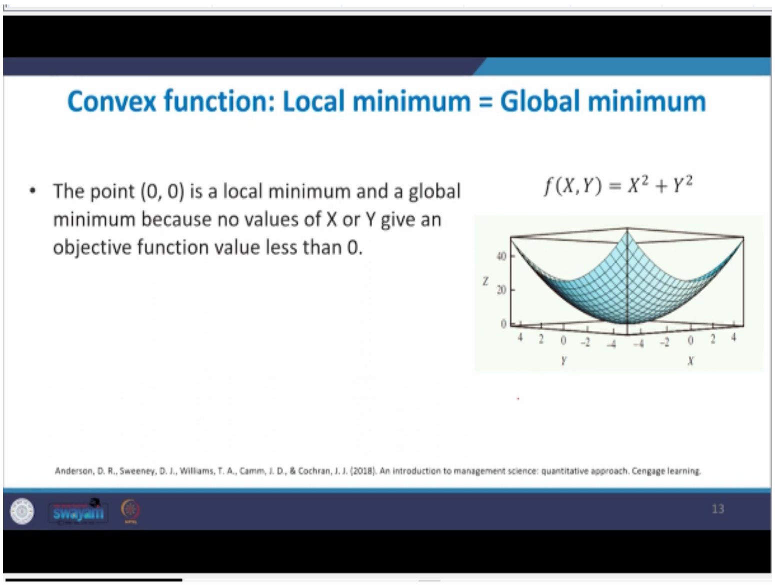


In other words, no values of X or Y result in an objective function value greater than 0. The functions that are concave, such as f(X, Y), = - $X^2 - Y^2$, have a single local maximum that is also a global maximum. So, in a concave function, the local maxima are equal to the global maximum. This type of non-linear problem is relatively easy to maximize.

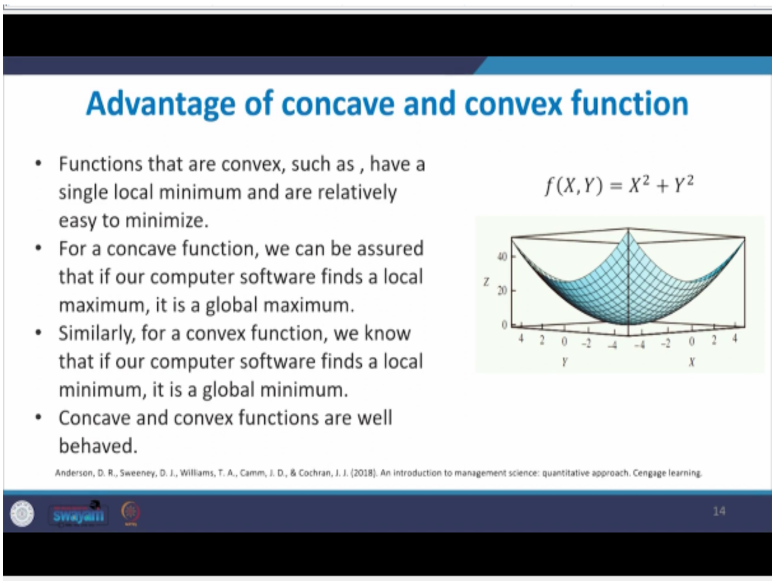


Now we are going to see another set of functions that is called convex function. So, in the convex function context, the local minimum is equal to the global minimum. As you have seen previously, if the function is concave, the local maximum is equal to the global maximum, but if the function is convex, the local minimum is equal to the global minimum. I will explain how it is. Let us now consider another type of function with a single local optimum that is also a global optimum.

The function f(X, Y), = $X^2 + Y^2$. The shape of this function is illustrated in the figure, so the bowl is shaped up and called the convex function; this looks like a bowl upside; this is a convex function. So, in the convex function, the minimum value for this function is 0, here, it is 0, and the point (0, 0) gives the minimum value of 0.

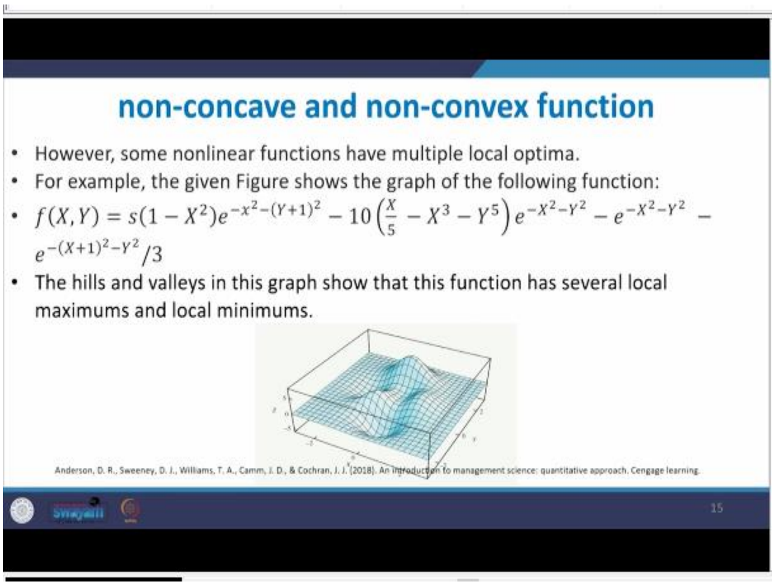


The point (0, 0) in each figure is a local minimum and a global minimum because no values of X or Y give an objective function value less than 0. So, learning from the convex function concept the local minimum is equal to the global minimum.



The advantages of concave and convex functions, as I discussed before, are that functions that are convex have a single local minimum and are relatively easy to minimize, and there is a guarantee that if you recommend a solution that is a local minimum at the same time, that is global minimum also. For a concave function, we can be assured that if our computer software finds a local maximum, it is also a global maximum.

Similarly, for a convex function, we know that if our computer software finds a local minimum, it is a global minimum, also. So, concave and convex functions are well-behaved and easy to solve. That is why before solving any non-linear problems, we have to test whether the function is convex or concave.

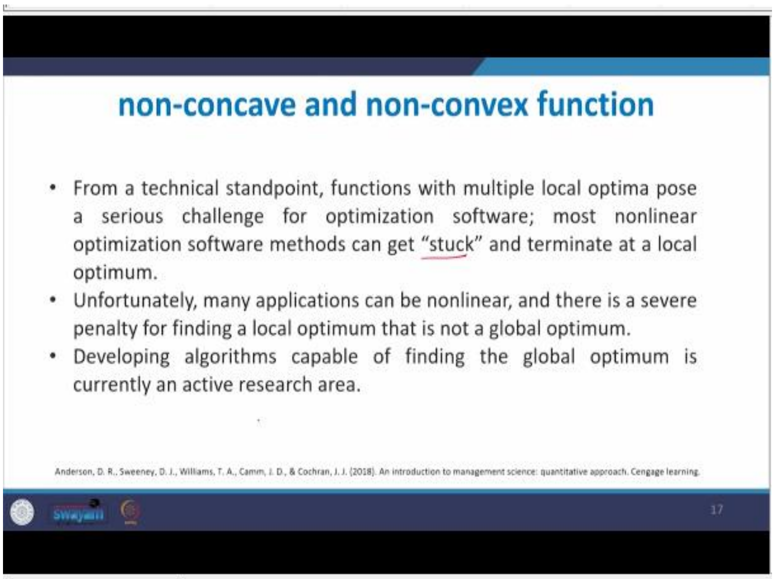


What will happen non concave and nonconvex functions? Some nonlinear functions have multiple local optima; for example, in the given figure, you see that there are different optimum values and different maximum values, different minimum values.

The figure shows that the graph of the following function

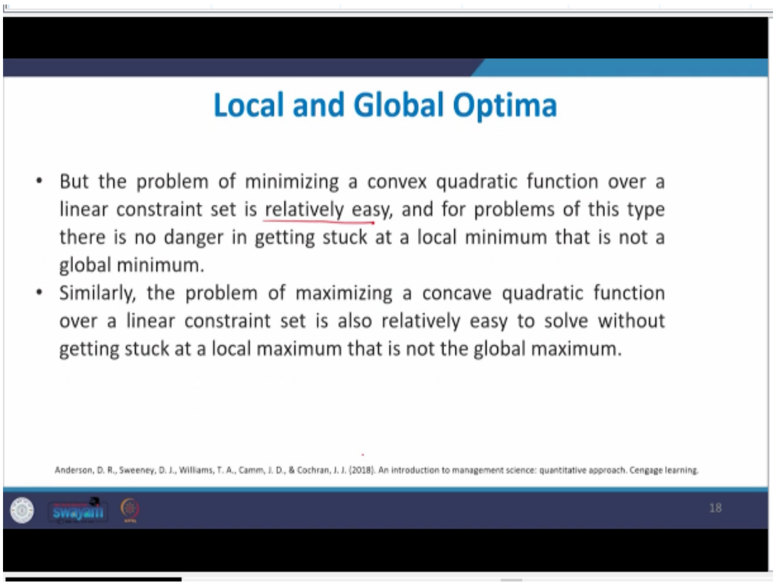
$$f(X,Y)=s(1-X^2)\, e\hat{}(-x^2-(Y+1)^2)-10(X/5-X^3-Y^5)\, e\hat{}(-X^2-Y^2)-e\hat{}(-X^2-Y^2)$$
$$-e\hat{}(-(X+1)^2-Y^2)/3$$

So, the hills and valleys in this graph show that this function has several local maximum and local minimums. So, solving these kind of non-linear optimization problems is very difficult and difficult in the sense we can provide a solution but there is no guarantee that the solution is global minimum or global maximum.

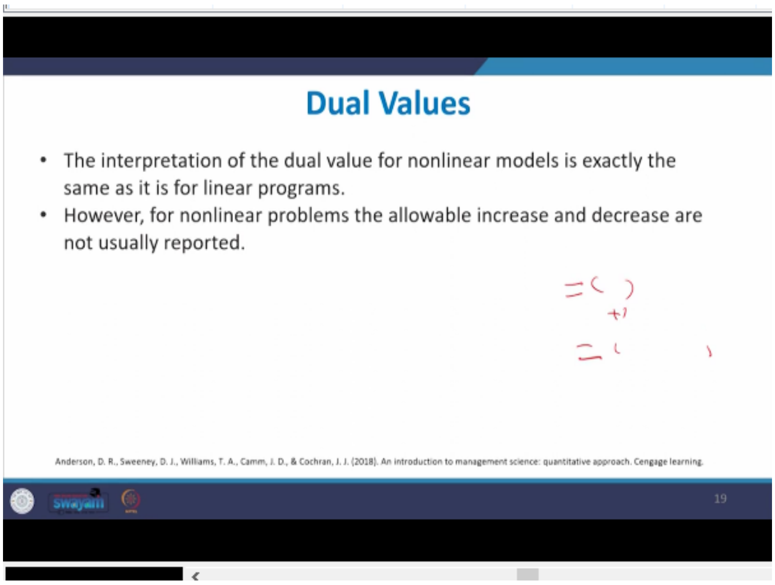


So, it indicates two local minimums and three local maximums. See that there are two local minimums and 3, 1, 2, and 3 local maximums. So, one of the local minima is also the global minimum, so this one, this point. Similarly, one of the local maximums is also the global maximum. So, from a technical standpoint, functions with multiple local optima pose a serious challenge for optimization software.

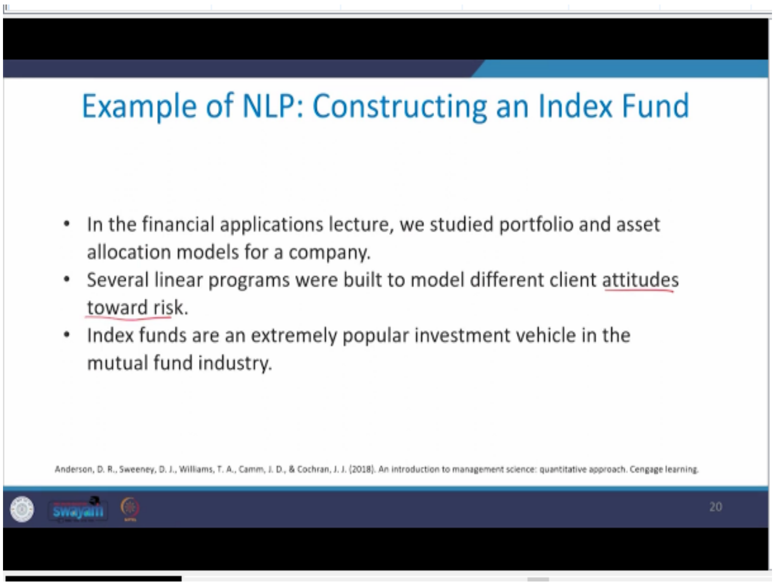
So, most non-linear optimization software methods can get stuck and terminate at a local optimum. Unfortunately, many applications can be non-linear, and there is a severe penalty for finding a local optimum that is not a global optimum. So, developing algorithms capable of finding the global optimum is currently an active research area.



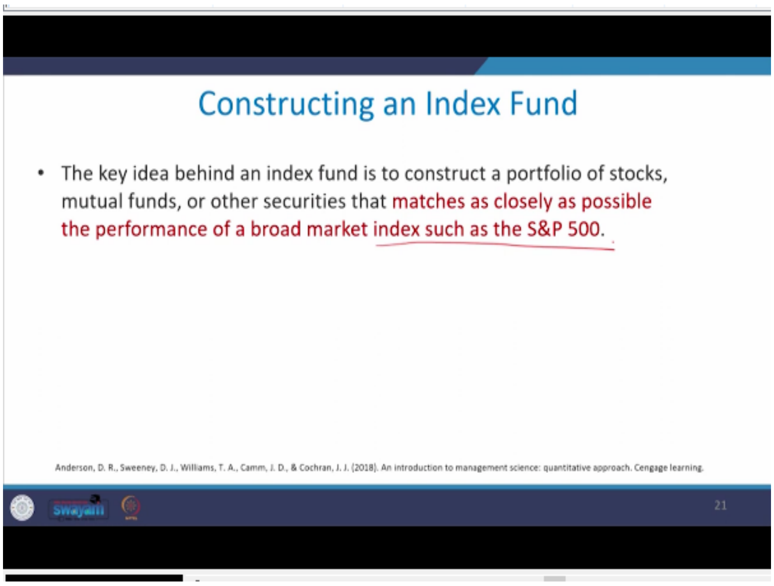
However, the problem of minimizing a convex quadratic function over a linear constraint set is relatively easy. Why it is easy? Where the local minimum is equal to the global minimum, and for a problem of this type, there is no danger of getting stuck at a local minimum. That is not a global minimum. Similarly, the problem of maximizing a concave quadratic function over a linear constraint, linear constraints it is also relatively easy to solve without getting stuck at a local maximum that is not the global maximum because we know if a function is a concave function where the local maximum is equal to the global maximum. Suppose the function is convex in nature, where the local minimum is equal to the global minimum.



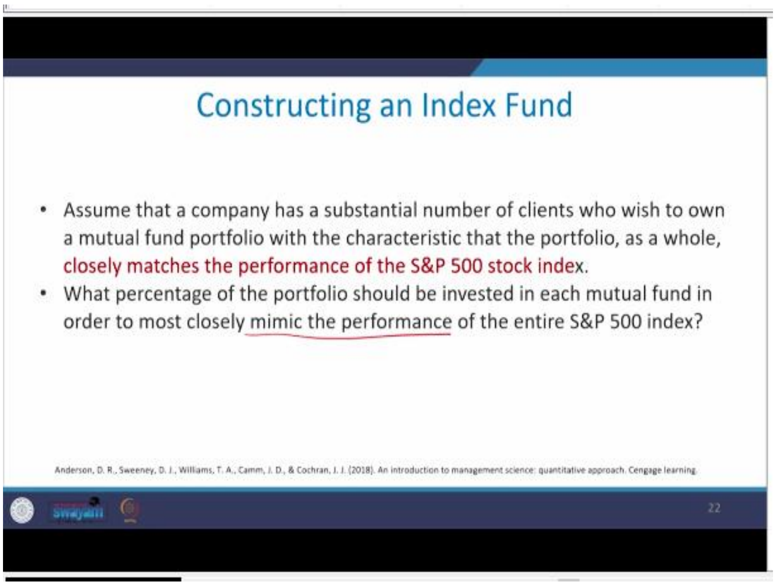
The next is the interpretation of the dual value in non-linear problems. So, what is the dual value of the right-hand side constraint? If the right-hand side of the constraint is increased by 1 unit, what is that corresponding effect on our objective function? So, in the non-linear programming context the interpretation is also the same. So, the interpretation of dual values for a non-linear model is the same as it is for linear programs. However, the non-linear problems that allowable increase may be a right-hand side constraint. The allowable increase and degrees are not usually reported in the software outputs.



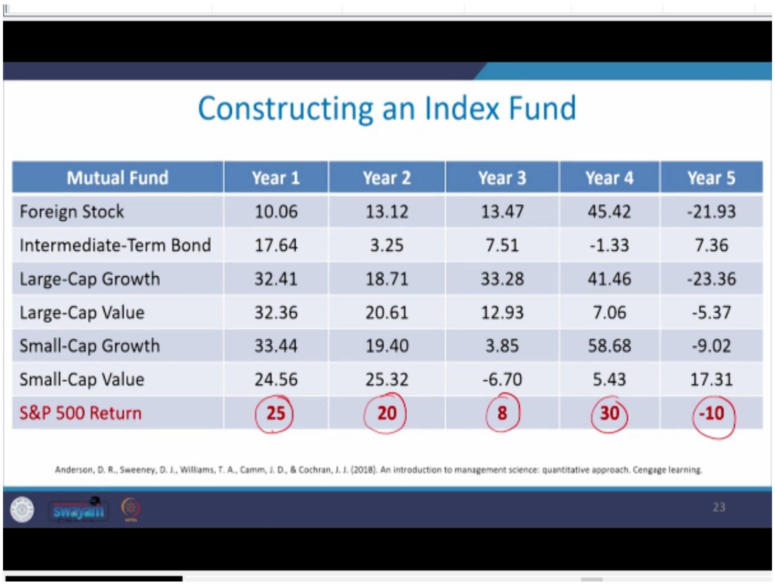
So, an example of a non-linear programming problem is constructing an index fund. In the financial applications lecture, we studied portfolio and asset allocation models for a company. Several linear programs were built to model different client attitudes towards risk. So, index funds are an extremely popular investment vehicle in the mutual fund industry.



The key idea behind an index fund is to construct your portfolio of stocks, mutual funds, or other securities that match as closely as possible the performance of a broad market index such as the S&P500. So, the idea is that we have to construct a portfolio that matches the index of the S&P500.

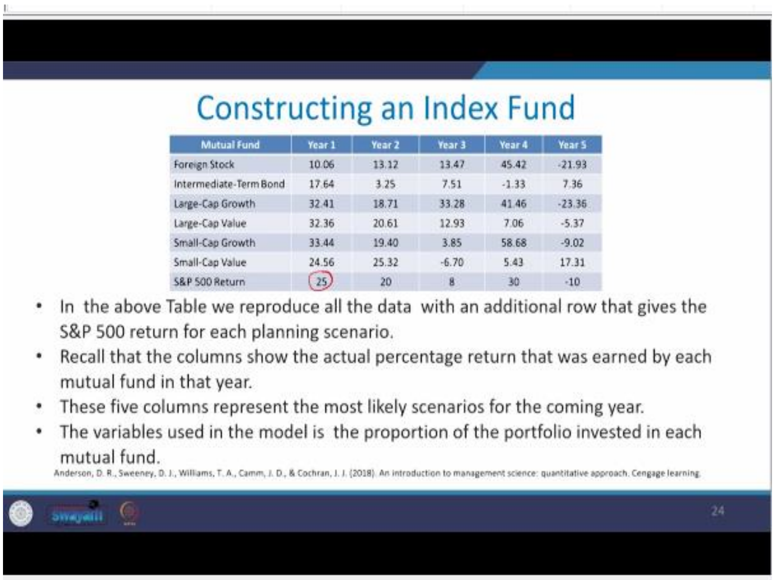


Assume that a company has a substantial number of clients who wish to own a mutual fund portfolio with the characteristics that the portfolio closely matches the performance of the S&P500 stock index. So, they were saying that these investors want to have a portfolio that has to match the performance of the S&P500 stock indexes. So, what percentage of the portfolio should be invested? In each mutual fund in order to most closely mimic the performance of the entire S and P 500 index. So, what do we have to suggest? We have to suggest what percentage of the portfolio should be invested. This will mimic the performance of the entire S&P500 index.



Mutual Fund	Year 1	Year 2	Year 3	Year 4	Year 5
Foreign Stock	10.06	13.12	13.47	45.42	-21.93
Intermediate-Term Bond	17.64	3.25	7.51	-1.33	7.36
Large-Cap Growth	32.41	18.71	33.28	41.46	-23.36
Large-Cap Value	32.36	20.61	12.93	7.06	-5.37
Small-Cap Growth	33.44	19.40	3.85	58.68	-9.02
Small-Cap Value	24.56	25.32	-6.70	5.43	17.31
S&P 500 Return	25	20	8	30	-10

So, here we have solved this already; see, there are different mutual funds, foreign stock, immediate-term funds, large-cap growth, large-cap value, small-cap growth, and small-cap value. Year 1, year 2, year 3, year 4, year 5 are different scenarios. What extra things added in this table is that standard and poor 500 written index is given. So, here for scenario 1 in the index is 25. For year 2, this is a different scenario. The index is 20. So, here, the index is 8, 30, and -10. So, what we are required to do? We have to suggest the portfolio, so that it closely matches with the index provided by this S and P company.



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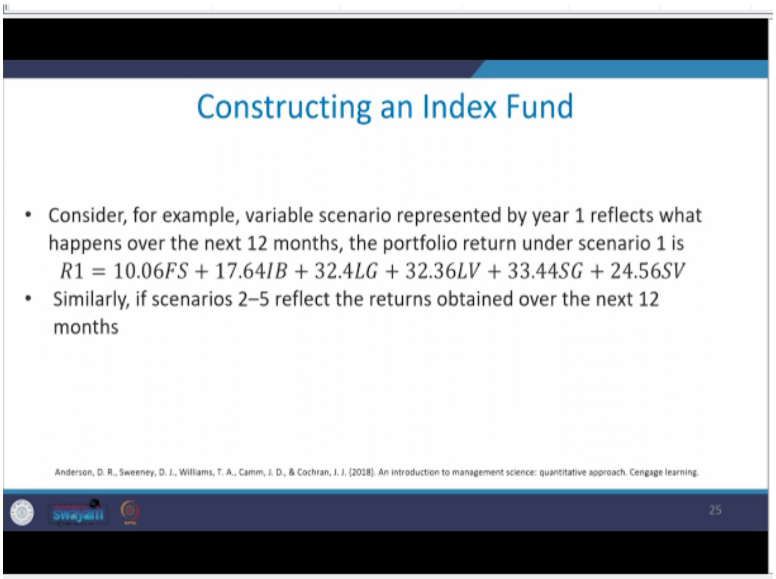
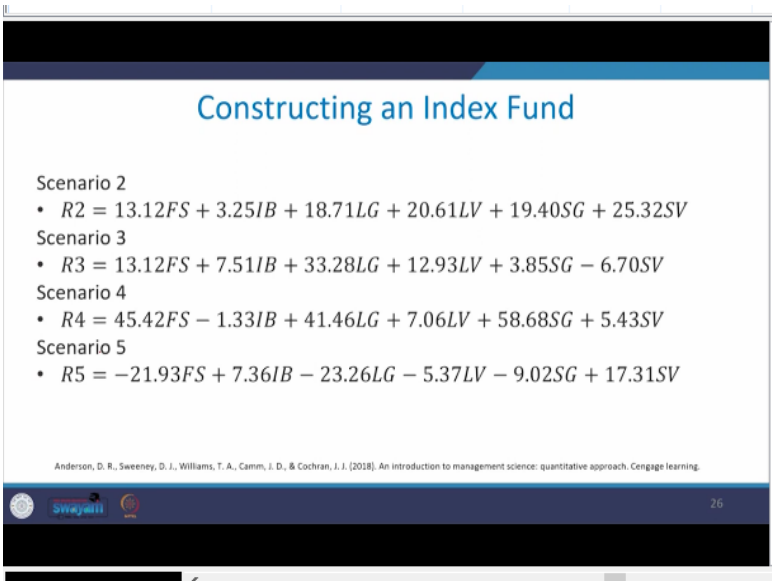
In the above table, we reproduce all the data, the data which you have discussed in the previous lecture on portfolio, but we have introduced an additional row that gives S and P 500 returns for each planning scenario. So, this was our additional row. Recall that columns show the actual percentage return that was earned by each mutual fund in that year. So, these 5 columns represent the most likely scenario for the coming year. The variable used in the model is the proportion of the portfolio that should be invested in each mutual fund, which was our decision variable.

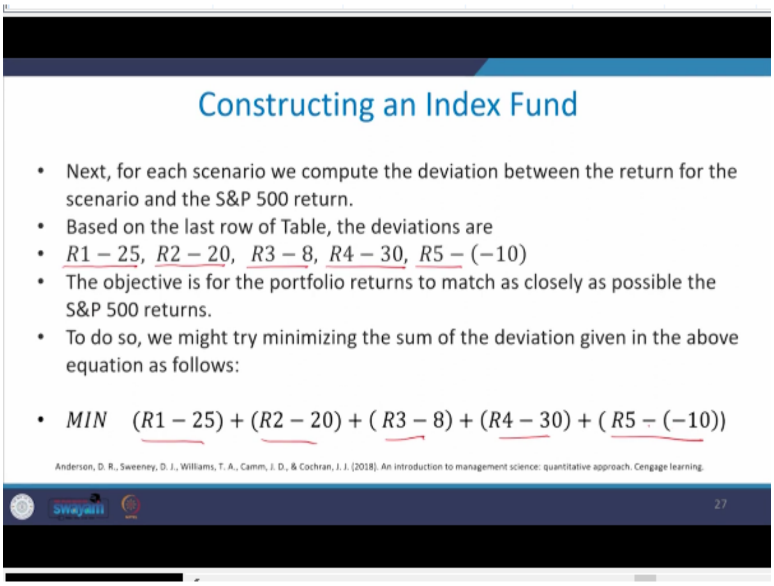


For example, the variable scenario represented by year 1 reflects what happens over the next 12 months. The portfolio return under scenario 1 is R1 = 10.06FS. How did we get this

10.06? When you return this one, 10.06FS, then 7.64IB, LG, LV, SG, and SV. Similarly, we have repeated. This is written one if scenario one is repeated. Similarly, R2, R3, R4, and R5 also have been written for the other 5 scenarios.

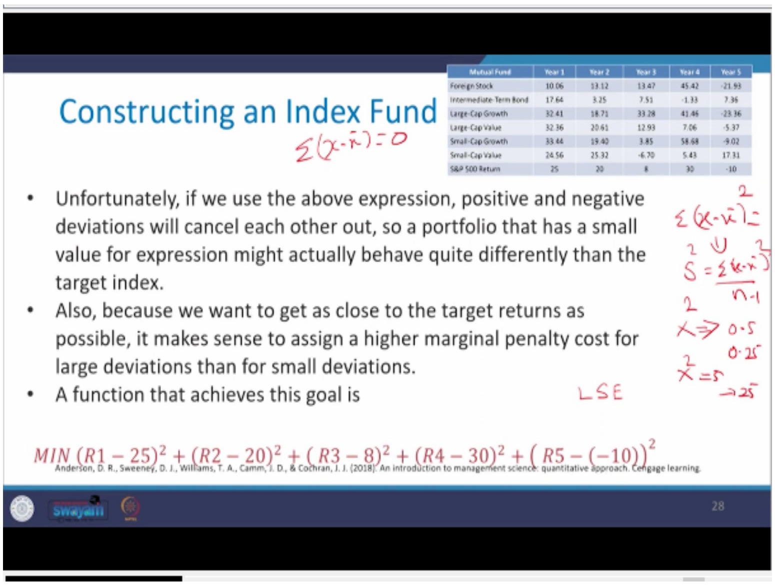


So, this slide shows scenario 2, scenario 3, scenario 4, and scenario 5.



Next, for each scenario we compute the deviation between the return for the scenario and the S&P 500 return. So, based on the last row of the table, the deviations are seen. For example, if scenario 1 is repeated, the deviation will be (R1 – 25), (R2 – 20), (R3 – 8), (R4 – 30), and (R5 – (-10)). So, the objective is for the portfolio returns to match as closely as possible the S and P 500 returns. So, this has to be matched.

That means this deviation has to be minimized. To do so we might try to minimize the sum of deviations given in the above equation. So, we have some of the deviations that has to be minimized, so what will happen to minimize? So, this is a deviation 1, deviation 2, deviation 3, deviation 4, deviation 5 for all 5 scenarios.



Unfortunately, we know that if the $\sum(x - \bar{x})$, when you sum it, it will be 0. So, what will happen if you use the above expression that is just a summation of deviation? The positive and negative deviations will cancel each other. So, a portfolio that has a smaller value for expression might actually behave quite differently than the target index. There is a logic behind this, and in the next lecture, we are going to study the mean-variance theorem.

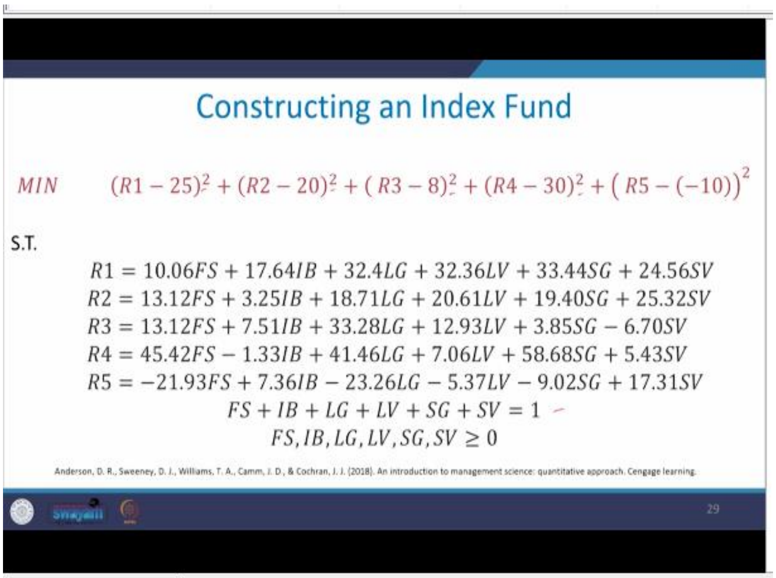
So, that is $\sum(x - \bar{x}) = 0$. So, what do you have to do? You have to square this.
This logic is also behind the variance formula:

$$\frac{\sum(x-\bar{x})^2}{n-1}$$

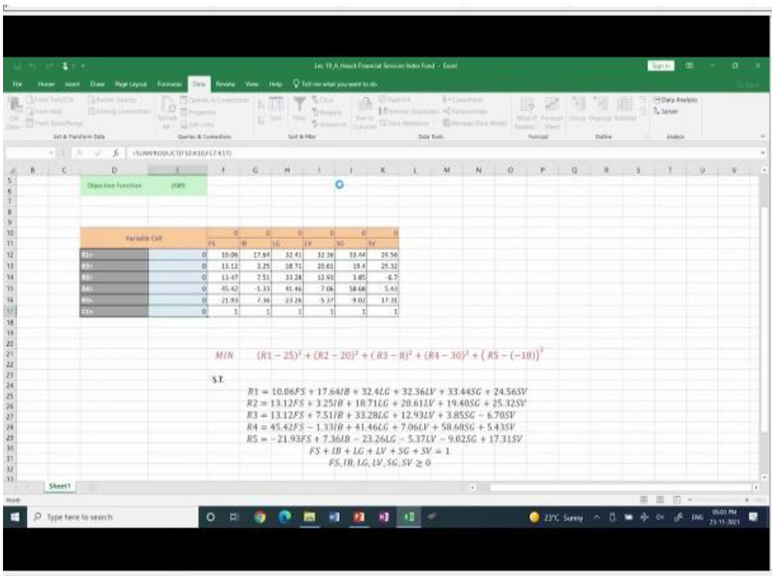
So, we have to square this sigma of the deviation has to be squared otherwise it will become 0. So, also, because we want to get as close to the target return as possible, it makes sense to assign a higher marginal penalty cost for larger deviations than for smaller deviations.

So, what will happen? So this is called the squared transformation. What is the meaning of the square transformation? If the deviation is 0.5 when you square it, it will be only 0.25. Suppose the deviation is higher, for example, 5, so when you square it, it will become 25. So, what is the logic behind this when you square the deviation, we are giving a larger penalty for a larger deviation and a lesser penalty for a lesser deviation.

That is the logic behind squaring the deviation, so when you square the deviation, this will become $(R1 - 25)^2$ and $(R2 - 20)^2$. This is like the least square estimate in your regression; what are we used to doing there? We must minimize some of the squares of the error the same thing. So, what we are doing here is minimized. So, what is the square here that is the (R1 – 25) is the deviation? There is nothing but error. So, we are all the squared error; when you sum it, that has to be minimized. So, if you want to minimize, then what should be our portfolio strategy? That is what we are going to do.



Now, I have brought up the complete problem of index funds. So, how can we know this is non-linear? You see this power is 2. So, now, this problem is a non-linear problem. What are the constraints? The R1 is this much, R2, R3, and other things one more constraint because it is a percentage the sum of the percentage; some of the probability should be 1. So, this is the final complete non-linear problem that I will solve with the help of a solver.



Now, I am going to solve this non-linear problem with the help of a solver. So, I have brought a screenshot of my formulation of this problem.

So, minimize $(R1 – 25)^2 + (R2 – 20)^2$,

and there are constraints is there. Now I have formulated. Now, I am going to explain how I have formulated it cells F10 to K10 are the decision variables where we are going to get the answer. Now look at this E12; E12 is a sum product of F10 to K10 and F12 to K12. That is R1, R2, R3, R4, R5.

The last cell that is E17 is some of the probabilities, so F10 to K10 and F17 to K17 because it is 1; it is just some of the probabilities, some product that will be equal to some of the probabilities. Now, I am going to date. So, solver, now I am going to explain what our objective function is. Object function is written on E5. Where is E5? Please look at this E5.

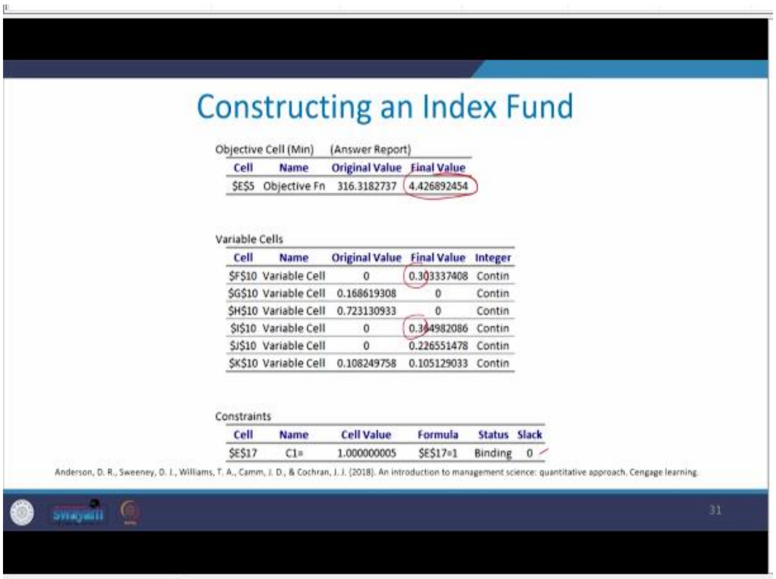
So, if E5 is $(R1 – 25)^2$, what is R1 here?

E12 is my R1, so $(R1 – 25)^2$ and $(R2 – 20)^2$, what is R2, U13

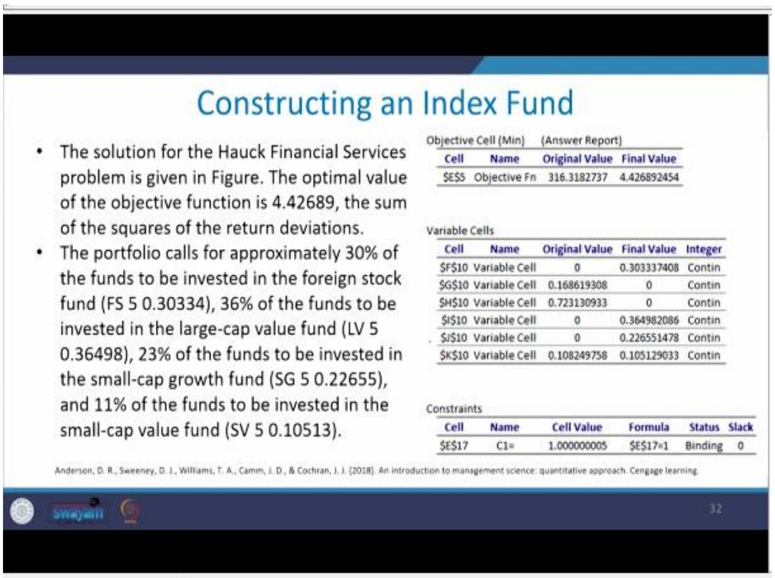
then $(E14 – 8)^2 + (E15 – 30)^2 + (E16 – (-10))^2$

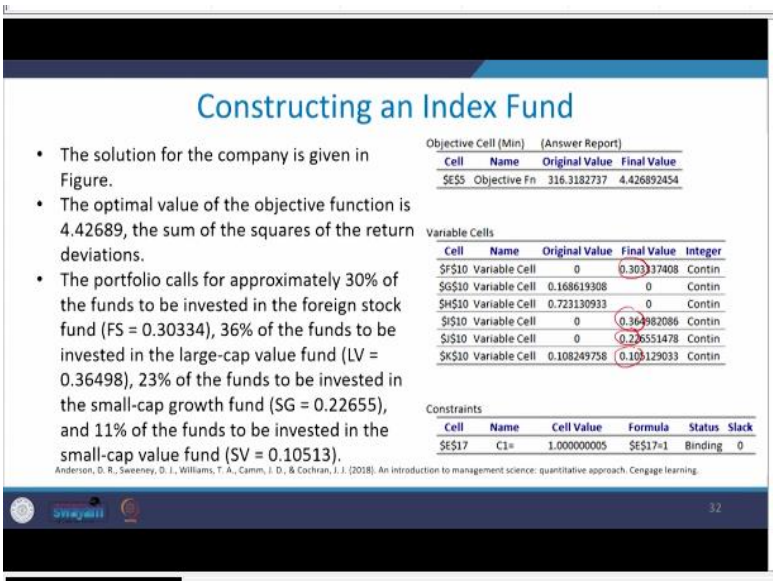
That is my objective function. Now, when I go to Solver, what are the constraints here? The constraint is that the sum of the probability should be 1, and the changing cell should be F10 to K10. So, here I am selecting non-linear options instead of simplex. I am choosing non-linear GRG, which is non-linear.

When I solved it, so I needed to answer everything I needed, now I got that in the FS, 30% of the stock should be invested in FS 36% of the stock should be invested in LV, 22% of the stock should be invested in SG, and 10.513% should be invested on SV. So, that is the case. The sum of the squares of the deviations here is 4.42. So, our deviations will be minimized, so it is on par with the index provided by S and P company. Now, I have taken this output in the presentation; there, I am going to interpret the result.

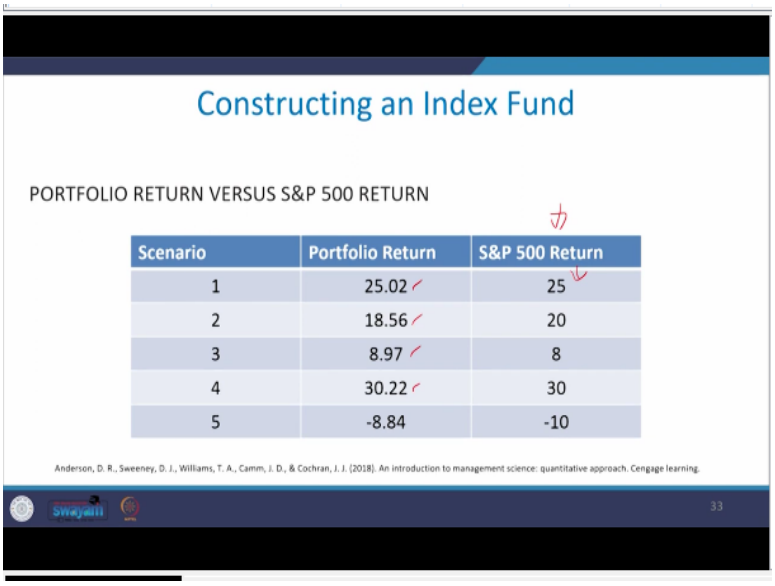


Yes, here my objective function is 4.42. That is the sum of the squares of the deviations. So, my constraint on the slack is 0; it is fully satisfied. So, we are getting this F10 at 30%, 36%, 22%, and 10%.

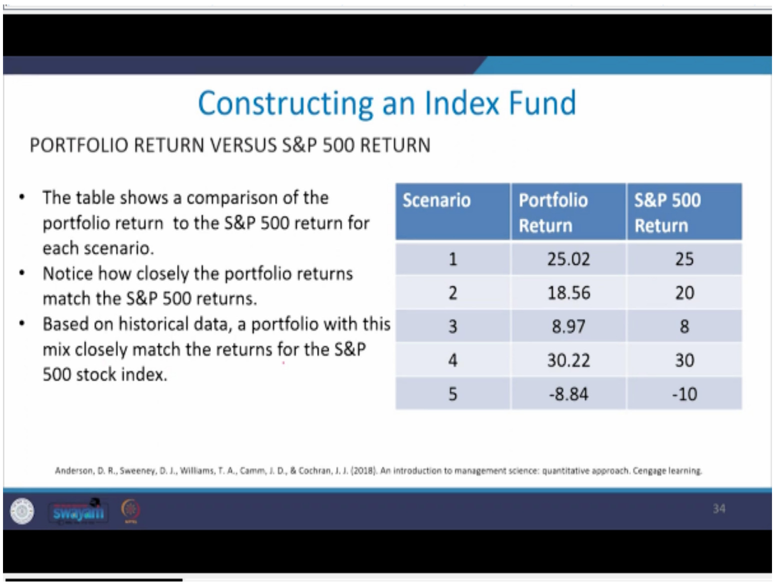




The solution for the company is given in this figure. The optimal value for the objective function is 4.42, which is the sum of the square of return deviations. The portfolio calls for 30% of funds here, 30% of funds to be invested in the foreign stock FS and 36% this one, 36% of the funds to be invested in the large-cap value and 23, see 22.6% of funds to be invested in small-cap growth and 11% this one, 11% of the funds to be invested in the small-cap value. That is why SV = 0.10.



So, we got these proportions when you substitute these proportions in the return expression function. We are getting R1 value of 25, R2 of 18, R3 of 8.97, and R4 of 30.22. You see the S and P returns given by this indexing company. Now, we are finding that if you follow that portfolio strategy, there will be a close match between the index provided by this company and what we will be getting the return from our own company.



So, what are we interpreting? This table shows a comparison of the portfolio return to the S&P 500 for each scenario. Notice how closely the portfolio returns match the S&P 500 returns. So, based on historical data, the portfolio with this mix closely matches the return of the S&P 500 stock index. In this lecture, I have explained the concepts of convex and concave functions and their advantages. By using the concept of convex and concave function I explained the local optima and global optima. What is that?

If the function is a concave function, the local maximum is equal to the global maximum. If the function is a convex function, the local minimum is equal to the global minimum. Then, I explained the meaning of dual values in the context of non-linear problems. After that, I took a sample problem to construct an index fund. So, that problem is a non-linear problem that I have solved using a solver, and then I have interpreted the result. In the next class, we are going to have an interesting problem, which is Markowitz portfolio models; we will see you in the next class; thank you.