Decision Making With Spreadsheet
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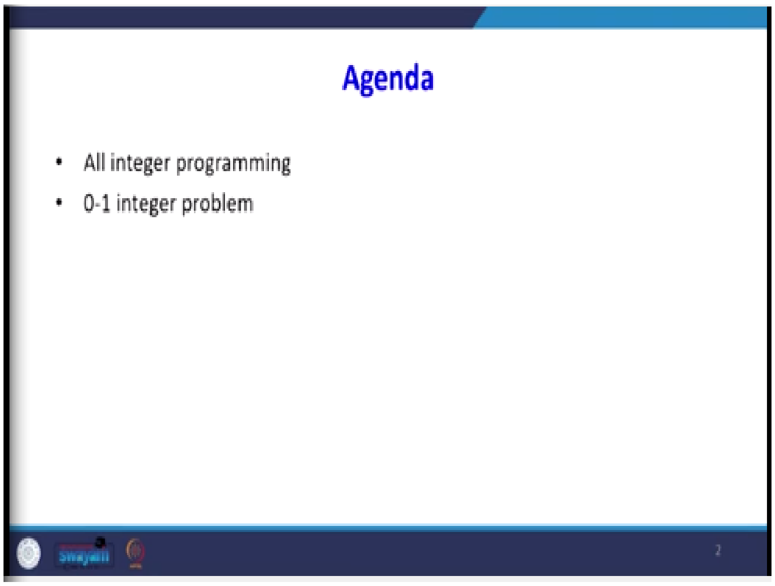
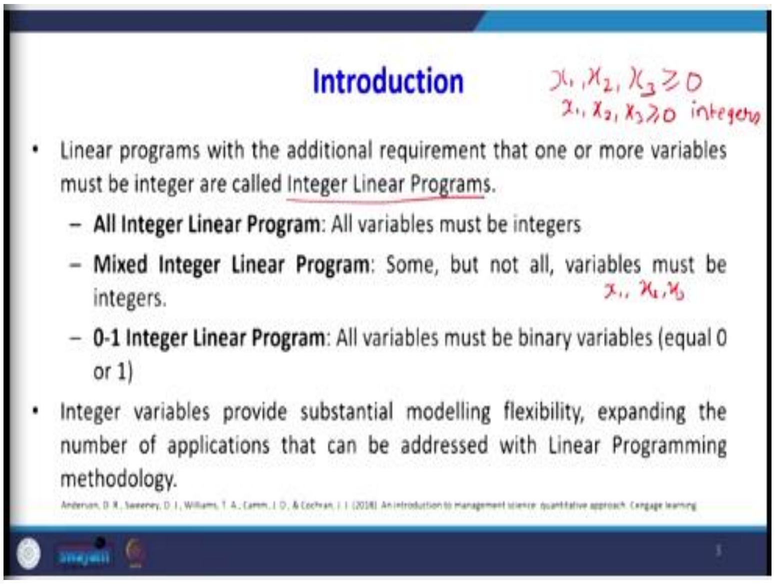
Lecture-22
Integer Programming

Dear students, in the last class, I discussed the maximum flow problem. In this lecture we will start a new topic called integer programming. The agenda for this lecture is, first, I will discuss all integer programming.



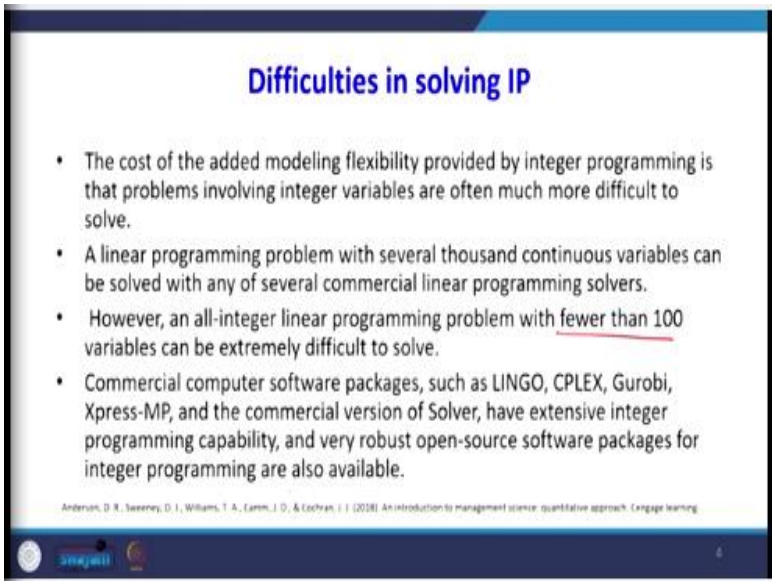
After that, a special case is called 0-1 integer programming. So, I will explain how to model how to solve, and how to interpret the result.



The introduction about linear programming is linear programs with the additional requirement that 1 or more variables must be integers called integer linear programming. Many times we are discussing, say, X_1, X_2, X_3. These are decision variables; we assume that our constraints are nonnegative. But apart from this, we are going to add some integer constraints, and then it will be called integer linear programming.

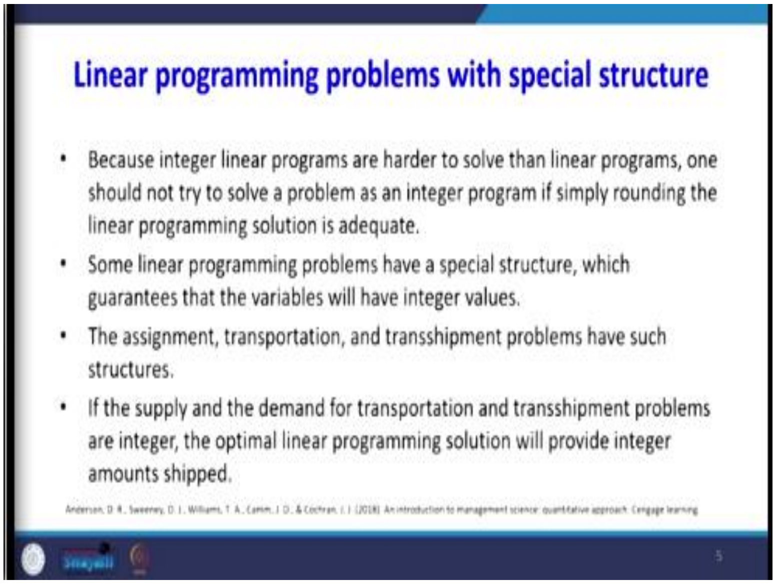
So, what is the meaning? This X_1, X_2, X_3 >= 0 and integers.
So, there are three possibilities there, all integer linear programming. So, in that, what will happen? All the variables X_1, X_2, and X3 all the variables will be the integer. Sometimes, in mixed integer linear programming, some but not all variables must be integers. For exampleX_1, X_2, X_3, some variables may be integers, maybe X_1 may be integers, and X_2, X_3 may not be integers, but it is non-negative.

The third case is a 0-1 integer linear program; all the variables must be binary variables. Integer variables provide substantial modeling flexibility, expanding the number of applications that can be addressed with the LP methodology. So, there are some applications to LP methodology, but when you introduce integer variables, it provides flexibility in our traditional linear programming methodology.



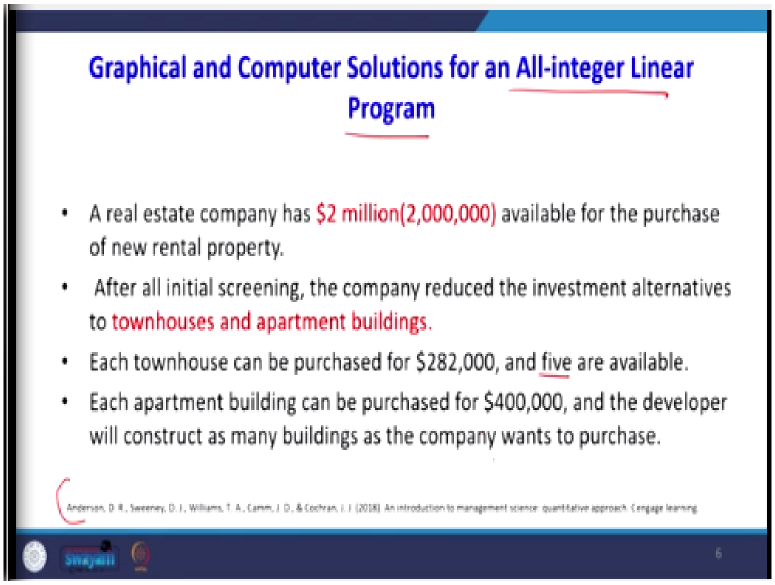
Difficulties in solving integer programming. We have seen that because of integer programming, there is modeling flexibility so that flexibility also comes with a cost. The cost of cost is, in the sense, difficult. So, the cost of added modeling flexibility provided by integer programming is that the problems involving integer variables are often much more challenging to solve. Yes, there is flexibility in introducing integer programming, but it is challenging to solve.

All linear programming problems with several 1000s continuous variables can be solved with any of several commercial LP solvers. However, all integer linear programming problems with fewer than 100 variables can be extremely difficult to solve. So, solving integer programming is more complex. Commercial computer packages such as LINGO, CPLEX, Gurobi, Xpress-MP, and the commercial version of Solver have extensive integer programming capability. Very robust open-source software packages for integer programming are also available.



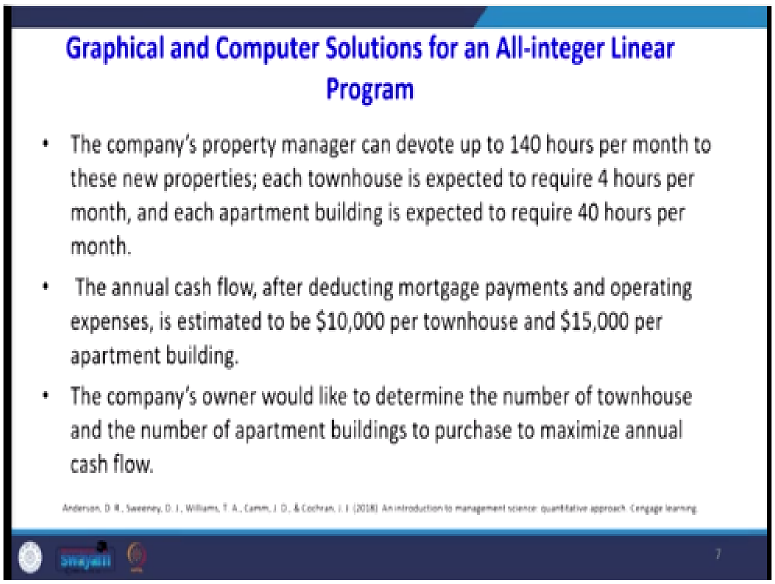
Linear programming problem with a unique structure. Because integer linear programs are harder to solve than linear programs, one should not try to solve a problem as an integer program if simply rounding the linear programming solution is adequate. Some linear programming problems have a special structure that guarantees that variables will have integer values. For example, assignment, transportation, and transshipment problems have such structure, what is the meaning?

In the assignment transportation and transshipment, if the variables and the cost and the coefficient of the objective function are linear integers, then your solution also will be an integer. If the supply and the demand for transportation and transshipment problems are integers, then the optimal linear programming solution will provide integer amounts shipped. So, that time, you need not introduce that integer constraint because when you bring an integer constraint, there is more difficulty in solving.



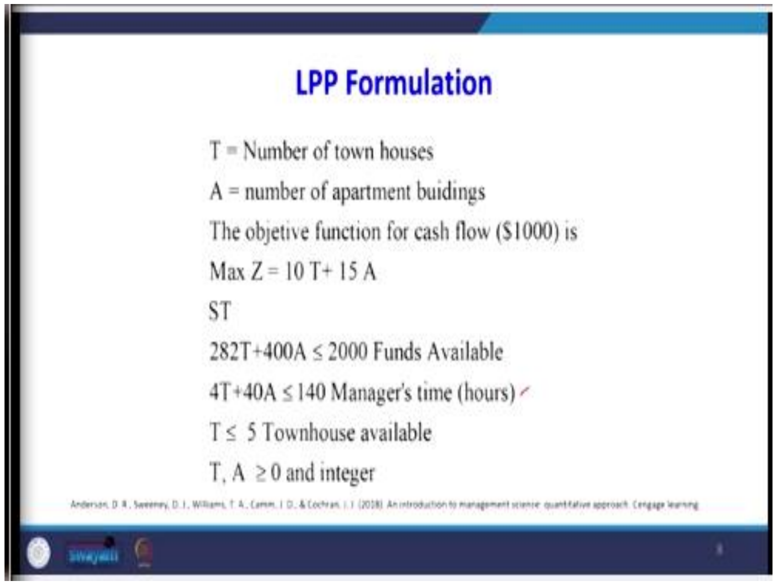
Now graphical and computer solution for All-integer linear programming. So, we are going to have one problem with All-integer linear programming. That problem is taken from this book by Anderson et al., so after formulating, we will solve it with Excel. A real estate company has 2 million dollars available for the purchase of new rental property. After all initial screening, the company reduced the investment alternative to 2 alternatives to townhouses and apartment buildings.

They are going to construct 2 types of buildings: one is a townhouse, and the other one is an apartment building. Each townhouse can be purchased for 282000 dollars, and five are available. Each apartment building can be purchased for 400,000 dollars, and the developer will construct as many buildings as the company wants to purchase.

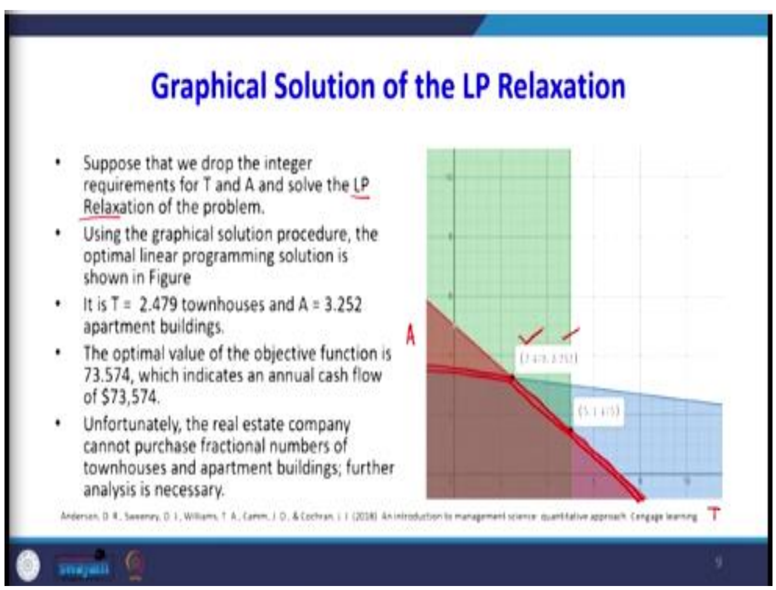


The company's property managers can devote up to 140 hours per month to these new properties. Each townhouse is expected to require 4 hours per month and each apartment building is expected to require 40 hours per month. The annual cash flow after detecting mortgage payments and operating expenses is estimated to be 10000 10,000 per townhouse and 15000 dollars per apartment building; this will be the coefficient of our objective function.

The company's owner would like to determine the number of townhouses and the number of apartment buildings to purchase to maximize annual cash flow. There are 2 alternatives: one is a townhouse, the other is an apartment building. There is a restriction on the amount available, and there is also a restriction on the number of times available for the managers. So, the problem with the manager is how many apartment buildings and how many townhouse buildings have to be constructed.



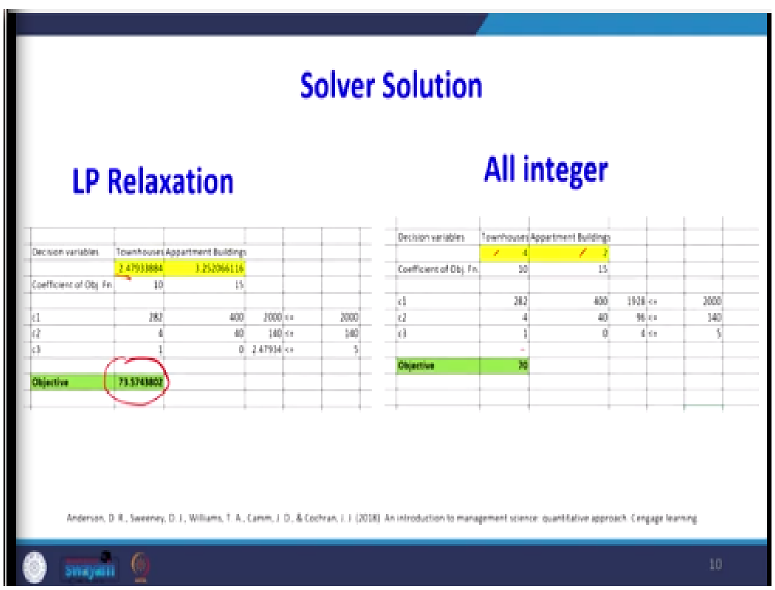
So, we will consider the decision variable: T represents the number of townhouses, and A represents the number of apartment buildings. See the objective function for cash flow in terms of 1000 is maximize Z 10, 10000 townhouse + 15000 apartment houses. The funds available to construct 1 townhouse will consume 282,000 dollars + 400,000 dollars for an apartment house, which should be less than or equal to the 2000 dollar funds available. The times when the manager can devote the townhouse to 4 hours and the apartment house 40 hours; the maximum time available is 140 hours. Their requirement is there should be a minimum of five townhouses required, so T less than or equal to 5. And T and A are non-negative integers.



First, we will solve this problem graphically. I have solved this problem with the help of Desmos. So, I have plotted all the constraints, so the feasible region is this region. So, when I solve graphically, the solution I get is 2.479; on the x-axis, it is a townhouse, and on the y-axis, an apartment building. Now, when you look at this result, you see that it is not an integer; we are getting it in terms of decimals.

But constructing this decimal point and interpreting it is very difficult; now, we will go back to the presentation. Suppose that we drop the integer requirements for T and A and solve the LP relaxation of the problem. What is the meaning of LP relaxation? Initially, it was an integer, so you relaxed that integer constraint, and then you solved, as usual, a simple LP problem called LP relaxation. Using the graphical solution procedure, which I solved with the help of Desmos, the optimal linear programming solution is shown in the figure. What is that?

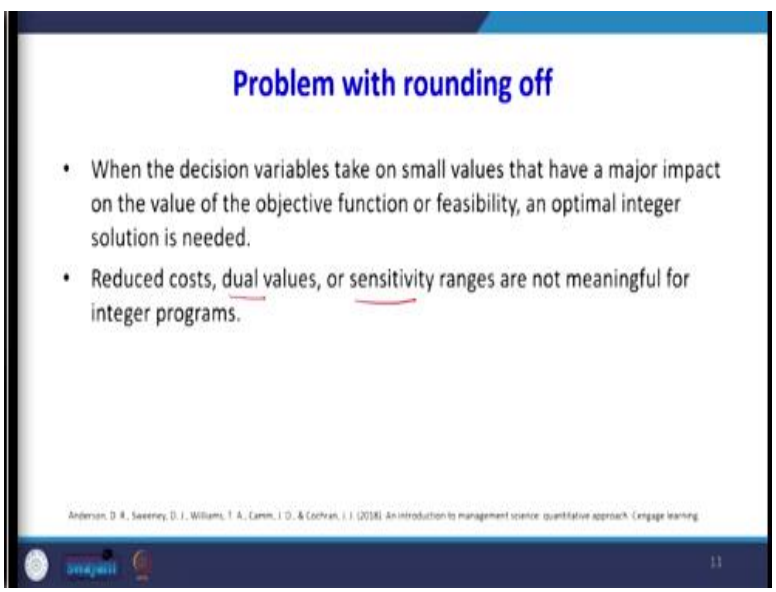
The T is 2.479, this value and A is 3.252, the optimal value of the objective function is 73.574, which indicates the annual cash flow of 73574 dollars because all the values are in terms of 1000. Unfortunately, the real estate company cannot purchase fractional numbers of townhouses and apartment buildings, so further analysis is required. That is why we are going to impose integer constraints.



Now, I solved the same problem with the help of a solver. So, when I solve the solver, you see that on the left-hand side, we are getting LP relaxation problem 2.47 townhouse and 3.25 apartment house. The objective function is 73.57 when I impose integer constraints, so the number of townhouses is 4, and the number of apartment buildings is 2; now, the objective function is 70. I will explain how I got this result in Solver.

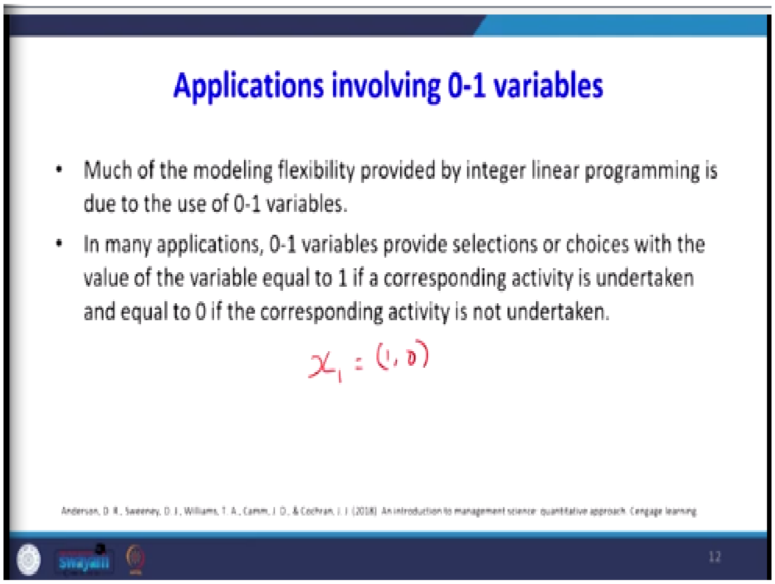
Now we have solved the problem graphically, and I will solve it with the help of a solver, so I have entered the decision variables and constraints and objective function. So, I go to data, solver, so my objective function is maximization, so here I am not imposing integer constraints. So, this problem is an LP relaxation problem when I solve it. Now you see the number of townhouses is 2.47, and the number of apartment buildings is 3.25. This is practical, so it is impractical to implement it.

Because the number of houses cannot be in the form of a fraction, and the objective function is 73.57. The same problem is I am going to impose an integer constraint, go to the solver, now you add a new constraint, you select E5 and F5, then there is an option no int integer, ok, then ok, then you solve it, yes. Now, this solution is which can be implemented, so we have to go for 4 townhouses and 2 apartment buildings; the objective function is 70. This is a very important point we have in terms of fractions. You should not round it off. So, rounding the region may be infeasible, and the rounding of the solution to make it an integer is not recommended.

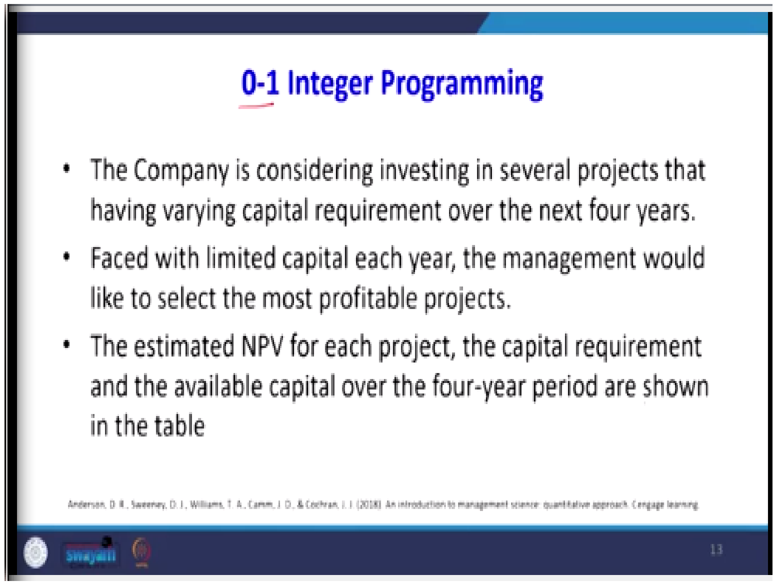


The problem with rounding off is that when the decision variables take on small values that have a major impact on the value of objective function or feasibility, an optimal integer solution is needed. So, when you round it off, it will not change proportionally, and it becomes infeasible also, so what you have to do if in an ordinary LP relaxation problem if you are getting the solution in the form of a decimal that cannot be rounded.

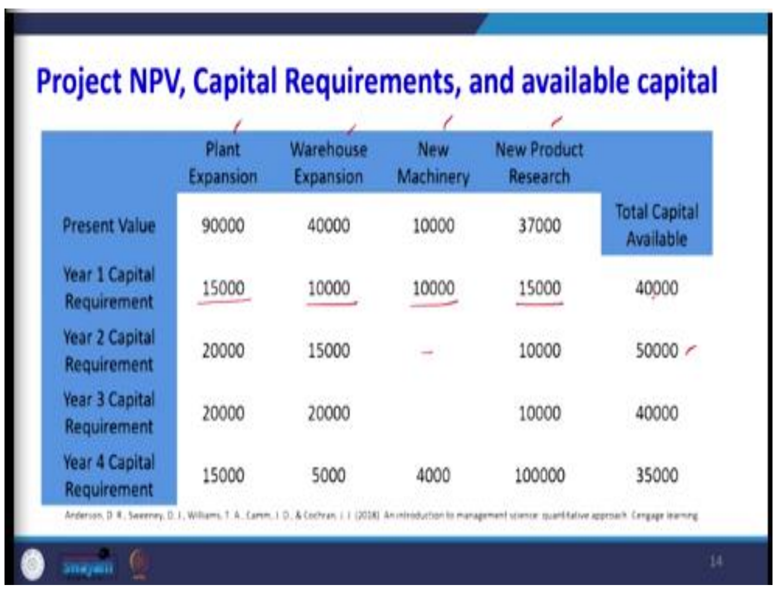
So, here, the reduced cost, dual value, and sensitivity ranges are not meaningful in individual programs. So, we cannot do sensitivity analysis, then we can do, but there is no meaning for these terms, which have seen our ordinary LP problem. Such as reduced cost, dual value, and sensitivity analysis. That is why when you solve with the help of a solver, also when you put an integer constraint, there will not be any option for sensitivity analysis to look.



So, in the application involving 0-1 integer variables, much of the modeling flexibility provided by integer linear programming is due to the use of 0-1 variables. In many applications, 0-1 variables provide selections or choices with the value of the variable equal to 1 if a corresponding activity is undertaken and equal to 0 if the corresponding activity is not undertaken. So, the value of, for example, x is a binary 1; it can have either 1 or 0; 1 represents the presence of this variable, and 0 represents the absence of this variable.



Now, we will take 1 sample problem; then I will introduce how to use 0-1 IP. This problem is taken from the book by Anderson et al. The company is considering investing in several projects that have varying capital requirements over the next 4 years. Faced with limited capital each year, the management would like to select the most profitable projects. The estimated NPV net present value for each project, the capital requirement, and the available capital for the 4 year period are shown in the table.


Project NPV, Capital Requirements, and available capital

	Plant Expansion	Warehouse Expansion	New Machinery	New Product Research	Total Capital Available
Present Value	90000	40000	10000	37000	
Year 1 Capital Requirement	15000	10000	10000	15000	40000
Year 2 Capital Requirement	20000	15000	–	10000	50000
Year 3 Capital Requirement	20000	20000		10000	40000
Year 4 Capital Requirement	15000	5000	4000	100000	35000

Anderson, D. R., Sweeney, D. J., Williams, T. A., Camm, J. D., & Cochran, J. J. (2018). An introduction to management science: quantitative approach. Cengage learning

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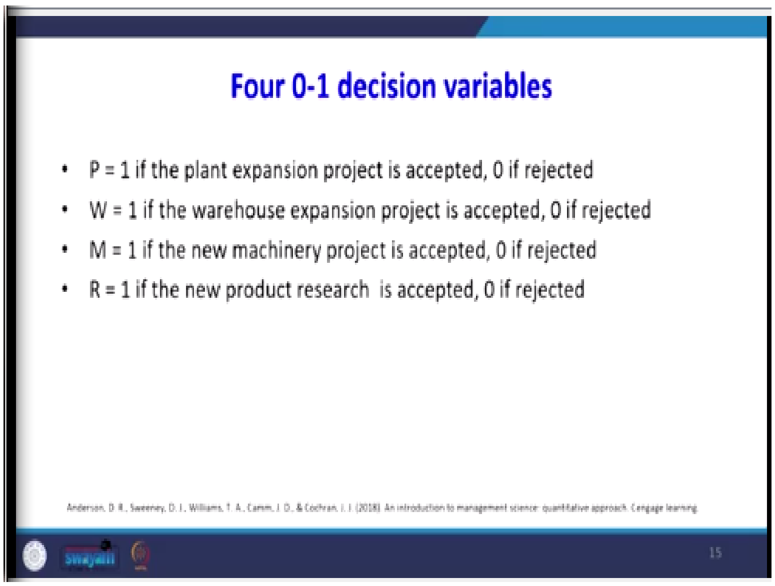
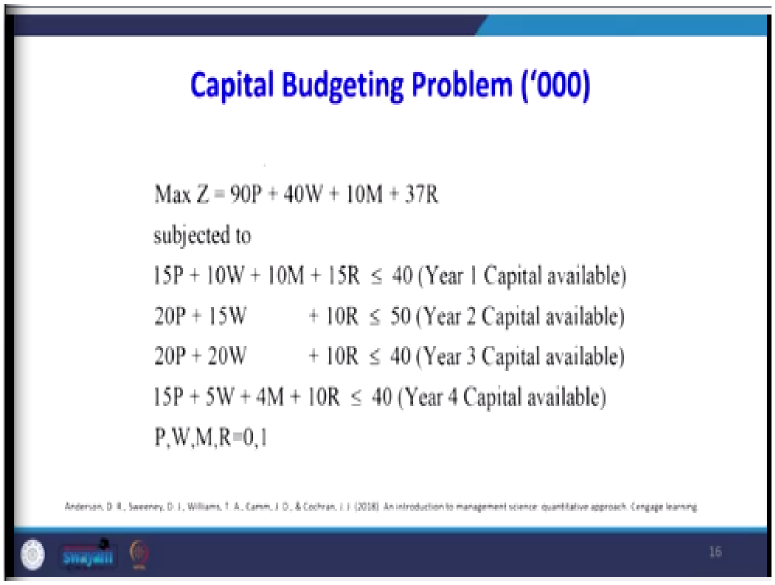
So, this table shows that there are 4 projects what are the 4 projects plant expansion, warehouse expansion, new machinery, or new product research. The net present value of plant expansion is 90,000 dollars, 40,000 dollars for warehouse expansion, 10,000 dollars for new machinery, and 30,000 dollars for new product research. Obviously, it is the net present value, so we have to choose a project, not a project. Out of these four, we have to suggest to the company what projects have to be chosen so that our net present value is maximized.

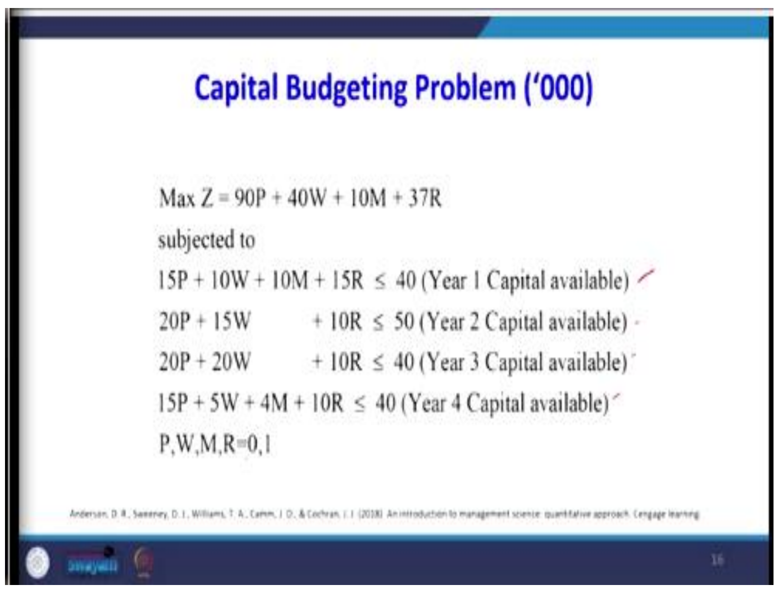
Here the capital requirement for year 1 for each project is given. For example, 15000 dollars for plant expansion, 10,000 dollars for warehouse expansion again, 10,000 dollars for new machinery, and 15,000 dollars for new product research. But in the first year, the capital availability is only 40,000 dollars. Similarly, the second-day requirement for these 4 projects is 20,000 dollars and 15,000 dollars, and there is no money required for new machinery. Because we are not going to buy machinery every year, then 10000 and the total requirement is 50,000, similarly for year 3 and year 4.



Now we are going to introduce 0-1 decision variables. If P = 1 if the planned expansion project is accepted, 0 if rejected. W = 1 if the warehouse expansion project is accepted, 0 if rejected. M = 1 if the new machinery project is accepted, 0 if rejected, so R = 1 if the new product research is accepted, 0 if rejected. Now these decision variables are binary.



Now, in the capital budgeting problem, what do we have to do? We have to maximize our NPV net present value, as we have seen previously.

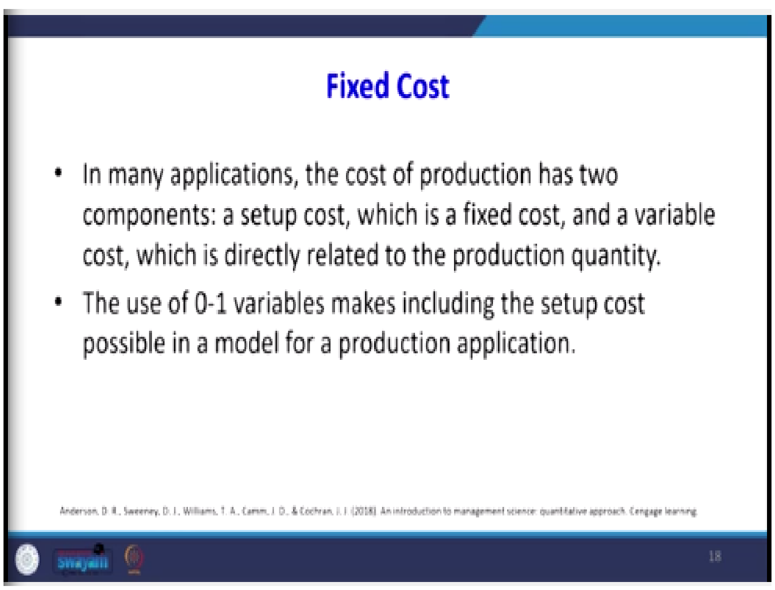


If you go for the first variable, that is, a P represents a plant expansion, the NPV is 90 in terms of 1000. Everything is in terms of 1000, 90P + 40W + 10M + 37R. The capital available in each year for year 1 is 15P + 10W + 10M + 15R less than or equal to 40; similarly, I have written for all 4 years. Now you see these variables are binary. Now, I am going to solve the formulated problem in Excel.

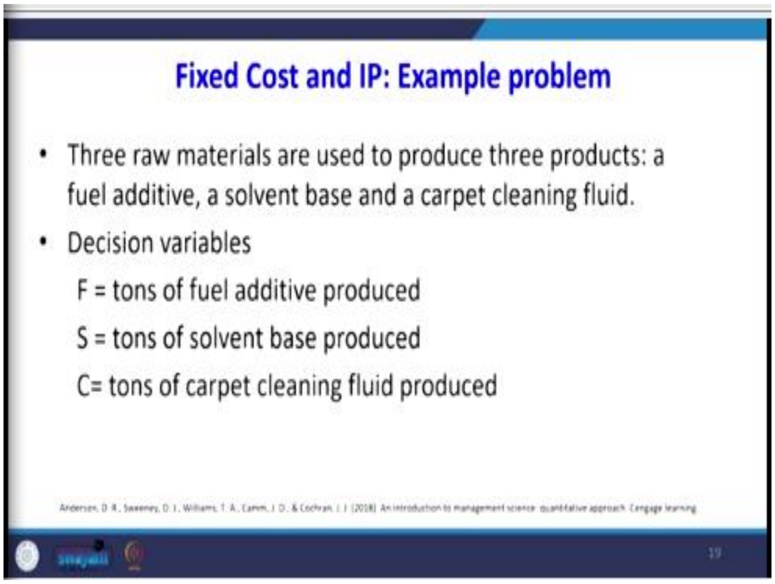
So, there are 4 projects: plant expansion, warehouse expansion, new machinery, and new product research; there are 4 constraints for each year. Now go to data and solver, so here objective function is maximizing the NPV, the changing cell is D5 to G5, here the constraint see D5 to G5 is binary, that is an additional constraint which you added. Then, the capital requirement constraint from H8 to H11 is less than equal to J8 to J11; all others are as usual.

Only the additional constraint that you have added is a binary constraint. How to add this binary constraint? So, I am selecting this binary, selecting that constraint, and now it is going to change. Look at this: There are different options. There: select that, drop-down, say there is an integer, binary; this is a is there that we will explain after some time. So, it means integer, and bin means binary, so now I am keeping it binary, so ok.

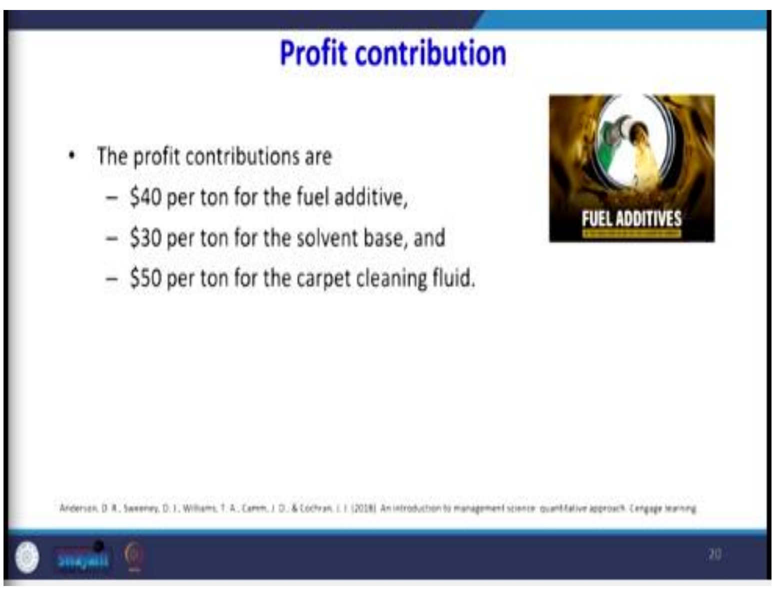
So, when I solve it, you see that one more thing you see here: since the individual programming in the only answer report is available, there is no sensitivity analysis, so press ok. Now, when I look at the result, you see I have to go for only 3 options; what are they? I have to go for plant expansion, warehouse expansion, and purchase new machinery. I should not go for new product research, so this is an example of 0-1 integer programming.



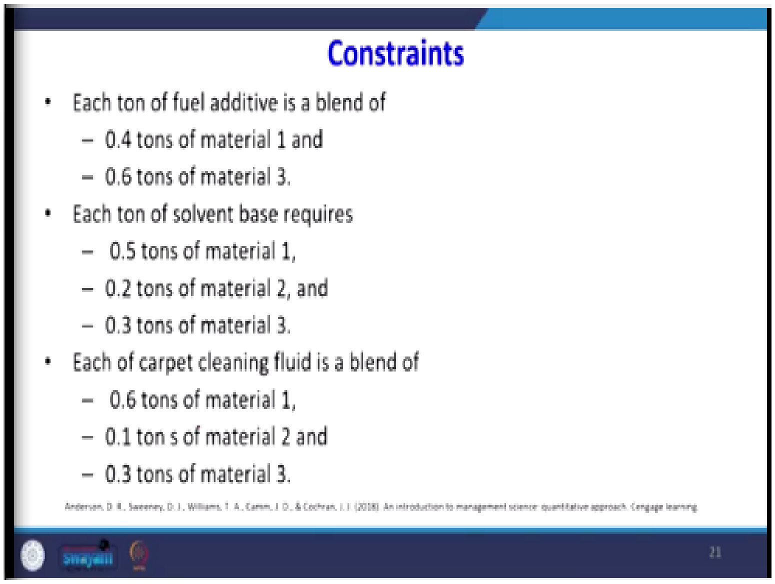
Now we are going to consider the fixed cost. In many applications, the cost of production has 2 components: one setup cost, which is a fixed cost, and a variable cost, which is directly related to production quantity. The use of 0-1 variables makes including the setup cost possible in a model for a production application. So, so far, what have we considered in LP models? We did not consider the fixed cost; we only considered the variable cost. So, because of the 0-1 individual variables, we have the flexibility of considering the fixed cost, which I will explain with the help of an example.



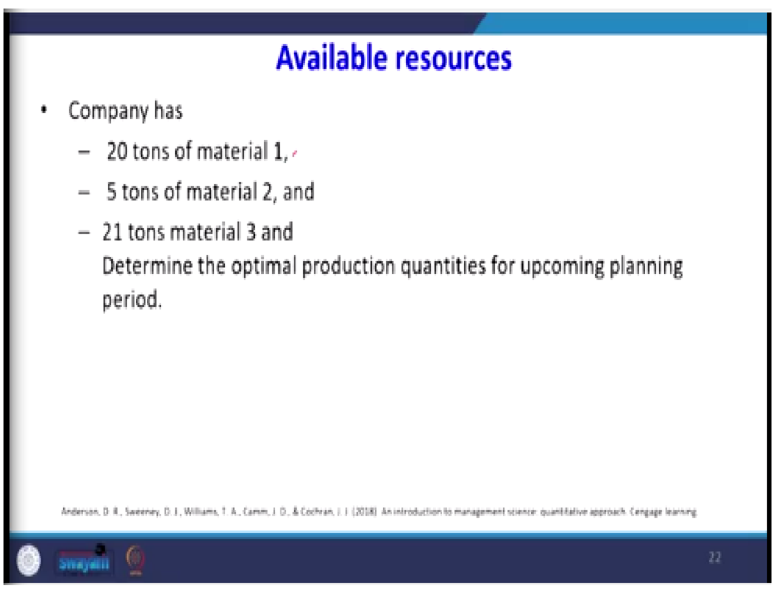
The example problem is taken from this book by Anderson et al.: Three raw materials are used to produce 3 products what are the 3 products a fuel additive, a solvent base, and a carpet cleaning fluid. So, 3 decision variables are F, which represents tons of fuel additive produced, S represents tons of solvent base produced, and C, which represents tons of carpet cleaning fluid produced.



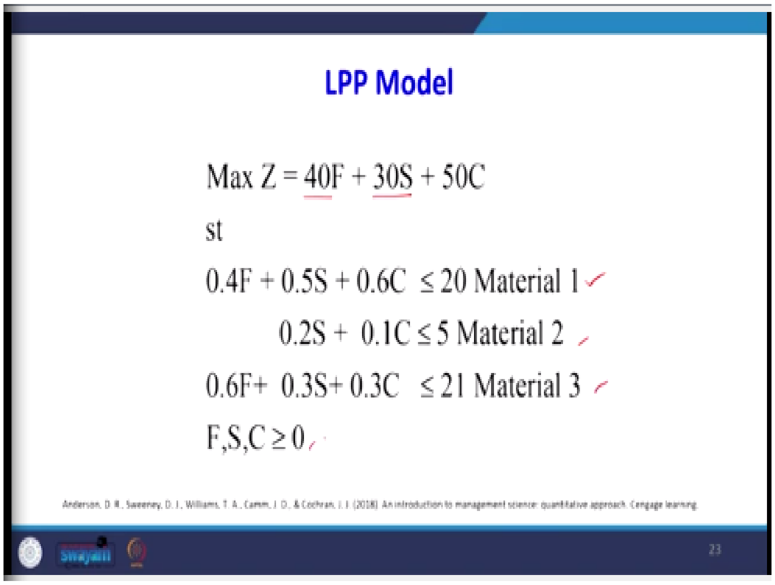
The profit contributions due to these 3 products are 40 dollars per ton of fuel additive, 30 dollars per ton of the solvent base, and 50 dollars per ton of carpet cleaning fluid. The fuel additive is added to the field to increase the efficiency of the fuel.



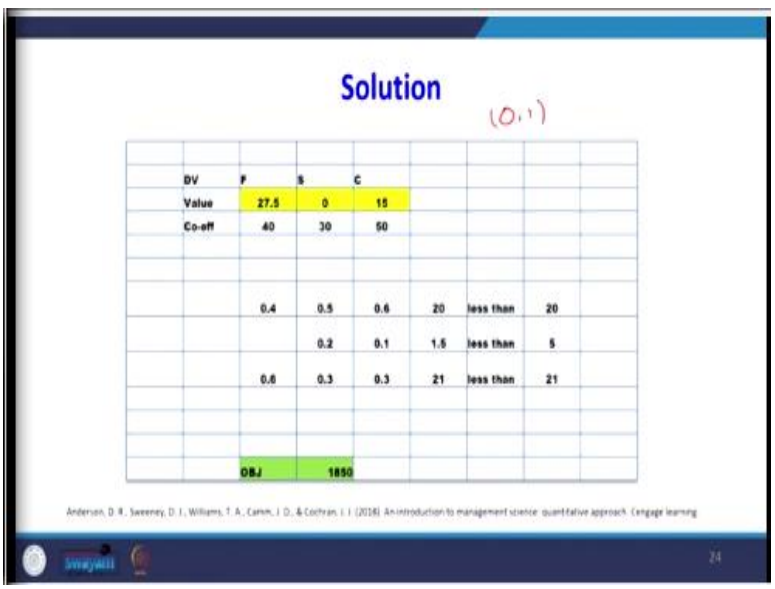
We can see about the constraint. So, each ton of fuel additive is a blend of 0.4 tons of material 1 and 0.6 tons of material 3. Similarly, the second product, called solvent base, required 0.5 tons of material 1, 0.2 tons of material 2, and 0.3 tons of material 3. So, the third product for a carpet cleaning fluid required 0.6 tons of material 1 0.1 tons of material 2, and 0.3 tons of material 3.



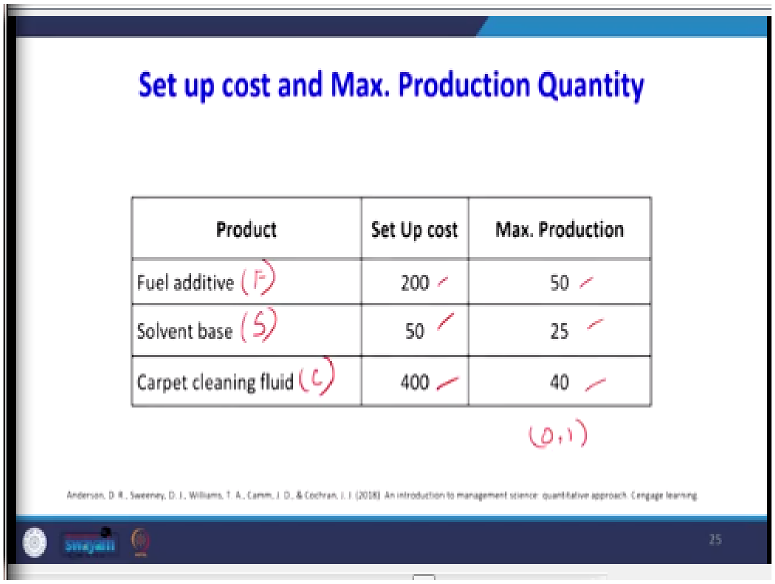
The maximum available of these 3 materials is 20 tons are available for material 1, 5 tons for material 2, and 21 tons for material 3 are available. So, the problem for a manager is that he must determine the optimal production quantities for the upcoming planning period. So, he must suggest, or he should know, there are 3 products. What should the production quantities for these 3 products be?



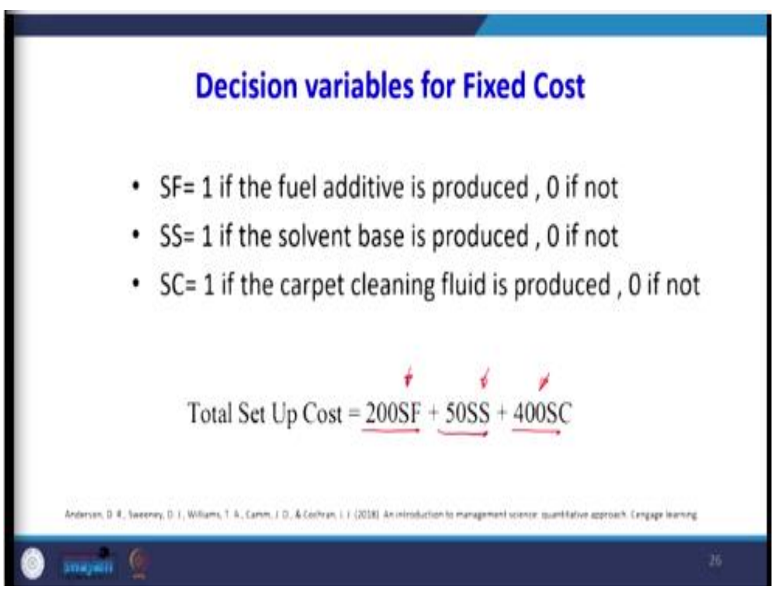
So, the objective function is maximization problem 40F, 30S, 50C, so the constraint is 0.4F + 0.5S + 0.6C, available material is 20 similarly for 0.2S + 0.1C less than or equal to 5 for material 2, 0.6F + 0.3S + 0.3C less than or equal to 21 material 3, these are non-negative.



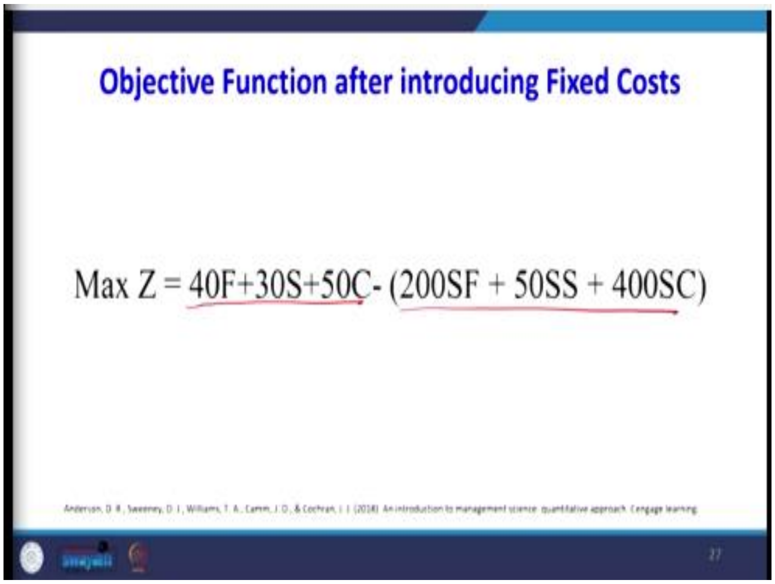
When I solve with the help of Excel, I get the following solution; what is that? The objective function is 1850, so the F is 27.5, S is 0, and C is 15. In this problem, we did not consider the fixed cost; in the next stage of this problem, we are going to consider fixed cost with the help of our 0-1 integer variables and binary variables, and then we are going to solve the same problem.



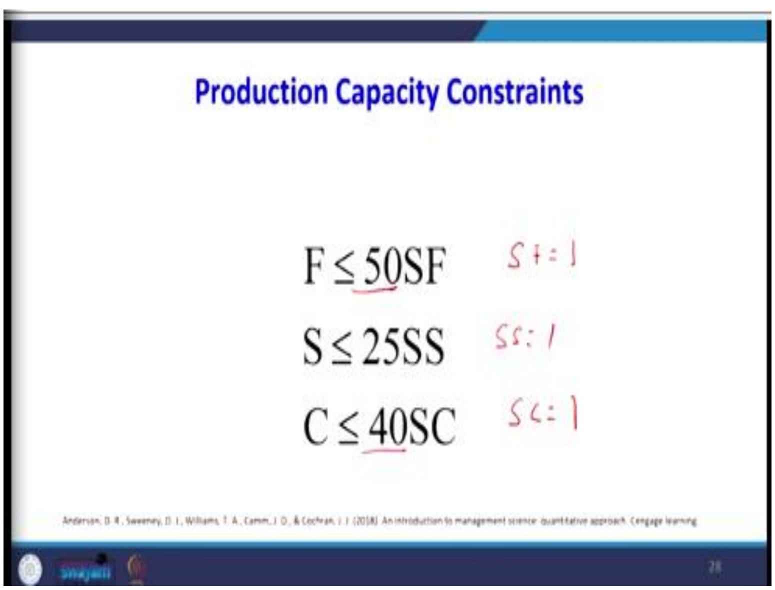
Now look at this: there is a fixed cost given, and the production capacity is also given; what is that? The fuel additive product is F, ok, setup cost, fixed cost is 200 dollars, maximum production quantity is 50. For solvent-based, the second product, the fixed cost is 50, and the maximum production quantity is 25. Carpet cleaning fluid, so the setup cost is 400 and the maximum production is 40. So, what are we going to do now? We are going to consider setting up cost and the production capacity with the help of these binary integer variables.



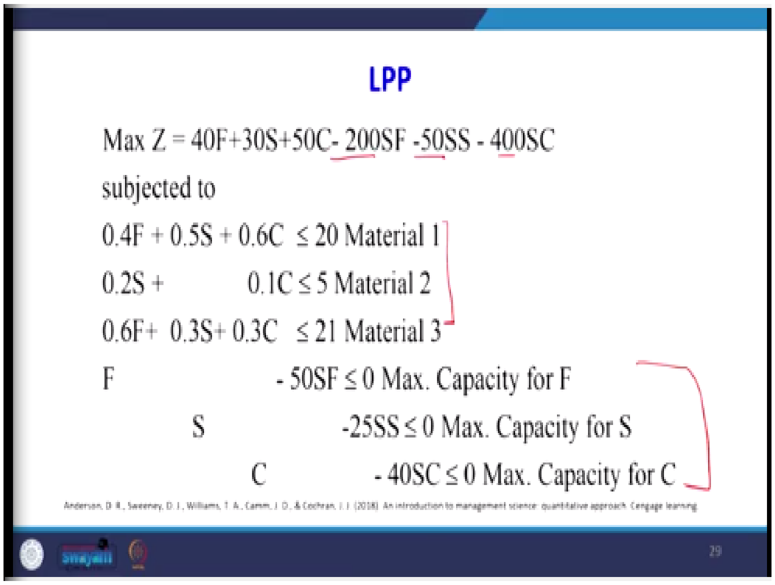
So, now I am going to introduce the binary variables, what is the SF? S means setup cost for product F = 1 if the fuel additive is produced; otherwise, 0. SS = 1 setup cost for solvent base is produced, SC = 1 if the carpet cleaning fluid is produced, otherwise 0. So, we know from the previous table the table which is there in the previous slide the setup cost is 200 for fuel additive, 50 for the solvent base, and 400 for cleaning fluid. So, if I am going for production of this product F, I have to incur the fixed cost of 200. If I am going for production of this product solvent base, I have to incur the fixed cost of 50. Similarly, if I am going to produce carpet cleaning fluid, I have to incur a fixed cost of 400.



So, now I am going to consider our objective function. Look at this, previously this component was profit, but this is cost, so I am going to subtract revenue minus the cost, which will be your profit, so it is the maximization of F. So, we must add the minus sign because the fixed cost is the cost element, the last 3 variables cost element.



The next part is that it is very interesting to introduce capacity constraint. So, F is less than 50 SF; what is the meaning of this one if the value of SF is 1, and the maximum number of products in the F category is only 50? If the value of SS = 1, the maximum production possible is only 25, which is why S is less than equal to 25SS. So, C is less than equal to 40; if I am going for SC = 1, that means if I am opening this plant, the maximum that I can produce is only 40. So, this is the best way to capture the capacity of the plant.



Now I have brought the modified linear programming problem; what are the modifications there? I have included the setup cost; these 3 were there already, but these 3 constraints are my capacity constraints; what are they? I brought in the left-hand side,

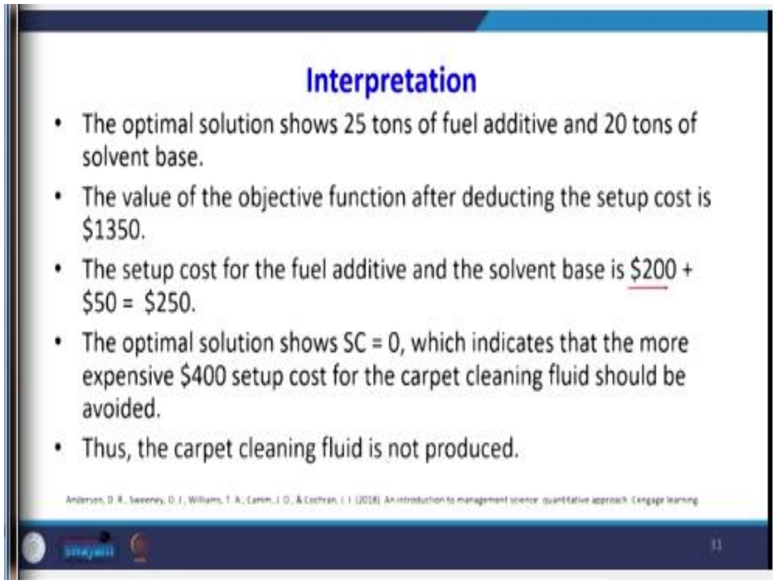
so F - 50SF <= 0,

S - 25SS <= 0.

Here, this S of SS, SC is a binary variable, and other variables like F, S, and C are non-negative variables; there are no integer constraints there, so I am going to solve with the help of a solver.

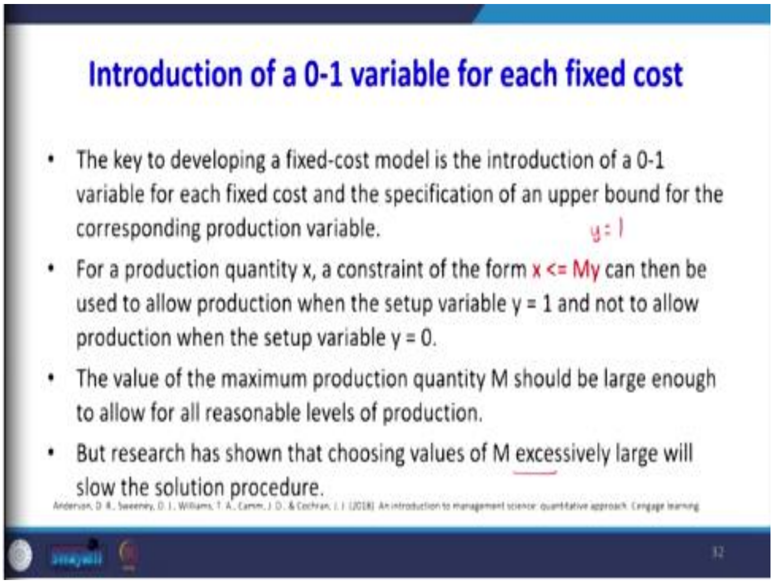
Now, we are going to solve the problem with the help of a solver. First, I have returned the decision variables, for example, D5; yes, this is my decision variable; the below that is highlighted in yellow color is the changing cells. The objective function is highlighted in green colour where the D15 is there. Similarly, below column J, I have written the resources utilized; you see that this is the resources utilized, and I am going to solve this with the help of a solver.

Go to data, then solver, and look at this constraint first. D15 is my objective function; this is the maximization problem. The changing variable cell is D5 to I5; this is an extension. Previously, I did not consider SF, SS, or SC; now, I have considered them. Note that the G5 to I5 is the binary an additional constraint is added, and then I have written all other constraints. So, yes, when I solve it, look at this F is 25 and SF is 1 S = 20, SS = 1. Here, the C value is 0; I am not producing any C, so I need not incur any fixed cost for SC. That is why it is 0; the value of the objective function is 1350.



Now, we will interpret the output of our Excel. The optimal solution shows 25 tons of fuel additive and 20 tons of solvent base. The value of the objective function after detecting the setup cost is 1350. The setup cost for the fuel additive and the solvent base is 200 + 50= 250 because we are not going to produce the carpet fluid, so there will not be any setup cost. The optimal solution shows SC = 0, which indicates that the more expensive 400-dollar setup cost for carpet cleaning fluid should be avoided. Why are we not getting SC = 1?

Because the setup cost is high, the problem is taken care not to incur more setup cost. Previously, we considered only the profit contribution, but now, in this problem, we are considering the setup cost. Also, the problem was not considered the product, which incurs a higher setup cost. That is why we did not get the value for SC. So, the carpet cleaning fluid is not produced.



One important point in the introduction of 0-1 variable for each fixed cost. The key to developing a fixed cost model is the introduction of a 0-1 variable for each fixed cost and the specification of an upper bound for the corresponding production variable. This specification of the upper bound represents your production capacity. So, for a production quantity x, a constraint in the form x less than equal to My can then be used to allow the production when the setup cost is y = 1. So, when y = 1 the maximum quantity that can be produced is M.

The value of maximum production quantity M should be large enough to allow for all reasonable levels of production. However, research has shown that choosing the value of M excessively large will slow the solution procedure. So, in this lecture, I have started a new topic called integer programming; I solved 3 types of problems: one is all integer programming. The second problem is 0-1 integer programming for a project selection problem; the third one is a 0-1 integer problem for considering fixed cost and plant capacity. In the next lecture, I will continue with the 0-1 integer problem for solving a distribution problem; thank you very much.