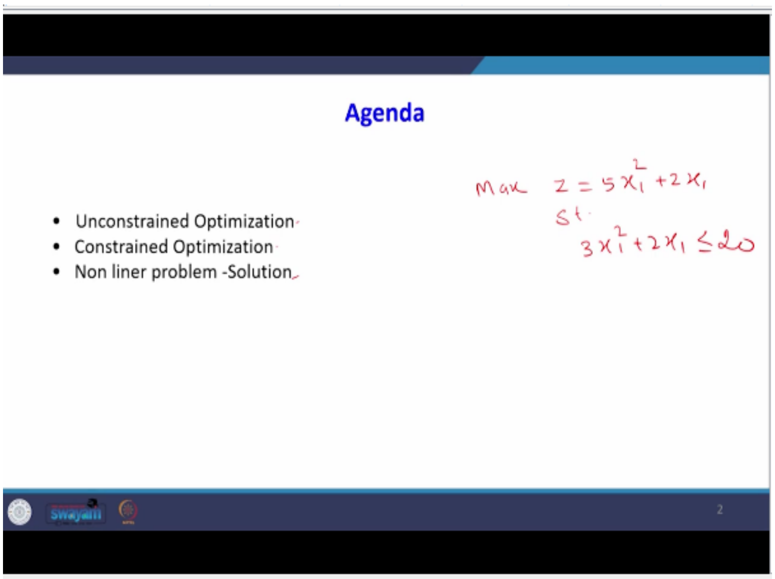
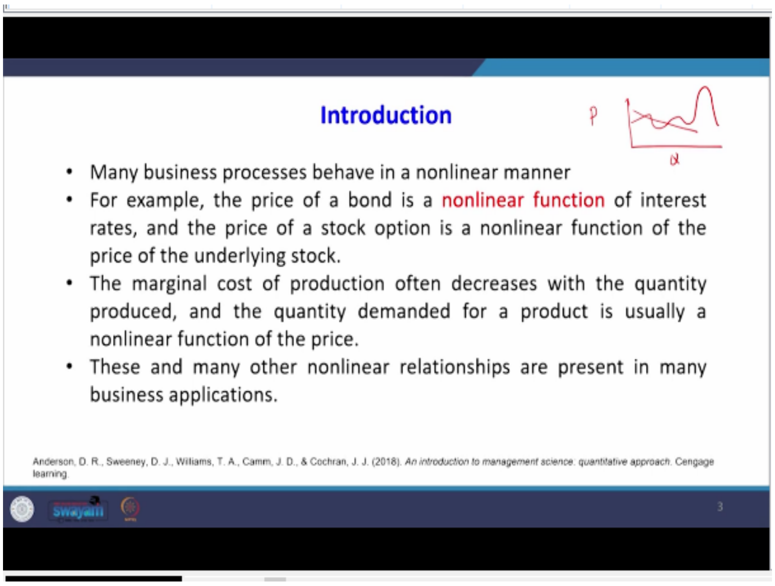
Decision Making With Spreadsheet
Prof. Ramesh Anbanandam
Department of Management Studies
Indian Institute of Technology-Roorkee

Lecture-24
Non-linear Optimization Models

Dear students, so far, we have discussed linear programming problems. When I say linear programming problems the objective function is in linear form and the constraint also in the linear form. But in this lecture, I am going to discuss non-linear programming problems. That is called non-linear optimization models. So, what is the meaning of this non-linear optimization?

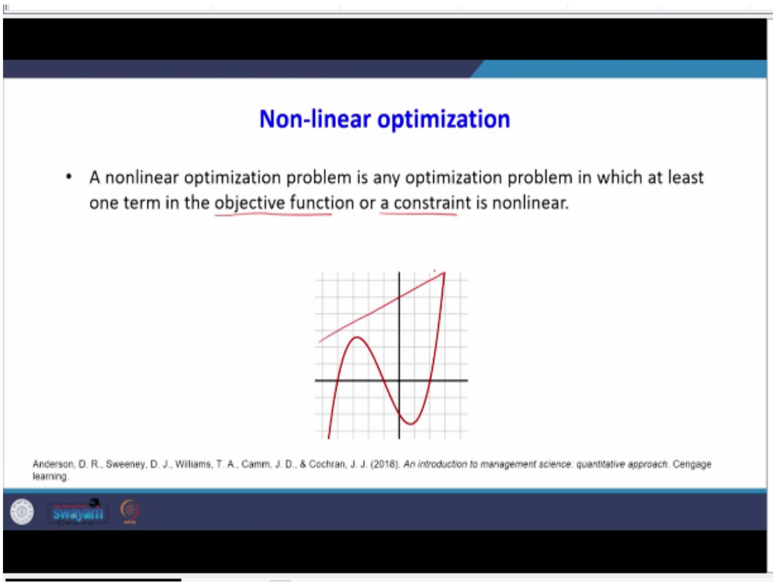


For example, when I write maximize, for example, maximize z equal to, say, $5x_1^2 + 2x_1$, you see that the power of x_1 is more than 1, 2. So, now I can call this objective function in non-linear form. Similarly, say this is subjected to, say, $3x_1^2 + 2x_1 = 20$. Now, what is happening here is the constraint; in the constraint, there is also a non-linear form. So, this kind of problem is a non-linear programming problem. So, in this lecture what I am going to discuss, I am going to explain unconstrained optimization and constrained optimization. Then, I will take a sample problem in a non-linear context and solve it with the help of a solver.

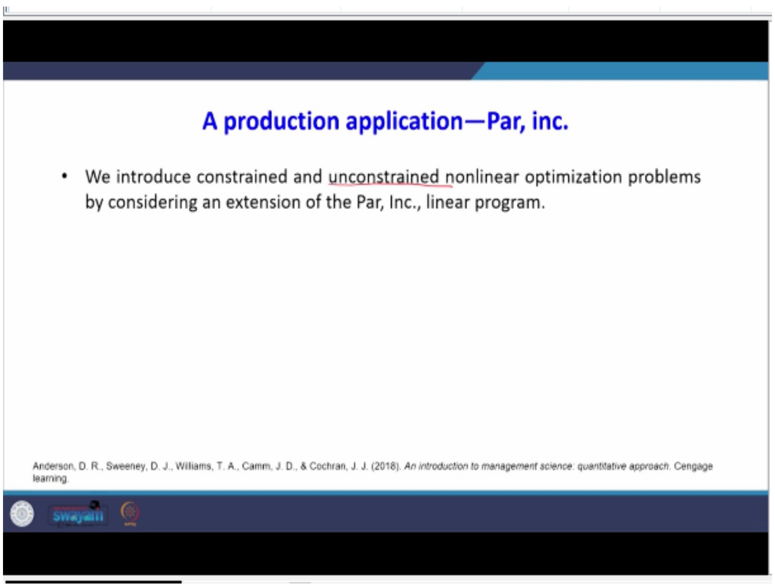


Introduction: Many business processes behave in a non-linear manner. So, far, the basic assumption for our linear programming model is the objective function, and the constraint should be in the linear form, but that is not the case. For example, the price of a bond is a non-linear function of interest rates because the interest rate is not in the linear form, it is a non-linear form, and the price of a stock option is a non-linear function of the price of the underlying stock.

Another example is the marginal cost of production often decreases with the quantity produced. So, the rate of decrease is in the non-linear form. Another example is the quantity demanded for a product, which is usually a nonlinear function of the price. See, we see the law of demand is like this. Here is the price in quantity; the law of demand says in the linear form, but in many situations, it would be in the linear form, it will be in the non-linear form, it may be like this, like this. So, these and many other non-linear relationships are present in many business applications.

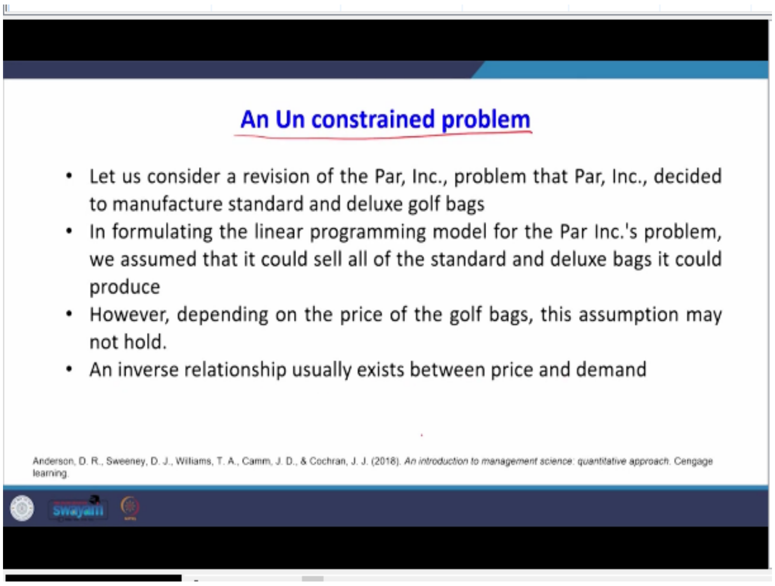


Then, what is non-linear optimization? A non-linear optimization problem is an optimization problem in which at least one term in the objective function or a constraint is non-linear. If any in one term, as I explained previously, if it is in nonlinear form, then we can say it is a non-linear optimization problem. This picture is an example of a non-linear form. What is the linear form? Linear form is a straight line.



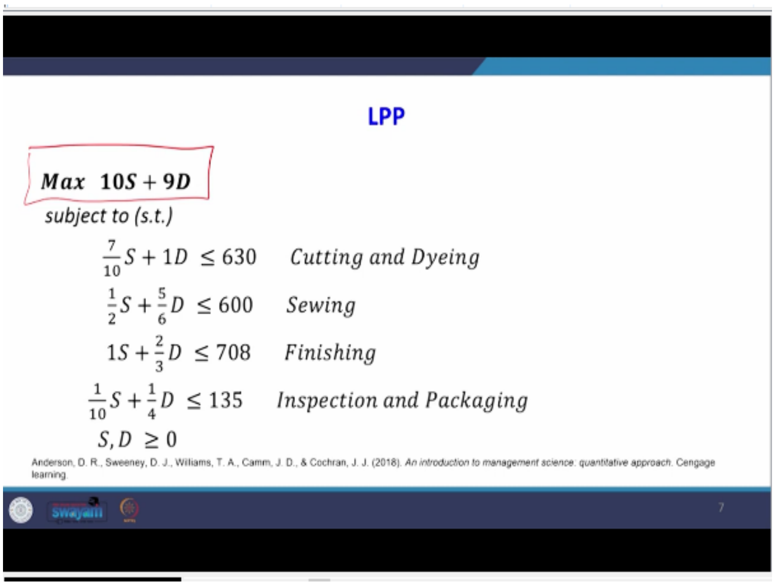
We introduce a constraint and unconstrained non-linear optimization problem by considering an extension of the Par, incorporation, linear programming problem. This problem is taken from the book Anderson et al. Already we have solved this problem in the linear form. Now, we are going to modify that problem in a non-linear context, and then we are going to solve it. So the first one is called a constrained and unconstrained non-linear optimization problem.

If there is no constraint in the problem, then we can say it is an unconstrained nonlinear optimization problem. That means there will be only an objective function; if you incorporate, if you include constraint into the problem, then we can say it is called a constrained non-linear optimization problem.

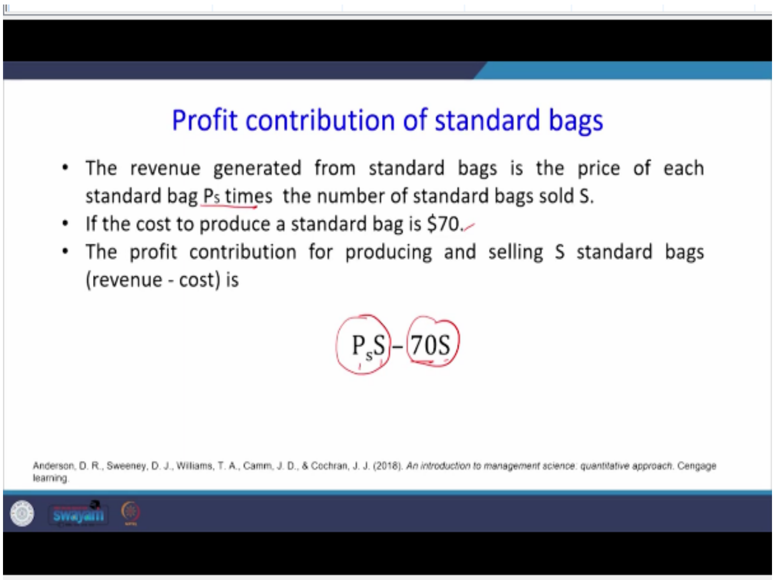


First, we will discuss an unconstrained problem. What is the meaning of unconstrained? There would not be any constraint. Let us consider the revision of the problem which you discussed. So, that company decided to manufacture standard and deluxe golf bags. In formulating the linear programming model for the problem, we assumed it could sell all of the standard and deluxe bags it could produce. That was the basic assumption in the LP model. What is that? Whatever they produce will be sold in the market.

However, depending on the price of the golf bags, this assumption may not hold because we know that in the law of demand, there is an inverse relationship between price and demand. So, an inverse relationship usually exists between price and demand. This was the problem.



What is that problem maximized 10 years + 9D? Their company produces two bags, 1 is the standard bag and the deluxe bag. There were four constraints: cutting and tying constraints, sewing constraints, finishing constraints, and inspection packaging constraints. This was the original problem. Now, we are going to modify it. What are we going to modify? Only in the objective function, we are going to introduce the concept of non-linearity.



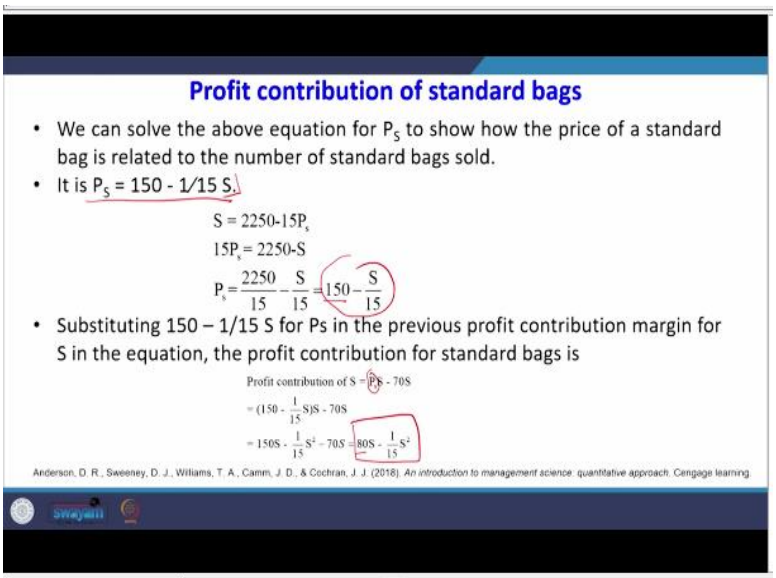
What is that? As the price goes up, the quantity demand goes down. That is what the law of demand says, what is that one? If this is price, this is quantity; if the price goes up, the quantity demanded goes down. Let P_S denote the price that the company charges for each standard bag; the price of the standard bag is called P_S, and P_D is the price of your deluxe bags. Assume that the demand for standard bag S and the demand for deluxe bags is given by these equations.

This is called your demand function. What is the demand function that connects your quantity and the price? So,

$S = 2250 - 15\ P_S$ is the demand function for the standard bag.

$D = 1500 - 5\ P_D$ is the demand function for the deluxe bag. Now we are going to find out the profit contribution of the standard bag. The revenue generated from the standard bags is the price of each standard back P_S times and the number of standard bags it sells. So, what will happen when you multiply the price, and the number of bags sold so you will get the revenue?

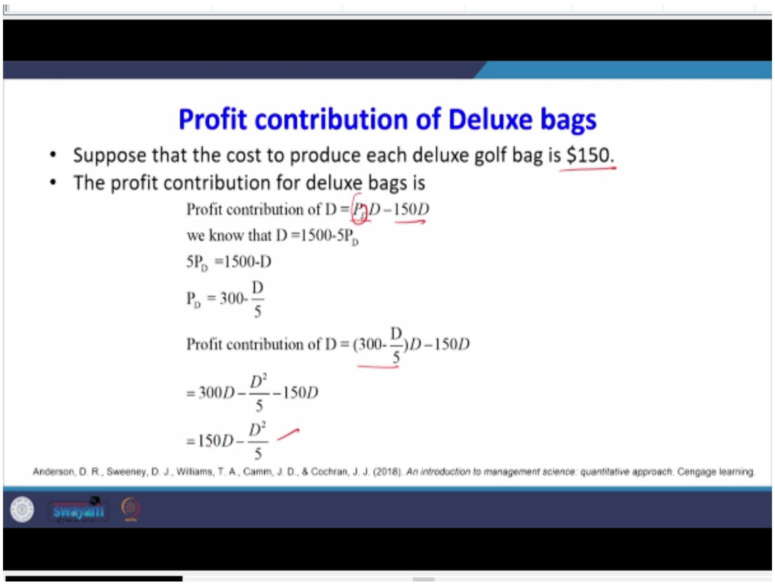
From the revenue, when you subtract the cost, you will get your profit contribution. If the cost to produce a standard bag is, say, 70 dollars. So, the profit contribution for producing and selling S number of standard bags is equal to revenue – cost. What is the revenue? Price multiplied by demand for standard bag - this cost multiplied by demand of standard bags. So, the profit contribution is this component is revenue, and this component is a cost. So, this will give you the profit contribution for the standard bag.



Similarly, we can solve the above equation for PS to show how the price of the standard bag is related to the number of standard bags sold. How? For example, I will come back to how we got into this relationship. For example, we know the demand function for the standard bag is S = 2250 - 15S. From this, I can find out the P_S. What is the P_S? What is the price of the standard bag? So, when you bring on the left-hand side, $15P_S = 2250 – S$.

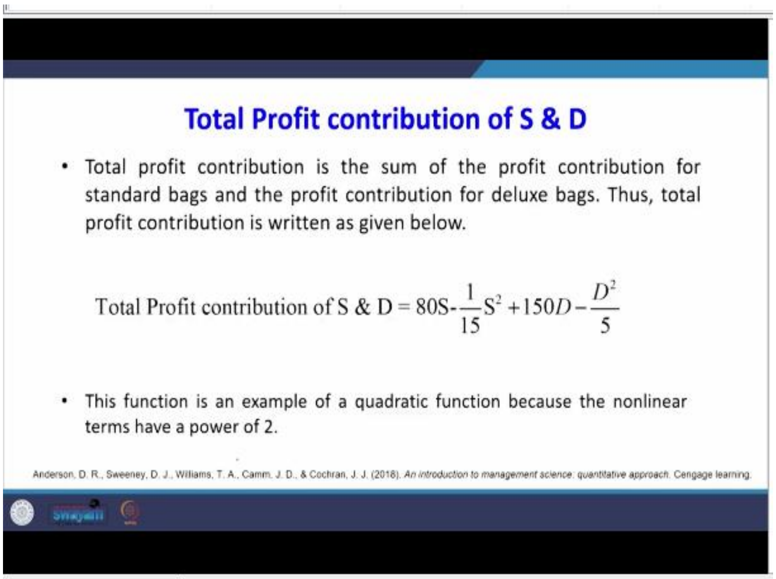
Then, when you simplify P_S, the whole component must be divided by 15. So, (2250/15) - S/15. So, that will be equal to 150 – S/15. That is what we got. P_S is 150 – S/15. Now we are going to substitute this P_S. So, by substituting (150 - S/15) for P_S in the previous profit contribution margin for S in the equation, the profit contribution for the standard bag is what we already know in the equation for the profit contribution of S.

So, instead of this P S, we are going to substitute this 150 – (1/15)S. So, if you 150 – (1/15)S, then you multiply (150S – S^2/15 – 70S). Again, if you further simplify, so 150 – 70 is (80S S^2/15). So, this is the profit contribution of our standard bag. Similarly, we can find the profit contribution for the deluxe bags. How?

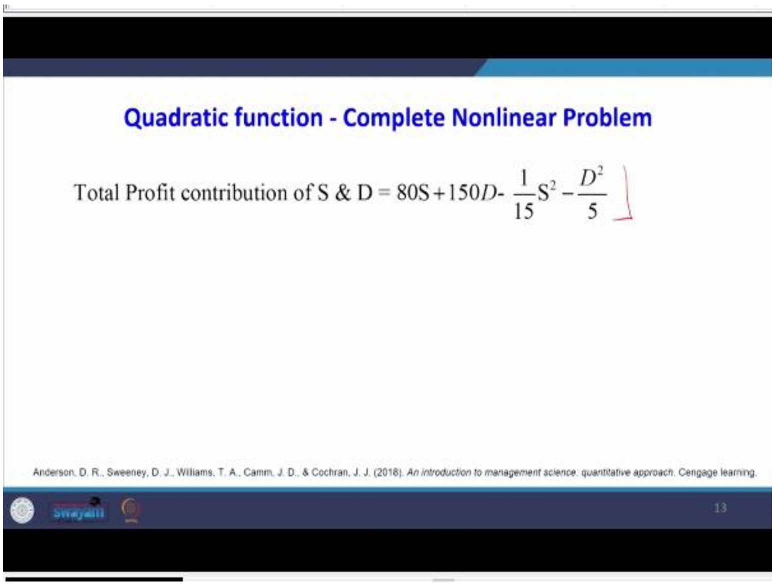


We know that the cost of producing each deluxe golf bag is, say, 150 dollars. Again we will come back to the finding profit contribution for the deluxe bag. So, profit contribution is revenue-cost. What is the revenue? Price multiplied by quantity demand, quantity sold or demanded - the cost of the product. So, here is what we are going to do. From our demand function, we know our demand function is D = 1500 - 5P_D. If I simplify, when you bring on the left-hand side, 5P_D = 1500 - D.

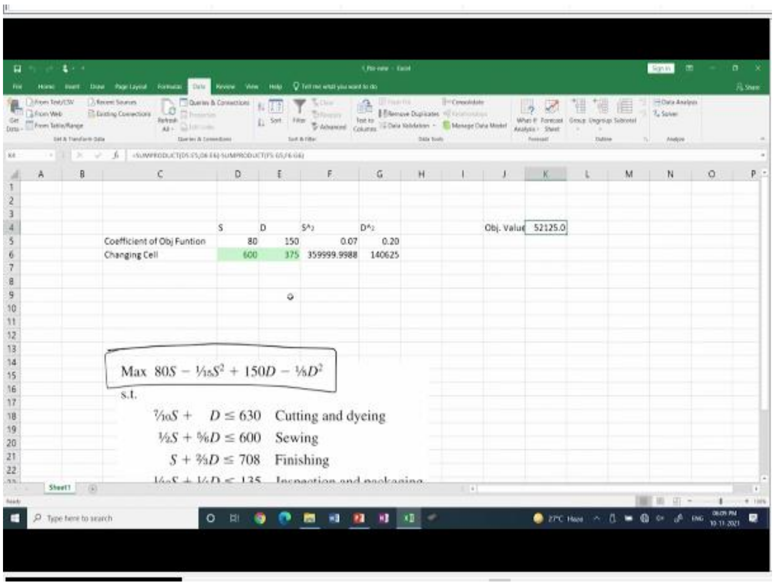
Then P D when you divide by 5 on both sides, so 300 - D upon S. So, instead of this P D, I am going to substitute 300 - D upon 5. So, when I substitute here, when I again simplify, I get this result. What is this result? (150D – D^2/5). Now, this is the profit contribution for deluxe bags. Previously we made profit contribution for the standard bag. Now we have the deluxe bag also.



Now, we are going to substitute this one, and then we can find out the total profit contribution of standard and deluxe bags. So, the total profit contribution is the sum of the profit contribution for standard bags and the profit contribution for deluxe bags. Thus, the total profit contribution is written like this: what is that? So, the total profit contribution of the standard bag and the deluxe bag is equal to $80S – (1/15) S^2 + 150D – D^2/5$. This function is an example of a quadratic function because the non-linear terms have a power of 2 if the power is 2, then we can say it is a non-linear function.



Again, I have brought this as a quadratic function. This objective function is a non-linear form. Now you see that we are not going to consider any constraint here. Now, with the help of a solver, I am going to solve this equation. What is that equation? $80S – (1/15) S^2 + 150D – D^2/5$.

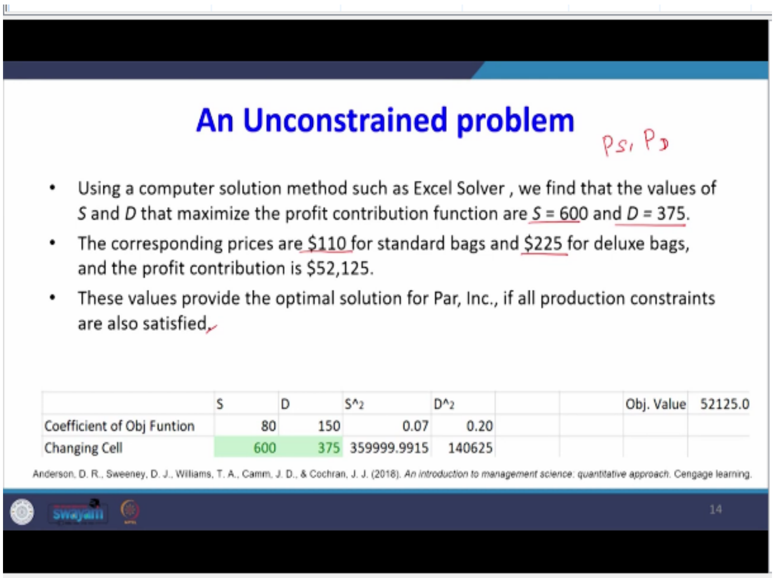


Now, I am going to solve this function, which maximizes 80S – (1/15) S^2+ 150D – D^2/5. Look at the formulation in Excel. So, I have written S in D5, I have written the coefficient of S. Similarly, in E5, I have written the coefficient of D, then in F5, when I click F5, the coefficient of (1/15) S^2. If I click F5, I see that it is one upon 15. Similarly, if I click G5, you see this is 1 upon 5. It is 0.2.

Then look at the value of F6; F6 is a square of D6; similarly, G6 is a D^2 that is a square of your E6. If you click on the objective function, the K3 will see this. So, what I have as a d1 sum product of first, I am multiplying the coefficient of S minus because both the ##d term has the minus, some product of the linear function minus some product of nonlinear function. That will be the value of my objective function. So, what point I want to say is we are optimizing only the S; we are not bothered about the S^2.

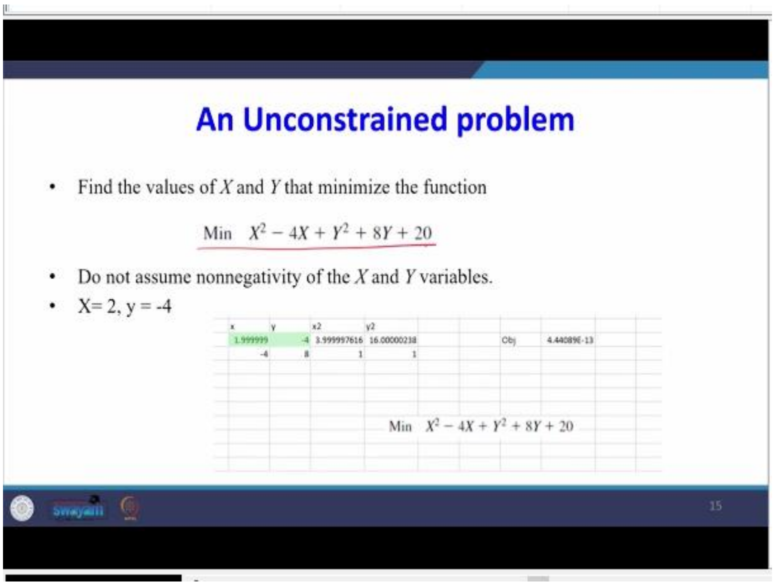
So, here, the changing cell is only in the S and D, which is a linear form. Now we go to solver home data solver, so the value of the objective function is K4; this is a maximization problem. The changing cells are only D6 and E6, with only the values of S and D. Note that there is no constraint here because this problem is unconstrained. We are not going to consider the constraint.

Then you see that I am selecting a method called GRG non-linear instead of simplex. There are different options there, simplex and non-linear. So, I am clicking non-linear, and then when I solve it, yes. You see that I got the value of S, which is 600, and the value of D, which is 375. So, the value of the objective function is 52125. This is the way to solve an unconstrained optimization problem.

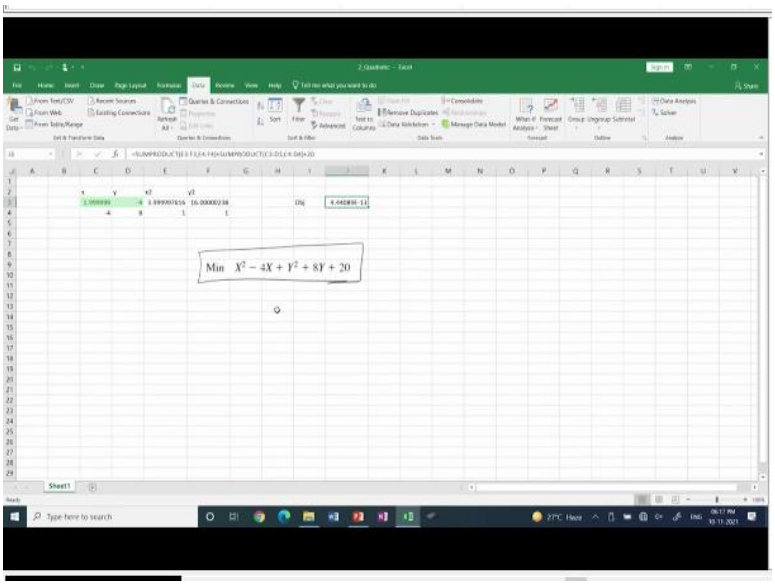


Now, I will interpret the solver's output. So, using a computer solution method such as Excel Solver, we find that the value of S and D that maximizes the profit contribution functions are S = 600 and D = 375. So, the number of units that should be sold for the standard bags is 600, and for the deluxe bags, is 375. So, when we substitute this value in our demand function, we will get the value for our P S and P_D. What is the P_S?

The price of the standard bag. For example, the price of the standard bag will be 110. The price of the deluxe bag will be 225. Then, the profit contribution will be 52125. So, these values provide the optimal solution for the problem, which you consider if all production constraints are satisfied. But at present, we do not consider the constraint because it is an unconstrained problem.



Now, I am going to explain how to use a solver for solving non-linear equations, where there is no constraint. For example, the problem is like this. Find the value of X and Y that minimizes the function. The previous problem is we consider the maximization problem. Now we have to find the value of X and Y which minimizes this function. What is this function? $X^2 – 4X + Y^2 + 8Y + 20$. Here, we are not going to consider the nonnegativity constraint. So, what it says is that do not assume the nonnegativity of X and Y. So, if the problem is like this, how to use a solver? So, now I will go to the solver I will explain how to solve this equation.

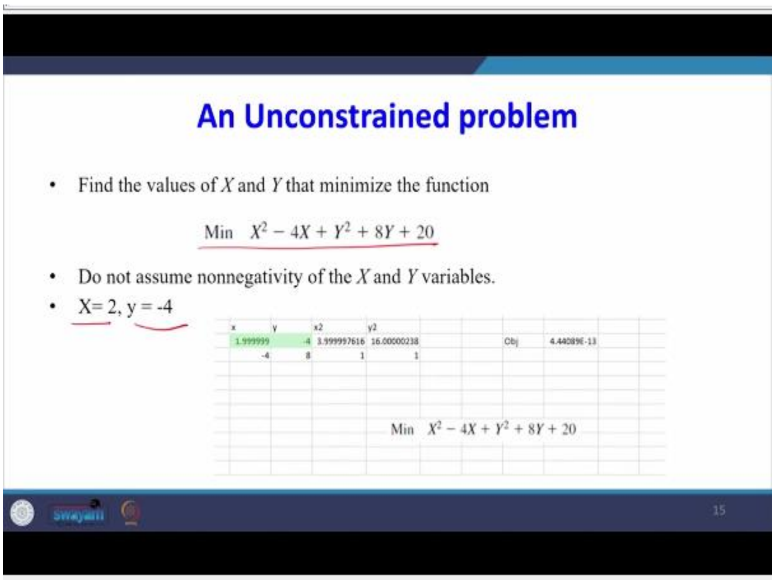


Now, the equation for which we need a solution is this one: what is that $X^2 – 4X + Y^2 + 8Y$? First, I have written only the values of X and Y. The coefficient of X is -4I, the coefficient of Y is +8 I have written. Then the coefficient of X^2 is 1, and the coefficient of Y^2 is also 1 + 20.

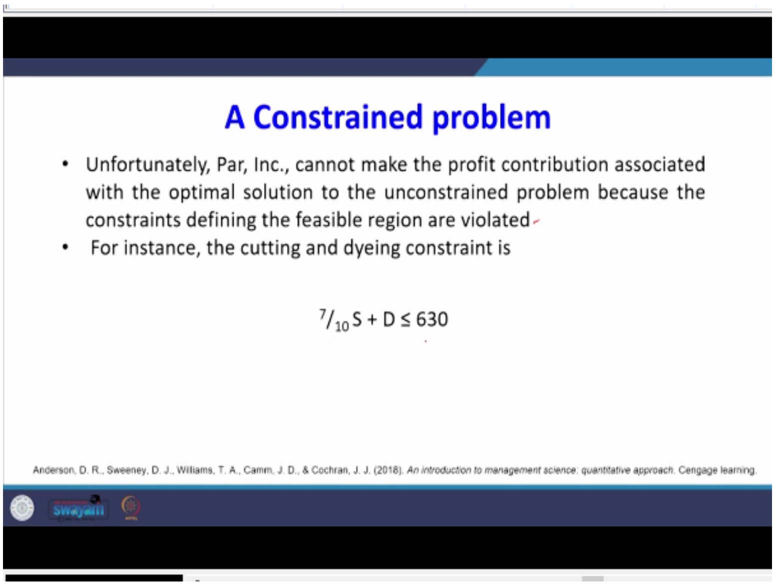
Now you see how I am writing the equations for this. So, if I click C3, see that C3 is the cell where I need where I am going to get the value of X.

Similarly, D3 is the place where I am going to get the value of Y. Now if I click on E3, I see that E3, E3 is the square of X that is C3 square. Similarly, F3 sees that, which is D3 square that is nothing but the value of Y^2. Now there is one constant also is there 20. Now, I am going to incorporate these 20 into my objective function. Now you see this: some product, the linear combinations, plus some product, the non-linear combinations plus 20.

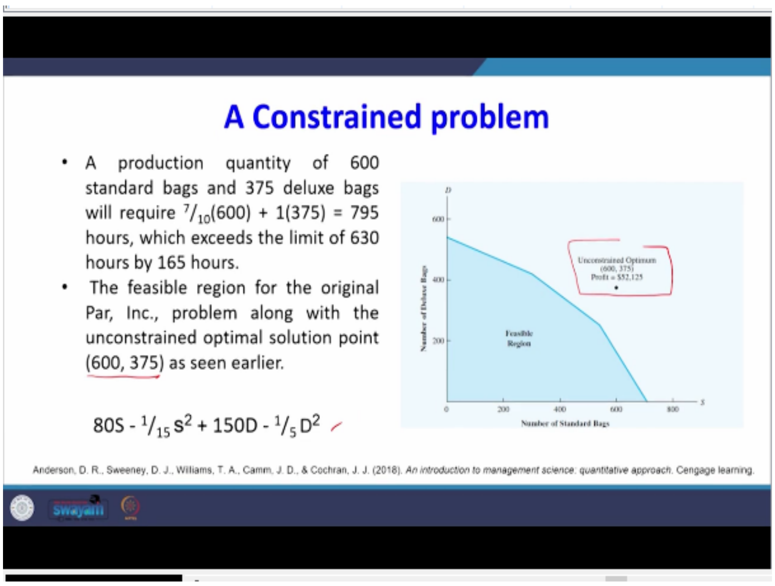
So, the constant is what I have added to the objective function itself. Now I go back to the solver data. Now you see that the minimization function, the variable cell is C3 and D3. So, I am going to solve nonlinear. You see that I did not check this box. So, that means I am allowing you to get a negative value also; when you check this box, you will get only a positive value. So, now I can get the negative value also. When I solve it, yes, now you see the value of C3 is 1.99, which is a 2, and the value of D3 is –4. So, this is the method to solve a non-linear equation with the help of a solver.



Now I have got the value of X is 1.99 say X = 2, the value of Y is -4. This is an example of how to use a solver.

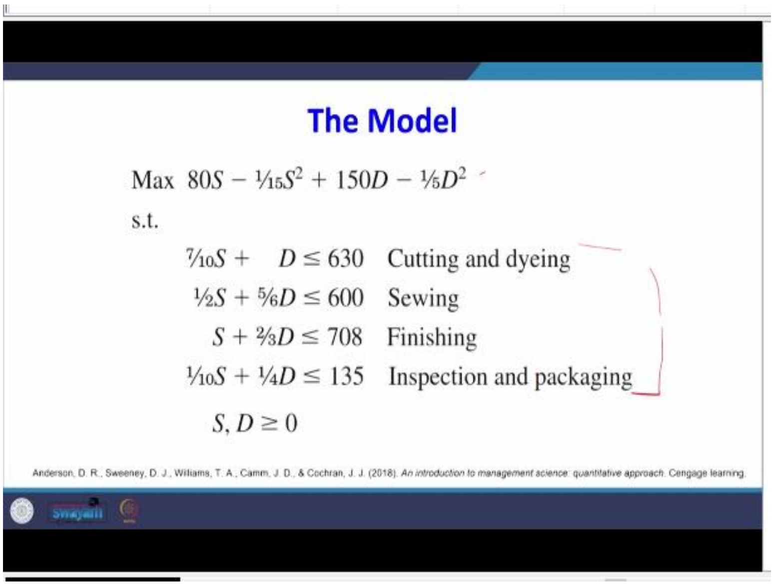


Now, we are going to see the second category, which is called a constrained problem. See, unfortunately, that company cannot make the profit contribution associated with the optimal solution to the unconstrained problem because the constraints defining the feasible region are violated. How it is violated will be because previously, we did not consider the constraint. For instance, the cutting and dying constraint is like this: 7 upon 10S + D less than equal to 630.

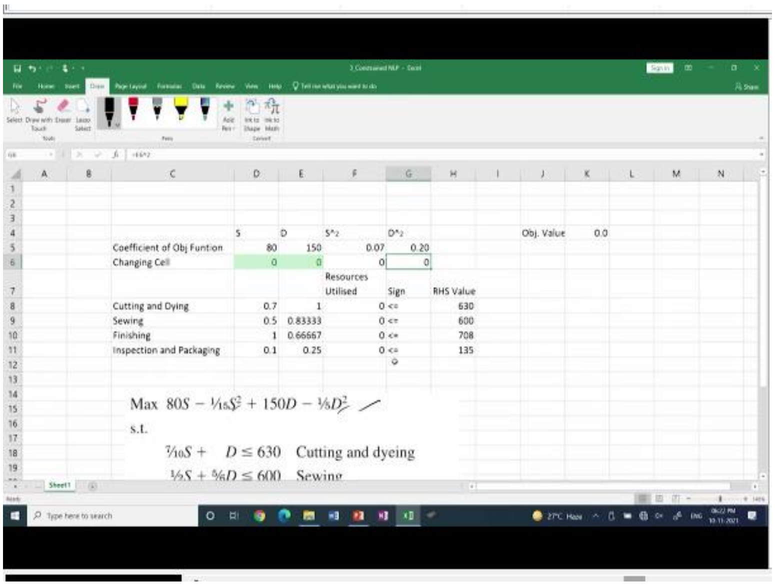


We got the values of S and D. The value of S is 600, and the value of D is 375, but when you put the total, it becomes 795, which is against our constraint. Our limit is 630, but it exceeds 165. Now, the values of 600 and 375 are lying here. But our feasible region is this region. Now, the solution we shall consider 600 and 375 is not an optimal solution because it violates our constraint.

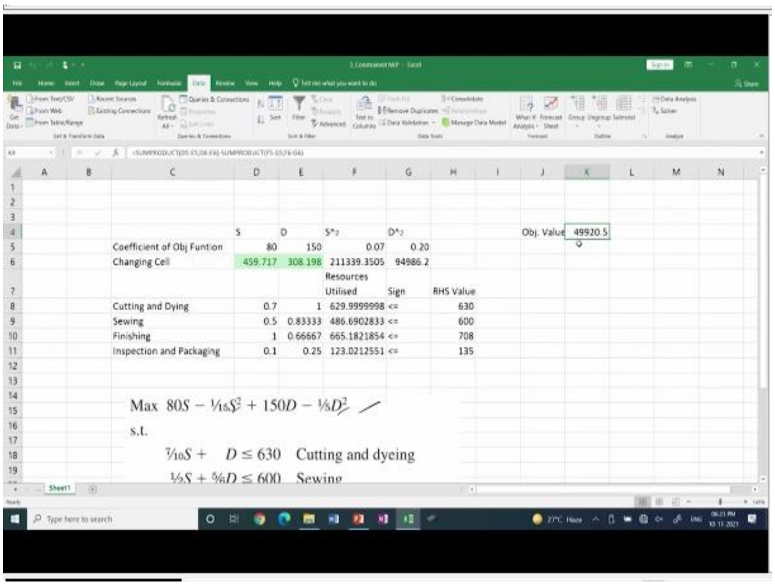
So, now what are we going to do? We will consider all the constraints, and then again, we will solve this problem. Such a problem is called a constrained non-linear optimization problem. So, the feasible region for the original problem, along with the unconstrained optimal solution point, for example, 600, 375 for this objective function, is like this. That is, it is violating our constraints.



Now, what am I going to do? I am going to consider all the constraints on this objective function. Again, I am going to solve with the help of a solver.

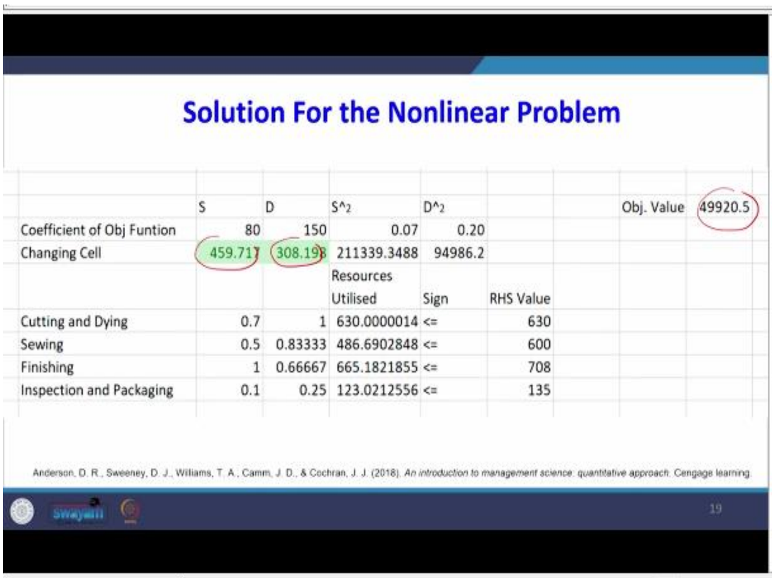


First, I will explain the formulation of this problem in Excel. So, I have written the values of S and D, and then I have written the coefficient of objective function 80S; please see this 80S + 150D – (1/15) S^2 and similarly (-1/ 5)D^2. So, D6 is the value of S. Similarly, E6 is the value of D, but F6 is the value of S^2. Similarly, G6 is the value of D^2. As usual, I have written all the other constraints because, at the bottom, all the constraints are linear.



So, again, I have written that if I click on F8, for example, F8, you see that some products are D6 to E6 and D8 to E8 because, in the constraint, there is no square and no nonlinear form. So, I am considering only the linear form. Now, we look at the objective function of the K4. The K4 has some products for positive coefficients, that is, for linear coefficients minus some products for non-linear coefficients. Now, if I go to the data solver, yes, this is a maximization problem.

Then, the changing cell is D6 to E6; here, you see that I am considering all the constraints. So, now, it will become a constrained non-linear optimization problem. So, when I solve it, yes, now I am getting the value of the objective function as 49920. Then I got the value of S, which is 459, and the value of D, which is 308. Now, I will interpret this result in the lecture.



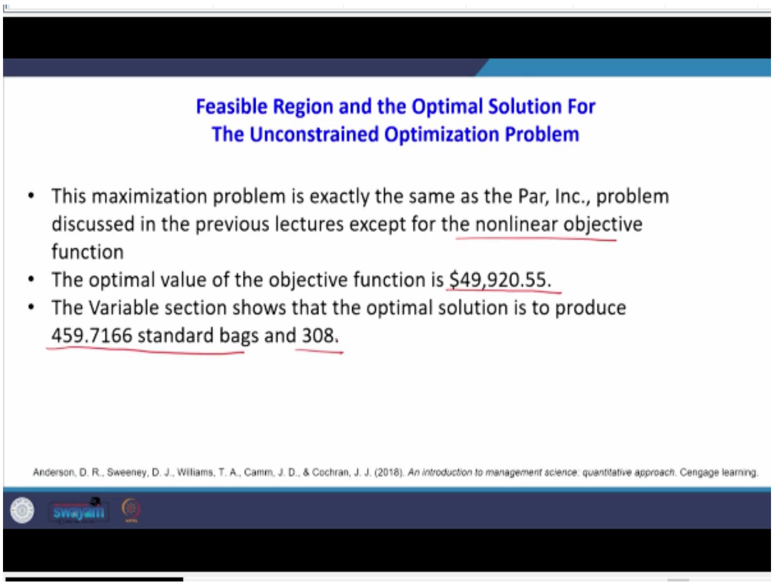
Solution For the Nonlinear Problem

	S	D	S^2	D^2		Obj. Value	49920.5
Coefficient of Obj Funtion	80	150	0.07	0.20			
Changing Cell	459.717	308.198	211339.3488	94986.2			
			Resources Utilised	Sign	RHS Value		
Cutting and Dying	0.7	1	630.0000014	<=	630		
Sewing	0.5	0.83333	486.6902848	<=	600		
Finishing	1	0.66667	665.1821855	<=	708		
Inspection and Packaging	0.1	0.25	123.0212556	<=	135		

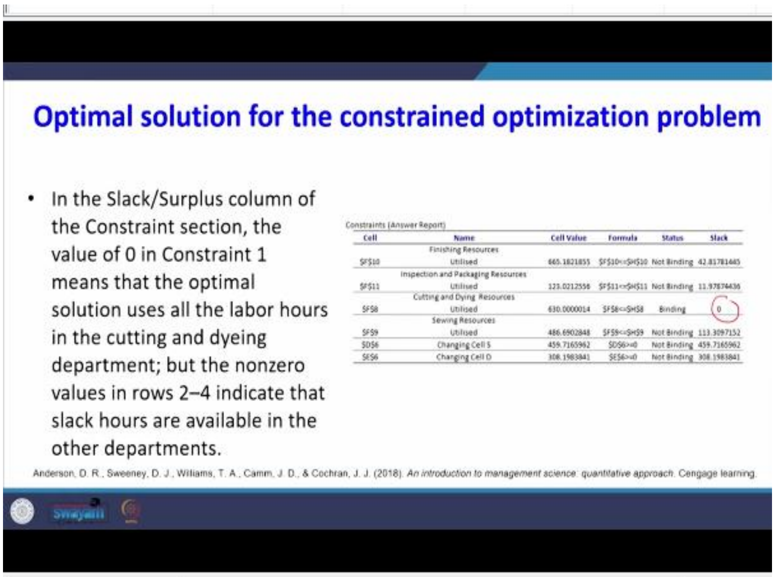
Anderson, D. R., Sweeney, D. J., Williams, T. A., Camm, J. D., & Cochran, J. J. (2018). *An introduction to management science: quantitative approach*. Cengage learning.

19

Now, I have brought the solution for the non-linear problem, in which I got the value of S, the value of D, and the value of the objective function. You see that the value of the objective function is decreased when we consider the constraint. Previously it was above 50000; now, by considering the constraint, the value of the objective function is decreased.

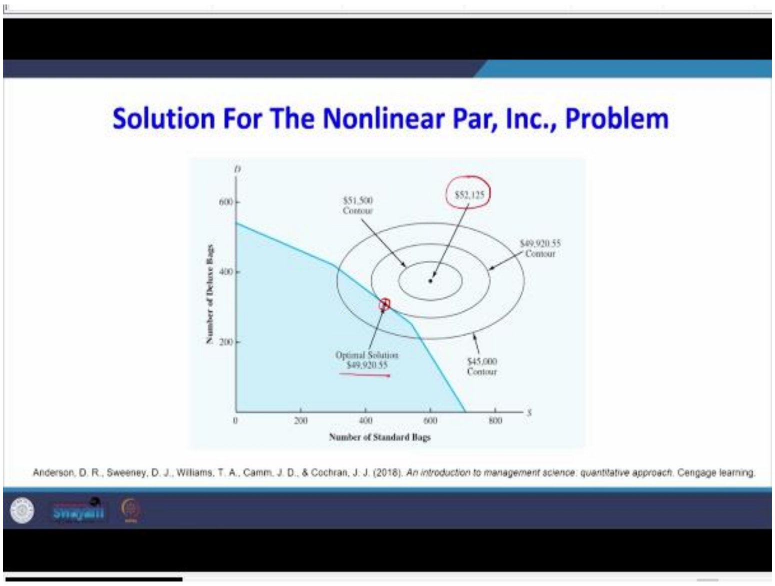


So, this maximization problem is exactly the same as the problem that you have discussed in the previous lecture except the non-linear objective function. So, the optimal value for the objective function is 49920 dollars; the variable section shows that the optimal solution is to produce 459 standard bags and 308 deluxe bags.

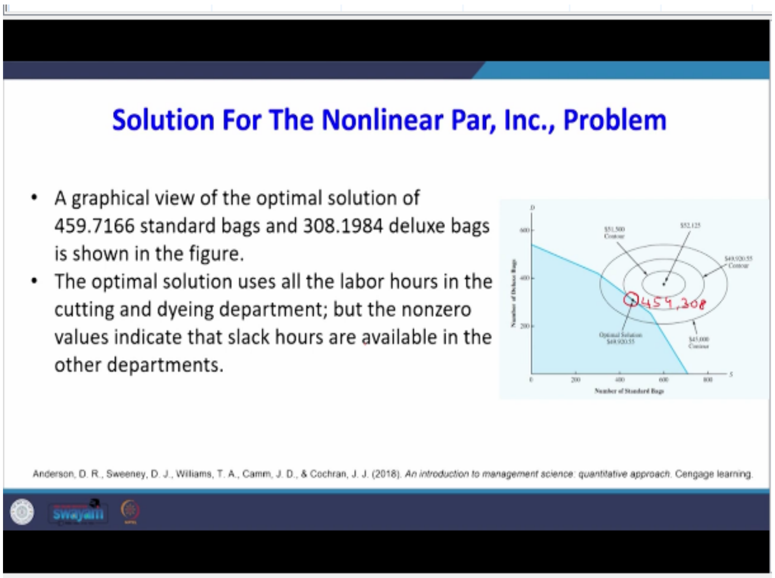


Cell	Name	Cell Value	Formula	Status	Slack
	Finishing Resources				
$F$10	Utilised	665.1821855	$F$10<=$H$10	Not Binding	42.81781445
	Inspection and Packaging Resources				
$F$11	Utilised	123.0212556	$F$11<=$H$11	Not Binding	11.97874436
	Cutting and Dying Resources				
$F$8	Utilised	630.0000014	$F$8<=$H$8	Binding	0
	Sewing Resources				
$F$9	Utilised	486.6902848	$F$9<=$H$9	Not Binding	113.3097152
$D$6	Changing Cell S	459.7165962	$D$6>=0	Not Binding	459.7165962
$E$6	Changing Cell D	308.1983841	$E$6>=0	Not Binding	308.1983841

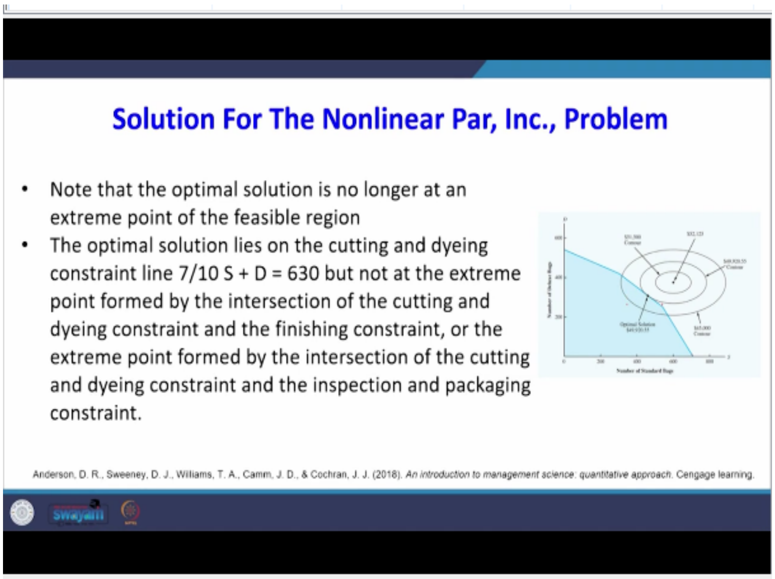
Here we can interpret the meaning of slack and surplus also. Here, all the constraints are less than or equal to type, so we can say all the variables are slack variables. So, like our linear programming problem. In the slack surplus column of the constrained section, the value of 0 in constraint 1 here means that the optimal solution uses all the labor hours in the cutting and dying department. However, nonzero values indicate that the slack hours are available in other departments. So, the interpretation of Slack is like what we have interpreted in the LP model linear programming model.



Now, we will analyze this solution and see when we do not at all consider any constraint, the value of the objective function is this one 52125; you see that this is an ellipse shape. But now we have found that the optimal solution is here when we consider the constraint. There is another constraint also there; there is another ellipse also appearing when the value of the objective function is 45000. So, the value of the objective function is 49920, where it intersects the feasible region at this point. So, this point is our optimal value for a non-linear problem. We will interpret this further in the coming slides.

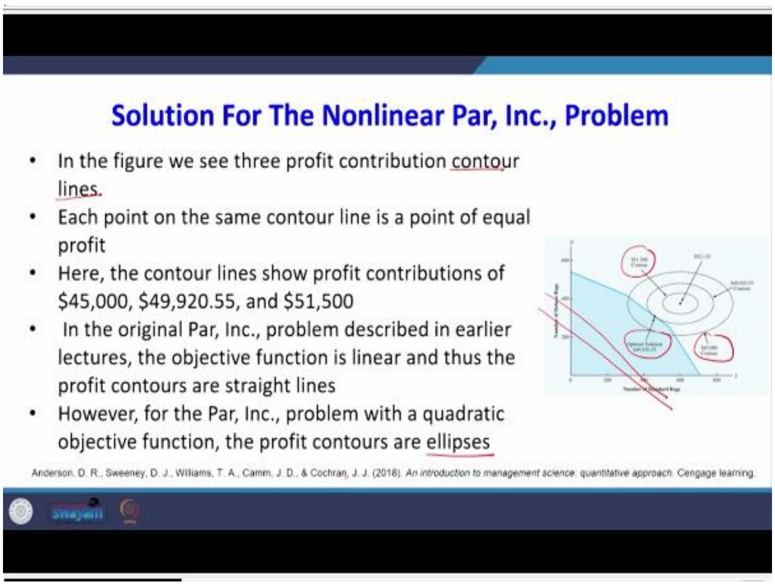


You see the graphical view of the optimal solution of 449; here this point is 459, 308, which point, this point, this point, the optimal solution uses all the labor hours in the cutting and dying department, but non zero value indicates that slack hours are available in the other departments.



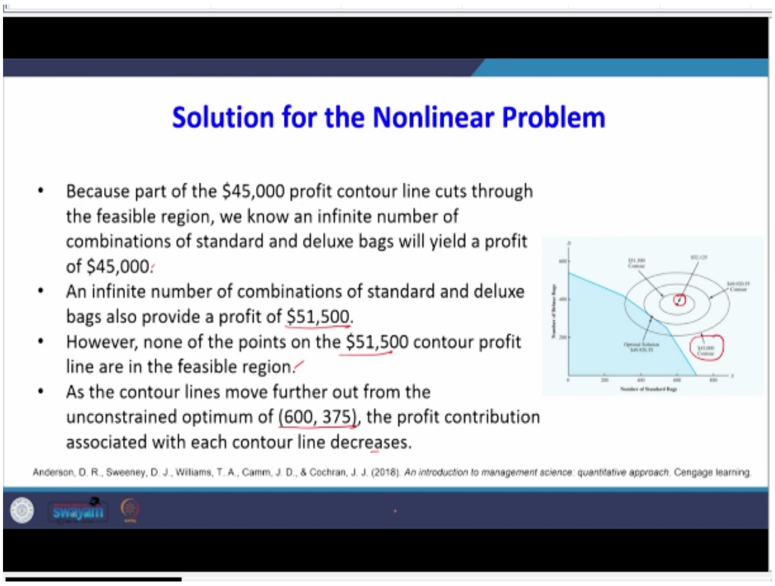
Note that the optimal solution is no longer at the extreme point of the feasible region. When we solve a linear programming problem, the optimal solution lies on the corner point, but that is not the case here in the non-linear problem. So, the optimal solution lies in the cutting and dying constraint $(7/10)S + D = 630$.

But not at the extreme point formed by the intersection of cutting and dying constraint and finishing constraint or the extreme point formed by the intersection of cutting and dying constraint and inspection packaging constraint. So, what we understand here is that the optimal solution does not need to lie on the intersection points or the corner points. The optimal point may lie anywhere; sometimes, the optimal point may lie inside the feasible region also, which we will see in the next slide.



In the figure, we see three profit contribution contour lines. So, this ellipse is nothing but contour lines. The meaning of a contour line is all the points on the contour points will provide the same value of the objective function. This means each point on the same contour line is the point of equal profit. Here, the contour line shows the profit contribution of 4500 this one, 4920 this one, and 51500 this one 51500.

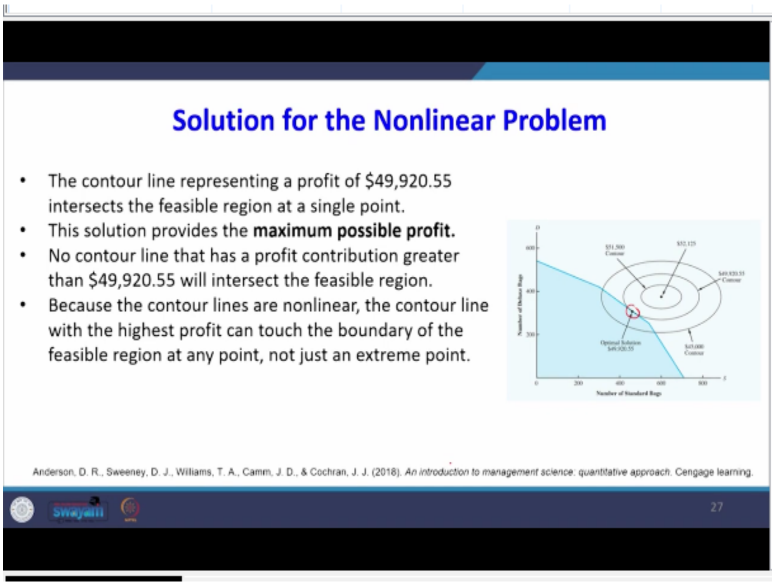
In the original problem described in earlier lectures, the objective function is linear, and thus, the profit contours are straight lines. Previously, in the linear models, the profit lines were linear, but now the contour lines are linear. But now, the contour lines are not linear. However, the problem with the quadratic objective function is that the profit contours are ellipses. That is a point we have to note here.



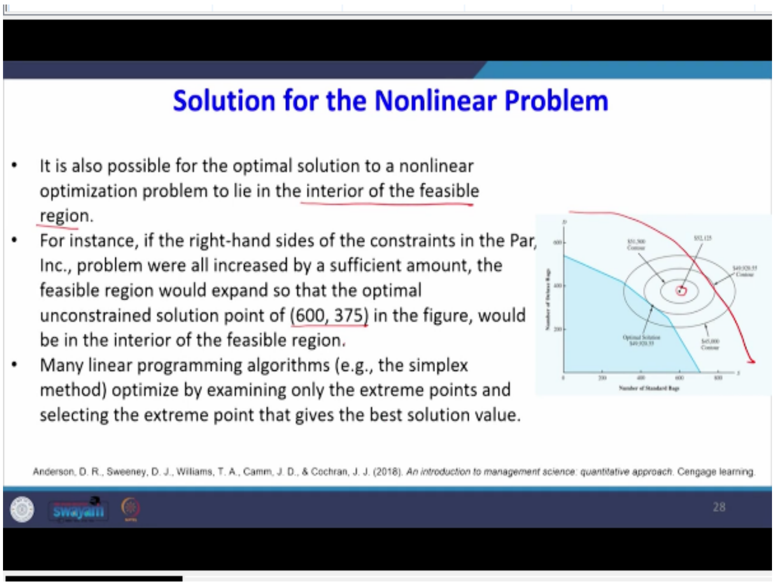
Because part of the 40000 dollar profit contour lines cuts through the feasible regions, we know that the infinite number of combinations of standard and deluxe bags will yield a profit of 45000. So, when we say the 45000, there may be an infinite number of combinations of our S and D. Similarly, an infinite number of combinations of standard and deluxe bags is also provided when the profit function is 51500.

However, none of the points when the value of the objective function is 51500 here, none of the points the contour profit lines are in the feasible region; it goes beyond the feasible region as the contour line moves further out from the unconstrained optimum; this is the unconstrained optimum value. So, the profit contribution associated with each contour line decreases; you see, initially, the solution was this one.

When we move away from these unconstrained optimal values, the value of the objective function is decreased, but where it is exactly touching when it is touching this point, we get the maximum value for the objective function, which is 49920. This 2 point that is the value of S and D is our optimal solution.

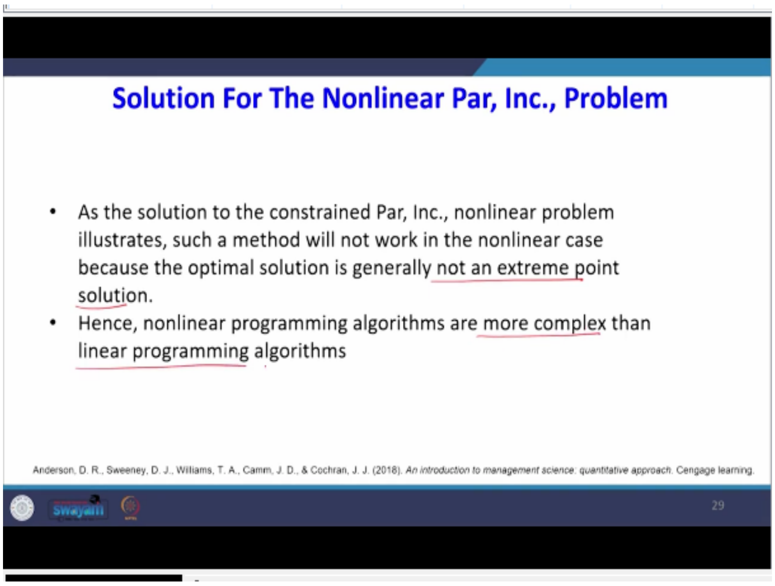


The contour line representing a profit of 49920 intersects the feasible region at a single point at this point. So, this solution provides the maximum possible profit. How much? 4920, no contour line that has a profit contribution greater than 4920 will intersect the feasible region. Because the contour lines are non-linear, the contour line with the highest profit can touch the boundary of the feasible region at any point not just an extreme point. The extreme point was the case of linear programming problems.

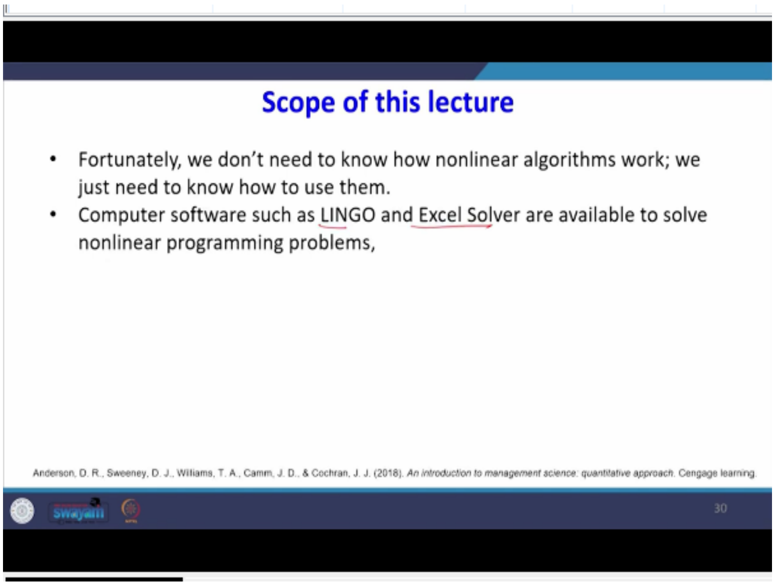


It is also possible for the optimal solution to a non-linear optimization problem to lie in the interior of the feasible region; another important point should be noted: it is not necessary on the boundary. Sometimes, your optimal solution may be inside the feasible region. For instance, if the right-hand side of the constraint in the problem were all increased by enough, the feasible region would expand so that the optimal unconstrained solution point, which is 600, 375 in the figure, would be in the interior of the feasible region.

So, what happens if the constraint on the right-hand side increases? What will happen to the feasible region that may become like this? When the feasible region becomes like this, our optimal solutions lie inside the feasible region. That is the point that I am trying to make. Many linear programming algorithms, for example, the simplex method, optimize examining only the extreme points and selecting the extreme point that gives the best solution value, but that is not the case for nonlinear problems.



As the solution to the constrained problem, what we have discussed in the nonlinear context, that method will not work in the nonlinear case; which method finding the extreme point then will be the optimal solution because the optimal solution is generally not an extreme point solution. Hence, non-linear programming algorithms are more complex than linear programming algorithms.



The important point I want to emphasize here the scope of this lecture is not on the different algorithms. So, fortunately, we do not need to know how non-linear algorithms work; we just need to know how to use the different algorithms. Especially in the management context, we should know how to use this algorithm. So, computer software such as LINGO and Excel Solver are available to solve non-linear programming problems.

However, in this course, we are going to use only Excel solver to solve non-linear problems. Dear students, in this lecture, I have discussed the concept of non-linear optimization problems. So, what I have discussed here is what an unconstrained optimization problem is where there would not be any constraint at all. Then, I have discussed the constrained optimization problem where we will consider different constraints. Then I took a sample problem that I explained with the help of a solver how to get the solution, and then I interpreted it. Thank you very much.